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Research article

Finite element approximation of time fractional optimal control problem with integral state constraint

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Abstract: In this paper we investigate the finite element approximation of time fractional optimal control problem with integral state constraint. A space-time finite element scheme for the control problem is developed with piecewise constant time discretization and piecewise linear spatial discretization for the state equation. A priori error estimate for the space-time discrete scheme is derived. Projected gradient algorithm is used to solve the discrete optimal control problem. Numerical experiments are carried out to illustrate the theoretical findings.

Keywords: time fractional optimal control problem; integral state constraint; space time finite element method; a priori error estimate
Mathematics Subject Classification: 65K10, 65N30

1. Introduction

This paper aims to develop a space-time finite element approximation of optimal control problem governed by time fractional diffusion equation and subjected to state constraint in integral form. Let Ω be a bounded domain of $R^d (d = 1, 2, 3)$ with sufficiently smooth boundary $\partial\Omega$. Set $\Omega_T = \Omega \times (0, T)$. We consider the following optimal control problem governed by time-fractional diffusion equations:

$$\min_{(y,u)\in\mathcal{K}\times L^2(\Omega_T)} J(y, u) := \frac{1}{2}\int_{\Omega_T} (y(x, t) - y_d(x, t))^2 dxdt + \frac{\gamma}{2}\int_{\Omega_T} u^2(x, t)dxdt. \tag{1.1}$$

subject to

$$\begin{cases} {}_0D_t^\alpha y(x, t) - \Delta y(x, t) = f(x, t) + u(x, t), & (x, t) \in \Omega_T, \\ y(x, t) = 0, & (x, t) \in \Gamma_T, \\ y(x, 0) = 0, & x \in \Omega \end{cases} \tag{1.2}$$

and

$$\mathcal{K} = \Big\{ v \in L^1(\Omega_T); \int_{\Omega_T} v dx dt \leq \delta \Big\}. \tag{1.3}$$

Here y_d is the desired state. $\gamma > 0$ is the regularization constant. $f \in L^2(\Omega_T)$ is a given function.

The research of control problem governed by differential equation forms a hot topic in the past decades. Lots of literatures are devoted to developing theoretical analysis or numerical methods([1–7]). As we know, fractional calculus has a very long history and was applied into many fields of science and engineering. For instance, the fractional PDEs arise in many engineering applications such as anomalous diffusion on fractals and fractional random walk([8,9]), and contaminant transport in groundwater flow([10, 11]). Over the past decades lots of numerical methods and algorithms are developed to discretize fractional derivative and solve the fractional differential equations, including finite difference methods [12,13], finite element methods [14–17], spectral methods [18,19] and so on.

In recent years optimal control problem governed by fractional PDEs has attracted the researcher's attention with the rapid development of numerical methods for fractional PDEs. For optimal control problem governed by time fractional PDEs we refer to [20–24] for finite element approximation, [25,28] for spectral Galerkin discretization. For optimal control problem governed by space fractional PDEs we refer to [26] for spectral Galerkin discretization, and [27] for finite element approximation. As far as we know, the researches on optimal control problem governed by fractional PDEs mainly focus on control constrained problem. The study with respect to state constrained fractional optimal control problem is very limited. In [28] spectral Galerkin approximation of time fractional optimal control problem with integral state constraint was discussed. In [29] an optimal control problem governed by fractional elliptic equation with pointwise state constrained was investigated. In [30] the authors discussed finite element approximation of space fractional optimal control problem with integral state constraint.

In the present paper we consider an optimal control problem governed by time fractional diffusion equation with state constraint in integral form. The first order optimality condition is derived and the corresponding regularity is discussed. A space-time finite element discrete scheme is built up based on piecewise constant discontinuous Galerkin discretization for temporal discretization and conforming linear finite elements for spatial discretization. A priori error estimate for state, adjoint state and control variables is derived. Numerical example is given to verify the theoretical findings.

The structure of this paper is as follows. In Section 2, the preliminaries of fractional derivative and Sobolev space are introduced. In Section 3 the space-time finite element approximation of the control problem is discussed. In Section 4 a priori error estimates are proved. Numerical example is given to verify the theoretical results in Section 5.

2. Preliminaries

In this section we introduce some basic knowledge of the fractional calculus and the Sobolev space. For $0 < \alpha < 1$, the left and right Riemann-Liouville fractional derivative are defined as follows

$${}_0D_t^\alpha v = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{v(s)}{(t-s)^\alpha} ds,$$

$$ {}_tD_T^{\alpha}v = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^T \frac{v(s)}{(s-t)^{\alpha}}ds. $$

Let $H^{\delta}(I)$ and $H_0^{\delta}(I)$ denote the usual Sobolev space of order δ on I with norm $\|\cdot\|_{H^{\delta}(I)}$ and semi-norm $|\cdot|_{H^{\delta}(I)}$. According to [14], we have the following properties.

Lemma 2.1. *Assume that $-\infty < a < b < \infty$. If $v \in H^{\alpha}(a,b)$, then the following estimates hold*

$$ \|{}_aD_t^{\frac{\alpha}{2}}v\|_{L^2(a,b)} \le C|v|_{H^{\frac{\alpha}{2}}(a,b)}, $$
$$ \|{}_tD_b^{\frac{\alpha}{2}}v\|_{L^2(a,b)} \le C|v|_{H^{\frac{\alpha}{2}}(a,b)}, $$
$$ \langle {}_aD_t^{\frac{\alpha}{2}}v, {}_tD_b^{\frac{\alpha}{2}}v\rangle_{(a,b)} \ge \mathbb{C}_0|v|^2_{H^{\frac{\alpha}{2}}(a,b)}. $$

