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Research article

Singular elliptic equations with directional diffusion†

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Abstract: We investigate conditions for the existence and uniqueness of viscosity solutions of the Dirichlet problem for a degenerate elliptic equation describing a stationary diffusion, which may take place in a partial number of spatial directions, with a possibly singular reaction term.

Keywords: degenerate elliptic operators; singular equations; Dirichlet problem; viscosity solutions

To Italo with great esteem and friendship.

1. Introduction and main results

In this paper we are interested to the Dirichlet problem

$$\begin{cases} \mathcal{D}_k(D^2u) + f(u) = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

in a bounded domain of $\mathbb{R}^n$, for an integer $k \in \{1, \dots, n\}$, where

$$\mathcal{D}_k(D^2u) = \sum_{i=1}^{k} \frac{\partial^2 u}{\partial x_i^2} \tag{1.2}$$

and $f(u)$ is a positive nonlinearity on $(0, \infty)$, which can go to infinity as $u \to 0$.

The above problem has a positive solution for uniformly elliptic equations ($k = n$) with a positive polynomial nonlinearity with negative exponent:

$$\Delta u + pu^{-\gamma} = 0 \tag{1.3}$$

See for instance Lazer-McKenna [31] and Crandall-Rabinowitz-Tartar [24]. For $n = 1$ equations of this kind arise in the theory of pseudoplastic fluids, appearing as generalized Emden-Fowler equations with negative exponent. See Nachman-Callegari [32].

Existence and uniqueness results results have also been obtained for equations with advection terms. See Giarrusso-Porru [27] and Porru-Vitolo [34].

Partial diffusion operators have been considered with respect to the maximum principle in cylindrical domains in [19–21], which had been previously investigated for linear and fully nonlinear uniformly elliptic equations in [4, 9, 10, 16, 19, 37–39].

In the present paper, we will show that similar results hold in the case of degenerate, not uniform ellipticity ($k < n$). For nonlinearities as in equation (1.3) in degenerate elliptic cases see also [5].

We need in our case a geometric condition (**G**) introduced by Blanc and Rossi [8], which says for the present use that Ω is a strictly convex domain. See Section 4.

Theorem 1.1. *Let Ω be a bounded domain of $\mathbb{R}^n$ satisfying condition* (**G**). *Let $f : (0, \infty) \to (0, \infty)$ be a continuous non-increasing function. Then there exists an unique solution, positive in Ω, of the Dirichlet problem* (1.1).

It is plain that Theorem 1.1 holds more generally in the case of the Dirichlet problem

$$\begin{cases} \mathcal{D}_{k,\delta}(D^2u) + f(u) = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

for anisotropic partial diffusion operators

$$\mathcal{D}_{k,\delta}(D^2u) = \sum_{i=1}^{k} \delta_i \frac{\partial^2 u}{\partial x_i^2} \tag{1.5}$$

with $\delta_i > 0$, $i = 1, \dots, k$.

The paper is organized as follows. In Section 2 we introduce the different notions of ellipticity and the viscosity solutions, then we establish comparison principles and recall existence, uniqueness and regularity results. In Section 3 we study an associate ODE and construct radial solutions of partial Laplace equations with positive non-increasing reaction terms. In Section 4 we prove the existence of Dirichlet problems for the smallest and the largest Hessian eigenvalue equation with positive non-increasing reaction terms. In Section 5 we finally prove Theorem 1.1.

2. Definitions and auxiliary results

Let $\mathcal{F}$ be a mapping from $\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$ to $\mathbb{R}$, where Ω is an open connected set of $\mathbb{R}^n$ and $\mathcal{S}^n$ the vector space of real $n \times n$ symmetric matrices. The fully nonlinear operator $\mathcal{F}$ acts on $u \in C^2(\Omega)$ as follows:

$$\mathcal{F}[u](x) = \mathcal{F}(x, u(x), Du(x), D^2u(x)), \quad x \in \Omega, \tag{2.1}$$

where Du is the gradient and D^2u the Hessian matrix of the function u.

As usual, $\mathcal{S}^n$ is endowed with the partial order relationship: $X \leq Y$ if and only if $X - Y$ is semidefinite positive. Moreover, if $\mathcal{F}$ is constant with respect to a variable, we omit such a variable.

For instance, the operator considered in Eq (1.1) can be represented as

$$\mathcal{F}[u] = \mathcal{F}(u, D^2u), \tag{2.2}$$

where

$$\mathcal{F}(t, X) = \sum_{i=1}^{k} X_{ii} + f(t). \tag{2.3}$$

Definition 1. *We say that $\mathcal{F}$ is degenerate elliptic in Ω if and only if*

$$\mathcal{F}(x, t, \xi, X) \leq \mathcal{F}(x, t, \xi, Y) \tag{2.4}$$

for all $X, Y \in \mathcal{S}^n$ such that $X \leq Y$ and $(x, t, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$.

Therefore the operator (2.3) under consideration is degenerate elliptic for all $k = 1, \dots, n$. Furthermore, let

$$\Pi_k = \sum_{i=1}^{k} e_i \otimes e_i \tag{2.5}$$

be the projection matrix on the vector subspace generated by the first k vectors $e_1, \dots, e_k$ of the canonical basis of $\mathbb{R}^n$, so that in particular $\Pi_n = I$, the $n \times n$ identity matrix. Setting

$$\mathcal{D}_k(X) = \sum_{i=1}^{k} X_{ii} = \mathrm{Tr}(\Pi_k X), \tag{2.6}$$

then

$$\mathcal{F}(t, X) = \mathcal{D}_k(X) + f(t). \tag{2.7}$$

Definition 2. *We say that $\mathcal{F}$ is proper in Ω if and only if it is degenerate elliptic and*

$$\mathcal{F}(x, t, \xi, X) \leq \mathcal{F}(x, s, \xi, Y) \tag{2.8}$$

for all $s, t \in \mathbb{R}$ such that $s \leq t$ and $(x, \xi) \in \Omega \times \mathbb{R}^n \times \mathcal{S}^n$.

