http://www.aimspress.com/journal/mine

Mathematics in Engineering, 3(4): 1–43.
DOI:10.3934/mine.2021036
Received: 28 July 2020
Accepted: 23 August 2020
Published: 08 September 2020

Research article

On the initial-boundary value problem for a Kuramoto-Sinelshchikov type equation

Giuseppe Maria Coclite[1,*] **and Lorenzo di Ruvo**[2]

[1] Dipartimento di Meccanica, Matematica e Management, Politecnico di Bari, via E. Orabona 4, 70125 Bari, Italy

[2] Dipartimento di Matematica, Università di Bari, via E. Orabona 4, 70125 Bari, Italy

* **Correspondence:** Email: giuseppemaria.coclite@poliba.it.

Abstract: The Kuramoto-Sinelshchikov equation describes the evolution of a phase turbulence in reaction-diffusion systems or the evolution of the plane flame propagation, taking in account the combined influence of diffusion and thermal conduction of the gas on the stability of a plane flame front. In this paper, we prove the well-posedness of the classical solutions for the initial-boundary value problem for this equation, under appropriate boundary conditions.

Keywords: existence; uniqueness; stability; Kuramoto-Sinelshchikov type equation; initial-boundary value problem

1. Introduction

In this paper, we investigate the well-posedness of the classical solutions for the equation:

$$\partial_t u + \kappa\partial_x u^2 + q\partial_x u^3 + \nu\partial_x^2 u + \delta\partial_x^3 u + \beta^2\partial_x^4 u = 0, \tag{1.1}$$

with κ, q, ν, δ, $\beta \in \mathbb{R}$ and $\beta \neq 0$.

We are interested in the initial-boundary value problem for this equation. More precisely we consider the following boundary conditions

$$\begin{cases} u(t,0) = g(t), & t > 0, \\ \partial_x u(t,0) = h(t), & t > 0, \end{cases} \qquad g,\ h \in W^{1,\infty}(0,\infty), \quad g(0) = u_0(0), \quad q = 0, \tag{1.2}$$

$$\begin{cases} u(t,0) = g(t), & t > 0, \\ \partial_x^2 u(t,0) = 0, & t > 0, \end{cases} \qquad g \in W^{1,\infty}(0,\infty), \quad g(0) = u_0(0), \quad q = 0, \tag{1.3}$$

$$\begin{cases} \partial_x u(t,0) = 0, & t > 0, \\ \partial_x^3 u(t,0) = 0, & t > 0, \end{cases} \qquad q = -a^2, \quad \delta = 0, \tag{1.4}$$

$$\begin{cases} \partial_x^2 u(t,0) = 0, & t > 0, \\ \partial_x^3 u(t,0) = 0, & t > 0, \end{cases} \qquad q = -a^2 \neq 0, \tag{1.5}$$

$$\begin{cases} u(t,0) = 0, & t > 0, \\ \partial_x u(t,0) = 0, & t > 0. \end{cases} \tag{1.6}$$

Moreover, we augment (1.1) with the following initial datum:

$$u(0,x) = u_0(x), \quad x > 0, \tag{1.7}$$

on which assume

$$u_0 \in H^2(0,\infty). \tag{1.8}$$

Assuming $q = 0$, (1.1) reads

$$\partial_t u + \kappa \partial_x u^2 + \nu \partial_x^2 u + \delta \partial_x^3 u + \beta^2 \partial_x^4 u = 0, \tag{1.9}$$

(1.9) arises in interesting physical situations, for example as a model for long waves on a viscous fluid owing down an inclined plane [50]. and to derive drift waves in a plasma [20]. (1.9) was derived also independently by Kuramoto [30–32] as a model for phase turbulence in reaction-diffusion systems and by Sivashinsky [47] as a model for plane flame propagation, describing the combined influence of diffusion and thermal conduction of the gas on the stability of a plane flame front.

Equation (1.9) also describes incipient instabilities in a variety of physical and chemical systems [7, 23, 33]. Moreover, (1.9), which is also known as the Benney-Lin equation [4, 41], was derived by Kuramoto in the study of phase turbulence in the Belousov-Zhabotinsky reaction [38].

The dynamical properties and the existence of exact solutions for (1.9) have been investigated in [21,26,28,43,44,51]. In [3,6,22], the control problem for (1.9) with periodic boundary conditions, and on a bounded interval are studied, respectively. In [8], the problem of global exponential stabilization of (1.9) with periodic boundary conditions is analyzed. In [24], it is proposed a generalization of optimal control theory for (1.9), while in [42] the problem of global boundary control of (1.9) is considered. In [45], the existence of solitonic solutions for (1.9) is proven. In [5, 18, 48], the well-posedness of the Cauchy problem for (1.9) is proven, using the energy space technique, a priori estimates together with an application of the Cauchy-Kovalevskaya and the fixed point method, respectively. In particular, in [18], the well-posedness of the Cauchy problem for (1.1) is proven. In [12], following [9, 10, 37, 46] it is proven that, when ν, δ, β^2 go to zero, the solution of (1.9) converges to the unique entropy one of the Burgers equation. Finally, the initial-boundary value problem for (1.9), under the conditions (1.2) is analyzed in [39,40], in a quarter plane and in a bounded domain, respectively, using the energy space technique, under appropriate assumptions on κ, ν, δ, β.

Taking $q = \nu = \beta = 0$ in (1.1), we have the Korteweg-de Vries equation [27]

$$\partial_t u + \kappa \partial_x u^2 + \delta \partial_x^3 u = 0, \tag{1.10}$$

that has a very wide range of applications, such as magnetic fluid waves, ion sound waves, and longitudinal astigmatic waves.

From a mathematical point of view, in [16, 19, 25], the Cauchy problem for (1.10) is studied, while in [29], the author reviewed the travelling wave solutions for (1.10). Moreover, in [14, 37, 46], the convergence of the solution of (1.10) to the unique entropy one of the Burgers equation is proven.

Taking $\kappa = r = \delta = \mu = \beta = 0$, (1.1) becomes

$$\partial_t u + q\partial_x u^3 + \delta\partial_x^3 u = 0, \tag{1.11}$$

which is known as the modified Korteweg-de Vries equation.

[1, 2, 15, 34–36] show that (1.11) is a non-slowly-varying envelope approximation model that describes the physics of few-cycle-pulse optical solitons. In [19, 25], the Cauchy problem for (1.11) is studied, while, in [13, 46], the convergence of the solution of (1.11) to the unique entropy solution of the following scalar conservation law

$$\partial_t u + q\partial_x u^3 = 0. \tag{1.12}$$

The main result of this paper is the following theorem.

Theorem 1.1. *Fix $T > 0$. The initial value problems* (1.1)-(1.2)-(1.7)*,* (1.1)-(1.3)-(1.7)*,* (1.1)-(1.4)-(1.7)*,* (1.1)-(1.5)-(1.7)*,* (1.1)-(1.6)-(1.7)*, admit an unique solution*

$$u \in H^1((0,T)\times(0,\infty)) \cap L^\infty(0,T;H^2(0,\infty)). \tag{1.13}$$

Moreover, if u_1 and u_2 are two solutions of the same initial-boundary value problem for (1.1)*, we have*

$$\|u_1(t,\cdot) - u_2(t,\cdot)\|_{L^2(0,\infty)} \le e^{C(T)t}\left\|u_{1,0} - u_{2,0}\right\|_{L^2(0,\infty)}, \tag{1.14}$$

for some suitable $C(T) > 0$, and every $\le t \le T$.

Compared to [39], Theorem 1.1 gives the well-posedness of the initial-boundary value problem (1.1)-(1.2)-(1.7), without any additional assumption on the constants. The proof of Theorem 1.1 relies on deriving suitable a priori estimates together with an application of the Cauchy-Kovalevskaya Theorem [49].

The paper is organized as follows. In Sections 2, 3, 4, 5, 6, we prove Theorem 1.1 for (1.1)-(1.2)-(1.7), (1.1)-(1.3)-(1.7), (1.1)-(1.4)-(1.7), (1.1)-(1.5)-(1.7), (1.1)-(1.6)-(1.7), respectively.

2. Proof of the Theorem 1.1 for (1.1)-(1.2)-(1.7)

In this section, we prove Theorem 1.1 for (1.1)-(1.2)-(1.7).

Let us prove some a priori estimates on u, denoting with C_0 the constants which depend only on the data, and with $C(T)$, the constants which depend also on T.

Following [11, 17], we introduce the auxiliary variable:

$$v(t,x) = u(t,x) - g(t)e^{-x} - \left[g(t) + h(t)\right]xe^{-x}. \tag{2.1}$$

Observe that

$$\begin{aligned}\partial_t v(t,x) =& \partial_t u(t,x) - g'(t)e^{-x} - \left[g'(t) + h'(t)\right]xe^{-x},\\ \partial_x v(t,x) =& \partial_x u(t,x) - h(t)e^{-x} + \left[g(t) + h(t)\right]xe^{-x},\end{aligned} \tag{2.2}$$

$$\begin{aligned}
\partial_x^2 v(t,x) =&\partial_x^2 u(t,x) + 2h(t)e^{-x} + g(t)e^{-x} - \left[g(t)+h(t)\right] xe^{-x},\\
\partial_x^3 v(t,x) =&\partial_x^3 u(t,x) - 3h(t)e^{-x} - 2g(t)e^{-x} + \left[g(t)+h(t)\right] xe^{-x},\\
\partial_x^4 v(t,x) =&\partial_x^4 u(t,x) + 4h(t)e^{-x} + 3g(t)e^{-x} - \left[g(t)+h(t)\right] xe^{-x}.
\end{aligned}$$

In particular, thanks to (1.1)-(1.2)-(1.7), (2.1) and (2.2),

$$v(t,0) = u(t,0) - g(t) = 0, \quad \partial_x v(t,0) = \partial_x u(t,0) - h(t) = 0. \tag{2.3}$$

Moreover, thanks to (1.2) and (2.2), we have that

$$\|v_0\|^2_{L^2(0,\infty)} \le \|u_0\|^2_{L^2(0,\infty)}. \tag{2.4}$$

Again by (1.1)-(1.2)-(1.7), we have the following equation for v:

$$\begin{aligned}
&\partial_t v + 2\kappa v\partial_x v + \nu\partial_x^2 v + \delta\partial_x^3 v + \beta^2\partial_x^4 v\\
&\quad = -g'(t)e^{-x} + \left[g'(t)+h'(t)\right] xe^{-x} - 2\kappa h(t)e^{-x}v + 2\kappa\left[g(t)+h(t)\right] xe^{-x}v\\
&\qquad - 2\kappa g(t)e^{-x}\partial_x v - 2\kappa g(t)h(t)e^{-2x} + 2\kappa g(t)\left[g(t)+h(t)\right] xe^{-2x}\\
&\qquad - 2\kappa\left[g(t)+h(t)\right] xe^{-x}\partial_x v - 2\kappa h(t)\left[g(t)+h(t)\right] xe^{-2x}\\
&\qquad + 2\kappa\left[g(t)+h(t)\right]^2 x^2e^{-2x} + 2h(t)e^{-x} + g(t)e^{-x} - \left[g(t)+h(t)\right] xe^{-x}\\
&\qquad - 3h(t)e^{-x} - 2g(t)e^{-x} + \left[g(t)+h(t)\right] xe^{-x} + 4h(t)e^{-x} + 3g(t)e^{-x}\\
&\qquad - \left[g(t)+h(t)\right] xe^{-x}.
\end{aligned} \tag{2.5}$$

Lemma 2.1. *Fix $T > 0$. There exists a constant $C(T) > 0$, such that*

$$\|v(t,\cdot)\|^2_{L^2(0,\infty)} + \frac{\beta^2 e^{C_0 t}}{2}\int_0^t e^{-C_0 s}\left\|\partial_x^2 v(s,\cdot)\right\|^2_{L^2(0,\infty)} ds \le C(T), \tag{2.6}$$

for every $0 \le t \le T$. In particular, we have that

$$\int_0^t \|\partial_x v(s,\cdot)\|^2_{L^2(0,\infty)} ds \le C(T), \tag{2.7}$$

for every $0 \le t \le T$.

Proof. Let $0 \le t \le T$. We begin by observing that, thanks to (2.3),

$$\begin{aligned}
4\kappa\int_0^\infty v^2\partial_x v dx =&0,\\
2\delta\int_0^\infty v\partial_x^3 v dx =&-2\delta\int_0^\infty \partial_x v\partial_x^2 v = 0,\\
2\beta\int_0^\infty v\partial_x^4 v dx =&-2\beta^2\int_0^\infty \partial_x v\partial_x^3 v dx = 2\beta^2\left\|\partial_x^2 v(t,\cdot)\right\|^2_{L^2(0,\infty)}.
\end{aligned} \tag{2.8}$$

Therefore, by (2.8), multiplying (2.5) by $2v$, an integration on $(0,\infty)$ gives

$$\frac{d}{dt}\|v(t,\cdot)\|^2_{L^2(0,\infty)} + 2\beta^2\left\|\partial_x^2 v(t,\cdot)\right\|^2_{L^2(0,\infty)}$$

$$
\begin{aligned}
&= -2\nu \int_0^\infty v\partial_x^2 v dx + 2g'(t)\int_0^\infty e^{-x} v dx + 2\left[g'(t)+h'(t)\right]\int_0^\infty x e^{-x} v dx \\
&+ 4\kappa\left[g(t)+h(t)\right]\int_0^\infty x e^{-x} v^2 dx - 4\kappa g(t)\int_{\mathbb{R}} e^{-x} v\partial_x v dx \\
&- 4\kappa\left[g(t)+h(t)\right]\int_0^\infty x e^{-x} v dx - 4\kappa\left[g(t)+h(t)\right]\int_0^\infty x e^{-x} v\partial_x v dx \\
&- 4\kappa h(t)\left[g(t)+h(t)\right]\int_0^\infty x e^{-2x} v dx + 4\kappa\left[g(t)+h(t)\right]^2\int_0^\infty x^2 e^{-2x} v dx \\
&+ 4h(t)\int_0^\infty e^{-x} v dx + 2g(t)\int_0^\infty e^{-x} v dx \\
&- 2\left[g(t)+h(t)\right]\int_0^\infty x e^{-x} v dx - 6h(t)\int_0^\infty e^{-x} v dx \\
&- 4g(t)\int_0^\infty e^{-x} v dx + 2\left[g(t)+h(t)\right]\int_0^\infty x e^{-x} v dx \\
&+ 8h(t)\int_0^\infty e^{-x} v dx + 6g(t)\int_0^\infty e^{-x} v dx - 2\left[g(t)+h(t)\right]\int_0^\infty x e^{-x} v dx.
\end{aligned} \tag{2.9}
$$

Observe that, for each $x \in (0, \infty)$,

$$
\begin{aligned}
&e^{-x} \le 1, \quad x e^{-x} \le e, \quad \int_0^\infty e^{-2x} dx = \frac{1}{2} \\
&\int_0^\infty x^2 e^{-4x} dx = \frac{1}{32}, \quad \int_0^\infty x^2 e^{-2x} dx = \frac{1}{4}, \quad \int_0^\infty x^4 e^{-4x} dx = \frac{3}{128}.
\end{aligned} \tag{2.10}
$$