Further, if $v, w \in H^{\frac{\alpha}{2}}(a,b)$, then

$$ \langle {}_aD_t^{\frac{\alpha}{2}}v, {}_tD_b^{\frac{\alpha}{2}}w\rangle_{(a,b)} \le C|v|_{H^{\frac{\alpha}{2}}(a,b)} \cdot |w|_{H^{\frac{\alpha}{2}}(a,b)}, $$
$$ \langle {}_aD_t^{\alpha}v, w\rangle_{H^{\alpha}(a,b)} = \langle {}_aD_t^{\frac{\alpha}{2}}v, {}_tD_b^{\frac{\alpha}{2}}w\rangle_{(a,b)} = \langle v, {}_tD_b^{\alpha}w\rangle_{H^{\alpha}(a,b)}. $$

We introduce the definition of space $\dot{H}^s(\Omega)$ following [31]. Let $\{\lambda_i\}_{i=0}^{\infty}$ and $\{\phi_i\}_{i=0}^{\infty}$ denote eigenvalues and $L^2(\Omega)$-orthogonal eigenfunctions of $-\Delta$ operator on domain Ω with zero Dirichlet boundary condition. For every $v \in L^2(\Omega)$ we have the representation $v = \sum_{i=0}^{\infty} a_i\phi_i$, where $a_i = (v, \phi_i)$. For $s \ge 0$, let $\dot{H}^s(\Omega) \subset L^2(\Omega)$ defined by

$$ \dot{H}^s(\Omega) := \Big\{ v = \sum_{i=0}^{\infty} a_i\phi_i : |v|_s = (\sum_{i=0}^{\infty} a_i^2\lambda_i^s)^{1/2} < \infty \Big\}. $$

To define the weak formulation of state equation we also need to introduce the space

$$ V = H^{\frac{\alpha}{2}}(0,T;L^2(\Omega)) \cap L^2(0,T;\dot{H}^1(\Omega)) $$

and endow this space with the norm

$$ \|\cdot\|_V = \Big(|\cdot|^2_{H^{\frac{\alpha}{2}}(0,T;L^2(\Omega))} + \|\cdot\|^2_{L^2(0,T;\dot{H}^1(\Omega))} \Big)^{\frac{1}{2}}. $$

3. Space-time finite element discretization

In this section we consider space-time finite element discretization of optimal control problem. To this end we begin with deriving the first order optimality condition and regularity of the solution.

3.1. First order optimality and regularity

Theorem 3.1. *Assume that* (y,u) *is the solution of control problem* (1.1) *and* (1.2)*. Then there exist a real number $\xi \ge 0$ and the adjoint state p satisfying the following first order optimality system:*

$$ \begin{cases} {}_0D_t^{\alpha}y - \Delta y = f + u, & (x,t) \in \Omega_T, \\ y(x,t) = 0, & (x,t) \in \Gamma_T, \\ y(x,0) = 0, & x \in \Omega, \end{cases} \tag{3.1} $$

$$\begin{cases} {}_tD_T^{\alpha}p - \Delta p = y - y_d + \xi, & (x,t) \in \Omega_T, \\ p(x,t) = 0, & (x,t) \in \Gamma_T, \\ p(x,T) = 0, & x \in \Omega, \end{cases} \tag{3.2}$$

$$(\xi, v - y)_{\Omega_T} \le 0, \forall v \in \mathcal{K} \tag{3.3}$$

and

$$\gamma u + p = 0. \tag{3.4}$$

Proof. According to [17], for given $f + u \in L^2(\Omega_T)$ the following weak formulation admits a unique solution $y \in V$

$$\langle {}_0D_t^{\alpha}y, v\rangle_{H^{\frac{\alpha}{2}}(0,T;L^2(\Omega))} + (\nabla y, \nabla v)_{\Omega_T} = (f + u, v)_{\Omega_T}, \forall v \in V.$$

We introduce the operator S mapping form $L^2(\Omega_T)$ to V, and the operator B mapping form V to $L^1(\Omega_T)$ ($V \hookrightarrow L^1(\Omega_T)$). We define $G : L^2(\Omega_T) \to L^1(\Omega_T)$ by $G = S \circ B$. Then we can rewrite the optimal control problem as the following optimization problem

$$\min_{u \in L^2(\Omega_T)} \hat{J}(u) := J(Su, u) \tag{3.5}$$

subject to

$$Gu \in \mathcal{K}.$$

In order to derive the first order optimality condition, we set $H(u) := \int_{\Omega_T} y(u)dxdt - \delta$. Here $y(u) := Su$ is the solution of the state equation. According to [32, 33], there exists a real number $\xi \ge 0$ such that

$$\xi H(u) = 0 \tag{3.6}$$

and

$$G_u^{'}(u)(v - u) = 0, \forall v \in L^2(\Omega_T). \tag{3.7}$$

Here $G(u, \xi) = \hat{J}(u) + \xi H(u)$ denotes the Lagrange functional with ξ being the Lagrange multiplier.

By (3.6), we obtain

$$\begin{aligned} 0 &= \xi H(u) \\ &= \xi(\int_{\Omega_T} y(u)dxdt - \delta) \\ &= \xi(\int_{\Omega_T} (y(u) - v)dxdt) + \xi(\int_{\Omega_T} vdxdt - \delta). \end{aligned}$$

Then we have

$$(\xi, v - y(u)) = \xi(\int_{\Omega_T} vdxdt - \delta) \le 0.$$

By a simple calculation we derive

$$\begin{aligned}\hat{J}'(u)(v-u) &= \lim_{t\to 0^+}\frac{\hat{J}(u+t(v-u))-\hat{J}(u)}{t}\\ &= \int_{\Omega_T}(y(u)-y_d)y'(u)(v-u)dxdt+\gamma\int_{\Omega_T}u(v-u)dxdt\end{aligned}$$

and

$$\begin{aligned}H'(u)(v-u) &= \lim_{t\to 0^+}\frac{H(u+t(v-u))-H(u)}{t}\\ &= \int_{\Omega_T}y'(u)(v-u)dxdt.\end{aligned}$$