Assuming that f is a non-increasing function, the operator (2.7) is proper. Under the same condition on f, the operator (2.7) is proper when considering any matrix $A \geq 0$ instead of Π_k.

Definition 3. *Let λ and Λ be positive constants such that $\lambda \leq \Lambda$. We say that $\mathcal{F}$ is uniformly elliptic in Ω with ellipticity constants λ and Λ if and only if*

$$\mathcal{F}(x, t, \xi, Y) - \Lambda \mathrm{Tr}(Y - X) \leq \mathcal{F}(x, t, \xi, X) \leq \mathcal{F}(x, t, \xi, Y) - \lambda \mathrm{Tr}(Y - X) \tag{2.9}$$

for all $X, Y \in \mathcal{S}^n$ such that $X \leq Y$ and $(x, t, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$.

It is plain that the uniform ellipticity is stronger than the degenerate ellipticity. Moreover, let $\mathcal{M}^{\pm}_{\lambda,\Lambda}(X)$, $X \in \mathcal{S}^n$, be the extremal Pucci operators:

$$\begin{aligned}\mathcal{M}^{+}_{\lambda,\Lambda}(X) &= \Lambda \mathrm{Tr}(X^+) - \lambda \mathrm{Tr}(X^-) = \sup_{\lambda I \le A \le \Lambda I} \mathrm{Tr}(AX)\\ \mathcal{M}^{-}_{\lambda,\Lambda}(X) &= \lambda \mathrm{Tr}(X^+) - \Lambda \mathrm{Tr}(X^-) = \inf_{\lambda I \le A \le \Lambda I} \mathrm{Tr}(AX).\end{aligned} \tag{2.10}$$

Then $\mathcal{M}^{\pm}_{\lambda,\Lambda}$ are uniformly elliptic and (2.10) can be restated as

$$\mathcal{M}^{-}_{\lambda,\Lambda}(Y-X) \le \mathcal{F}(x,t,\xi,Y) - \mathcal{F}(x,t,\xi,X) \le \mathcal{M}^{+}_{\lambda,\Lambda}(Y-X). \tag{2.11}$$

It is easy to check that the operators (2.7) are uniform elliptic only for $k = n$, when the second-order part is the Laplace operator Δu.

For $k < n$ (directional diffusion) the operators (2.6) are degenerate elliptic (proper if f is non-increasing), but not uniformly elliptic. See [41].

Even though a degenerate elliptic fully nonlinear operator $\mathcal{F}[u]$ is defined in the classical sense only when u is twice differentiable, nonetheless the equation $\mathcal{F}[u] = 0$ makes (a weaker) sense also when u is less regular.

Here we will consider the viscosity sense. Let D be a locally compact subset of $\mathbb{R}^n$, the spaces of upper and lower semicontinuous functions in D will be indicated with usc(D) and lsc(D), respectively.

Let $x_0 \in D$ and $u : D \to \mathbb{R}$. We denote by $J^{2,\pm}_D u(x_0)$ the second order superjet and subjet of u at x_0:

$$\begin{aligned}J^{2,+}_D u(x_0) = \{(\xi, X) \in \mathbb{R}^n \times \mathcal{S}^n : u(x_0+h) \le u(x_0) + \langle \xi, h\rangle + \tfrac{1}{2}\langle Xh, h\rangle\\ + o(|h|^2) \text{ as } h \to 0\Big\};\\ J^{2,-}_D u(x_0) = \{(\xi, X) \in \mathbb{R}^n \times \mathcal{S}^n : u(x_0+h) \ge u(x_0) + \langle \xi, h\rangle + \tfrac{1}{2}\langle Xh, h\rangle\\ + o(|h|^2) \text{ as } h \to 0\Big\}.\end{aligned} \tag{2.12}$$

Definition 4. *Let u be in usc(D), resp. in lsc(D). We say that u is a viscosity subsolution, resp. supersolution, in Ω of the equation $\mathcal{F}[u] = 0$ if and only if for all $x_0 \in D$*

$$\mathcal{F}(x_0, u(x_0), \xi, X) \ge 0 \;\; \forall\, (\xi, X) \in J^{2,+}_O u(x_0), \tag{2.13}$$

resp.

$$\mathcal{F}(x_0, u(x_0), \xi, X) \le 0 \;\; \forall\, (\xi, X) \in J^{2,-}_O u(x_0), \tag{2.14}$$

A function $u \in C(D)$, which is both a subsolution and a supersolution of the equation $\mathcal{F}[u] = 0$ is called a viscosity solution of such equation.

We will also use the notations $\mathcal{F}[u] \ge 0$, resp. $\mathcal{F}[u] \ge 0$, to say that u is a subsolution u, resp. a supersolution, of the equation $\mathcal{F}[u] = 0$.

For more properties of viscosity solutions we refer to [12, 22, 23, 30].

An important role is played by the maximum principle for subsolutions and, more generally, the comparison principle between upper semicontinuous subsolutions and lower semicontinuous supersolutions. See [23, 25, 41].