Due (1.2), (2.10) and the Young inequality,

$$
\begin{aligned}
2|\nu|\int_0^\infty |v||\partial_x^2 v| dx &= 2\int_0^\infty \left|\frac{\nu v}{\beta}\right| \left|\beta\partial_x^2 v\right| dx \\
&\le \frac{\nu^2}{\beta^2}\|v(t,\cdot)\|^2_{L^2(0,\infty)} + \beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}, \\
2|g'(t)|\int_0^\infty e^{-x}|v| dx &\le 2C_0\int_0^\infty e^{-x}|v| dx \\
&\le C_0 + C_0\|v(t,\cdot)\|^2_{L^2(0,\infty)}, \\
2\left|g'(t)+h'(t)\right|\int_0^\infty x e^{-x}|v| dx &\le 2C_0\int_0^\infty x e^{-x}|v| dx \\
&\le C_0 + C_0\|v(t,\cdot)\|^2_{L^2(0,\infty)}, \\
4|\kappa|\left|g(t)+h(t)\right|\int_0^\infty x e^{-x} v^2 dx &\le C_0\int_0^\infty x e^{-x} v^2 dx \\
&\le C_0\|v(t,\cdot)\|^2_{L^2(0,\infty)}, \\
4|\kappa||g(t)|\int_0^\infty e^{-x}|v||\partial_x v| dx &\le 2C_0\int_0^\infty e^{-x}|v||\partial_x v| dx \\
&\le 2C_0\int_0^\infty |v||\partial_x v| dx \le C_0\|v(t,\cdot)\|^2_{L^2(0,\infty)} + C_0\|\partial_x v(t,\cdot)\|^2_{L^2(0,\infty)},
\end{aligned}
$$

$$
\begin{aligned}
&4|\kappa|\,|g(t)+h(t)|\int_0^\infty xe^{-x}|v|dx \le 2C_0\int_0^\infty xe^{-x}|v|dx\\
&\qquad \le C_0 + C_0\,\|v(t,\cdot)\|^2_{L^2(0,\infty)},\\
&4\kappa\,|g(t)+h(t)|\int_0^\infty xe^{-x}|v||\partial_x v|dx \le 2C_0\int_0^\infty xe^{-x}|v||\partial_x v|dx\\
&\qquad \le C_0\,\|v(t,\cdot)\|^2_{L^2(0,\infty)} + C_0\,\|\partial_x v(t,\cdot)\|^2_{L^2(0,\infty)}\\
&4|\kappa||h(t)|\,|g(t)+h(t)|\int_0^\infty xe^{-2x}|v|dx \le 2C_0\int_0^\infty xe^{-2x}|v|dx\\
&\qquad \le C_0 + C_0\,\|v(t,\cdot)\|^2_{L^2(0,\infty)},\\
&4|\kappa|\,[g(t)+h(t)]^2\int_0^\infty x^2e^{-2x}|v|dx \le 2C_0\int_0^\infty x^2e^{-2x}|v|dx\\
&\qquad \le C_0 + C_0\,\|v(t,\cdot)\|^2_{L^2(0,\infty)},\\
&4|h(t)|\int_0^\infty e^{-x}|v|dx \le 2C_0\int_0^\infty e^{-x}|v|dx\\
&\qquad \le C_0 + C_0\,\|v(t,\cdot)\|^2_{L^2(0,\infty)},\\
&2|g(t)|\int_0^\infty e^{-x}|v|dx \le 2C_0\int_0^\infty e^{-x}|v|dx\\
&\qquad \le C_0 + C_0\,\|v(t,\cdot)\|^2_{L^2(0,\infty)},\\
&2\,|g(t)+h(t)|\int_0^\infty xe^{-x}|v|dx \le 2C_0\int_0^\infty xe^{-x}|v|dx\\
&\qquad \le C_0 + C_0\,\|v(t,\cdot)\|^2_{L^2(0,\infty)},\\
&6|h(t)|\int_0^\infty e^{-x}|v|dx \le 2C_0\int_0^\infty e^{-x}|v|dx\\
&\qquad \le C_0 + C_0\,\|v(t,\cdot)\|^2_{L^2(0,\infty)},\\
&4|g(t)|\int_0^\infty e^{-x}|v|dx \le 2C_0\int_0^\infty e^{-x}|v|dx\\
&\qquad \le C_0 + C_0\,\|v(t,\cdot)\|^2_{L^2(0,\infty)},\\
&2\,|g(t)+h(t)]\,|\int_0^\infty xe^{-x}|v|dx \le 2C_0\int_0^\infty xe^{-x}|v|dx\\
&\qquad \le C_0 + C_0\,\|v(t,\cdot)\|^2_{L^2(0,\infty)},\\
&8|h(t)|\int_0^\infty e^{-x}|v|dx \le 2C_0\int_{\mathbb{R}} e^{-x}|v|dx\\
&\qquad \le C_0 + C_0\,\|v(t,\cdot)\|^2_{L^2(0,\infty)},\\
&6|g(t)|\int_0^\infty e^{-x}|v|dx \le 2C_0\int_0^\infty e^{-x}|v|dx\\
&\qquad \le C_0 + C_0\,\|v(t,\cdot)\|^2_{L^2(0,\infty)},\\
&2\,|g(t)+h(t)|\int_0^\infty xe^{-x}|v|dx \le 2C_0\int_0^\infty xe^{-x}|v|dx
\end{aligned}
$$

$$\leq C_0 + C_0 \|v(t,\cdot)\|^2_{L^2(0,\infty)}.$$

It follows from (2.9) that

$$\begin{aligned}\frac{d}{dt}\|v(t,\cdot)\|^2_{L^2(0,\infty)} + \beta^2 \left\|\partial_x^2 v(t,\cdot)\right\|^2_{L^2(0,\infty)}& \\ \leq C_0 \|v(t,\cdot)\|^2_{L^2(0,\infty)} + C_0 \|\partial_x v(t,\cdot)\|^2_{L^2(0,\infty)} + C_0.&\end{aligned} \tag{2.11}$$

Thanks to (2.3),

$$C_0 \|\partial_x v(t,\cdot)\|^2_{L^2(0,\infty)} = C_0 \int_0^\infty \partial_x v \partial_x v dx = -C_0 \int_0^\infty v \partial_x^2 v dx.$$

Therefore, by the Young inequality,

$$\begin{aligned}C_0 \|\partial_x v(t,\cdot)\|^2_{L^2(0,\infty)} \leq& \int_0^\infty \left|\frac{C_0 v}{\beta}\right| \left|\beta \partial_x^2 v\right| dx \\ \leq& C_0 \|v(t,\cdot)\|^2_{L^2(0,\infty)} + \frac{\beta^2}{2} \left\|\partial_x^2 v(t,\cdot)\right\|^2_{L^2(0,\infty)}.\end{aligned} \tag{2.12}$$

Consequently, by (2.11),

$$\frac{d}{dt}\|v(t,\cdot)\|^2_{L^2(0,\infty)} + \frac{\beta^2}{2} \left\|\partial_x^2 v(t,\cdot)\right\|^2_{L^2(0,\infty)} \leq C_0 \|v(t,\cdot)\|^2_{L^2(0,\infty)} + C_0.$$

By the Gronwall Lemma and (2.4), we have

$$\begin{aligned}\|v(t,\cdot)\|^2_{L^2(0,\infty)} + \frac{\beta^2 e^{C_0 t}}{2} \int_0^t e^{-C_0 s} \left\|\partial_x^2 v(s,\cdot)\right\|^2_{L^2(0,\infty)} ds& \\ \leq C_0 e^{C_0 t} + C_0 t \leq C(T),&\end{aligned}$$

which gives (2.6).

Finally, we prove (2.7). By (2.6), and (2.12),

$$C_0 \|\partial_x v(t,\cdot)\|^2_{L^2(0,\infty)} \leq C(T) + \frac{\beta^2}{2} \left\|\partial_x^2 v(t,\cdot)\right\|^2_{L^2(0,\infty)}.$$

Integrating on $(0,t)$, by (2.6), we have

$$C_0 \int_0^t \|\partial_x v(s,\cdot)\|^2_{L^2(0,\infty)} ds \leq C(T)t + \frac{\beta^2}{2} \int_0^t \left\|\partial_x^2 v(s,\cdot)\right\|^2_{L^2(0,\infty)} ds \leq C(T),$$

which gives (2.7). □

Lemma 2.2. *Fix $T > 0$. There exists a constant $C(T) > 0$, such that*

$$\|u(t,\cdot)\|_{L^2(0,\infty)} \leq C(T), \tag{2.13}$$

$$\int_0^t \|\partial_x u(s,\cdot)\|^2_{L^2(0,\infty)} ds \leq C(T), \tag{2.14}$$

$$\int_0^t \left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(0,\infty)} ds \leq C(T), \tag{2.15}$$

for every $0 \leq t \leq T$.

Proof. Let $0 \leq t \leq T$. We begin by proving (2.13). Observe that, by (2.1),

$$\begin{aligned} u^2(t,x) = &\left[v(t,x) + g(t)e^{-x} + \left[g(t)+h(t)\right]xe^{-x}\right] \\ =&v^2(t,x) + g^2(t)e^{-2x} + \left[g(t)+h(t)\right]x^2e^{-2x} \\ &+ 2g(t)e^{-x}v(t,x) + 2\left[g(t)+h(t)\right]xe^{-x}v + 2g(t)\left[g(t)+h(t)\right]xe^{-2x}. \end{aligned}$$

Consequently, an integration on $(0,\infty)$ gives

$$\begin{aligned} \|u(t,\cdot)\|^2_{L^2(0,\infty)} = &\|v(t,\cdot)\|^2_{L^2(0,\infty)} + g^2(t)\int_0^\infty e^{-2x}dx + \left[g(t)+h(t)\right]^2\int_0^\infty x^2e^{-2x}dx \\ &+ 2g(t)\int_0^\infty e^{-x}vdx + 2\left[g(t)+h(t)\right]\int_0^\infty xe^{-x}vdx \\ &+ 2g(t)\left[g(t)+h(t)\right]\int_0^x xe^{-2x}dx. \end{aligned} \tag{2.16}$$

Observe that

$$\int_0^\infty xe^{-2x} = \frac{1}{4}. \tag{2.17}$$

Due to (1.2), (2.10), (2.17) and the Young inequality,

$$\begin{aligned} &g^2(t)\int_0^\infty e^{-2x}dx \leq C_0\int_0^\infty e^{-2x}dx \leq C_0, \\ &\left[g(t)+h(t)\right]^2\int_0^\infty x^2e^{-2x}dx \leq C_0\int_0^\infty x^2e^{-2x}dx \leq C_0, \\ &2|g(t)|\int_0^\infty e^{-x}|v|dx \leq 2C_0\int_0^\infty e^{-x}|v|dx \leq C_0 + C_0\|v(t,\cdot)\|^2_{L^2(0,\infty)} \\ &2\left|g(t)+h(t)\right|\int_0^\infty xe^{-x}|v|dx \leq 2C_0\int_0^\infty xe^{-x}|v|dx \leq C_0 + C_0\|v(t,\cdot)\|^2_{L^2(0,\infty)} \\ &2|g(t)|\left|g(t)+h(t)\right|]\int_0^x xe^{-2x}dx \leq C_0\int_0^x xe^{-2x}dx \leq C_0. \end{aligned}$$

Therefore, by (2.6) and (2.16),

$$\|u(t,\cdot)\|^2_{L^2(0,\infty)} \leq C_0\|v(t,\cdot)\|^2_{L^2(0,\infty)} + C_0 \leq C(T),$$

that is (2.13).

We prove (2.14). By (2.5), we have that

$$\begin{aligned} (\partial_x u(t,x))^2 = &\left[\partial_x v(t,x) + h(t)e^{-x} - \left[g(t)+h(t)\right]xe^{-x}\right]^2 \\ =&(\partial_x v(t,x))^2 + h^2(t)e^{-2x} + \left[g(t)+h(t)\right]^2x^2e^{-2x} \\ &+ 2h(t)e^{-x}\partial_x v(t,x) - 2\left[g(t)+h(t)\right]xe^{-x}\partial_x v(t,x) - 2h(t)\left[g(t)+h(t)\right]xe^{-2x}. \end{aligned}$$

Therefore, an integration on $(0,\infty)$ gives

$$\|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} = \|\partial_x v(t,\cdot)\|^2_{L^2(0,\infty)} + h^2(t)\int_0^\infty e^{-2x}dx + \left[g(t)+h(t)\right]^2\int_0^\infty x^2e^{-2x}dx$$

$$+ 2h(t)\int_0^\infty e^{-x}\partial_x v dx - 2\left[g(t)+h(t)\right]\int_0^\infty xe^{-x}\partial_x v dx \tag{2.18}$$
$$- 2h(t)\left[g(t)+h(t)\right]\int_0^\infty xe^{-2x}dx.$$

Due to (1.2), (2.10), (2.17) and the Young inequality,

$$\begin{aligned}
&h^2(t)\int_0^\infty e^{-2x}dx \le C_0\int_0^\infty e^{-2x}dx \le C_0,\\
&\left[g(t)+h(t)\right]^2\int_0^\infty x^2e^{-2x}dx \le C_0\int_0^\infty x^2e^{-2x}dx \le C_0,\\
&2|h(t)|\int_0^\infty e^{-x}|\partial_x v|dx \le 2C_0\int_0^\infty e^{-x}|\partial_x v|dx \le C_0 + C_0\left\|\partial_x v(t,\cdot)\right\|^2_{L^2(0,\infty)},\\
&2\left|g(t)+h(t)\right|\int_0^\infty xe^{-x}|\partial_x v|dx \le 2C_0\int_0^\infty xe^{-x}|\partial_x v|dx \le C_0 + C_0\left\|\partial_x v(t,\cdot)\right\|^2_{L^2(0,\infty)},\\
&2|h(t)|\left|g(t)+h(t)\right|\int_0^\infty xe^{-2x}dx \le C_0\int_0^\infty xe^{-2x}dx \le C_0.
\end{aligned}$$

It follows from (2.18) that

$$\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(0,\infty)} \le C_0\left\|\partial_x v(t,\cdot)\right\|^2_{L^2(0,\infty)} + C_0.$$

Integrating on $(0,t)$, by (2.7), we have that

$$\int_0^t \left\|\partial_x u(s,\cdot)\right\|^2_{L^2(0,\infty)} ds \le C_0\int_0^t \left\|\partial_x v(s,\cdot)\right\|^2_{L^2(0,\infty)} ds + C_0 t \le C(T),$$

which gives (2.14).

Finally, we prove (2.15). By (2.2), we have that

$$\begin{aligned}
(\partial_x^2 u(t,x))^2 =&\left[\partial_x^2 v(t,x) - \left[2h(t)+g(t)\right]e^{-x} + \left[g(t)+h(t)\right]xe^{-x}\right]^2\\
=&(\partial_x^2 v(t,x))^2 + \left[2h(t)+g(t)\right]^2 e^{-2x} + \left[g(t)+h(t)\right]^2 x^2e^{-2x}\\
&- 2\left[2h(t)+g(t)\right]e^{-x}\partial_x^2 v + 2\left[g(t)+h(t)\right]xe^{-x}\partial_x^2 v\\
&- 2\left[2h(t)+g(t)\right]\left[g(t)+h(t)\right]xe^{-2x}.
\end{aligned}$$

An integration on $(0,\infty)$ gives

$$\begin{aligned}
\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} =&\left\|\partial_x^2 v(t,\cdot)\right\|^2_{L^2(0,\infty)} + \left[2h(t)+g(t)\right]^2\int_0^\infty e^{-2x}dx\\
&+ \left[g(t)+h(t)\right]^2\int_0^\infty x^2e^{-2x}dx - 2\left[2h(t)+g(t)\right]\int_0^\infty e^{-x}\partial_x^2 v dx \qquad (2.19)\\
&+ 2\left[g(t)+h(t)\right]\int_0^\infty xe^{-x}\partial_x^2 v dx\\
&- 2\left[2h(t)+g(t)\right]\left[g(t)+h(t)\right]\int_0^\infty xe^{-2x}dx.
\end{aligned}$$

Due to (1.2), (2.10), (2.17) and the Young inequality,

$$
\begin{aligned}
&[2h(t)+g(t)]^2\int_0^\infty e^{-2x}dx \le C_0\int_0^\infty e^{-2x}dx \le C_0,\\
&[g(t)+h(t)]^2\int_0^\infty x^2e^{-2x}dx \le C_0\int_0^\infty x^2e^{-2x}dx \le C_0,\\
&2\,|2h(t)+g(t)|\int_0^\infty e^{-x}|\partial_x^2 v|dx \le 2C_0\int_0^\infty e^{-x}|\partial_x^2 v|dx \le C_0 + C_0\left\|\partial_x^2 v(t,\cdot)\right\|^2_{L^2(0,\infty)},\\
&2\,|g(t)+h(t)|\int_0^\infty xe^{-x}|\partial_x^2 v|dx \le 2C_0\int_0^\infty xe^{-x}|\partial_x^2 v|dx \le C_0 + C_0\left\|\partial_x^2 v(t,\cdot)\right\|^2_{L^2(0,\infty)},\\
&2\,|2h(t)+g(t)|\,|g(t)+h(t)|\int_0^\infty xe^{-2x}dx \le C_0\int_0^\infty xe^{-2x}dx \le C_0.
\end{aligned}
$$

It follows from (2.19) that

$$
\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} \le \left\|\partial_x^2 v(t,\cdot)\right\|^2_{L^2(0,\infty)} + C_0.
$$

Integrating on $(0,t)$, by (2.6), we have that

$$
\int_0^t \left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(0,\infty)} ds \le \int_0^t \left\|\partial_x^2 v(s,\cdot)\right\|^2_{L^2(0,\infty)} ds + C_0 t \le C(T),
$$

which gives (2.15). □

Lemma 2.3. *Fix $T>0$. There exists a constant $C(T)>0$, such that*

$$
\int_0^t \|\partial_x u(s,\cdot)\|^2_{L^\infty(0,\infty)} ds \le C(T), \tag{2.20}
$$

$$
\int_0^t \|u(s,\cdot)\partial_x u(s,\cdot)\|^2_{L^2(0,\infty)} ds \le C(T), \tag{2.21}
$$

for every $0\le t\le T$.