Thus we arrive at

$$\int_{\Omega_T}(y(u)-y_d)y'(u)(v-u)dxdt+\gamma\int_{\Omega_T}u(v-u)dxdt+\xi\int_{\Omega_T}y'(u)(v-u)dxdt=0. \tag{3.8}$$

By the state equation, we deduce

$$\begin{cases}{}_0D_t^{\alpha}y'(u)(v-u)-\Delta y'(u)(v-u)=v-u, & \text{in } \Omega_T,\\ y'(u)(v-u)=0, & \text{on } \Gamma_T,\\ y'(u)(v-u)(x,0)=0, & \text{in } \Omega_T.\end{cases}$$

To simplify (3.8), we introduce the adjoint state equation

$$\begin{cases}{}_tD_T^{\alpha}p-\Delta p=y-y_d+\xi, & \text{in } \Omega_T,\\ p(x,t)=0, & \text{on } \Gamma_T,\\ p(x,T)=0, & \text{in } \Omega_T.\end{cases}$$

By the integration by parts we deduce

$$\begin{aligned}&\int_{\Omega_T}(y-y_d+\xi)y'(u)(v-u)dxdt\\ =&\int_{\Omega_T}({}_tD_T^{\alpha}p-\Delta p)y'(u)(v-u)dxdt\\ =&\int_{\Omega_T}p\Big({}_0D_t^{\alpha}y'(u)(v-u)-\Delta y'(u)(v-u)\Big)dxdt\\ =&\int_{\Omega_T}p(v-u)dxdt.\end{aligned}$$

Combining this with (3.8), we have

$$\int_{\Omega_T}p(v-u)dxdt+\gamma\int_{\Omega_T}u(v-u)dxdt=0.$$

Thus we arrive at

$$G'_u(u)(v-u) = \int_{\Omega_T} (\gamma u + p)(v-u)dxdt = 0.$$

i.e.

$$\gamma u + p = 0.$$

□

Remark 3.2. *According to* (3.3)*, we can deduce that*

$$\begin{cases} \xi \geq 0, \int_{\Omega_T} ydxdt = \delta, \\ \xi = 0, \int_{\Omega_T} ydxdt < \delta. \end{cases}$$

In the following we investigate the regularity of the solution to the optimal control problem. Assume that η is the solution of the state equation with right hand term $g(x,t)$. Then according to [17] we have the following regularity estimates.

Lemma 3.3. *Assume that* $g \in L^2(0,T;L^2(\Omega)), 0 < \alpha < 1$. *Then we have*

$$\|\eta\|_{H^\alpha(0,T;L^2(\Omega))} + |\eta|_{H^{\frac{\alpha}{2}}(0,T;\dot{H}^1(\Omega))} + \|\eta\|_{L^2(0,T;\dot{H}^2(\Omega))} \leq C_\alpha \|g\|_{L^2(0,T;L^2(\Omega))}. \tag{3.9}$$

Furthermore, for $g \in H^{1-\alpha}(0,T;L^2(\Omega)), \frac{1}{2} < \alpha < 1$*, we have*

$$\|\eta\|_{H^1(0,T;L^2(\Omega))} + |\eta|_{H^{1-\frac{\alpha}{2}}(0,T;\dot{H}^1(\Omega))} + \|\eta\|_{L^2(0,T;\dot{H}^2(\Omega))} \leq C_{\alpha,T} \|g\|_{H^{1-\alpha}(0,T;L^2(\Omega))}. \tag{3.10}$$

Remark 3.4. *Note that ξ is a constant. Therefore the state and adjoint state have the same regularity to (3.9) for* $f, u, y_d \in L^2(0,T;L^2(\Omega)), 0 < \alpha < 1$.

Further we restrict that $f, y_d \in H^{1-\alpha}(0,T;L^2(\Omega)), \frac{1}{2} < \alpha < 1$. *Note that* $\gamma u = -p$ *and* $\frac{1}{2} < \alpha < 1$. *Then by (3.9) we have* $u \in H^\alpha(0,T;L^2(\Omega)) \subset H^{1-\alpha}(0,T;L^2(\Omega))$. *Combining with* $f \in H^{1-\alpha}(0,T;L^2(\Omega)), \frac{1}{2} < \alpha < 1$ *leads to an improved regularity for the state variable y:*

$$\|y\|_{H^1(0,T;L^2(\Omega))} + |y|_{H^{1-\frac{\alpha}{2}}(0,T;\dot{H}^1(\Omega))} + \|y\|_{L^2(0,T;\dot{H}^2(\Omega))} \leq C\|f+u\|_{H^{1-\alpha}(0,T;L^2(\Omega))}. \tag{3.11}$$

For $y_d \in H^{1-\alpha}(0,T;L^2(\Omega)), \frac{1}{2} < \alpha < 1$*, above result further improves the regularity of the adjoint state variable p, i.e.,*

$$\|p\|_{H^1(0,T;L^2(\Omega))} + |p|_{H^{1-\frac{\alpha}{2}}(0,T;\dot{H}^1(\Omega))} + \|p\|_{L^2(0,T;\dot{H}^2(\Omega))} \leq C\|y - y_d + \xi\|_{H^{1-\alpha}(0,T;L^2(\Omega))}. \tag{3.12}$$

3.2. Space-time discrete scheme

The weak formulation of the control problem can be characterized as:

$$\min_{(y,u)\in\mathcal{K}\times L^2(\Omega_T)} J(y,u) \tag{3.13}$$

subject to

$$({}_0D_t^\alpha y, v)_{\Omega_T} + (\nabla y, \nabla v)_{\Omega_T} = (f + u, v)_{\Omega_T}, \forall v \in L^2(0, T; \dot{H}^1(\Omega)). \tag{3.14}$$