The comparison principle for the operators considered in this paper depends on the non-totally degenerate ellipticity, which is intermediate between the degenerate and the uniform ellipticity. See [3, 41].

Definition 5. *Let $\mathcal{F}$ be a degenerate elliptic fully nonlinear operator. We say that $\mathcal{F}$ is non-totally degenerate elliptic in Ω if and only if there exists a continuous function $\lambda(x) > 0$ such that*

$$\mathcal{F}(x, t, \xi, X + \varepsilon I) - \mathcal{F}(x, t, \xi, X) \geq \lambda(x)\varepsilon \tag{2.15}$$

for all $\varepsilon > 0$ and $(x, t, \xi, X) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$.

It is easy to check, by linearity, that the operators (2.5) are non-totally degenerate elliptic with $\lambda(x) = k$: in fact,

$$\sum_{i=1}^{k}(X + \varepsilon I)_{ii} = \sum_{i=1}^{k} X_{ii} + \varepsilon k \tag{2.16}$$

We will also consider below other non-totally degenerate elliptic operators, the partial traces

$$\mathcal{P}_k^-(X) = \sum_{i=1}^{k} \lambda_i(X), \tag{2.17}$$

where $\lambda_1(X) \leq \cdots \leq \lambda_n(X)$ are the eigenvalues of the matrix $X \in \mathcal{S}^n$, and the dual partial trace operators

$$\mathcal{P}_k^+(X) = -\mathcal{P}_k^-(-X) = \sum_{i=n-k+1}^{n} \lambda_i(X). \tag{2.18}$$

Such operators arise for geometric problems of partial mean curvature [35, 36, 42] and in stochastic differential games [7, 8]. They have been firstly investigated by Harvey-Lawson [28] and Caffarelli-Li-Nireneberg [13–15], and then in subsequent papers among which for instance [1, 6–8, 25, 26, 40].

Is is also easy to check as well that the operators (2.17) and (2.18) are non-totally degenerate elliptic with $\lambda(x) = k$. For instance

$$\sum_{i=1}^{k} \lambda_i(X + \varepsilon I) = \sum_{i=1}^{k} \lambda_i(X) + \varepsilon k \tag{2.19}$$

Note also that all operators $\mathcal{D}_k$ and $\mathcal{P}_k^{\pm}$ coincide with the Laplace operator when $k = n$. Only in this case such operators are uniformly elliptic.

Lemma 2.1. *Let $\mathcal{F}(t, X) = \mathcal{M}(X) + f(t)$ be a degenerate elliptic operator, where f is a continuous and non-increasing positive function in $(0, \infty)$.*

Let $u \in \mathrm{usc}(\overline{\Omega})$ and $v \in \mathrm{lsc}(\overline{\Omega})$ be non-negative functions such that $\mathcal{F}[u] \geq 0$ and $\mathcal{F}[v] \leq 0$ in a bounded domain Ω. If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.

Proof. Since the function $u - v$ is upper semicontinuous, then $u - v$ has a maximum at a point $\hat{x} \in \overline{\Omega}$. We have to prove that, $u \leq v$ on $\partial\Omega$, then $u(\hat{x}) \leq v(\hat{x})$.

Suppose, by contradiction, that $\max_{\overline{\Omega}}(u - v) = \delta > 0.$, namely

$$u(\hat{x}) - v(\hat{x}) = \delta > 0, \tag{2.20}$$

Let B_d a ball centered at the origin such that $\overline{\Omega} \subset B_d$, and where $0 < \varepsilon < \frac{\delta}{d^2}$. Then set

$$u_\varepsilon(x) = u(x) + \frac{\varepsilon}{2}|x - \hat{x}|^2, \tag{2.21}$$

By the choice of ε, we have for $x \in \partial\Omega$:

$$\begin{aligned} u_\varepsilon(x) - v(x) + \frac{\delta}{2} &\le \frac{\varepsilon}{2}|x-\hat{x}|^2 + \frac{\delta}{2} \\ &\le \frac{\varepsilon}{2} d^2 + \frac{\delta}{2} \\ &\le \delta = u_\varepsilon(\hat{x}) - v(\hat{x}), \end{aligned} \tag{2.22}$$

so that the usc function $u_\varepsilon - v$ has a positive maximum at a point of Ω.

Following the proof of [23, Theorem 3.3], let $x_\alpha, y_\alpha \in \overline{\Omega}$ be such that

$$\begin{aligned} M_\alpha &= \max_{\overline{\Omega}\times\overline{\Omega}} \left(u_\varepsilon(x) - v(y) - \tfrac{\alpha}{2}|x-y|^2\right) \\ &= u_\varepsilon(x_\alpha) - v(y_\alpha) - \tfrac{\alpha}{2}|x_\alpha - y_\alpha|^2. \end{aligned} \tag{2.23}$$

Let $\hat{x}_\varepsilon \in \overline{\Omega}$ be a limit point of x_α. From [23, Lemma 3.1] we have

$$\begin{aligned} &\lim_{\alpha\to\infty} \alpha|x_\alpha - y_\alpha|^2 = 0; \\ &\lim_{\alpha\to\infty} M_\alpha = u_\varepsilon(\hat{x}_\varepsilon) - v(\hat{x}_\varepsilon) = \max_{\overline{\Omega}} (u_\varepsilon(x) - v(x)). \end{aligned} \tag{2.24}$$