Proof. Let $0\le t\le T$. We begin by proving (2.20). Thanks to (1.2) and the Young inequality,

$$
\begin{aligned}
(\partial_x u(t,x))^2 =& 2\int_0^x \partial_x u\partial_x^2 u dy + h^2(t) \le 2\int_0^\infty |\partial_x u||\partial_x^2 u|dx + C_0\\
\le& \|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} + \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + C_0.
\end{aligned}
$$

Hence,

$$
\|\partial_x u(t,\cdot)\|^2_{L^\infty(0,\infty)} \le \|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} + \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + C_0 \tag{2.22}
$$

Integrating on $(0,t)$, by (2.14) and (2.15), we have that

$$
\begin{aligned}
\int_0^t \|\partial_x u(s,\cdot)\|^2_{L^\infty(0,\infty)} ds \le& \int_0^t \|\partial_x u(s,\cdot)\|^2_{L^2(0,\infty)} ds\\
&+ \int_0^t \left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(0,\infty)} ds + C_0 t \le C(T),
\end{aligned}
$$

which gives (2.20).

Finally, we prove (2.21). Thanks to (2.13), we have that

$$\begin{aligned}\|u(t,\cdot)\partial_x u(s,\cdot)\|^2_{L^2(0,\infty)}\,ds &= \int_0^\infty u^2(\partial_x u)^2 dx\\ &\le \|\partial_x u(t,\cdot)\|^2_{L^\infty(0,\infty)}\|u(t,\cdot)\|^2_{L^2(0,\infty)}\\ &\le C(T)\|\partial_x u(t,\cdot)\|^2_{L^\infty(0,\infty)}.\end{aligned}$$

(2.21) follows from (2.20) and an integration on $(0,t)$. □

Lemma 2.4. *Fix $T>0$. There exists a constant $C(T)>0$, such that*

$$\begin{aligned}&\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \frac{\delta^2+24}{\beta^4}\|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)}\\ &+\frac{\beta^2}{2}\int_0^t \left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(0,\infty)}ds + \frac{\delta^2+24}{6\beta^2}\int_0^t\left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(0,\infty)}ds \le C(T),\end{aligned} \tag{2.23}$$

$$\int_0^t (\partial_x^2 u(s,0))^2 ds \le C(T), \quad \int_0^t (\partial_x^3 u(s,0))^2 ds \le C(T), \tag{2.24}$$

$$\|\partial_x u\|_{L^\infty((0,T)\times(0,\infty))} \le C(T), \tag{2.25}$$

$$\|u\|_{L^\infty((0,T)\times(0,\infty))} \le C(T), \tag{2.26}$$

for every $0\le t\le T$.

Proof. Let $0\le t\le T$. Consider an positive constant A, which will be specified later. Multiplying (1.1)-(1.2)-(1.7) by

$$2\partial_x^4 u - 2A\partial_x^2 u,$$

we have that

$$\begin{aligned}&\left(2\partial_x^4 u - 2A\partial_x^2 u\right)\partial_t u + 2\kappa\left(2\partial_x^4 u - 2A\partial_x^2 u\right)u\partial_x u\\ &\quad+\nu\left(2\partial_x^4 u - 2A\partial_x^2 u\right)\partial_x^2 u + \delta\left(2\partial_x^4 u - 2A\partial_x^2 u\right)\partial_x^3 u\\ &\quad+\beta^2\left(2\partial_x^4 u - 2A\partial_x^2 u\right)\partial_x^4 u = 0.\end{aligned} \tag{2.27}$$

Observe that, thanks to (1.1)-(1.2)-(1.7),

$$\begin{aligned}&\int_0^\infty \left(2\partial_x^4 u - 2A\partial_x^2 u\right)\partial_t u dx\\ &\quad= -2\partial_x^3 u(t,0)\partial_t u(t,0) - 2\int_0^\infty \partial_x^3 u\partial_t\partial_x u dx + 2A\partial_x u(t,0)\partial_t u(t,0)\\ &\quad+A\frac{d}{dt}\|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)}\\ &\quad= -2g'(t)\partial_x^3 u(t,0) + 2h'(t)\partial_x^2 u(t,0) + 2Ah(t)g(t)\\ &\quad+\frac{d}{dt}\left(\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + A\|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)}\right),\end{aligned} \tag{2.28}$$

$$- 2A\beta^2 \int_0^\infty \partial_x^2 u \partial_x^4 u dx = 2A\beta^2 \partial_x^2 u(t,0)\partial_x^3 u(t,0) + 2A\beta^2 \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.$$

Therefore, thanks to (2.28), an integration of (2.27) on $(0, \infty)$ gives

$$\begin{aligned}
&\frac{d}{dt}\left(\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + A\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(0,\infty)}\right)\\
&\quad + 2\beta^2 \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + 2A\beta^2 \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&= -4\kappa \int_0^\infty u\partial_x u \partial_x^4 u dx + 4A\kappa \int_0^\infty u\partial_x u\partial_x^2 u dx - 2\nu \int_0^\infty \partial_x^4 u \partial_x^2 u dx\\
&\quad - 2A\nu \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} - 2\delta \int_0^\infty \partial_x^4 u \partial_x^3 u dx + 2A\delta \int_0^\infty \partial_x^2 u \partial_x^3 u dx\\
&\quad + 2g'(t)\partial_x^3 u(t,0) - 2h'(t)\partial_x^2 u(t,0) - 2Ah(t)g(t) - 2A\beta^2 \partial_x^2 u(t,0)\partial_x^3 u(t,0).
\end{aligned} \tag{2.29}$$

Due to the Young inequality,

$$\begin{aligned}
4|\kappa| \int_0^\infty |u\partial_x u||\partial_x^4 u| dx &= 2\int_0^\infty \left|\frac{2\kappa u\partial_x u}{\beta\sqrt{D_1}}\right| \left|\beta\sqrt{D_1}\partial_x^4 u\right| dx\\
&\le \frac{4\kappa^2}{\beta^2 D_1}\left\|u(t,\cdot)\partial_x u(t,\cdot)\right\|^2_{L^2(0,\infty)} + D_1\beta^2 \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)},\\
4A|\kappa| \int_0^\infty |u\partial_x u||\partial_x^2 u| dx &= 2\int_0^\infty |2A\kappa u\partial_x u| \left|\partial_x^2 u\right| dx\\
&\le 4A^2\kappa^2 \left\|u(t,\cdot)\partial_x u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)},\\
2|\nu| \int_0^\infty |\partial_x^4 u||\partial_x^2 u| dx &= 2\int_0^\infty \left|\beta\sqrt{D_1}\partial_x^4 u\right| \left|\frac{\nu\partial_x^2 u}{\beta\sqrt{D_1}}\right| dx\\
&\le D_1\beta^2 \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \frac{\nu^2}{\beta^2 D_1}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)},\\
2|\delta| \int_0^\infty |\partial_x^4 u||\partial_x^3 u| dx &= 2\int_0^\infty \left|\beta\sqrt{D_1}\partial_x^4 u\right| \left|\frac{\delta\partial_x^3 u}{\beta\sqrt{D_1}}\right| dx\\
&\le D_1\beta^2 \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \frac{\delta^2}{\beta^2 D_1}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)},\\
2A|\delta| \int_0^\infty |\partial_x^2 u||\partial_x^3 u| dx &= A\int_0^\infty \left|\frac{\delta\partial_x^2 u}{\beta}\right| \left|\beta\partial_x^3 u\right| dx\\
&\le \frac{A\delta^2}{\beta^2}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + A\beta^2 \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)},
\end{aligned}$$

where D_1 is a positive constant, which will be specified later. It follows from (2.29) that

$$\begin{aligned}
&\frac{d}{dt}\left(\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + A\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(0,\infty)}\right)\\
&\qquad + \beta^2 (2 - 3D_1)\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \left(A\beta^2 - \frac{\delta^2}{\beta^2 D_1}\right)\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)}
\end{aligned}$$

$$
\begin{aligned}
&\leq 4\kappa^2\left(\frac{1}{\beta^2 D_1}+A^2\right)\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)}\\
&+\left(1+\frac{\nu^2}{\beta^2 D_1}+2A|\nu|+\frac{A\delta^2}{\beta^2}\right)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&+2|g'(t)||\partial_x^3 u(t,0)|+2|h'(t)||\partial_x^2 u(t,0)|+2A|h(t)||g(t)|+2A\beta^2|\partial_x^2 u(t,0)||\partial_x^3 u(t,0)|.
\end{aligned}
$$

Choosing $D_1 = 3$, we have that

$$
\begin{aligned}
&\frac{d}{dt}\left(\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}+A\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(0,\infty)}\right)\\
&\qquad+\beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)}+\left(A\beta^2-\frac{\delta^2}{3\beta^2}\right)\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\leq 4\kappa^2\left(\frac{1}{3\beta^2}+A^2\right)\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)}\\
&\qquad+\left(1+\frac{\nu^2}{3\beta^2}+2A|\nu|+\frac{A\delta^2}{\beta^2}\right)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\qquad+2|g'(t)||\partial_x^3 u(t,0)|+2|h'(t)||\partial_x^2 u(t,0)|\\
&\qquad+2A|h(t)||g(t)|+2A\beta^2|\partial_x^2 u(t,0)||\partial_x^3 u(t,0)|.
\end{aligned}
\tag{2.30}
$$

Due to (1.2) and the Young inequality,

$$
\begin{aligned}
&2|g'(t)||\partial_x^3 u(t,0)|\leq (g'(t))^2+(\partial_x^3 u(t,0))^2\leq C_0+(\partial_x^3 u(t,0))^2,\\
&2|h'(t)||\partial_x^2 u(t,0)|\leq (h'(t))^2+(\partial_x^2 u(t,0))^2\leq C_0+(\partial_x^2 u(t,0))^2,\\
&2A|h(t)||g(t)|\leq AC_0,\\
&2A\beta^2|\partial_x^2 u(t,0)||\partial_x^3 u(t,0)|\leq A^2\beta^4(\partial_x^2 u(t,0))^2+(\partial_x^3 u(t,0))^2.
\end{aligned}
$$

Consequently, by (2.30),

$$
\begin{aligned}
&\frac{d}{dt}\left(\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}+A\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(0,\infty)}\right)\\
&\qquad+\beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)}+\left(A\beta^2-\frac{\delta^2}{3\beta^2}\right)\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\leq 4\kappa^2\left(\frac{1}{3\beta^2}+A^2\right)\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)}\\
&\qquad+\left(1+\frac{\nu^2}{3\beta^2}+2A|\nu|+\frac{A\delta^2}{\beta^2}\right)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\qquad+2(\partial_x^3 u(t,0))^2+\left(1+A^2\beta^4\right)(\partial_x^2 u(t,0))^2+C_0(1+A).
\end{aligned}
\tag{2.31}
$$

Thanks to the Young inequality,

$$
\begin{aligned}
2(\partial_x^3 u(t,0))^2 &= -4\int_0^\infty \partial_x^3 u\partial_x^4 u dx \leq \int_0^\infty \left|\frac{4\partial_x^3 u}{\beta}\right|\left|\beta\partial_x^4 u\right| dx\\
&\leq \frac{8}{\beta^2}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)}+\frac{\beta^2}{2}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)},
\end{aligned}
\tag{2.32}
$$

$$\begin{aligned}\left(1+A^2\beta^4\right)(\partial_x^2u(t,0))^2=&-2\left(1+A^2\beta^4\right)\int_0^\infty \partial_x^2u\partial_x^3udx\\ \le&\int_0^\infty\left|\frac{2\left(1+A^2\beta^4\right)\partial_x^2u}{\beta\sqrt{A}}\right|\left|\beta\sqrt{A}\partial_x^3u\right|dx\\ \le&2\left(1+A^2\beta^4\right)^2\left\|\partial_x^2u(t,\cdot)\right\|^2_{L^2(0,\infty)}+\frac{A\beta^2}{2}\left\|\partial_x^3u(t,\cdot)\right\|^2_{L^2(0,\infty)}.\end{aligned}$$

It follows from (2.31) that

$$\begin{aligned}&\frac{d}{dt}\left(\left\|\partial_x^2u(t,\cdot)\right\|^2_{L^2(0,\infty)}+A\left\|\partial_xu(t,\cdot)\right\|^2_{L^2(0,\infty)}\right)\\ &\quad+\frac{\beta^2}{2}\left\|\partial_x^4u(t,\cdot)\right\|^2_{L^2(0,\infty)}+\left(\frac{A\beta^2}{2}-\frac{\delta^2+24}{3\beta^2}\right)\left\|\partial_x^3u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\ &\le 4\kappa^2\left(\frac{1}{3\beta^2}+A^2\right)\left\|u(t,\cdot)\partial_xu(t,\cdot)\right\|^2_{L^2(0,\infty)}\\ &\quad+\left(1+\frac{\nu^2}{3\beta^2}+2A|\nu|+\frac{A\delta^2}{\beta^2}+2\left(1+A^2\beta^4\right)^2\right)\left\|\partial_x^2u(t,\cdot)\right\|^2_{L^2(0,\infty)}+C_0(1+A).\end{aligned}$$

Choosing

$$A=\frac{\delta^2+24}{\beta^4},\tag{2.33}$$

we have that

$$\begin{aligned}&\frac{d}{dt}\left(\left\|\partial_x^2u(t,\cdot)\right\|^2_{L^2(0,\infty)}+\frac{\delta^2+24}{\beta^4}\left\|\partial_xu(t,\cdot)\right\|^2_{L^2(0,\infty)}\right)\\ &\quad+\frac{\beta^2}{2}\left\|\partial_x^4u(t,\cdot)\right\|^2_{L^2(0,\infty)}+\frac{\delta^2+24}{6\beta^2}\left\|\partial_x^3u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\ &\le C_0\left\|u(t,\cdot)\partial_xu(t,\cdot)\right\|^2_{L^2(0,\infty)}+C_0\left\|\partial_x^2u(t,\cdot)\right\|^2_{L^2(0,\infty)}+C_0.\end{aligned}$$

Integrating on $(0,t)$, by (1.8), (2.15) and (2.21), we get

$$\begin{aligned}&\left\|\partial_x^2u(t,\cdot)\right\|^2_{L^2(0,\infty)}+\frac{\delta^2+24}{\beta^4}\left\|\partial_xu(t,\cdot)\right\|^2_{L^2(0,\infty)}\\ &\quad+\frac{\beta^2}{2}\int_0^t\left\|\partial_x^4u(s,\cdot)\right\|^2_{L^2(0,\infty)}ds+\frac{\delta^2+24}{6\beta^2}\int_0\left\|\partial_x^3u(s,\cdot)\right\|^2_{L^2(0,\infty)}ds\\ &\le C_0+C_0\int_0^t\left\|u(s,\cdot)\partial_xu(s,\cdot)\right\|^2_{L^2(0,\infty)}ds+C_0\int_0^t\left\|\partial_x^2u(s,\cdot)\right\|^2_{L^2(0,\infty)}ds+C_0t\\ &\le C(T),\end{aligned}$$

which gives (2.23).

(2.24) follows from (2.15), (2.23), (2.33) and an integration of (2.32) on $(0,t)$, while (2.22) and (2.23) give (2.25).