Then the corresponding first order optimality system reads

$$({}_0D_t^\alpha y, v)_{\Omega_T} + (\nabla y, \nabla v)_{\Omega_T} = (f + u, v)_{\Omega_T}, \forall v \in L^2(0, T; \dot{H}^1(\Omega)), \tag{3.15}$$

$$({}_tD_T^\alpha p, w)_{\Omega_T} + (\nabla p, \nabla w)_{\Omega_T} = (y - y_d, w)_{\Omega_T} + \xi(1, w)_{\Omega_T}, \forall w \in L^2(0, T; \dot{H}^1(\Omega)), \tag{3.16}$$

$$\xi(1, v - y)_{\Omega_T} \leq \ 0, \forall v \in \mathcal{K} \tag{3.17}$$

and

$$\gamma u + p = 0. \tag{3.18}$$

Let $0 = t_0 < t_1 < ... < t_J = T$ be a partition of [0,T] with $\tau = \frac{T}{J}$. Set $I_j := (t_{j-1}, t_j)$ for each $1 \leq j \leq J$. Let $\mathcal{T}_h$ be a quasi-uniform triangulation of Ω. We denote by h the maximum diameter of the elements in $\mathcal{T}_h$ and define

$$\begin{aligned}
&\mathcal{M}_h = \{v \in H^1(\Omega) : v|_E \in P_1(E), \forall E \in \mathcal{T}_h\},\\
&\tilde{V}_h = \{v_h \in L^2(0, T; \mathcal{M}_h) : v_h(x, \cdot) \in \mathcal{M}_h, v_h(\cdot, t)|_{I_j} \in P_0, \forall 1 \leq j \leq J\},\\
&V_h = \{v_h \in L^2(0, T; \mathcal{M}_h \cap H_0^1(\Omega)) : v_h(x, \cdot) \in \mathcal{M}_h, v_h(\cdot, t)|_{I_j} \in P_0, \forall 1 \leq j \leq J\}.
\end{aligned}$$

Set $U_h = V_h \cap \mathcal{K}$. Then the space-time finite element approximation of control problem can be characterized as follows

$$\min_{(y_h,\ u_h) \in U_h \times \tilde{V}_h} J(y_h, u_h) \tag{3.19}$$

subject to

$$({}_0D_t^\alpha y_h, v_h)_{\Omega_T} + (\nabla y_h, \nabla v_h)_{\Omega_T} = (f + u_h, v_h)_{\Omega_T}, \forall v_h \in V_h. \tag{3.20}$$

In an analogous way to continuous case we can derive the discrete first order optimality system:

$$({}_0D_t^\alpha y_h, v_h)_{\Omega_T} + (\nabla y_h, \nabla v_h)_{\Omega_T} = (f + u_h, v_h)_{\Omega_T}, \forall v_h \in V_h, \tag{3.21}$$

$$({}_tD_T^\alpha p_h, w_h)_{\Omega_T} + (\nabla p_h, \nabla w_h)_{\Omega_T} = (y_h - y_d, w_h)_{\Omega_T} + \xi_h(1, w_h)_{\Omega_T}, \forall w_h \in V_h, \tag{3.22}$$

$$\xi_h(1, v_h - y_h)_{\Omega_T} \leq 0, \forall v_h \in U_h \tag{3.23}$$

and

$$\gamma u_h + p_h = 0. \tag{3.24}$$

Remark 3.5. *According to* (3.23)*, the discrete Lagrange multipliers satisfies*

$$\begin{cases} \xi_h \geq 0, \displaystyle\int_{\Omega_T} y_h dxdt = \delta, \\ \xi_h = 0, \displaystyle\int_{\Omega_T} y_h dxdt < \delta. \end{cases}$$

4. Error analysis

In this section we derive a priori error estimates for the space-time finite element discretization of the optimal control problem. For this purpose we need to introduce the following auxiliary problems for every $w \in L^2(0,T;\dot{H}^1(\Omega))$

$$({}_0D_t^\alpha y(u_h), w)_{\Omega_T} + (\nabla y(u_h), \nabla w)_{\Omega_T} = (f + u_h, w)_{\Omega_T}, \tag{4.1}$$

$$({}_tD_T^\alpha p(u_h), w)_{\Omega_T} + (\nabla p(u_h), \nabla w)_{\Omega_T} = (y(u_h) - y_d, w)_{\Omega_T} + \xi_h(1, w)_{\Omega_T}, \tag{4.2}$$

$$({}_tD_T^\alpha p(y_h), w)_{\Omega_T} + (\nabla p(y_h), \nabla w)_{\Omega_T} = (y_h - y_d, w)_{\Omega_T} + \xi_h(1, w)_{\Omega_T}. \tag{4.3}$$

For clarity we assume that $f, y_d \in L^2(0,T;L^2(\Omega))$ in the following analysis of Lemmas 4.1–4.3 and Theorems 4.4–4.5.

Lemma 4.1. *Assume that y and y_h are the solutions of* (3.15) *and* (3.21)*, respectively. Then we have*

$$\|y - y_h\|_{L^2(0,T;L^2(\Omega))} \leq C(h^2 + \tau^\alpha) + C\|u - u_h\|_{L^2(0,T;L^2(\Omega))}.$$

Proof. Combining (3.15) and (4.1) we have

$$({}_0D_t^\alpha (y - y(u_h)), w)_{\Omega_T} + (\nabla (y - y(u_h)), \nabla w)_{\Omega_T} = (u - u_h, w)_{\Omega_T}.$$

By Lemma 3.3 and the embedding theorem we have

$$\|y - y(u_h)\|_{L^2(0,T;L^2(\Omega))} \leq C\|u - u_h\|_{L^2(0,T;L^2(\Omega))}. \tag{4.4}$$

By (3.21) and (4.1) we get

$$({}_0D_t^\alpha (y(u_h) - y_h), v_h)_{\Omega_T} + (\nabla (y(u_h) - y_h), \nabla v_h)_{\Omega_T} = 0.$$