From the above limits:

$$\begin{aligned} &\lim_{\alpha\to\infty} x_\alpha = \hat{x}_\varepsilon = \lim_{\alpha\to\infty} y_\alpha; \\ &\lim_{\alpha\to\infty} (u_\varepsilon(x_\alpha) - v(y_\alpha)) = u_\varepsilon(\hat{x}_\varepsilon) - v(\hat{x}_\varepsilon). \end{aligned} \tag{2.25}$$

By the upper semicontinuity of u_ε and the lower semicontinuity of v, on a subsequence:

$$\lim_{\alpha\to\infty} u_\varepsilon(x_\alpha) = u_\varepsilon(\hat{x}_\varepsilon), \quad \lim_{\alpha\to\infty} v(y_\alpha) = v(\hat{x}_\varepsilon). \tag{2.26}$$

By (2.22) we have $\max_{\partial\Omega}(u_\varepsilon - v) < u_\varepsilon(\hat{x}_\varepsilon) - v(\hat{x}_\varepsilon)$. Therefore $\hat{x}_\varepsilon \in \Omega$, and $x_\alpha, y_\alpha \in \Omega$ for large α, by the upper semicontinuity of $u_\varepsilon - v$.

From [23, Theorem 3.2] we deduce that matrices $X_\alpha, Y_\alpha \in \mathcal{S}^n$ such that

$$\begin{aligned} &(\alpha(x_\alpha - y_\alpha), X_\alpha) \in \overline{J}_\Omega^{2,+} w(x_\alpha), \\ &(\alpha(x_\alpha - y_\alpha), Y_\alpha) \in \overline{J}_\Omega^{2,-} v(y_\alpha), \end{aligned} \tag{2.27}$$

and

$$X_\alpha \le Y_\alpha. \tag{2.28}$$

Since v is a viscosity subsolution, then

$$\mathcal{M}(Y_\alpha) + f(v(y_\alpha)) \le 0. \tag{2.29}$$

On the other hand, by the non-totally degenerate ellipticity, u_ε is a viscosity solution of the differential inequality

$$\mathcal{M}(D^2 u_\varepsilon(x)) + f\left(u_\varepsilon(x) - \tfrac{\varepsilon}{2}|x-\hat{x}|^2\right) \ge \lambda(x)\varepsilon \ \text{ in } \Omega \tag{2.30}$$

and therefore

$$\mathcal{M}(X_\alpha) + f\left(u_\varepsilon(x) - \tfrac{\varepsilon}{2}|x-\hat{x}|^2\right) \ge \lambda(x_\alpha)\varepsilon. \tag{2.31}$$

Combining (2.29) and (2.31), by the degenerate ellipticity:

$$\begin{aligned}\mathcal{M}(X_\alpha) + f\left(u_\varepsilon(x_\alpha) - \tfrac{\varepsilon}{2}|x_\alpha-\hat{x}|^2\right) &\ge \mathcal{M}(Y_\alpha) + f(v(y_\alpha)) + \lambda(x_\alpha)\varepsilon \\ &\ge \mathcal{M}(X_\alpha) + f(v(y_\alpha)) + \lambda(x_\alpha)\varepsilon,\end{aligned} \tag{2.32}$$

so that

$$f\left(u_\varepsilon(x_\alpha) - \tfrac{\varepsilon}{2}|x_\alpha-\hat{x}|^2\right) \ge f(v(y_\alpha)) + \lambda(x_\alpha)\varepsilon. \tag{2.33}$$

Let $\alpha \to \infty$ using (2.26). Then

$$f((u_\varepsilon(\hat{x}_\varepsilon) - \tfrac{\varepsilon}{2}|\hat{x}_\varepsilon-\hat{x}|^2) \ge f(v(\hat{x}_\varepsilon)) + \lambda(\hat{x})\varepsilon. \tag{2.34}$$

From this, by the non-increasing monotonicity of f we have therefore

$$u_\varepsilon(\hat{x}_\varepsilon) - v(\hat{x}_\varepsilon) \le \tfrac{\varepsilon}{2}|\hat{x}_\varepsilon-\hat{x}|^2. \tag{2.35}$$

On the other hand, recalling the assumption $u(\hat{x}) - v(\hat{x}) = \delta > 0$, we have:

$$u_\varepsilon(\hat{x}_\varepsilon) - v(\hat{x}_\varepsilon) = \max_{\overline{\Omega}}(u_\varepsilon - v) \ge \max_{\overline{\Omega}}(u-v) = \delta, \tag{2.36}$$

Together with (2.36), this implies $0 < \delta \le \frac{\varepsilon}{2}|\hat{x}_\varepsilon-\hat{x}|^2$, which cannot hold for small $\varepsilon > 0$. So the assumption $u(\hat{x}) > v(\hat{x})$ leads to a contradiction, and it turns aout that $u(\hat{x}) \le v(\hat{x})$. □

The above theorem works for all non-totally degenerate elliptic operators, in particular when $\mathcal{M}$ is the partial diffusion operator $\mathcal{D}_k$ as well as for the partial trace operators $\mathcal{P}_k^\pm$.

We will use in the sequel the interior Lipschitz estimate of [25, Lemma 5.5] for $\mathcal{P}_1^-(X) = \lambda_1(X)$ and the dual operator $\mathcal{P}_1^+(X) = \lambda_n(X)$. See also [6].