Finally, we prove (2.26). Due to (1.2), (2.13), (2.23) and the Hölder inequality,

$$u^2(t,x)=2\int_0^x u\partial_xudy+g^2(t)\le 2\int_0^\infty|u||\partial_xu|dx+C_0$$

$$\leq \|u(t,\cdot)\|_{L^2(0,\infty)} \|\partial_x u(t,\cdot)\|_{L^2(0,\infty)} + C_0 \leq C(T).$$

Hence,

$$\|u\|^2_{L^\infty((0,T)\times(0,\infty))} \leq C(T),$$

which gives (2.26). □

Lemma 2.5. *Fix $T > 0$. There exists a constant $C(T) > 0$, such that*

$$\int_0^t \|\partial_t u(s,\cdot)\|^2_{L^2(0,\infty)} ds \leq C(T), \tag{2.34}$$

for every $0 \leq t \leq T$.

Proof. Let $0 \leq t \leq T$. Multiplying (1.1)-(1.2)-(1.7) by $2\partial_t u$, an integration on $(0,\infty)$ gives

$$\begin{aligned} 2\|\partial_t u(t,\cdot)\|^2_{L^2(0,\infty)} = &-4\kappa \int_0^\infty u\partial_x u\partial_t u dx - 2\nu \int_0^\infty \partial_x^2 u\partial_t u dx \\ &- 2\delta \int_0^\infty \partial_x^3 u\partial_t u dx - 2\beta^2 \int_0^\infty \partial_x^4 u\partial_t u dx. \end{aligned} \tag{2.35}$$

Due to (2.23) and the Young inequality,

$$\begin{aligned} 4|\kappa| \int_0^\infty u\partial_x u\partial_t u dx &= 2\int_0^\infty \left|\frac{2\kappa u\partial_x u}{\sqrt{D_2}}\right| \left|\sqrt{D_2}\partial_t u\right| dx \\ &\leq \frac{4\kappa^2}{D_2} \|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} + D_2 \|\partial_t u(t,\cdot)\|^2_{L^2(0,\infty)}, \\ 2|\nu| \int_0^\infty |\partial_x^2 u||\partial_t u| dx &= 2\int_0^\infty \left|\frac{\nu\partial_x^2 u}{\sqrt{D_2}}\right| \left|\sqrt{D_2}\partial_t u\right| dx \\ &\leq \frac{\nu^2}{D_2} \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + D_2 \|\partial_t u(t,\cdot)\|^2_{L^2(0,\infty)} \\ &\leq \frac{C(T)}{D_2} + D_2 \|\partial_t u(t,\cdot)\|^2_{L^2(0,\infty)}, \\ 2|\delta| \int_0^\infty |\partial_x^3 u||\partial_t u| dx &= 2\int_0^\infty \left|\frac{\delta\partial_x^3 u}{\sqrt{D_2}}\right| \left|\sqrt{D_2}\partial_t u\right| dx \\ &\leq \frac{\delta^2}{D_2} \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + D_2 \|\partial_t u(t,\cdot)\|^2_{L^2(0,\infty)}, \\ 2\beta^2 \int_0^\infty |\partial_x^4 u||\partial_t u| dx &= 2\int_0^\infty \left|\frac{\beta^2\partial_x^4 u}{\sqrt{D_2}}\right| \left|\sqrt{D_2}\partial_t u\right| dx \\ &\leq \frac{\beta^4}{D_2} \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + D_2 \|\partial_t u(t,\cdot)\|^2_{L^2(0,\infty)}. \end{aligned}$$

Consequently, by (2.35),

$$2(1-2D_2)\|\partial_t u(t,\cdot)\|^2_{L^2(0,\infty)} \leq \frac{C(T)}{D_2} + \frac{4\kappa^2}{D_2} \|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)}$$

$$+\frac{\delta^2}{D_2}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)}+\frac{\beta^4}{D_2}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.$$

Choosing $D_2=\frac{1}{4}$, we have that

$$\begin{aligned}\|\partial_t u(t,\cdot)\|^2_{L^2(0,\infty)}\le& C(T)+16\kappa^2\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)}\\&+4\delta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)}+4\beta^4\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.\end{aligned}$$

Integrating on $(0,t)$, by (2.21) and (2.23), we get

$$\begin{aligned}\int_0^t\|\partial_t u(s,\cdot)\|^2_{L^2(0,\infty)}ds\le& C(T)t+16\kappa^2\int_0^t\|u(s,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)}ds\\&+4\delta^2\int_0^t\left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(0,\infty)}ds+4\beta^4\int_0^t\left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(0,\infty)}ds\\\le& C(T),\end{aligned}$$

which gives (2.34). □

Now, we prove Theorem 1.1.

Proof. Fix $T>0$. Thanks to Lemmas 2.2, 2.4, 2.5 and the Cauchy-Kovalevskaya Theorem [49], we have that u is solution of (1.1)-(1.2)-(1.7) and (1.13) holds.

We prove (1.14). Let u_1 and u_2 be two solutions of (1.1)-(1.2)-(1.7), which verify (1.13), that is

$$\begin{cases}\partial_t u_1+2\kappa u_1\partial_x u_1+\nu\partial_x^2 u_1+\delta\partial_x^3 u_1+\beta^2\partial_x^4 u_1=0, & t>0,\quad x>0,\\ u_1(t,0)=g(t), & t>0,\\ \partial_x u_1(t,0)=h(t), & t>0,\\ u_1(0,x)=u_{1,0}(x), & x>0,\end{cases}$$

$$\begin{cases}\partial_t u_2+2\kappa u_2\partial_x u_2+\nu\partial_x^2 u_2+\delta\partial_x^3 u_2+\beta^2\partial_x^4 u_2=0, & t>0,\quad x>0,\\ u_2(t,0)=g(t), & t>0,\\ \partial_x u_2(t,0)=h(t), & t>0,\\ u_2(0,x)=u_{2,0}(x), & x>0.\end{cases}$$

Then, the function

$$\omega=u_1-u_2 \tag{2.36}$$

solves the following initial-boundary value problem:

$$\begin{cases}\partial_t\omega+2\kappa\left(u_1\partial_x u_1-u_2\partial_x u_2\right)+\nu\partial_x^2\omega+\delta\partial_x^3\omega+\beta^2\partial_x^4\omega=0, & t>0,\quad x>0,\\ \omega(t,0)=0, & t>0,\\ \partial_x\omega(t,0)=0, & t>0,\\ \omega(0,x)=u_{1,0}(x)-u_{2,0}(x), & x>0.\end{cases} \tag{2.37}$$

Observe that, thanks to (2.36),

$$u_1\partial_x u_1-u_2\partial_x u_2=u_1\partial_x u_1-u_1\partial_x u_2+u_1\partial_x u_2-u_2\partial_x u_2=u_1\partial_x\omega+\partial_x u_2\omega.$$

Therefore, (2.37) is equivalent to the following equation:

$$\partial_t\omega + 2\kappa u_1\partial_x\omega + 2\kappa\partial_x u_2\omega + \nu\partial_x^2\omega + \delta\partial_x^3\omega + \beta^2\partial_x^4\omega = 0. \tag{2.38}$$

Moreover, since $u_1,\ u_2 \in L^\infty(0,T;H^2(0,\infty))$, we have that

$$\|\partial_x u_1\|_{L^\infty((0,T)\times(0,\infty))},\ \|\partial_x u_2\|_{L^\infty((0,T)\times(0,\infty))} \le C(T). \tag{2.39}$$

Observe again that, thanks to (2.37),

$$\begin{aligned} 4\kappa\int_0^\infty u_1\omega\partial_x\omega dx =& -2\kappa\int_0^\infty \partial_x u_1\omega^2 dx,\\ 2\delta\int_0^\infty \omega\partial_x^3\omega dx =& -2\delta\int_0^\infty \partial_x\omega\partial_x^2\omega dx = 0,\\ 2\beta^2\int_0^\infty \omega\partial_x^4\omega dx =& -2\beta^2\int_0^\infty \partial_x\omega\partial_x^3\omega dx = 2\beta^2\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(0,\infty)}. \end{aligned} \tag{2.40}$$

Therefore, multiplying (2.38) by 2ω, thanks to (2.40), an integration on $(0,\infty)$ gives

$$\begin{aligned} &\frac{d}{dt}\|\partial_x\omega(t,\cdot)\|^2_{L^2(0,\infty)} + 2\beta^2\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(0,\infty)}\\ &\qquad =2\kappa\int_0^\infty(\partial_x u_1 - 2\partial_x u_2)\,\omega^2 dx - 2\nu\int_0^\infty \omega\partial_x^2\omega dx. \end{aligned} \tag{2.41}$$

Due to (2.39) and the Young inequality,

$$\begin{aligned} 2|\kappa|\int_0^\infty |\partial_x u_1 - 2\partial_x u_2|\,\omega^2 dx &\le C(T)\|\omega(t,\cdot)\|^2_{L^2(0,\infty)},\\ 2|\nu|\int_0^\infty |\omega||\partial_x^2\omega| dx &= 2\int_0^\infty\left|\frac{\nu\omega}{\beta}\right|\left|\beta\partial_x^2\omega\right| dx\\ &\le \frac{\nu^2}{\beta^2}\|\omega(t,\cdot)\|^2_{L^2(0,\infty)} + \beta^2\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(0,\infty)}. \end{aligned}$$

Therefore, by (2.41),

$$\frac{d}{dt}\|\partial_x\omega(t,\cdot)\|^2_{L^2(0,\infty)} + \beta^2\left\|\partial_x^2\omega(t,\cdot)\right\|^2_{L^2(0,\infty)} \le C(T)\|\omega(t,\cdot)\|^2_{L^2(0,\infty)}.$$

The Gronwall Lemma and (2.37) gives

$$\begin{aligned} \|\partial_x\omega(t,\cdot)\|^2_{L^2(0,\infty)} &+\beta^2 e^{C(T)t}\int_0^t e^{-C(T)s}\left\|\partial_x^2\omega(s,\cdot)\right\|^2_{L^2(0,\infty)} ds\\ &\le e^{C(T)t}\|\omega_0\|^2_{L^2(0,\infty)}. \end{aligned} \tag{2.42}$$

(1.14) follows from (2.36) and (2.42). □

3. Proof of the Theorem 1.1 for (1.1)-(1.3)-(1.7)

In this section, we prove Theorem 1.1 for (1.1)-(1.3)-(1.7).

Let us prove some a priori estimates on u.

Following [11, 17], we consider the following function:

$$v(t,x) = u(t,x) - g(t)e^{-x}. \tag{3.1}$$

Observe that

$$\begin{aligned}
\partial_t v(t,x) =& \partial_t u(t,x) - g'(t)e^{-x},\\
\partial_x v(t,x) =& \partial_x u(t,x) + g(t)e^{-x},\\
\partial_x^2 v(t,x) =& \partial_x^2 u(t,x) - g(t)e^{-x},\\
\partial_x^3 v(t,x) =& \partial_x^3 u(t,x) + g(t)e^{-x},\\
\partial_x^4 v(t,x) =& \partial_x^4 u - g(t)e^{-x}.
\end{aligned} \tag{3.2}$$

By (1.1)-(1.3)-(1.7) and (3.2),

$$v(t,0) = u(t,0) - g(t) = 0, \quad \partial_x^2 v(t,0) = -g(t), \tag{3.3}$$

while, by (1.3) and (3.2), we have (2.4).

Again by (1.1)-(1.3)-(1.7) and (3.2), we have the following equation for v.

$$\begin{aligned}
&\partial_t v + 2\kappa v\partial_x v + \nu\partial_x^2 v + \delta\partial_x^3 v + \beta^2\partial_x^4 v\\
&= - g'(t)e^{-x} + 2\kappa g(t)e^{-x}v - 2\kappa g(t)e^{-x}\partial_x v + 2\kappa g^2(t)e^{-2x}\\
&\quad - \nu g(t)e^{-x} - \delta g(t)e^{-x} - \beta^2 g(t)e^{-x}.
\end{aligned} \tag{3.4}$$

We prove the following result.

Lemma 3.1. *Fix $T > 0$. There exists a constant $C(T) > 0$, such that*

$$\|v(t,\cdot)\|^2_{L^2(0,\infty)} + \frac{\beta^2 e^{C_0 t}}{42}\int_0^t e^{-C_0 s}\left\|\partial_x^2 v(s,\cdot)\right\|^2_{L^2(0,\infty)} ds \le C(T), \tag{3.5}$$

for every $0 \le t \le T$. In particular, we have (2.7). *Moreover,*

$$\int_0^t (\partial_x v(s,0))^2 ds \le C(T), \tag{3.6}$$

for every $0 \le t \le T$.

Proof. Let $0 \le t \le T$. We begin by observing that, thanks to (3.3),

$$\begin{aligned}
2\int_0^\infty v\partial_t v dx =& \frac{d}{dt}\|v(t,\cdot)\|^2_{L^2(0,\infty)},\\
2\delta\int_0^\infty v\partial_x^3 v =& -2\delta\int_0^\infty \partial_x v\partial_x^2 v dx,
\end{aligned} \tag{3.7}$$

$$
\begin{aligned}
2\beta^2 \int_0^\infty v\partial_x^4 v dx = & -2\beta^2 \int_0^\infty \partial_x v \partial_x^3 v dx \\
&= 2\beta^2 \partial_x v(t,0)\partial_x^2 v(t,0) + 2\beta^2 \left\|\partial_x^2 v(t,\cdot)\right\|^2_{L^2(0,\infty)} \\
&= 2\beta^2 g(t)\partial_x v(t,0) + 2\beta^2 \left\|\partial_x^2 v(t,\cdot)\right\|^2_{L^2(0,\infty)}.
\end{aligned}
$$

Consequently, multiplying (3.4) by $2v$, thanks to (3.7), an integration of (3.7) on $(0,\infty)$ gives

$$
\begin{aligned}
&\frac{d}{dt}\|v(t,\cdot)\|^2_{L^2(0,\infty)} + 2\beta^2 \left\|\partial_x^2 v(t,\cdot)\right\|^2_{L^2(0,\infty)} \\
&= -2g'(t)\int_0^\infty e^{-x} v dx + 4\kappa g(t)\int_0^\infty e^{-x} v^2 dx - 4\kappa g(t)\int_0^\infty e^{-x} v\partial_x v dx \\
&\quad + 4\kappa g^2(t)\int_0^\infty e^{-2x} v dx - 2\nu g(t)\int_0^\infty e^{-x} v dx - 2\delta g(t)\int_0^\infty e^{-x} v dx \\
&\quad - \beta^2 g(t)\int_0^\infty e^{-x} v dx - 2\nu\int_0^\infty v\partial_x^2 v dx + 2\delta\int_0^\infty \partial_x v\partial_x^2 v dx \\
&\quad - 2\beta^2 g(t)\partial_x v(t,0).
\end{aligned}
\tag{3.8}
$$