Since y_h is the finite element approximation of $y(u_h)$, according to [17] we have

$$\|y(u_h) - y_h\|_{L^2(0,T;L^2(\Omega))} \leq C(h^2 + \tau^\alpha). \tag{4.5}$$

By the triangle inequality we obtain

$$\begin{aligned} \|y - y_h\|_{L^2(0,T;L^2(\Omega))} &\leq \|y - y(u_h)\|_{L^2(0,T;L^2(\Omega))} + \|y(u_h) - y_h\|_{L^2(0,T;L^2(\Omega))} \\ &\leq C(h^2 + \tau^\alpha) + C\|u - u_h\|_{L^2(0,T;L^2(\Omega))}. \end{aligned}$$

□

Lemma 4.2. *Assume that (y, p, u, ξ) and (y_h, p_h, u_h, ξ_h) are the solutions of the optimality system* (3.15)–(3.18) *and the discrete counterpart, respectively. Then we have*

$$\|p - p_h\|_{L^2(0,T;L^2(\Omega))} \leq C(h^2 + \tau^\alpha) + C\Big(|\xi - \xi_h| + \|u - u_h\|_{L^2(0,T;L^2(\Omega))}\Big).$$

Proof. By (3.16) and (4.3) we have

$$({}_tD_T^{\alpha}(p-p(y_h)),w)_{\Omega_T}+(\nabla(p-p(y_h)),\nabla w)_{\Omega_T}=(y-y_h,w)_{\Omega_T}+(\xi-\xi_h,w)_{\Omega_T}.$$

By Lemma 2.1 and $w=p-p(y_h)$ we have

$$\begin{aligned}&\mathbb{C}_0|p-p(y_h)|^2_{H^{\frac{\alpha}{2}}(0,T;L^2(\Omega))}+\|\nabla(p-p(y_h))\|^2_{L^2(0,T;L^2(\Omega))}\\ \le\ &\Big(\|y-y_h\|_{L^2(0,T;L^2(\Omega))}+|\xi-\xi_h|\Big)\|p-p(y_h)\|_{L^2(0,T;L^2(\Omega))}.\end{aligned}$$

This implies

$$\|p-p(y_h)\|_{L^2(0,T;L^2(\Omega))}\le C\Big(\|y-y_h\|_{L^2(0,T;L^2(\Omega))}+|\xi-\xi_h|\Big).$$

Combining (3.22) and (4.3) we obtain

$$({}_tD_T^{\alpha}(p(y_h)-p_h),w_h)_{\Omega_T}+(\nabla(p(y_h)-p_h),\nabla w_h)_{\Omega_T}=0.$$

Note that p_h is the finite element approximation of $p(y_h)$. Then we can derive by [17]

$$\|p(y_h)-p_h\|_{L^2(0,T;L^2(\Omega))}\le C(h^2+\tau^{\alpha}). \tag{4.6}$$

Using the triangle inequality and Lemma 4.1 we obtain

$$\|p-p_h\|_{L^2(0,T;L^2(\Omega))}\le C(h^2+\tau^{\alpha})+C|\xi-\xi_h|+C\|u-u_h\|_{L^2(0,T;L^2(\Omega))}.$$

□

Next we are going to derive the estimate of $|\xi-\xi_h|$ and $\|u-u_h\|_{L^2(0,T;L^2(\Omega))}$.

Lemma 4.3. *Assume that* (y,p,u,ξ) *and* (y_h,p_h,u_h,ξ_h) *are the solutions of the optimality system* (3.15)–(3.18) *and the discrete counterpart, respectively. Then we have*

$$|\xi-\xi_h|\le C(h^2+\tau^{\alpha})+C\|u-u_h\|_{L^2(0,T;L^2(\Omega))}.$$

Proof. By (3.16) and (4.2) we have

$$({}_tD_T^{\alpha}(p-p(u_h)),w)_{\Omega_T}+(\nabla(p-p(u_h)),\nabla w)_{\Omega_T}=(y-y(u_h),w)_{\Omega_T}+(\xi-\xi_h,w)_{\Omega_T}. \tag{4.7}$$

Choosing $w=\psi\in V$ satisfy $\|\psi\|_V\le C$ and $\frac{1}{|\Omega_T|}\int_{\Omega_T}\psi dxdt=1$ and combining Lemma 3.3 we get

$$\begin{aligned}|\xi-\xi_h|\ &\le\ C|({}_tD_T^{\alpha}(p-p(u_h)),\psi)_{\Omega_T}|+C|(\nabla(p-p(u_h)),\nabla\psi)_{\Omega_T}|+C|(y-y(u_h),\psi)_{\Omega_T}|\\ &\le\ C|p-p(u_h)|_{H^{\frac{\alpha}{2}}(0,T;L^2(\Omega))}+C\|\nabla(p-p(u_h))\|_{L^2(0,T;L^2(\Omega))}\\ &\quad +C\|y-y(u_h)\|_{L^2(0,T;L^2(\Omega))}.\end{aligned} \tag{4.8}$$

Setting $P=\frac{1}{|\Omega_T|}\int_{\Omega_T}(p-p(u_h))dxdt$ and choosing $w=p-p(u_h)-P\psi$ in (4.7) leads to

$$\begin{aligned}&({}_tD_T^{\alpha}(p-p(u_h)),p-p(u_h))_{\Omega_T}+(\nabla(p-p(u_h)),\nabla(p-p(u_h)))_{\Omega_T}\\ =\ &({}_tD_T^{\alpha}(p-p(u_h)),P\psi)_{\Omega_T}+(\nabla(p-p(u_h)),\nabla(P\psi))_{\Omega_T}\end{aligned}$$

$$+(y-y(u_h),p-p(u_h)-P\psi)_{\Omega_T}+(\xi-\xi_h,p-p(u_h)-P\psi)_{\Omega_T}.$$

Since $(\xi-\xi_h,p-p(u_h)-P\psi)_{\Omega_T}=0$, then we can derive

$$\begin{aligned}&({}_tD_T^{\alpha}(p-p(u_h)),p-p(u_h))_{\Omega_T}+(\nabla(p-p(u_h)),\nabla(p-p(u_h)))_{\Omega_T}\\ =\ &({}_tD_T^{\alpha}(p-p(u_h)),P\psi)+(\nabla(p-p(u_h)),\nabla(P\psi))_{\Omega_T}+(y-y(u_h),p-p(u_h)-P\psi)_{\Omega_T}.\end{aligned}$$