Lemma 2.2. *Let $u \in usc(B_1)$ be a viscosity subsolution of the equation $\lambda_1(D^2u) = g(x)$ in B_1, a ball of unit radius. If g is a continuous function, bounded below in B_1, then u in B_1 is Lipschitz-continuous and the following interior Lipschitz estimate holds:*

$$\|Du\|_{L^\infty(B_{1/2})} \le C\left(\|u\|_{L^\infty(B_1)} + \|g^-\|_{L^\infty(B_1)}\right), \tag{2.37}$$

where $B_{1/2}$ is the ball of radius $1/2$ concentric with B_1 and C a positive costant depending on n.

By duality, let $u \in lsc(B_1)$ be a supersolution of the equation $\lambda_n(D^2u) = g(x)$ in B_1. If g is a continuous function, bounded above in B_1, then $u \in C^\alpha(B_1)$ and the following interior C^α estimate holds:

$$\|Du\|_{L^\infty(B_{1/2})} \le C\left(\|u\|_{L^\infty(B_1)} + \|g^+\|_{L^\infty(B_1)}\right), \tag{2.38}$$

It is also worth to recall a very interesting $C^{1,\alpha}$ regularity result proved in [33] for the highly degenerate elliptic operator λ_1, if we consider that, also in the uniform elliptic case, we have basically the C^α regularity, and this is the best we can have in the general case. For Hölder estimates in the uniform and degenerate elliptic case we refer for instance to [2, 11, 25]. For Lipschitz estimates see [6, 25, 29].

3. On the equation $\lambda_j(D^2u) + f(u) = 0$: radial solutions

In this section we investigate the qualitative properties of the radial solutions of the equation $\lambda_j(D^2u) + f(u) = 0$, where f is a continuous and non-increasing positive function in $(0, \infty)$.

Asin the case of entire solutions, we preliminarly discuss an associated ODE with suitable initial conditions. See [17, 18].

In the present case, for $j = 1$ we consider the Cauchy problem

$$\begin{cases} \varphi'' + f(\varphi) = 0, \quad r > 0, \\ \varphi(0) = t_0 > 0 \\ \varphi'(0) = 0. \end{cases} \tag{3.1}$$

and for $j \geq 2$

$$\begin{cases} \varphi' + rf(\varphi) = 0, \quad r > 0, \\ \varphi(0) = t_0 > 0. \end{cases} \tag{3.2}$$

Here below we state some useful properties of the classical solutions.

Lemma 3.1. *Let $f : (0, \infty) \to (0, \infty)$ be a continuous and non-increasing function. Then the local classical solution of problem* (3.1) *or* (3.2) *is strictly decreasing and concave.*

In case (3.1) *the solution φ is C^2 and*

$$\varphi''(r) \leq \frac{\varphi'(r)}{r}. \tag{3.3}$$

In case (3.2) *the solution φ is C^1. Assuming in addition that f is C^1, then φ is C^2, and (3.3) holds as well.*

Proof. For problem (3.1) we refer for instance Lemma 2.1 of [27] and Lemma 3 of [34]. Here we limit the discussion to problem (3.2).

Since f is positive and non-increasing, we have:

$$\begin{aligned} &\varphi'(0) = \lim_{r \to 0^+} \varphi'(r) = 0, \ \ \varphi'(r) = -rf(\varphi(r)) < 0 \ \text{ for } r > 0; \\ &\varphi'(r_2) - \varphi'(r_1) = -f(\varphi(r_2)) + f(\varphi(r_1)) \leq 0 \ \text{ if } r_1 < r_2. \end{aligned} \tag{3.4}$$

Supposing that φ is C^2,

$$\varphi''(r) = -f(\varphi(r)) - rf'(\varphi(r))\varphi'(r) \leq -f(\varphi(r)) = \frac{\varphi'(r)}{r}. \tag{3.5}$$

□

Lemma 3.2. *Let $f : (0, \infty) \to (0, \infty)$ be a non-increasing function. Let $[0, R)$ be the maximal positivity interval of the classical solution φ of problem* (3.1) *or* (3.2)*. Then $R_f(t_0) = R$ is finite. Moreover*

(i) $t_1 < t_2 \Rightarrow R_f(t_1) < R_f(t_2)$*;*

(ii) $\lim_{t_0 \to 0^+} R_f(t_0) = 0$*;*

(iii) if $g : (0, \infty) \to (0, \infty)$, then $f \leq g \Rightarrow R_g(t_0) \leq R_f(t_0)$.

Proof. We discuss the case of problem (3.2), referring to [27] and [34] for (3.1).
Since $\varphi'(0) = 0$ and $\varphi(r)$ is concave, then $R_f(t_0) < \infty$.
The remaining properties (i), (ii) and (iii) can be deduced observing that:

$$R_f(t_0) = \left(2\int_0^{t_0} \frac{dt}{f(t)}\right)^{1/2}. \tag{3.6}$$

□

4. On the equation $\lambda_j(D^2u) + f(u) = 0$: the Dirichlet problem

In this section we investigate the Dirichlet problem

$$\begin{cases} \lambda_j(D^2u) + f(u) = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \Omega \end{cases} \tag{4.1}$$

with f strictly decreasing in $(0, \infty)$.
We start solving by the Perron's method the approximate problems

$$\begin{cases} \lambda_j(D^2u) + f(u) = 0 & \text{in } \Omega \\ u = \varepsilon & \text{on } \Omega \end{cases} \tag{4.2}$$

with $\varepsilon > 0$.
In order to do this, we need some geometric property of the boundary $\partial\Omega$. Consider the Dirichlet problem

$$\begin{cases} \lambda_j(D^2u) = 0 & \text{in } \Omega \\ u = g & \text{on } \Omega \end{cases} \tag{4.3}$$

Definition 6. (condition (**G**)). *Given $y \in \partial\Omega$, for every $r > 0$ there exists $\delta > 0$ such that*

$$(x + \mathbb{R}\, v) \cap B_r(y) \cap \partial\Omega \neq \emptyset \tag{4.4}$$

for every $x \in B_\delta(y)$ and direction $v \in \mathbb{R}^n$ ($|v| = 1$).