Since

$$
\int_0^\infty e^{-4x} dx = \frac{1}{4}, \tag{3.9}
$$

thanks to (1.3), (2.10), (3.9) and the Young inequality,

$$
\begin{aligned}
&2|g'(t)|\int_0^\infty e^{-x}|v|dx \le 2C_0\int_0^\infty e^{-x}|v|dx \le C_0 + C_0\|v(t,\cdot)\|^2_{L^2(0,\infty)}, \\
&4|\kappa||g(t)|\int_0^\infty e^{-x}v^2 dx \le C_0\int_0^\infty e^{-x}v^2 dx \le C_0\|v(t,\cdot)\|^2_{L^2(0,\infty)}, \\
&4|\kappa||g(t)|\int_0^\infty e^{-x}|v||\partial_x v|dx \le 2C_0\int_0^\infty e^{-x}|v||\partial_x v|dx \\
&\qquad \le 2C_0\int_0^\infty |v||\partial_x v|dx \le C_0\|v(t,\cdot)\|^2_{L^2(0,\infty)} + C_0\|\partial_x v(t,\cdot)\|^2_{L^2(0,\infty)}, \\
&4|\kappa|g^2(t)\int_0^\infty e^{-2x}|v|dx \le 2C_0\int_0^\infty e^{-2x}|v|dx \le C_0 + C_0\|v(t,\cdot)\|^2_{L^2(0,\infty)}, \\
&2|\nu+\delta+\beta||g(t)|\int_0^\infty e^{-x}|v|dx \le 2C_0\int_0^\infty e^{-x}|v|dx \le C_0 + C_0\|v(t,\cdot)\|^2_{L^2(0,\infty)}, \\
&2|\nu|\int_0^\infty v|\partial_x^2 v|dx = 2\int_0^\infty \left|\frac{\nu v}{\beta}\right|\left|\beta\partial_x^2 v\right| dx \le \frac{\nu^2}{\beta^2}\|v(t,\cdot)\|^2_{L^2(0,\infty)} + \beta^2\left\|\partial_x^2 v(t,\cdot)\right\|^2_{L^2(0,\infty)}, \\
&2|\delta|\int_0^\infty |\partial_x v||\partial_x^2 v|dx = \int_0^\infty \left|\frac{2\delta\partial_x v}{\beta}\right|\left|\beta\partial_x^2 v\right| dx \\
&\qquad \le \frac{2\delta^2}{\beta^2}\|\partial_x v(t,\cdot)\|^2_{L^2(0,\infty)} + \frac{\beta^2}{2}\left\|\partial_x^2 v(t,\cdot)\right\|^2_{L^2(0,\infty)}, \\
&2\beta^2|g(t)||\partial_x v(t,0)| \le 2C_0|\partial_x v(t,0)| \le C_0 + C_0(\partial_x v(t,0))^2.
\end{aligned}
$$

It follows from (3.8) that

$$
\frac{d}{dt}\|v(t,\cdot)\|^2_{L^2(0,\infty)} + \frac{\beta^2}{2}\left\|\partial_x^2 v(t,\cdot)\right\|^2_{L^2(0,\infty)}
$$

$$\leq C_0 + C_0 \|v(t,\cdot)\|^2_{L^2(0,\infty)} + C_0 \|\partial_x v(t,\cdot)\|^2_{L^2(0,\infty)} + C_0(\partial_x v(t,0))^2. \tag{3.10}$$

Observe that, by the Young inequality,

$$\begin{aligned} C_0(\partial_x v(t,0))^2 =& -2C_0 \int_0^\infty \partial_x v \partial_x^2 v dx \leq 2C_0 \int_0^\infty |\partial_x v||\partial_x^2 v| dx \\ \leq& 2\int_0^\infty \left| \frac{\sqrt{3} C_0 \partial_x v}{\beta} \right| \left| \frac{\beta}{\sqrt{3}} \partial_x^2 v \right| dx \\ \leq& C_0 \|\partial_x v(t,\cdot)\|^2_{L^2(0,\infty)} + \frac{\beta^2}{3} \left\|\partial_x^2 v(t,\cdot)\right\|^2_{L^2(0,\infty)}. \end{aligned} \tag{3.11}$$

Consequently, by (3.10),

$$\begin{aligned} &\frac{d}{dt} \|v(t,\cdot)\|^2_{L^2(0,\infty)} + \frac{\beta^2}{6} \left\|\partial_x^2 v(t,\cdot)\right\|^2_{L^2(0,\infty)} \\ &\qquad \leq C_0 + C_0 \|v(t,\cdot)\|^2_{L^2(0,\infty)} + C_0 \|\partial_x v(t,\cdot)\|^2_{L^2(0,\infty)}. \end{aligned} \tag{3.12}$$

Observe that, thanks to (3.3),

$$C_0 \|\partial_x v(t,\cdot)\|^2_{L^2(0,\infty)} = C_0 \int_0^\infty \partial_x v \partial_x v dx = -C_0 \int_0^\infty v \partial_x v dx.$$

Therefore, by the Young inequality,

$$\begin{aligned} C_0 \|\partial_x v(t,\cdot)\|^2_{L^2(0,\infty)} \leq& 2\int_0^\infty \left| \frac{\sqrt{7} C_0 v}{2\beta} \right| \left| \frac{\beta \partial_x^2 v}{\sqrt{7}} \right| dx \\ \leq& C_0 \|v(t,\cdot)\|^2_{L^2(0,\infty)} + \frac{\beta^2}{7} \left\|\partial_x^2 v(t,\cdot)\right\|^2_{L^2(0,\infty)}. \end{aligned} \tag{3.13}$$

It follows from (3.12) that

$$\begin{aligned} &\frac{d}{dt} \|v(t,\cdot)\|^2_{L^2(0,\infty)} + \frac{\beta^2}{42} \left\|\partial_x^2 v(t,\cdot)\right\|^2_{L^2(0,\infty)} \\ &\qquad \leq C_0 + C_0 \|v(t,\cdot)\|^2_{L^2(0,\infty)}. \end{aligned} \tag{3.14}$$

By (2.4) and the Gronwall Lemma, we get

$$\begin{aligned} &\|v(t,\cdot)\|^2_{L^2(0,\infty)} + \frac{\beta^2 e^{C_0 t}}{42} \int_0^t e^{-C_0 s} \left\|\partial_x^2 v(s,\cdot)\right\|^2_{L^2(0,\infty)} ds \\ &\qquad \leq C_0 e^{C_0 t} + C_0 e^{C_0 t} \int_0^t e^{-C_0 s} ds \leq C(T), \end{aligned}$$

which gives (3.5).

We prove (2.7). By (3.5) and (3.13), we have that

$$C_0 \|\partial_x v(t,\cdot)\|^2_{L^2(0,\infty)} \leq C(T) + \frac{\beta^2}{7} \left\|\partial_x^2 v(t,\cdot)\right\|^2_{L^2(0,\infty)}.$$

Integrating on $(0,t)$, by (3.5), we have (2.7).

Finally, (3.6) follows from (2.7), (3.5), (3.11) and an integration on $(0,t)$. □

Lemma 3.2. *Fix $T > 0$. There exists a constant $C(T) > 0$, such that*

$$\int_0^t (\partial_x u(s,0))^2 ds \le C(T), \tag{3.15}$$

for every $0 \le t \le T$. In particular, (2.13)*,* (2.14)*,* (2.15)*,* (2.20)*,* (2.21) *hold.*

Proof. Let $0 \le t \le T$. We prove (3.15). We begin by observing that, thanks (3.2),

$$\partial_x u(t,0) = \partial_x v(t,0) - g(t).$$

Therefore, by (1.3) and the Young inequality,

$$\begin{aligned}(\partial_x u(t,0))^2 =&(\partial_x v(t,0))^2 + g^2(t) - 2g(t)\partial_x v(t,0) \le (\partial_x v(t,0))^2 + g^2(t) + 2|g(t)||\partial_x v(t,0)|\\ \le&2(\partial_x v(t,0))^2 + 2g^2(t) \le 2(\partial_x v(t,0))^2 + C_0,\end{aligned}$$

Integrating on $(0,t)$, by (3.6), we get

$$\int_0^t (\partial_x u(s,0))^2 ds \le 2\int_0^t (\partial_x v(s,0))^2 ds + C_0 t \le C(T),$$

which gives (3.15).

Arguing as in Lemma 2.2, we have (2.13), (2.14) and (2.15).

We prove (2.20). Thanks to the Young inequality,

$$\begin{aligned}(\partial_x u(t,x))^2 =&2\int_0^x |\partial_x u||\partial_x^2 u|dx + 2(\partial_x u(t,0))^2 \le 2\int_0^\infty |\partial_x u||\partial_x^2 u|dx + 2(\partial_x u(t,0))^2\\ \le& \|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} + \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + 2(\partial_x u(t,0))^2.\end{aligned}$$

Therefore,

$$\|\partial_x u(t,\cdot)\|^2_{L^\infty(0,\infty)} \le \|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} + \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + 2(\partial_x u(t,0))^2. \tag{3.16}$$

Integrating on $(0,t)$, by (2.14), (2.15) and (3.15), we have (2.20).

Finally, arguing as in Lemma 2.3, we have (2.20). □

Lemma 3.3. *Fix $T > 0$. There exists a constant $C(T) > 0$, such that*

$$\|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} + \beta^2 \int_0^t \left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(0,\infty)} ds \le C(T), \tag{3.17}$$

for every $0 \le t \le T$. In particular, (2.26) *holds.*

Proof. Let $0 \le t \le T$. We begin by observing that

$$\begin{aligned}-2\int_0^\infty u\partial_x^2 u dx =&2\partial_x u(t,0)\partial_t u(t,0) + \frac{d}{dt}\|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)},\\ -2\delta\int_0^\infty \partial_x^2 u\partial_x^3 u dx =&0,\\ -2\beta^2\int_0^\infty \partial_x^2 u\partial_x^4 u dx =&2\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.\end{aligned} \tag{3.18}$$

Since, thanks (1.1)-(1.3)-(1.7), $\partial_t u(t,0) = g'(t)$, multiplying (1.1)-(1.3)-(1.7) by $-2\partial_x^2 u$, thanks to (3.18), an integration on $(0,\infty)$ gives

$$\begin{aligned}\frac{d}{dt}\|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} &+ 2\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\ &= -2g'(t)\partial_x u(t,0) + 4\kappa\int_0^\infty u\partial_x u\partial_x^2 u dx + 2\nu\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.\end{aligned} \tag{3.19}$$

Due to (1.3) and the Young inequality,

$$\begin{aligned}&2|g'(t)||\partial_x u(t,0)| \le 2C_0|\partial_x u(t,0)| \le C_0 + C_0(\partial_x u(t,0))^2\\ &4|\kappa|\int_0^\infty |u\partial_x u||\partial_x^2 u| dx \le 2\kappa^2\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} + \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.\end{aligned}$$

Consequently, by (3.19),

$$\begin{aligned}&\frac{d}{dt}\|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} + 2\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\ &\quad\le C_0 + C_0(\partial_x u(t,0))^2 + 2\kappa^2\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} + (1+|\nu|)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.\end{aligned}$$

Integrating on $(0,t)$, by (1.8), (2.15) and (2.21), we get

$$\begin{aligned}&\|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} + 2\beta^2\int_0^t\left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(0,\infty)} ds\\ &\quad\le C_0 + C_0 t + C_0\int_0^t(\partial_x u(s,0))^2 ds + 2\kappa^2\int_0^t\|u(s,\cdot)\partial_x u(s,\cdot)\|^2_{L^2(0,\infty)} ds\\ &\qquad + (1+|\nu|)\int_0^t\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(0,\infty)} ds\\ &\quad\le C(T),\end{aligned}$$

which gives (3.17).

Finally, arguing as in Lemma 2.4, we have (2.26). □

Lemma 3.4. *Fix $T > 0$. There exists a constant $C(T) > 0$, such that*

$$\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \beta^2\int_0^t\left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(0,\infty)} ds \le C(T), \tag{3.20}$$

for ever $0 \le t \le T$. In particular, we have (2.25) and

$$\int_0^t(\partial_x^3 u(s,0))^2 ds \le C(T), \tag{3.21}$$

for every $0 \le t \le T$. Moreover, (2.34) holds.

Proof. Let $0 \le t \le T$. We begin by observing that, thanks to (1.1)-(1.3)-(1.7),

$$2\int_0^\infty \partial_x^4 u\partial_t u dx = -2\partial_x^3 u(t,0)\partial_t u(t,0) - 2\int_0^\infty \partial_x^3 u\partial_t\partial_x u dx$$

$$=-2\partial_x^3 u(t,0)\partial_t u(t,0)+\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}, \tag{3.22}$$

$$2\nu\int_0^\infty \partial_x^2 u\partial_x^4 u dx=-2\nu\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.$$

Since, thanks to (1.1)-(1.3)-(1.7), $\partial_t u(t,0)=g'(t)$, multiplying (1.1)-(1.3)-(1.7) by $2\partial_x^4 u$, thanks (3.22), an integration on $(0,\infty)$ gives

$$\begin{aligned}&\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}+2\beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\&\quad=2g'(t)\partial_x^3 u(t,0)-4\kappa\int_0^\infty u\partial_x u\partial_x^4 u dx+2\nu\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\&\quad-2\delta\int_0^\infty \partial_x^3 u\partial_x^4 u dx.\end{aligned} \tag{3.23}$$

Due to (1.3) and the Young inequality,

$$\begin{aligned}&2|g'(t)||\partial_x^3 u(t,0)|\le 2C_0|\partial_x^3 u(t,0)|\le C_0+C_0(\partial_x^3 u(t,0))^2,\\&4|\kappa|\int_0^\infty |u\partial_x u||\partial_x^4 u|dx=2\int_0^\infty\left|\frac{2\kappa u\partial_x u}{\beta}\right|\left|\beta\partial_x^4 u\right|dx\\&\qquad\le\frac{4\kappa^2}{\beta^2}\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)}+\beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\&2|\delta|\int_0^\infty|\partial_x^3 u||\partial_x^4 u|dx=\int_0^\infty\left|\frac{2\delta\partial_x^3 u}{\beta}\right|\left|\beta\partial_x^4 u\right|dx\\&\qquad\le\frac{2\delta^2}{\beta^2}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)}+\frac{\beta^2}{2}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.\end{aligned}$$

It follows from (3.23) that

$$\begin{aligned}&\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}+\frac{\beta^2}{2}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\&\qquad\le C_0+C_0(\partial_x^3 u(t,0))^2+\frac{4\kappa^2}{\beta^2}\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)}\\&\qquad\quad+\left(2|\nu|+\frac{2\delta^2}{\beta^2}\right)\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.\end{aligned} \tag{3.24}$$

Thanks to the Young inequality,

$$\begin{aligned}C_0(\partial_x^3 u(t,0))^2&=-2C_0\int_0^\infty \partial_x^3 u\partial_x^4 u dx\le 2\int_0^\infty\left|\frac{\sqrt{3}C_0\partial_x^3 u}{\beta}\right|\left|\frac{\beta\partial_x^4 u}{\sqrt{3}}\right|dx\\&\le C_0\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)}+\frac{\beta^2}{3}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.\end{aligned} \tag{3.25}$$

Consequently, by (3.24),

$$\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}+\frac{\beta^2}{6}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)}$$

$$\leq C_0 + \frac{4\kappa^2}{\beta^2} \|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} + C_0 \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.$$

Integrating on $(0,t)$, by (1.8), (2.21) and (3.17), we get

$$\begin{aligned}
&\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \frac{\beta^2}{6}\int_0^t \left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(0,\infty)} \\
&\quad\leq C_0 + C_0 t + \frac{4\kappa^2}{\beta^2}\int_0^t \|u(s,\cdot)\partial_x u(s,\cdot)\|^2_{L^2(0,\infty)} ds + C_0 \int_0^t \left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(0,\infty)} ds \\
&\quad\leq C(T),
\end{aligned}$$

which gives (3.20).

We prove (2.25). Thanks to the Hölder inequality,

$$\begin{aligned}
(\partial_x u(t,0))^2 =& -2\int_0^\infty \partial_x u \partial_x^2 u dx \leq 2\int_0^\infty |\partial_x u||\partial_x^2 u| dx \\
&\leq 2\|\partial_x u(t,\cdot)\|_{L^2(0,\infty)} \left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(0,\infty)}.
\end{aligned} \tag{3.26}$$

Therefore, by (3.17) and (3.20),

$$(\partial_x u(t,0))^2 \leq C(T). \tag{3.27}$$

By (3.16), (3.17) and (3.20), we obtain that

$$\|\partial_x u\|^2_{L^\infty((0,T)\times(0,\infty))} \leq C(T),$$

which gives (2.25).

Finally, (3.21) follows from (3.17), (3.20), (3.25) and an integration on $(0,t)$, while arguing as in Lemma 2.5, we have (2.34). □

Arguing as in Section 2, we have Theorem 1.1.

4. Proof of the Theorem 1.1 for (1.1)-(1.4)-(1.7)

In this section, we prove Theorem 1.1 for (1.1)-(1.4)-(1.7).

Let us prove some a priori estimates on u.

We begin by proving the following lemma.