Using Lemma 2.1 we have

$$\begin{aligned}&\mathbb{C}_0|p-p(u_h)|^2_{H^{\frac{\alpha}{2}}(0,T;L^2(\Omega))}+\|\nabla(p-p(u_h))\|^2_{L^2(0,T;L^2(\Omega))}\\ \leq\ &C\Big(|p-p(u_h)|_{H^{\frac{\alpha}{2}}(0,T;L^2(\Omega))}|P|+\|\nabla(p-p(u_h))\|_{L^2(0,T;L^2(\Omega))}|P|\\ &+\|y-y(u_h)\|_{L^2(0,T;L^2(\Omega))}|P|\Big)+\|y-y(u_h)\|_{L^2(0,T;L^2(\Omega))}\|p-p(u_h)\|_{L^2(0,T;L^2(\Omega))}.\end{aligned}$$

Note that $\|p-p(u_h)\|_{L^2(0,T;L^2(\Omega))}\leq C\|\nabla(p-p(u_h))\|_{L^2(0,T;L^2(\Omega))}$. Further we have by Young inequality

$$\begin{aligned}&\mathbb{C}_0|p-p(u_h)|^2_{H^{\frac{\alpha}{2}}(0,T;L^2(\Omega))}+\|\nabla(p-p(u_h))\|^2_{L^2(0,T;L^2(\Omega))}\\ \leq\ &C\Big(|p-p(u_h)|_{H^{\frac{\alpha}{2}}(0,T;L^2(\Omega))}|P|+\|\nabla(p-p(u_h))\|_{L^2(0,T;L^2(\Omega))}|P|\\ &+\|y-y(u_h)\|_{L^2(0,T;L^2(\Omega))}\|\nabla(p-p(u_h))\|_{L^2(0,T;L^2(\Omega))}+\|y-y(u_h)\|_{L^2(0,T;L^2(\Omega))}|P|\Big)\\ \leq\ &\frac{1}{2}\mathbb{C}_0|p-p(u_h)|^2_{H^{\frac{\alpha}{2}}(0,T;L^2(\Omega))}+C|P|^2+\frac{1}{4}\|\nabla(p-p(u_h))\|^2_{L^2(0,T;L^2(\Omega))}+C|P|^2\\ &+\frac{1}{4}\|\nabla(p-p(u_h))\|^2_{L^2(0,T;L^2(\Omega))}+C\|y-y(u_h)\|^2_{L^2(0,T;L^2(\Omega))}+\frac{1}{2}\|y-y(u_h)\|^2_{L^2(0,T;L^2(\Omega))}+\frac{1}{2}|P|^2\\ \leq\ &\frac{1}{2}\mathbb{C}_0|p-p(u_h)|^2_{H^{\frac{\alpha}{2}}(0,T;L^2(\Omega))}+C|P|^2+\frac{1}{2}\|\nabla(p-p(u_h))\|^2_{L^2(0,T;L^2(\Omega))}\\ &+C\|y-y(u_h)\|^2_{L^2(0,T;L^2(\Omega))}.\end{aligned}$$

Then we derive

$$\begin{aligned}&\mathbb{C}_0|p-p(u_h)|^2_{H^{\frac{\alpha}{2}}(0,T;L^2(\Omega))}+\|\nabla(p-p(u_h))\|^2_{L^2(0,T;L^2(\Omega))}\\ \leq\ &C|P|^2+C\|y-y(u_h)\|^2_{L^2(0,T;L^2(\Omega))}.\end{aligned}$$

By definition of P, (3.18), (3.24) and (4.5) we have

$$\begin{aligned}|P|\ =\ &\Big|\frac{1}{|\Omega_T|}\int_{\Omega_T}(p-p(u_h))dxdt\Big|\\ \leq\ &C\|p-p_h\|_{L^2(0,T;L^2(\Omega))}+C\|p_h-p(u_h)\|_{L^2(0,T;L^2(\Omega))}\\ \leq\ &C\|u-u_h\|_{L^2(0,T;L^2(\Omega))}+C\|p_h-p(y_h)\|_{L^2(0,T;L^2(\Omega))}+C\|p(y_h)-p(u_h)\|_{L^2(0,T;L^2(\Omega))}.\end{aligned}$$

Combining (4.2) and (4.3) we get

$$({}_tD_T^{\alpha}(p(y_h)-p(u_h)),w)_{\Omega_T}+(\nabla(p(y_h)-p(u_h)),\nabla w)_{\Omega_T}=(y_h-y(u_h),w)_{\Omega_T}.$$

By Lemma 2.1 and $w = p(y_h) - p(u_h)$ we have

$$\|p(y_h) - p(u_h)\|_{L^2(0,T;L^2(\Omega))} \leq C\|y_h - y(u_h)\|_{L^2(0,T;L^2(\Omega))}.$$

Combined with the above inequality, we get

$$\begin{aligned} |P| &\leq C\|u - u_h\|_{L^2(0,T;L^2(\Omega))} + C\|p_h - p(y_h)\|_{L^2(0,T;L^2(\Omega))} + C\|y(u_h) - y_h\|_{L^2(0,T;L^2(\Omega))} \\ &\leq C(h^2 + \tau^\alpha) + C\|u - u_h\|_{L^2(0,T;L^2(\Omega))}. \end{aligned}$$

Then we derive

$$\begin{aligned} &\sqrt{\mathbb{C}_0}|p - p(u_h)|_{H^{\frac{\alpha}{2}}(0,T;L^2(\Omega))} + \|\nabla(p - p(u_h))\|_{L^2(0,T;L^2(\Omega))} \\ \leq\ & C|P| + C\|y - y(u_h)\|_{L^2(0,T;L^2(\Omega))} \\ \leq\ & C(h^2 + \tau^\alpha) + C\|u - u_h\|_{L^2(0,T;L^2(\Omega))}. \end{aligned}$$

Inserting the above estimate into (4.8) we obtain

$$|\xi - \xi_h| \leq C(h^2 + \tau^\alpha) + C\|u - u_h\|_{L^2(0,T;L^2(\Omega))}. \tag{4.9}$$

□

Finally we need to estimate $\|u - u_h\|_{L^2(0,T;L^2(\Omega))}$.