This is a necessary and sufficient to solve the Dirichlet problem (4.3) for all $j = 1, \dots, n$ and all continuous boundary data g.
Such condition can be weakened to require that (4.4) holds for only one direction v in each subspace of dimension j and $n - j$ to solve (4.3) for a singòe j such that $1 < j < n$. See Blanc and Rossi [8, Theorem 1].

Lemma 4.1. *Let Ω be a bounded open set satisfying the geometric condition* (**G**)*, and $f : (0, \infty) \to (0, \infty)$ be a continuous and strictly decreasing function. For every $\varepsilon > 0$ there exists an unique continuous viscosity solution of problem* (4.2) *such that $u(x) \geq \varepsilon$.*

Proof. As announced before, we use the Perron's method [23, Theorem 4.1].
The comparison principle is provided by Lemma 2.1. We need to find a subsolution $\underline{u}$ and a supersolution $\overline{u}$ such that $\underline{u} = \varepsilon = \overline{u}$ on $\partial\Omega$.

The function $\underline{u} \equiv \varepsilon$ is plainly a subsolution.

To find a supersolution we solve the problem

$$\begin{cases} \lambda_1(D^2 v) = 0 & \text{in } \Omega \\ v(x) = -\varepsilon - \frac{1}{2} f(\varepsilon)|x|^2 & \text{on } \partial\Omega \end{cases} \tag{4.5}$$

See for instance [8,28,40]. By duality $\overline{u}(x) = -v(x) - \frac{1}{2} f(\varepsilon)|x|^2$ is a viscosity solution of the problem

$$\begin{cases} \lambda_n(D^2\overline{u}) + f(\varepsilon) = 0 & \text{in } \Omega \\ \overline{u}(x) = \varepsilon & \text{on } \partial\Omega. \end{cases} \tag{4.6}$$

Since $\lambda_n(D^2\overline{u}) \le 0$ in Ω and $\overline{u} \ge \varepsilon$ on $\partial\Omega$, by the maximum principle [1,26,28] we have $\overline{u} \ge \varepsilon$ in Ω. By the non-increasing monotonicity of f, then $f(\overline{u}) \le f(\varepsilon)$. On the other hand $\lambda_j \le \lambda_n$, so that $\overline{u}$ can be chosen as the supersolution we are searching for:

$$\begin{cases} \lambda_j(D^2\overline{u}) + f(\overline{u}) \le 0 & \text{in } \Omega \\ \overline{u}(x) = \varepsilon & \text{on } \partial\Omega. \end{cases} \tag{4.7}$$

By [23, Theorem 4.1] we conclude that there exists an unique continuous viscosity solution of problem (4.2).

Furthermore, by the comparison principle, being $u \equiv \varepsilon$ a subsolution, we have $u \ge \varepsilon$ in Ω. □

From the solutions of problems (4.2), we obtain as limit the solution of problem (4.1) for $j = 1$ and $j = n$.

Firstly we show that such limit is a positive function.

Lemma 4.2. *Let Ω be a bounded open set satisfying the geometric condition* (**G**)*, and $f : (0,\infty) \to (0,\infty)$ be a continuous non-increasing function.*

For $h \in \mathbb{N}$, let u_h be the solutions of problems (4.2) *with $\varepsilon = \frac{1}{h}$ for $j = 1$. Then for every compact subset K of Ω there exists a number $t_K > 0$ such that*

$$u_h(x) \ge t_K \quad \forall\, x \in K \ \ \textit{and} \ \ \forall\, h \in \mathbb{N} \tag{4.8}$$

Suppose in addition that f is C^1. Then the same holds for the solutions of problems (4.2) *with $\varepsilon = \frac{1}{h}$ for $j = 2, \dots, n$.*

Proof. Let K be a compact subset of Ω.

In the case $j = 1$, we consider a maximal positive solution φ of the Cauchy problem (3.1). We choose an initial condition $\varphi(0) = t_0 > 0$ such that the maximal positivity radius R is small enough (see Lemma 3.2), say $0 < R \le \frac{1}{2}\,\text{dist}(K, \partial\Omega)$.

For any $x_0 \in K$, let $\phi(x) = \varphi(|x - x_0|)$. Then ϕ is C^2, and by (3.3) :

$$\lambda_1(D^2\phi(x)) = \varphi''(|x - x_0|). \tag{4.9}$$

Thus ϕ is a classical solution of the Dirichlet problem:

$$\begin{cases} \lambda_1(D^2\phi) + f(\phi) = 0 & \text{in } B_R(x_0) \\ \phi = 0 & \text{on } \partial B_R(x_0). \end{cases} \tag{4.10}$$

For $j \geq 2$, we consider a maximal positive solution φ of the Cauchy problem (3.2), and choose the initial condition $\varphi(0) = t_0 > 0$ as before, depending on the maximal positivity radius R.