Lemma 4.1. *Fix $T > 0$. There exists a constant $C(T) > 0$, such that*

$$\begin{aligned}
\|u(t,\cdot)\|^2_{L^(0,\infty)} &+ \beta^2 e^{C_0 t}\int_0^t e^{-C_0 s}\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(0,\infty)} ds \\
&+ a^2 e^{C_0 t}\int_0^t e^{-C_0 s} u^4(s,0) ds \leq C(T).
\end{aligned} \tag{4.1}$$

for every $0 \leq t \leq T$. In particular, we have (2.14), (2.20), (2.21) and

$$\int_0^t u^2(s,0) ds \leq C(T), \tag{4.2}$$

for every $0 \leq t \leq T$.

Proof. Let $0 \le t \le T$. Multiplying (1.1)-(1.4)-(1.7) by 2ν, thanks to (1.1)-(1.4)-(1.7), an integration on $(0,\infty)$ give

$$\begin{aligned}
\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(0,\infty)} =& 2\int_0^\infty u\partial_t u dx \\
=& -4\kappa\int_0^\infty u^2\partial_x u dx + 6a^2\int_0^\infty u^3\partial_x u dx - 2\nu\int_0^\infty u\partial_x^2 u dx \\
& -2\beta^2\int_0^\infty u\partial_x^4 u dx \\
=& \frac{4\kappa}{3}u^3(t,0) - \frac{3a^2}{2}u^4(t,0) - 2\nu\int_0^\infty u\partial_x^2 u dx \\
& +2\beta^2\int_0^\infty \partial_x u\partial_x^3 u dx \\
=& \frac{4\kappa}{3}u^3(t,0) - \frac{3a^2}{2}u^4(t,0) - 2\nu\int_0^\infty u\partial_x^2 u dx \\
& -2\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
&\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(0,\infty)} + 2\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \frac{3a^2}{2}u^4(t,0) \\
&\qquad = \frac{4\kappa}{3}u^3(t,0) - 2\nu\int_0^\infty u\partial_x^2 u dx.
\end{aligned} \tag{4.3}$$

Due to the Young inequality,

$$\begin{aligned}
\frac{4|\kappa|}{3}|u(t,0)|^3 =& 2\left(\frac{2|\kappa u(t,0)|}{3\sqrt{D_3}}\right)\left(\sqrt{D_3}u^2\right) \\
\le& \frac{4\kappa^2}{9D_3}u^2(t,0) + D_3u^4(t,0) = 2\left(\frac{2\kappa^2}{9D_3\sqrt{D_3}}\right)\left(\sqrt{D_3}u^2(t,0)\right) + D_3u^4(t,0) \\
\le& \frac{4\kappa^4}{81D_3} + 2D_3u^4(t,0), \\
2|\nu|\int_0^\infty |u||\partial_x^2 u| dx \le& 2\int_0^\infty \left|\frac{\nu u}{\beta}\right|\left|\beta\partial_x^2 u\right| dx \\
\le& \frac{\nu^2}{\beta^2}\|u(t,\cdot)\|^2_{L^2(0,\infty)} + \beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)},
\end{aligned}$$

where D_3 is a positive constant, which will be specified later. Consequently, by (4.3),

$$\begin{aligned}
&\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(0,\infty)} + \beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \left(\frac{3a^2}{2} - 2D_3\right)u^4(t,0) \\
&\qquad \le \frac{4\kappa^4}{81D_3} + \frac{\nu^2}{\beta^2}\|u(t,\cdot)\|^2_{L^2(0,\infty)}.
\end{aligned}$$

Choosing $D_3 = \frac{a^2}{2}$, we have that

$$\begin{aligned}\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(0,\infty)} + 2\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + a^2u^4(t,0)&\\ \le C_0 + C_0\|u(t,\cdot)\|^2_{L^2(0,\infty)}.&\end{aligned}$$

By the Gronwall Lemma and (1.8), we get

$$\begin{aligned}\|u(t,\cdot)\|^2_{L^2(0,\infty)} + 2\beta^2 e^{C_0t}\int_0^t e^{-C_0s}\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(0,\infty)}ds + a^2e^{C_0t}\int_0^t e^{-C_0s}u^4(s,0)ds&\\ \le C_0e^{C_0t} + C_0e^{C_0t}\int_0^t e^{-C_0s}ds \le C(T),&\end{aligned}$$

which gives (4.1).

We prove (2.14). Thanks to (1.1)-(1.4)-(1.7), (4.1) and the Young inequality,

$$\begin{aligned}\|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} =& \int_0^\infty \partial_x u\partial_x u dx = -\int_0^\infty u\partial_x^2 u dx\\ \le& \int_0^\infty |u||\partial_x^2 u|dx \le \frac{1}{2}\|u(t,\cdot)\|^2_{L^2(0,\infty)} + \frac{1}{2}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\ \le& C(T) + \frac{1}{2}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.\end{aligned}$$

(2.14) follows from (4.1) and an integration on $(0,t)$.

We prove (2.20). By (3.16) and (1.1)-(1.4)-(1.7), we have that

$$\|\partial_x u(t,\cdot)\|^2_{L^\infty(0,\infty)} \le \|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} + \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} \tag{4.4}$$

Therefore, an integration on $(0,t)$, (2.14) and (4.1) gives (2.20).

Arguing as in Lemma 2.3, we have (2.21).

Finally, we prove (4.2). We begin by observing that, by the Young inequality,

$$u^2(t,0) \le \frac{1}{2} + \frac{1}{2}u^4(t,0).$$

(4.2) follows from (4.1) and an integration on $(0,t)$. □

Lemma 4.2. *Fix $T > 0$. There exists a constant $C(T) > 0$, such that* (2.26) *holds. In particular, we have that*

$$\|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} + 2\beta^2\int_0^t\left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(0,\infty)}ds \le C(T), \tag{4.5}$$

$$\int_0^t(\partial_x^2 u(s,0))^2ds \le C(T), \tag{4.6}$$

for every $0 \le t \le T$.

Proof. Let $0 \le t \le T$. Multiplying (1.1)-(1.4)-(1.7) by $-2\partial_x^2 u$, thanks to (1.1)-(1.4)-(1.7), an integration on $(0,\infty)$ gives

$$\begin{aligned}
\frac{d}{dt}\|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} =& -2\int_0^\infty \partial_x^2 u \partial_t u dx\\
=&4\kappa\int_0^\infty u\partial_x u\partial_x^4 u dx - 6a^2\int_0^\infty u^2\partial_x u\partial_x^2 u dx + 2\nu\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&+ 2\beta^2\int_0^\infty \partial_x^2 u\partial_x^4 u dx\\
=&4\kappa\int_0^\infty u\partial_x u\partial_x^4 u dx - 6a^2\int_0^\infty u^2\partial_x u\partial_x^2 u dx + 2\nu\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&- 2\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
&\frac{d}{dt}\|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} + 2\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\quad= 4\kappa\int_0^\infty u\partial_x u\partial_x^2 u dx - 6a^2\int_0^\infty u^2\partial_x u\partial_x^2 u dx + 2\nu\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.
\end{aligned}\tag{4.7}$$

Due to the Young inequality,

$$\begin{aligned}
&4|\kappa|\int_0^\infty |u\partial_x u||\partial_x^4 u| dx \le 2\kappa^2\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} + 2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)},\\
&6a^2\int_0^\infty |u^2\partial_x u||\partial_x^2 u| dx \le 3a^4\int_0^\infty u^4(\partial_x u)^2 dx + 3\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\quad\le 3a^2\|u\|^2_{L^\infty((0,T)\times(0,\infty))}\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} + 3\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.
\end{aligned}$$

It follows from (4.7) that

$$\begin{aligned}
&\frac{d}{dt}\|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} + 2\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\quad\le\left(2\kappa^2 + 3a^2\|u\|^2_{L^\infty((0,T)\times(0,\infty))}\right)\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)}\\
&\qquad+ (2|\nu| + 4)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.
\end{aligned}$$

Integrating on $(0,t)$, by (1.8), (2.21) and (4.1), we have that

$$\begin{aligned}
&\|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} + 2\beta^2\int_0^t\left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(0,\infty)} ds\\
&\quad\le C_0 + \left(2\kappa^2 + 3a^2\|u\|^2_{L^\infty((0,T)\times(0,\infty))}\right)\int_0^t\|u(s,\cdot)\partial_x u(s,\cdot)\|^2_{L^2(0,\infty)} ds\\
&\qquad+ (2|\nu| + 4)\int_0^t\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(0,\infty)} ds\\
&\quad\le C(T)\left(1 + \|u\|^2_{L^\infty((0,T)\times(0,\infty))}\right).
\end{aligned}\tag{4.8}$$

We prove (2.26). Thanks to (4.1), (4.8) and the Hölder inequality,

$$
\begin{aligned}
u^2(t,x) =& 2\int_0^x u\partial_x u dy - 2\int_0^\infty u\partial_x u dx \le 2\int_0^\infty |u||\partial_x u| dx \\
\le& 2\|u(t,\cdot)\|_{L^2(0,\infty))}\|\partial_x u(t,\cdot)\|_{L^2(0,\infty)} \le C(T)\sqrt{\left(1+\|u\|^2_{L^\infty((0,T)\times(0,\infty))}\right)}.
\end{aligned}
$$

Therefore,

$$
\|u\|^4_{L^\infty((0,T)\times(0,\infty))} - C(T)\|u\|^2_{L^\infty((0,T)\times(0,\infty))} - C(T) \le 0, \tag{4.9}
$$

which gives (2.26).

(4.5) follows from (2.26) and (4.8).

Finally, we prove (4.6). Thanks to the Young inequality,

$$
\begin{aligned}
(\partial_x^2 u(t,0))^2 =& -2\int_0^\infty \partial_x^2 u\partial_x^3 u dx \le 2\int_0^\infty |\partial_x^2 u||\partial_x^3 u| dx \\
\le& \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.
\end{aligned}
$$

(4.1), (4.5) and an integration on $(0,t)$ give (4.6). □

Lemma 4.3. *Fix $T>0$. There exists a constant $C(T)>0$, such that*

$$
\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \beta^2\int_0^t \left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(0,\infty)} ds \le C(T), \tag{4.10}
$$

for every $0\le t\le T$. In particular, (2.25) *holds. Moreover, we have* (2.34).

Proof. Let $0\le t\le T$. Since, by (1.1)-(1.4)-(1.7), $\partial_{tx}^2 u(t,0)=0$, multiplying (1.1)-(1.4)-(1.7) by $2\partial_x^4 u$, thanks to (1.1)-(1.4)-(1.7), an integration on $(0,t)$ gives

$$
\begin{aligned}
\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} =& 2\int_0^\infty \partial_x^4 u\partial_t u dx \\
=& -4\kappa\int_0^\infty u\partial_x u\partial_x^4 u dx + 6a^2\int_0^\infty u^2\partial_x u\partial_x^4 u dx \\
& - 2\nu\int_0^\infty \partial_x^2 u\partial_x^4 u dx - 2\beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)} \\
=& -4\kappa\int_0^\infty u\partial_x u\partial_x^4 u dx + 6a^2\int_0^\infty u^2\partial_x u\partial_x^4 u dx \\
& + 2\nu\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)} - 2\beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + 2\beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)} \\
&\quad = -4\kappa\int_0^\infty u\partial_x u\partial_x^4 u dx + 6a^2\int_0^\infty u^2\partial_x u\partial_x^4 u dx + 2\nu\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.
\end{aligned} \tag{4.11}
$$

Due to (2.26), (4.5) and the Young inequality,

$$
\begin{aligned}
4|\kappa| \int_0^\infty |u||\partial_x u||\partial_x^4 u| dx &\leq 4|\kappa| \|u\|_{L^\infty((0,T)\times(0,\infty))} \int_0^\infty |\partial_x u||\partial_x^4 u| dx \\
&\leq C(T) \int_0^\infty |\partial_x u||\partial_x^4 u| dx = \int_0^\infty \left| \frac{C(T)\partial_x u}{\beta} \right| \left|\beta \partial_x^4 u\right| dx \\
&\leq C(T) \|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} + \frac{\beta^2}{2} \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)} \\
&\leq C(T) + \frac{\beta^2}{2} \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)}, \\
6a^2 \int_0^\infty u^2 |\partial_x u||\partial_x^4 u| dx &\leq 6a^2 \|u\|^2_{L^\infty((0,T)\times(0,\infty))} \int_0^\infty |\partial_x u||\partial_x^4 u| dx \\
&\leq C(T) \int_0^\infty |\partial_x u||\partial_x^4 u| dx = \int_0^\infty \left| \frac{C(T)\partial_x u}{\beta} \right| \left|\beta \partial_x^4 u\right| dx \\
&\leq C(T) \|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} + \frac{\beta^2}{2} \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)} \\
&\leq C(T) + \frac{\beta^2}{2} \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.
\end{aligned}
$$

It follows from (4.11) that

$$
\begin{aligned}
&\frac{d}{dt} \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \beta^2 \left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)} \\
&\qquad \leq C(T) + 2|\nu| \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.
\end{aligned}
$$

Integrating on $(0,t)$, by (1.8) and (4.5), we get

$$
\begin{aligned}
&\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \beta^2 \int_0^t \left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(0,\infty)} ds \\
&\qquad \leq C_0 C(T) t + 2|\nu| \int_0^t \left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(0,\infty)} ds \leq C(T),
\end{aligned}
$$

which gives (4.10).

(2.25) follows from (4.4), (4.5) and (4.10).

Finally, we prove (2.34). Multiplying (1.1)-(1.4)-(1.7) by $2\partial_t u$, an integration on $(0,\infty)$ give

$$
\begin{aligned}
2\|\partial_t u(t,\cdot)\|^2_{L^2(0,\infty)} =& -4\kappa \int_0^\infty u \partial_x u \partial_t u dx + 6a^2 \int_0^\infty u^2 \partial_x u \partial_t u dx \\
& - 2\nu \int_0^\infty \partial_x^2 u \partial_t u dx - 2\beta^2 \int_0^\infty \partial_x^4 u \partial_t u dx.
\end{aligned}
\tag{4.12}
$$

Due to (2.26), (4.5), (4.10) and the Young inequality,

$$
\begin{aligned}
4|\kappa| \int_0^\infty |u||\partial_x u||\partial_t u| dx &\leq 4|\kappa| \|u\|_{L^\infty((0,T)\times(0,\infty))} \int_0^\infty |\partial_x u||\partial_t u| dx \\
&\leq 2C(T) \int_0^\infty |\partial_x u||\partial_t u| dx = 2\int_0^\infty \left| \frac{C(T)\partial_x u}{\sqrt{D_4}} \right| \left| \sqrt{D_4} \partial_t u \right| dx
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{C(T)}{D_4}\|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} + D_4\|\partial_t u(t,\cdot)\|^2_{L^2(0,\infty)}\\
&\leq \frac{C(T)}{D_4} + D_4\|\partial_t u(t,\cdot)\|^2_{L^2(0,\infty)},\\
6a^2\int_0^\infty u^2|\partial_x u||\partial_t u|dx &\leq 6a^2\|u\|^2_{L^\infty((0,T)\times(0,\infty))}\int_0^\infty |\partial_x u||\partial_t u|dx\\
&\leq 2C(T)\int_0^\infty |\partial_x u||\partial_t u|dx = 2\int_0^\infty \left|\frac{C(T)\partial_x u}{\sqrt{D_4}}\right|\left|\sqrt{D_4}\partial_t u\right|dx\\
&\leq \frac{C(T)}{D_4}\|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} + D_4\|\partial_t u(t,\cdot)\|^2_{L^2(0,\infty)}\\
&\leq \frac{C(T)}{D_4} + D_4\|\partial_t u(t,\cdot)\|^2_{L^2(0,\infty)},\\
2|\nu|\int_0^\infty |\partial_x^2 u||\partial_t u|dx &= 2\int_0^\infty \left|\frac{\nu\partial_x^2 u}{\sqrt{D_4}}\right|\left|\sqrt{D_4}\partial_t u\right|dx\\
&\leq \frac{\nu^2}{D_4}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + D_4\|\partial_t u(t,\cdot)\|^2_{L^2(0,\infty)}\\
&\leq \frac{C(T)}{D_4} + D_4\|\partial_t u(t,\cdot)\|^2_{L^2(0,\infty)},\\
2\beta^2\int_0^\infty |\partial_x^4 u||\partial_t u|dx &= 2\int_0^\infty \left|\frac{\beta^2\partial_x^4 u}{\sqrt{D_2}}\right|\left|\sqrt{D_4}\partial_t u\right|dx\\
&\leq \frac{\beta^4}{D_4}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + D_4\|\partial_t u(t,\cdot)\|^2_{L^2(0,\infty)}.
\end{aligned}
$$

Therefore, by (4.12),

$$
2(1-2D_4)\|\partial_t u(t,\cdot)\|^2_{L^2(0,\infty)} \leq \frac{C(T)}{D_4} + \frac{\beta^4}{D_4}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.
$$

Choosing $D_4 = \frac{1}{4}$, we have

$$
\|\partial_t u(t,\cdot)\|^2_{L^2(0,\infty)} \leq C(T) + 4\beta^4\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.
$$

An integration on $(0,t)$ and (4.10) give (2.34). □

Arguing as in Section 2, we have Theorem 1.1.