Theorem 4.4. *Assume that* (y, p, u, ξ) *and* (y_h, p_h, u_h, ξ_h) *be the solutions of* (3.15)–(3.18) *and the discrete counterpart, respectively. Then the following estimate holds*

$$\|u - u_h\|_{L^2(0,T;L^2(\Omega))} \leq C(h^2 + \tau^\alpha).$$

Proof. By (3.18) and (3.24) we have

$$\begin{aligned} &\gamma\|u - u_h\|^2_{L^2(0,T;L^2(\Omega))} \\ =\ & (\gamma u - \gamma u_h, u - u_h)_{\Omega_T} \\ =\ & (p_h - p, u - u_h)_{\Omega_T} \\ =\ & (p(y_h) - p, u - u_h)_{\Omega_T} - (p(y_h) - p_h, u - u_h)_{\Omega_T}. \end{aligned}$$

Combining (3.15) and (4.1) and choosing $v = p(y_h) - p$ leads to

$$\begin{aligned} &({}_0D_t^\alpha(y - y(u_h)), p(y_h) - p)_{\Omega_T} + (\nabla(y - y(u_h)), \nabla(p(y_h) - p))_{\Omega_T} \\ =\ & (u - u_h, p(y_h) - p)_{\Omega_T}. \end{aligned}$$

Using (3.16) and (4.3), and setting $w = y - y(u_h)$ yields

$$\begin{aligned} &({}_tD_T^\alpha(p(y_h) - p), y - y(u_h))_{\Omega_T} + (\nabla(p(y_h) - p), \nabla(y - y(u_h)))_{\Omega_T} \\ =\ & (y_h - y, y - y(u_h))_{\Omega_T} + (\xi_h - \xi, y - y(u_h))_{\Omega_T}. \end{aligned}$$

By using Lemma 2.1 we derive

$$({}_0D_t^\alpha(y - y(u_h)), p(y_h) - p)_{\Omega_T} + (\nabla(y - y(u_h)), \nabla(p(y_h) - p))_{\Omega_T}$$

$$= (y_h - y, y - y(u_h))_{\Omega_T} + (\xi_h - \xi, y - y(u_h))_{\Omega_T}.$$

Using above inequalities we further derive

$$\begin{aligned}
& \gamma\|u - u_h\|^2_{L^2(0,T;L^2(\Omega))} \\
= & (y_h - y, y - y(u_h))_{\Omega_T} + (\xi_h - \xi, y - y(u_h))_{\Omega_T} - (p(y_h) - p_h, u - u_h)_{\Omega_T} \\
= & (y_h - y(u_h), y - y(u_h))_{\Omega_T} + (y(u_h) - y, y - y(u_h))_{\Omega_T} + (\xi_h - \xi, y_h - y(u_h))_{\Omega_T} \\
& -(\xi_h - \xi, y_h - y)_{\Omega_T} - (p(y_h) - p_h, u - u_h)_{\Omega_T}.
\end{aligned}$$

Further we have

$$\begin{aligned}
& \gamma\|u - u_h\|^2_{L^2(0,T;L^2(\Omega))} + \|y - y(u_h)\|^2_{L^2(0,T;L^2(\Omega))} \\
= & (y_h - y(u_h), y - y(u_h))_{\Omega_T} + (\xi - \xi_h, y(u_h) - y_h)_{\Omega_T} - (\xi - \xi_h, y - y_h)_{\Omega_T} \\
& -(p(y_h) - p_h, u - u_h)_{\Omega_T}.
\end{aligned}$$

Indeed, we have

$$\begin{cases}
\xi(1, y - y_h) \geq 0, \displaystyle\int_{\Omega_T} y dxdt = \delta, \\
\xi(1, y - y_h) = 0, \displaystyle\int_{\Omega_T} y dxdt < \delta
\end{cases}$$

and

$$\begin{cases}
-\xi_h(1, y - y_h) \geq 0, \displaystyle\int_{\Omega_T} y_h dxdt = \delta, \\
-\xi_h(1, y - y_h) = 0, \displaystyle\int_{\Omega_T} y_h dxdt < \delta.
\end{cases}$$

Then we have $(\xi - \xi_h, y - y_h)_{\Omega_T} \geq 0$. By Lemma 4.3 and Young inequality, we deduce

$$\begin{aligned}
& \gamma\|u - u_h\|^2_{L^2(0,T;L^2(\Omega))} + \|y - y(u_h)\|^2_{L^2(0,T;L^2(\Omega))} \\
\leq & C(\varepsilon)\Big(\|y_h - y(u_h)\|^2_{L^2(0,T;L^2(\Omega))} + \|p(y_h) - p_h\|^2_{L^2(0,T;L^2(\Omega))} + (h^2 + \tau^\alpha)^2\Big) \\
& +\varepsilon\Big(\|y - y(u_h)\|^2_{L^2(0,T;L^2(\Omega))} + \|u - u_h\|^2_{L^2(0,T;L^2(\Omega))}\Big).
\end{aligned}$$

Using (4.4)–(4.6) we conclude

$$\|u - u_h\|_{L^2(0,T;L^2(\Omega))} \leq C(h^2 + \tau^\alpha).$$

□

Based on above estimates, we can obtain the following result.