In this case, we need to assume that f is C^1 in order that the radial function $\phi(x) = \varphi(|x - x_0|)$ is C^2, according to Lemma 3.1, and by (3.3) :

$$\lambda_j(D^2\phi(x)) = \varphi'(|x - x_0|)/|x - x_0|. \tag{4.11}$$

Thus we have found as well a classical solution ϕ of the Dirichlet problem:

$$\begin{cases} \lambda_j(D^2\phi) + f(\phi) = 0 & \text{in } B_R(x_0) \\ \phi = 0 & \text{on } \partial B_R(x_0). \end{cases} \tag{4.12}$$

From now on, we can proceed with the same argument for all $j = 1, \ldots, n$.

We note that $u_h \geq 1/h > 0 = \phi$ on $\partial B_R(x_0)$, for every $h \in \mathbb{N}$. Comparing the solutions ϕ and u_h we have $u_h(x) \geq \phi_(x) = \varphi(|x - x_0|)$ in $B_R(x_0)$.

In particular $u_h(x_0) \geq \varphi(0) = t_0$, which proves the assert with $t_K = t_0$. □

Theorem 4.3. *Let Ω be a bounded open set satisfying the geometric condition* (**G**)*, and $f : (0, \infty) \to (0, \infty)$ be a continuous non-increasing function. Then for $j = 1$ there exists an unique continuous positive viscosity solution of problem* (4.1).

Proof. For $h \in \mathbb{N}$, let u_h be the solutions of the Dirichlet problems (4.2) with $\varepsilon = \frac{1}{h}$ for $j = 1$. By the comparison principle, this is a non-increasing sequence. Therefore the u_h converge pointwise, as $h \to \infty$, to a function u in Ω such that $u = 0$ on $\partial\Omega$.

From Lemma 4.2 the function u has a positive lower bound every compact subset K of Ω, namely $u(x) \geq t_K > 0$ for all $x \in K$.

We will show that the u_h converge locally uniformly in Ω. Thus, by the stability theorems for viscosity solutions [12, 23], u is a solution of problem (4.1), which is unique by the comparison principle.

We are left therefore with proving the local uniform convergence of the u_h.

To this end, we firstly observe that, being $u_1 \geq u_h > 0$ on Ω for all $h \in \mathbb{N}$, the u_h are equi-bounded in Ω: setting $M = \max_\Omega u_1$,

$$0 \leq u_h(x) \leq M. \tag{4.13}$$

Next, let us fix a compact subset K of Ω, and consider a finite covering of K with balls of type $B_{R/4}(x_0)$, where $x_0 \in K$ and $0 < R \leq \frac{1}{2}\,\mathrm{dist}(K, \partial\Omega)$.

As in the proof of Lemma 4.2, we choose $t_0 > 0$ be such that R is the maximal positivity radius of Cauchy problem (3.1). Letting φ be the maximal positive solution, we obtain a radial function $\phi(x) = \varphi(|x - x_0|)$, which is a classical solution of the Dirichlet problem (4.12).

By the comparison principle $u_h(x) \geq \phi(x)$ in $B_R(x_i)$. In particular, setting $t^*_K = \varphi(R/2)$, and using the decreasing monotonicity of φ, we have:

$$u_h(x) \geq \varphi(R/2) = t^*_K \ \text{ in } \overline{B}_{R/2}(x_0), \ \ \forall\, h \in \mathbb{N}. \tag{4.14}$$

As a consequence:

$$f(u_h(x)) \leq f(t^*_K) \equiv M_K \ \ \forall\, x \in \overline{B}_{R/2}(x_0), \ \ \forall\, h \in \mathbb{N}. \tag{4.15}$$

Then, for all $x_0 \in K$:

$$\lambda_1(D^2u_h) \geq -M_K \text{ in } B_{R/2}(x_0) \tag{4.16}$$

Therefore, by (4.16), (2.37) and (4.13):

$$\|Du_h\|_{L^\infty(B_{R/4}(x_0))} \leq C\,(M + M_K)\,. \tag{4.17}$$

This inequality, together with (4.13), shows that the u_h are equi-continuous and equi-bounded on K. By Ascoli-Arzelà therefore $u_h \to u$ as $h \to \infty$ uniformly on K, as it was to be proved. □

We prove the same result for the Dirichlet problem (4.1) with $j = n$. We follow the same lines of the proof of Theorem 4.3 but an additional approximation argument is needed since φ is C^2 provided f is C^1, and we want to show the result under the weaker assumption that f is C^0.

Theorem 4.4. *Let Ω be a bounded open set satisfying the geometric condition* (**G**)*, and $f : (0,\infty) \to (0,\infty)$ be a continuous non-increasing function. Then for $j = n$ there exists an unique continuous positive viscosity solution of problem* (4.1).

Proof. Case 1: f is C^1 non-increasing

We can repeat step by step the proof of Theorem 4.3 if we assume that f is C^1, referring to (4.2) for $j = n$ instead of $j = 1$. We construct as there a non-increasing sequence u_h, converging to a function u in Ω.

The sequence u_h is equi-bounded as (4.13) in Ω.

We obtain easier way for all $x_0 \in \Omega$ the inequality

$$\lambda_n(D^2u_h) \leq 0 \text{ in } B_{R/2}(x_0), \tag{4.18}$$

Let K be a compact subset of Ω, and choose R as in the proof of Theorem 4.3.

From (4.18), by (2.38) and (4.13), we deduce for all $x_0 \in K$:

$$\|Du_h\|_{L^\infty(B_{R/4}(x_0))} \leq CM. \tag{4.19}$$

We conclude as before that the u_h are equi-bounded and equi-continuous on K, so that u is a continuous positive solution of problem (4.4) in this case.