5. Proof of the Theorem 1.1 for (1.1)-(1.5)-(1.7)

In this section, we prove Theorem 1.1 for (1.1)-(1.5)-(1.7).

Let us prove some a priori estimates on u.

We begin by proving the following lemma.

Lemma 5.1. *Fix $T > 0$. There exists a constant $C(T) > 0$, such that*

$$
\begin{aligned}
\|u(t,\cdot)\|^2_{L^(0,\infty)} &+ \frac{\beta^2 e^{C_0 t}}{6}\int_0^t e^{-C_0 s}\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(0,\infty)}ds\\
&+ \frac{a^2 e^{C_0 t}}{2}\int_0^t e^{-C_0 s}u^4(s,0)ds \leq C(T).
\end{aligned}
\tag{5.1}
$$

for every $0 \le t \le T$. *In particular, we have* (2.14), (2.20), (2.21) *and* (4.2).

Proof. Let $0 \le t \le T$. We begin by observing that, thanks to (1.1)-(1.5)-(1.7),

$$\begin{aligned} 4\kappa \int_0^\infty u^2 \partial_x u dx =& \frac{4\kappa}{3} u^3(t,0),\\ -6a^2 \int_0^\infty u^2 \partial_x u dx =& \frac{3a^2}{2} u^4(t,0),\\ 2\delta \int_0^x u \partial_x^3 u =& -2\delta \int_0^\infty \partial_x u \partial_x^2 u dx,\\ 2\beta^2 \int_0^\infty u \partial_x^4 u dx =& -2\beta^2 \int_0^\infty u \partial_x^3 u dx = 2\beta^2 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}. \end{aligned} \tag{5.2}$$

Therefore, multiplying (1.1)-(1.5)-(1.7) by $2u$, thanks to (5.2), an integration on $(0,\infty)$ gives

$$\begin{aligned} &\frac{d}{dt} \|u(t,\cdot)\|^2_{L^2(0,\infty)} + 2\beta^2 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \frac{3a^2}{2} u^4(t,0)\\ &\quad= \frac{4\kappa}{3} u^2(t,0) - 2\nu \int_0^\infty u \partial_x^2 u dx + 2\delta \int_0^\infty \partial_x u \partial_x^2 u dx. \end{aligned} \tag{5.3}$$

Due to the Young inequality,

$$\begin{aligned} \frac{4|\kappa|}{3} u^2(t,0) \le& \frac{8\kappa^2}{9a^2} u^2(t,0) + \frac{a^2}{2} u^4(t,0),\\ 2|\nu| \int_0^\infty |u||\partial_x^2 u| dx =& \int_0^\infty \left|\frac{2\nu u}{\beta}\right| \left|\beta \partial_x^2 u\right| dx\\ \le& \frac{2\nu^2}{\beta^2} \|u(t,\cdot)\|^2_{L^2(0,\infty)} + \frac{\beta^2}{2} \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)},\\ 2|\delta| \int_0^\infty |\partial_x u||\partial_x^2 u| dx =& \int_0^\infty \left|\frac{2\delta \partial_x u}{\beta}\right| \left|\beta \partial_x^4 u\right| dx\\ \le& \frac{2\delta^2}{\beta^2} \|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} + \frac{\beta^2}{2} \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}. \end{aligned}$$

It follows from (5.3) that

$$\begin{aligned} &\frac{d}{dt} \|u(t,\cdot)\|^2_{L^2(0,\infty)} + \beta^2 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + a^2 u^4(t,0)\\ &\quad\le \frac{8\kappa^2}{9a^2} u^2(t,0) + \frac{2\nu^2}{\beta^2} \|u(t,\cdot)\|^2_{L^2(0,\infty)} + \frac{2\delta^2}{\beta^2} \|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)}\\ &\quad\le C_0 u^2(t,0) + C_0 \|u(t,\cdot)\|^2_{L^2(0,\infty)} + C_0 \|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)}. \end{aligned} \tag{5.4}$$

Observe that, by the Hölder inequality,

$$\begin{aligned} \|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} =& \int_0^\infty \partial_x u \partial_x u dx = -u(t,0)\partial_x u(t,0) - \int_0^\infty u \partial_x^2 u dx\\ \le& |u(t,0)||\partial_x u(t,0)| + \int_0^\infty |u||\partial_x^2 u| dx\\ \le& |u(t,0)||\partial_x u(t,0)| + \|u(t,\cdot)\|_{L^2(0,\infty)} \left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(0,\infty)}. \end{aligned} \tag{5.5}$$

Moreover, by the Young inequality,

$$|u(t,0)||\partial_x u(t,0)| \le \frac{1}{2D_5}u^2(t,0) + \frac{D_5}{2}(\partial_x u(t,0))^2, \tag{5.6}$$

where D_5 is a positive constant, which will be specified later. Observe that, by the Young inequality,

$$\begin{aligned}\frac{D_5}{2}(\partial_x u(t,0))^2 =& - D_5\int_0^\infty \partial_x u\partial_x^2 u dx \le \int_0^\infty |\partial_x u| \left|D_5\partial_x^2 u\right| dx\\ \le&\frac{1}{2}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(0,\infty)} + D_5^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.\end{aligned} \tag{5.7}$$

It follows from (5.5), (5.6) and (5.7) that

$$\begin{aligned}\frac{1}{2}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(0,\infty)} \le&\frac{1}{2D_5}u^2(t,0) + D_5^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\ &+ \left\|u(t,\cdot)\right\|_{L^2(0,\infty)}\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(0,\infty)},\end{aligned}$$

that is

$$\begin{aligned}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(0,\infty)} \le&\frac{1}{D_5}u^2(t,0) + 2D_5^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\ &+ 2\left\|u(t,\cdot)\right\|_{L^2(0,\infty)}\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(0,\infty)}.\end{aligned} \tag{5.8}$$

Due to the Young inequality,

$$2\left\|u(t,\cdot)\right\|_{L^2(0,\infty)}\left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(0,\infty)} \le \frac{1}{D_6}\left\|u(t,\cdot)\right\|^2_{L^2(0,\infty)} + D_6\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)},$$

where D_6 is a positive constant, which will be specified later. It follows from (5.8) that

$$\begin{aligned}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(0,\infty)} \le&\frac{1}{D_5}u^2(t,0) + 2D_5^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\ &+ \frac{1}{D_6}\left\|u(t,\cdot)\right\|^2_{L^2(0,\infty)} + D_6\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.\end{aligned} \tag{5.9}$$

Consequently, by (5.4) and (5.9),

$$\begin{aligned}&\frac{d}{dt}\left\|u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \left(\beta^2 - 2D_5^2 - D_6\right)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + a^2u^4(t,0)\\ &\le \left(C_0 + \frac{1}{D_5}\right)u^2(t,0) + \left(C_0 + \frac{1}{D_6}\right)\left\|u(t,\cdot)\right\|^2_{L^2(0,\infty)}.\end{aligned}$$

Choosing

$$D_5 = \frac{|\beta|}{2}, \quad D_6 = \frac{1}{3}, \tag{5.10}$$

we have that

$$\begin{aligned}&\frac{d}{dt}\left\|u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \frac{\beta^2}{6}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + a^2u^4(t,0)\\ &\le C_0u^2(t,0) + C_0\left\|u(t,\cdot)\right\|^2_{L^2(0,\infty)}.\end{aligned} \tag{5.11}$$

Due to the Young inequality,

$$C_0u^2(t,0) \le C_0 + \frac{a^2}{2}u^4(t,0).$$

It follows from (5.11) that

$$\begin{aligned}\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(0,\infty)} + \frac{\beta^2}{6}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \frac{a^2}{2}u^4(t,0)\\ \le C_0 + C_0\|u(t,\cdot)\|^2_{L^2(0,\infty)}.\end{aligned}$$

By the Gronwall Lemma and (1.8), we get

$$\begin{aligned}\|u(t,\cdot)\|^2_{L^2(0,\infty)} + \frac{\beta^2e^{C_0t}}{6}\int_0^t e^{-C_0s}\left\|\partial_x^2u(s,\cdot)\right\|^2_{L^2(0,\infty)}ds + \frac{a^2e^{C_0t}}{2}\int_0^t e^{-C_0s}u^4(s,0)ds\\ \le C_0 + C_0e^{C_0t}\int_0^t e^{-C_0s}ds \le C(T),\end{aligned}$$

which gives (5.1).

Arguing as in Lemma 4.1, we have (4.2), while (4.2), (5.1), (5.9), (5.10) and an integration on $(0,t)$ give (2.14).

We prove (2.20). By (3.16), (5.7) and (5.10), we have that

$$\|\partial_x u(t,\cdot)\|^2_{L^\infty(0,\infty)} \le C_0\|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} + C_0\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.$$

(2.14), (5.1) and an integration on $(0,t)$ give (2.20).

Arguing as in Lemma 2.3, we have (2.21), while (3.15) follows from (2.14), (5.1), (5.7), (5.10) and an integration on $(0,t)$. □

Lemma 5.2. *Fix* $T>0$. *There exists a constant* $C(T)>0$*, such that* (2.26) *holds. In particular,*

$$\left\|\partial_x^2u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \frac{\beta^2}{2}\int_0^t\left\|\partial_x^4u(s,\cdot)\right\|^2_{L^2(0,\infty)}ds \le C(T), \tag{5.12}$$

for every $0\le t\le T$. *Moreover, we have* (2.25),

$$\|\partial_x u(t,\cdot)\|_{L^2(0,\infty)} \le C(T), \tag{5.13}$$

$$\int_0^t\left\|\partial_x^3u(s,\cdot)\right\|^2_{L^2(0,\infty)}ds \le C(T), \tag{5.14}$$

for every $0\le t\le T$.

Proof. Let $0\le t\le T$. We begin by observing that, thanks (1.1)-(1.5)-(1.7),

$$\begin{aligned}&2\int_0^\infty \partial_x^4u\partial_tudx = -2\int_0^\infty \partial_x^3u\partial_t\partial_xudx = \frac{d}{dt}\left\|\partial_x^2u(t,\cdot)\right\|^2_{L^2(0,\infty)},\\ &2\delta\int_0^\infty \partial_x^3u\partial_x^4udx = 0.\end{aligned} \tag{5.15}$$

Therefore, multiplying (1.1)-(1.5)-(1.7) by $2\partial_x^4 u$, by (5.15) an integration on $(0,\infty$ gives

$$\begin{aligned} &\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + 2\beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)} \\ &\quad = -4\kappa\int_0^\infty u\partial_x u\partial_x^4 u dx + 6a^2\int_0^\infty u^2\partial_x u\partial_x^4 u dx - 2\nu\int_0^\infty \partial_x^2 u\partial_x^4 u dx. \end{aligned} \tag{5.16}$$

Due to the Young inequality,

$$\begin{aligned} 4|\kappa|\int_0^\infty |u\partial_x u||\partial_x^4 u|dx &= \int_0^\infty \left|\frac{4\kappa u\partial_x u}{\beta}\right|\left|\beta\partial_x^4 u\right| dx \\ &\le \frac{8\kappa^2}{\beta^2}\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} + \frac{\beta^2}{2}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)}, \\ 6a^2\int_0^\infty |u^2\partial_x u||\partial_x^4 u|dx &= \int_0^\infty \left|\frac{6a^2u^2\partial_x u}{\beta}\right|\left|\beta\partial_x^4 u\right| dx \\ &\le \frac{18a^4}{\beta^2}\int_0^\infty u^4(\partial_x u)^2 dx + \frac{\beta^2}{2}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)} \\ &\le \frac{18a^4}{\beta^2}\|u\|^2_{L^\infty((0,\infty)\times(0\, infty))}\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} + \frac{\beta^2}{2}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)}, \\ 2|\nu|\int_0^\infty |\partial_x^2 u||\partial_x^4 u|dx &= \int_0^\infty \left|\frac{2\nu\partial_x^2 u}{\beta}\right|\left|\beta\partial_x^4 u\right| dx \\ &\le \frac{2\nu^2}{\beta^2}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \frac{\beta^2}{2}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)}. \end{aligned}$$

It follows from (5.16) that

$$\begin{aligned} &\frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \frac{\beta^2}{2}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)} \\ &\quad \le C_0\left(1+\|u\|^2_{L^\infty((0,\infty)\times(0,\infty))}\right)\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} \\ &\qquad + C_0\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}. \end{aligned}$$

Integrating on $(0,\infty)$, by (1.8), (2.21) and (5.1), we have that

$$\begin{aligned} &\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \frac{\beta^2}{2}\int_0^t \left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(0,\infty)} ds \\ &\quad \le C_0 + C_0\left(1+\|u\|^2_{L^\infty((0,\infty)\times(0,\infty))}\right)\int_0^t \|u(s,\cdot)\partial_x u(s,\cdot)\|^2_{L^2(0,\infty)} ds \\ &\qquad + C_0\int_0^t \left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(0,\infty)} ds \\ &\quad \le C(T)\left(1+\|u\|^2_{L^\infty((0,\infty)\times(0,\infty))}\right). \end{aligned} \tag{5.17}$$

We prove that

$$\|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} \le C(T)\left(1+\|u\|^2_{L^\infty((0,\infty)\times(0,\infty))}\right). \tag{5.18}$$

We begin by observing that, by (5.1), (5.9) and (5.10),

$$\|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} \leq C_0 u^2(t,0) + C(T)\left(1 + \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\right). \tag{5.19}$$

Due to (5.1) and the Young inequality,

$$\begin{aligned} C_0 u^2(t,0) =& -2C_0\int_0^\infty u\partial_x u dx \leq 2C_0 \int_0^\infty |u||\partial_x u| dx \\ \leq& C_0 \|u(t,\cdot)\|_{L^2(0,\infty)} + \frac{1}{2}\|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} \leq C(T) + \frac{1}{2}\|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)}. \end{aligned}$$

Therefore, by (5.18),

$$\|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} \leq C(T)\left(1 + \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\right). \tag{5.20}$$

(5.17) and (5.19) give (5.20).

We prove (2.26). Thanks to (5.1), (5.19) and the Hölder inequality, we have (4.9), which gives (2.26).

(5.12) and (5.13) follows from (2.26), (5.17) and (5.18), respectively, while (3.16), (5.7), (5.10) (5.12) and (5.13) gives (2.25).

Finally, we prove (5.14). We begin by observing that, thanks to (1.1)-(1.5)-(1.7),

$$\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)} = \int_0^\infty \partial_x^3 u \partial_x^3 u dx = -\int_0^\infty \partial_x^2 u \partial_x^3 u dx.$$

Therefore, by (5.12) and the Young inequality,

$$\begin{aligned} \left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)} \leq& \int_0^\infty |\partial_x^2 u||\partial_x^3 u| dx \\ \leq& \frac{1}{2}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \frac{1}{2}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)} \\ \leq& C(T) + +\frac{1}{2}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)}. \end{aligned}$$

(5.14) follows from (5.12) and an integration on $(0,t)$. □

Lemma 5.3. *Fix $T > 0$. There exists a constant $C(T) > 0$, such that* (2.34) *holds.*

Proof. Let $0 \leq t \leq T$. Arguing as in Lemma 4.3, we have that

$$\|\partial_t u(t,\cdot)\|^2_{L^2(0,\infty)} \leq C(T) + 4\beta^4\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + 2|\delta|\int_0^\infty |\partial_x^3 u||\partial_t u| dx. \tag{5.21}$$

Due to the Young inequality,

$$2|\delta|\int_0^\infty |\partial_x^3 u||\partial_t u| dx \leq 2\delta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \frac{1}{2}\|\partial_t u(t,\cdot)\|^2_{L^2(0,\infty)}.$$

Therefore, by (5.21), we have

$$\frac{1}{2}\|\partial_t u(t,\cdot)\|^2_{L^2(0,\infty)} \leq C(T) + 4\beta^4\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + 2\delta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.$$

(2.34) follows from (5.12), (5.14) and an integration on $(0,t)$. □

Arguing as in Section 2, we have Theorem 1.1.

6. Proof of the Theorem 1.1 for (1.1)-(1.6)-(1.7)

In this section, we prove Theorem 1.1 for (1.1)-(1.6)-(1.7).

Let us prove some a priori estimates on u.

We begin by proving the following lemma.

Lemma 6.1. *Fix $T > 0$. There exists a constant $C(T) > 0$, such that*

$$\|u(t,\cdot)\|^2_{L^2(0,\infty)} + \frac{\beta^2 e^{C_0 t}}{2}\int_0^t e^{-C_0 s}\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(0,\infty)} ds \le C(T), \tag{6.1}$$

for every $0 \le t \le T$. In particular, we have (2.14), (2.20) *and* (2.21).

Proof. Let $0 \le t \le T$. We begin by observing that, thanks to (1.1)-(1.6)-(1.7),

$$\begin{aligned}
4\kappa\int_0^\infty u^2\partial_x u dx =&0,\\
6q\int_0^\infty u^3\partial_x u dx =&0,\\
2\nu\int_0^\infty u\partial_x^2 u dx =&-2\nu\|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)},\\
2\delta\int_0^\infty u\partial_x^3 u dx =&-2\delta\int_0^\infty \partial_x u\partial_x^2 u dx,\\
2\beta^2\int_0^\infty u\partial_x^4 u dx =&-2\beta^2\int_0^\infty \partial_x u\partial_x^3 u dx = 2\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.
\end{aligned} \tag{6.2}$$

Therefore, multiplying (1.1)-(1.6)-(1.7) by 2ν, thanks to (6.2), an integration on $0,\infty)$ gives

$$\begin{aligned}
&\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(0,\infty)} + 2\beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\qquad=2\nu\|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} - 2\delta\int_0^\infty \partial_x u\partial_x^2 u dx.
\end{aligned} \tag{6.3}$$

Due to the Young inequality,

$$\begin{aligned}
2|\delta|\int_0^\infty |\partial_x u||\partial_x^2 u| dx =& 2\int_0^\infty \left|\frac{\delta\partial_x u}{\beta}\right|\left|\beta\partial_x^2 u\right| dx\\
\le& \frac{\delta^2}{\beta^2}\|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} + \beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.
\end{aligned}$$

Consequently, by (6.3),

$$\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(0,\infty)} + \beta^2\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} \le C_0\|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)}. \tag{6.4}$$

Observe that, by (5.5) and (1.1)-(1.6)-(1.7),

$$C_0\|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} \le C_0\|u(t,\cdot)\|_{L^2(0,\infty)}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}$$

Therefore, by the Young inequality,

$$C_0 \|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)} \le C_0 \|u(t,\cdot)\|^2_{L^2(0,\infty)} + \frac{\beta^2}{2} \left\|\partial_x^2 u(t,\cdot)\right\|_{L^2(0,\infty)}. \tag{6.5}$$

It follows from (6.4) that

$$\frac{d}{dt}\|u(t,\cdot)\|^2_{L^2(0,\infty)} + \frac{\beta^2}{2}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} \le C_0 \|u(t,\cdot)\|^2_{L^2(0,\infty)}.$$

By the Gronwall Lemma and (1.8).

$$\|u(t,\cdot)\|^2_{L^2(0,\infty)} + \frac{\beta^2 e^{C_0 t}}{2}\int_0^t e^{-C_0 s}\left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(0,\infty)} ds \le C_0 e^{C_0 t} \le C(T),$$

which gives (6.1).

(2.14) follows from (6.1) and an integration on $(0,t)$, while, arguing as in Lemma 2.3, we have (2.20) and (2.21). □

Lemma 6.2. *Fix* $T > 0$. *There exists a constant* $C(T) > 0$*, such that* (2.26) *holds. In particular,*

$$\begin{aligned}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} &+ \frac{4(4\delta^2+4)}{\beta^4}\|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)}\\ &+ \frac{\beta^2}{2}\int_0^t \left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(0,\infty)} ds + \frac{4\delta^2+2}{\beta^2}\int_0^t \left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(0,\infty)} ds \le C(T),\end{aligned} \tag{6.6}$$

for every $0 \le t \le T$. *Moreover, we have* (2.24), (2.25) *and* (2.34).

Proof. Let $0 \le t \le T$. Consider an positive constant B, which will be specified later. Observing that, since, thanks to (1.1)-(1.6)-(1.7), $\partial_t u(t, 0 = \partial_t\partial_x u(t,0)$, we have that

$$\begin{aligned} 2\int_0^\infty \partial_x^4 u \partial_t u dx &= -2\int_0^\infty \partial_x^3 u \partial_t \partial_x u dx = \frac{d}{dt}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)},\\ -2B\int_0^\infty \partial_x^2 u\partial_t u dx &= B\frac{d}{dt}\|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)},\\ -2B\beta^2\int_0 \partial_x^2 u \partial_x^4 u dx &= 2B\beta^2\partial_x^2 u(t,0)\partial_x^3 u(t,0) + 2B\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.\end{aligned} \tag{6.7}$$

Therefore, multiplying (1.1)-(1.6)-(1.7) by $2\partial_x^4 u$, thanks to (6.7), an integration on $(0,\infty)$ gives

$$\begin{aligned} &\frac{d}{dt}\left(\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + B\|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)}\right)\\ &\qquad + 2\beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + 2B\beta^2\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\ &\quad = 2B\beta^2\partial_x^2 u(t,0)\partial_x^3 u(t,0) - 4\kappa\int_0^\infty u\partial_x u\partial_x^4 u dx + 4B\kappa\int_0^\infty u\partial_x u\partial_x^2 u dx\\ &\qquad - 6q\int_0^\infty u^2\partial_x u\partial_x^4 u dx + 6Bq\int_0^\infty u^2\partial_x u\partial_x^2 u dx - 2\nu\int_0^\infty \partial_x^2 u\partial_x^4 u dx\end{aligned} \tag{6.8}$$

$$+\,2B\nu\left\|\partial_x^2u(t,\cdot)\right\|^2_{L^2(0,\infty)}-2\delta\int_0^\infty\partial_x^3u\partial_x^4udx+2B\delta\int_0^\infty\partial_x^3u\partial_x^2udx.$$

Due to the Young inequality,

$$\begin{aligned}
&2B\beta^2\partial_x^2u(t,0)\partial_x^3u(t,0)\le\beta^4B^2(\partial_x^2u(t,0)^2+(\partial_x^3u(t,0))^2\\
&4|\kappa|\int_0^\infty|u\partial_xu||\partial_x^4u|dx=2\int_0^\infty\left|\frac{2\kappa u\partial_xu}{\beta\sqrt{D_7}}\right|\left|\beta\sqrt{D_7}\partial_x^4u\right|dx\\
&\qquad\le\frac{4\kappa^2}{\beta^2D_7}\left\|u(t,\cdot)\partial_xu(t,\cdot)\right\|^2_{L^2(0,\infty)}+D_7\beta^2\left\|\partial_x^4u(t,\cdot)\right\|^2_{L^2(0,\infty)},\\
&4B|\kappa|\int_0^\infty|u\partial_xu||\partial_x^2u|dx\le\left\|u(t,\cdot)\partial_xu(t,\cdot)\right\|^2_{L^2(0,\infty)}+4B^2\kappa^2\left\|\partial_x^2u(t,\cdot)\right\|^2_{L^2(0,\infty)},\\
&6|q|\int_0^\infty|u^2\partial_xu||\partial_x^4u|dx=2\int_0^\infty\left|\frac{3qu^2\partial_xu}{\beta\sqrt{D_7}}\right|\left|\beta\sqrt{D_7}\partial_x^4u\right|dx\\
&\qquad\le\frac{9q^2}{\beta^2D_7}\int_0^\infty u^4(\partial_xu)^2dx+D_7\beta^2\left\|\partial_x^4u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\qquad\le\frac{9q^2}{\beta^2D_7}\left\|u\right\|^2_{L^\infty((0,T)\times(0,\infty))}\left\|u(t,\cdot)\partial_xu(t,\cdot)\right\|^2_{L^2(0,\infty)}+D_7\beta^2\left\|\partial_x^4u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&6B|q|\int_0^\infty|u^2\partial_xu||\partial_x^2u|dx\le\int_0^\infty u^4(\partial_xu)^2dx+9q^2B^2\left\|\partial_x^2u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\qquad\le\left\|u\right\|^2_{L^\infty((0,T)\times(0,\infty))}\left\|u(t,\cdot)\partial_xu(t,\cdot)\right\|^2_{L^2(0,\infty)}+9q^2B^2\left\|\partial_x^2u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&2|\nu|\int_0^\infty|\partial_x^2u||\partial_x^4u|dx=2\int_0^\infty\left|\frac{\nu\partial_x^2u}{\beta\sqrt{D_7}}\right|\left|\beta\sqrt{D_7}\partial_x^4u\right|dx\\
&\qquad\le\frac{\nu^2}{\beta^2}\left\|\partial_x^2u(t,\cdot)\right\|^2_{L^2(0,\infty)}+\beta^2D_7\left\|\partial_x^4u(t,\cdot)\right\|^2_{L^2(0,\infty)},\\
&2|\delta|\int_0^\infty|\partial_x^3u||\partial_x^4u|dx=2\int_0^\infty\left|\frac{\delta\partial_x^3u}{\beta\sqrt{D_7}}\right|\left|\beta\sqrt{D_7}\partial_x^4u\right|dx\\
&\qquad\le\frac{\delta^2}{\beta^2D_7}\left\|\partial_x^3u(t,\cdot)\right\|^2_{L^2(0,\infty)}+\beta^2D_7\left\|\partial_x^4u(t,\cdot)\right\|^2_{L^2(0,\infty)},\\
&2B|\delta|\int_0^\infty|\partial_x^3u||\partial_x^2u|dx=2B\int_0^\infty\left|\beta\partial_x^3u\right|\left|\frac{\delta\partial_x^2u}{\beta}\right|dx\\
&\qquad\le B\beta^2\left\|\partial_x^3u(t,\cdot)\right\|^2_{L^2(0,\infty)}+\frac{B\delta^2}{\beta^2}\left\|\partial_x^2u(t,\cdot)\right\|^2_{L^2(0,\infty)},
\end{aligned}$$

where D_7 is a positive constant, which will be specified later. It follows from (6.8) that

$$\begin{aligned}
&\frac{d}{dt}\left(\left\|\partial_x^2u(t,\cdot)\right\|^2_{L^2(0,\infty)}+B\left\|\partial_xu(t,\cdot)\right\|^2_{L^2(0,\infty)}\right)\\
&\qquad+(2-4D_7)\beta^2\left\|\partial_x^4u(t,\cdot)\right\|^2_{L^2(0,\infty)}+\left(B\beta^2-\frac{\delta^2}{\beta^2D_7}\right)\left\|\partial_x^3u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\quad\le C_0B^2(\partial_x^2u(t,0)^2+(\partial_x^3u(t,0))^2+C_0\left(1+\frac{1}{4}+B^2+B\right)\left\|\partial_x^2u(t,\cdot)\right\|^2_{L^2(0,\infty)}
\end{aligned}$$

$$+ C_0\left(1+\frac{1}{D_7}+\left(1+\frac{1}{D_7}\right)\|u\|^2_{L^\infty((0,T)\times(0,\infty))}\right)\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)}.$$

Choosing $D_7 = \frac{1}{4}$, we have that

$$\begin{aligned}
&\frac{d}{dt}\left(\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + B\|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)}\right)\\
&\qquad + \beta^2\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \left(B\beta^2 - \frac{4\delta^2}{\beta^2}\right)\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\quad \le B^2(\partial_x^2 u(t,0)^2 + (\partial_x^3 u(t,0))^2 + C_0\left(1+B^2+B\right)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\qquad + C_0\left(1+\|u\|^2_{L^\infty((0,T)\times(0,\infty))}\right)\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)}.
\end{aligned} \tag{6.9}$$

Observe that by the Young inequality,

$$\begin{aligned}
B^2(\partial_x^2 u(t,0)^2 =& - B^2\int_0^\infty \partial_x^2 u\partial_x^3 u dx \le \int_0^\infty \left|\frac{2B\sqrt{B}\partial_x^2 u}{\beta}\right|\left|\sqrt{B}\beta\partial_x^3 u\right| dx\\
&\le \frac{B^3}{\beta^2}\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \frac{B\beta^2}{2}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)},\\
(\partial_x^3 u(t,0))^2 =& -2\int_0^\infty \partial_x^3 u\partial_x^4 u dx \le= \int_0^\infty \left|\frac{2\partial_x^3 u}{\beta}\right|\left|\beta\partial_x^4 u\right| dx\\
&\le \frac{2}{\beta^2}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \frac{\beta^2}{2}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.
\end{aligned}$$

Consequently, by (6.9),

$$\begin{aligned}
&\frac{d}{dt}\left(\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + B\|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)}\right)\\
&\qquad + \frac{\beta^2}{2}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \left(\frac{B\beta^2}{2} - \frac{4\delta^2+2}{\beta^2}\right)\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\quad \le C_0\left(1+\|u\|^2_{L^\infty((0,T)\times(0,\infty))}\right)\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)}\\
&\qquad + C_0\left(1+B^3+B^2+B\right)\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.
\end{aligned}$$

Choosing

$$B = \frac{4(4\delta^2+4)}{\beta^4},$$

we have that

$$\begin{aligned}
&\frac{d}{dt}\left(\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \frac{4(4\delta^2+4)}{\beta^4}\|\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)}\right)\\
&\qquad + \frac{\beta^2}{2}\left\|\partial_x^4 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \frac{4\delta^2+2}{\beta^2}\left\|\partial_x^3 u(t,\cdot)\right\|^2_{L^2(0,\infty)}\\
&\quad \le C_0\left(1+\|u\|^2_{L^\infty((0,T)\times(0,\infty))}\right)\|u(t,\cdot)\partial_x u(t,\cdot)\|^2_{L^2(0,\infty)}
\end{aligned}$$

$$+ C_0 \left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)}.$$

Integrating on $(0,t)$, by (1.8), (2.21) and (6.1), we get

$$\begin{aligned}
&\left\|\partial_x^2 u(t,\cdot)\right\|^2_{L^2(0,\infty)} + \frac{4(4\delta^2+4)}{\beta^4}\left\|\partial_x u(t,\cdot)\right\|^2_{L^2(0,\infty)} \\
&\quad + \frac{\beta^2}{2}\int_0^t \left\|\partial_x^4 u(s,\cdot)\right\|^2_{L^2(0,\infty)} ds + \frac{4\delta^2+2}{\beta^2}\int_0^t \left\|\partial_x^3 u(s,\cdot)\right\|^2_{L^2(0,\infty)} ds \\
&\le C_0 + C_0\left(1+\|u\|^2_{L^\infty((0,T)\times(0,\infty))}\right)\int_0^t \|u(s,\cdot)\partial_x u(s,\cdot)\|^2_{L^2(0,\infty)} ds \\
&\quad + C_0\int_0^t \left\|\partial_x^2 u(s,\cdot)\right\|^2_{L^2(0,\infty)} ds \\
&\le C(T)\left(1+\|u\|^2_{L^\infty((0,T)\times(0,\infty))}\right).
\end{aligned} \tag{6.10}$$

We prove (2.26). Thanks to (6.1), (6.10) and the Hölder inequality, we have (4.9), which gives (2.26).

(6.6) follows from (2.26) and (6.10).

Finally, arguing as in Lemma 2.4, we have (2.24) and (2.25), while arguing as in Lemma 5.3, (2.34) holds. □

Arguing as in Section 2, we have Theorem 1.1.

Acknowledgments

The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

Conflict of interest

The authors declare that there are no conflict of interest.

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