Case 2: f is continuous non-increasing

We approximate f with a sequence f_i of C^1 non-increasing functions such that $f_i \to f$ as $i \to \infty$ locally uniformly in $(0,\infty)$ and $f_{i+1} \leq f_i$ for all $i \in \mathbb{N}$.

Then we solve, by Case 1, the following Dirichlet problems:

$$\begin{cases} \lambda_n(D^2u_i) + f_i(u_i) = 0 & \text{in } \Omega \\ u_i(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.20}$$

Recall that the viscosity solutions u_i are positive in Ω, for $i \in \mathbb{N}$.

Since $f_{i+1} \leq f_i$, then $\lambda_n(D^2u_{i+1}) + f_i(u_{i+1}) \geq \lambda_n(D^2u_{i+1}) + f_{i+1}(u_{i+1}) = 0$, so that by comparison $u_i \geq u_{i+1}$.

Then the u_i converge to a function u in Ω such that $u = 0$ on $\partial\Omega$.

The u_i are equi-bounded in Ω. In fact

$$0 \le u_i(x) \le M, \tag{4.21}$$

where $M = \max_\Omega u_1$.

Let K be a compact subset of Ω.

Since $\lambda_n(D^2 u_i) \le 0$ in $B_R(x_0)$, then (2.38) and (4.21) imply, as in the proof of Case 1,

$$\|Du_i\|_{L^\infty(B_{R/4}(x_0))} \le CM. \tag{4.22}$$

Therefore the u_i are equi-bounded and equi-continuous on K, and by Ascoli-Arzelà the u_h converge uniformly to u.

Next, choose $t_0 > 0$ such that the maximal positive radius $R = R_f(t_0)$ of the Cauchy problem (3.2) is small enough, in order that $0 < R < \frac{1}{2}\,\mathrm{dist}(K, \partial\Omega)$.

Let φ be the maximal positive solution, and let φ_i be the maximal positive solutions of the following Cauchy problems

$$\begin{cases} \varphi_i'(r) + f_i(\varphi_i(r)) = 0 & \text{in } \Omega \\ \varphi_i(0) = t_0 & \text{on } \partial\Omega. \end{cases} \tag{4.23}$$

Since $f \le f_i \le f_1$, then $R_1 \le R_i \equiv R_{f_i}(t_0) \le R$ and $\varphi_1 \le \varphi_i \le \varphi$ in $[0, R_1]$.

For $x_0 \in K$, then $\phi_i(x) = \varphi_i(|x - x_0|)$ is a classical radial solution of the Dirichlet problem

$$\begin{cases} \lambda_n(D^2\phi_i) + f_i(\phi_i) = 0 & \text{in } B_R(x_0) \\ \phi_i(x) = 0 & \text{on } \partial B_{R_i}(x_0), \end{cases} \tag{4.24}$$

Since $u_i > 0$ in Ω, comparing u_i and ϕ_i on $B_{R_i}(x_0)$, we get $u_i \ge \phi_i$ in $B_{R_i}(x_0)$.

In particular, recalling that the φ_i are decreasing, we get

$$u_i(x) \ge \varphi_i(R_1/2) \ge \varphi_1(R_1/2) \equiv t'_K > 0 \ \text{ in } B_{R_1/2}(x_0). \tag{4.25}$$

Therefore the f_i uniformly converge to f in $\{u_i(x) : x \in K, i \in \mathbb{N}\} \subset [t'_K, M]$. We have already shown that the u_i converge uniformly on K.

By the aforementioned stability results for viscosity solutions, then u is a viscosity solution of the Dirichlet problem (4.2) with $j = n$, positive in Ω by (4.25). □

5. Proof of Theorem 1.1

We have to solve problem (1.1) with f positive and non-increasing in $(0, \infty)$.

Thanks to the results of the previous section, we will use once again the Perron's method of [23, Theorem 4.1].

The comparison principle is provided by Lemma 2.1, recalling that $\mathcal{D}_k$ is non-totally degenerate elliptic.

Next we solve with Theorems 4.3 and 4.4 the following Dirichlet problems:

$$\begin{cases} \lambda_1(D^2\underline{u}) + \frac{1}{k} f(\underline{u}) = 0 & \text{in } \Omega \\ \underline{u}(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

and

$$\begin{cases} \lambda_n(D^2\overline{u}) + \frac{1}{k} f(\overline{u}) = 0 & \text{in } \Omega \\ \overline{u}(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

Observe that

$$k\lambda_1(X) \le \sum_{i=1}^{k} X_{ii} \le k\lambda_n(X), \tag{5.3}$$

Then $\underline{u}$ and $\overline{u}$ are continuous viscosity subsolution and supersolution, respectively, positive on Ω, of the equation

$$\mathcal{D}_k(D^2u) + f(u) \equiv \sum_{i=1}^{k} \frac{\partial^2 u}{\partial x_i^2} + f(u) = 0 \ \text{ in } \Omega \tag{5.4}$$

such that $\underline{u} = 0 = \overline{u}$ on $\partial\Omega$.

Theorem 4.1 of [23] implies therefore that there exists an unique positive viscosity solution of the problem

$$\begin{cases} \displaystyle\sum_{i=1}^{k} \frac{\partial^2 u}{\partial x_i^2} + f(u) = 0 & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.5}$$

as wanted.

Conflict of interest

The authors declare no conflict of interest.

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