Theorem 4.5. *Assume that* (y, p, u, ξ) *and* (y_h, p_h, u_h, ξ_h) *are the solution of* (3.15)–(3.18) *and* (3.21)–(3.24)*, respectively. Then the following estimate holds*

$$\|y - y_h\|_{L^2(0,T;L^2(\Omega))} + \|p - p_h\|_{L^2(0,T;L^2(\Omega))} + |\xi - \xi_h| \leq C(h^2 + \tau^\alpha).$$

Proof. Combining Lemma 4.1–4.4 leads to the theorem results. □

Remark 4.6. *If $f, y_d \in H^{1-\alpha}(0,T;L^2(\Omega))$, according to [17], we get $\|y(u_h) - y_h\|_{L^2(0,T;L^2(\Omega))} \le C(h^2+\tau)$ and $\|p(y_h) - p_h\|_{L^2(0,T;L^2(\Omega))} \le C(h^2+\tau)$. By analogy with the above error estimate derivation process, we get the following error estimates*

$$\|y - y_h\|_{L^2(0,T;L^2(\Omega))} + \|p - p_h\|_{L^2(0,T;L^2(\Omega))} + |\xi - \xi_h| \le C(h^2+\tau).$$

5. Numerical experiments

In this section, we use the projected gradient algorithm ([5]) to effectively solve the optimal control problem.

5.1. Projected Gradient Algorithm

Set the objective gradient function $d^k = \gamma u_h^k + p_h^k$. Then we have

$$u_h^{k+1} = u_h^k - \rho d^k, \rho > 0.$$

Here

$$({}_0D_t^\alpha y_h^k, v_h)_{\Omega_T} + (\nabla y_h^k, \nabla v_h)_{\Omega_T} = (f + u_h^k, v_h)_{\Omega_T}, \forall v_h \in V_h$$

and

$$({}_tD_T^\alpha p_h^k, w_h)_{\Omega_T} + (\nabla p_h^k, \nabla w_h)_{\Omega_T} = (y_h^k - y_d, w_h)_{\Omega_T} + \xi_h^k(1, w_h)_{\Omega_T}, \forall w_h \in V_h.$$

We set $p_h^k = \hat{p}_h^k + \xi_h^k \psi_h$ with

$$({}_tD_T^\alpha \hat{p}_h^k, w_h)_{\Omega_T} + (\nabla \hat{p}_h^k, \nabla w_h)_{\Omega_T} = (y_h^k - y_d, w_h)_{\Omega_T}, \forall w_h \in V_h$$

and

$$({}_tD_T^\alpha \psi_h, w_h)_{\Omega_T} + (\nabla \psi_h, \nabla w_h)_{\Omega_T} = (1, w_h)_{\Omega_T}, \forall w_h \in V_h.$$

Further, we define $\hat{y}_h^{k+1}$ by

$$({}_0D_t^\alpha \hat{y}_h^{k+1}, v_h)_{\Omega_T} + (\nabla \hat{y}_h^{k+1}, \nabla v_h)_{\Omega_T} = (f + \hat{u}_h^{k+1}, v_h)_{\Omega_T}, \forall v_h \in V_h.$$

Here $\hat{u}_h^{k+1} = u_h^k - \rho(\gamma u_h^k + \hat{p}_h^k)$. Note that

$$({}_0D_t^\alpha y_h^{k+1}, v_h)_{\Omega_T} + (\nabla y_h^{k+1}, \nabla v_h)_{\Omega_T} = (f + u_h^{k+1}, v_h)_{\Omega_T}, \forall v_h \in V_h.$$

Then we have

$$y_h^{k+1} = \hat{y}_h^{k+1} - \rho \xi_k^h \varphi_h.$$

Here φ_h satisfies

$$({}_0D_t^\alpha \varphi_h, w_h)_{\Omega_T} + (\nabla \varphi_h, \nabla w_h)_{\Omega_T} = (\psi_h, w_h)_{\Omega_T}, \forall w_h \in V_h.$$

In order to guarantee the state constraints, we choose ξ_h^k as follows

$$\xi_h^k = \frac{1}{\rho \int_{\Omega_T} \varphi_h dxdt} \max\Big\{ \int_{\Omega_T} \hat{y}_h^{k+1} dxdt - \delta, 0 \Big\}.$$

It is easy to check that $\int_{\Omega_T} y_h^{k+1} dxdt \leq \delta$. The projected gradient algorithm is described in detail.

Algorithm 1:

1: Given initial value $u_h^0 = 0$, and parameter $\rho > 0$. Solve the following equation:

$$\begin{aligned}
({}_tD_T^\alpha \psi_h, w_h)_{\Omega_T} + (\nabla \psi_h, \nabla w_h)_{\Omega_T} &= (1, w_h)_{\Omega_T}, \forall w_h \in V_h, \\
({}_0D_t^\alpha \varphi_h, w_h)_{\Omega_T} + (\nabla \varphi_h, \nabla w_h)_{\Omega_T} &= (\psi_h, w_h)_{\Omega_T}, \forall w_h \in V_h.
\end{aligned}$$

2: Solve the following equation:

$$\begin{aligned}
({}_0D_t^\alpha y_h^k, v_h)_{\Omega_T} + (\nabla y_h^k, \nabla v_h)_{\Omega_T} &= (f + u_h^k, v_h)_{\Omega_T}, \forall v_h \in V_h, \\
({}_tD_T^\alpha \hat{p}_h^k, w_h)_{\Omega_T} + (\nabla \hat{p}_h^k, \nabla w_h)_{\Omega_T} &= (y_h^k - y_d, w_h)_{\Omega_T}, \forall w_h \in V_h.
\end{aligned}$$

3: Compute $\hat{u}_h^{k+1} = u_h^k - \rho(\gamma u_h^k + \hat{p}_h^k)$ and solve the following equation:

$$({}_0D_t^\alpha \hat{y}_h^{k+1}, v_h)_{\Omega_T} + (\nabla \hat{y}_h^{k+1}, \nabla v_h)_{\Omega_T} = (f + \hat{u}_h^{k+1}, v_h)_{\Omega_T}, \forall v_h \in V_h.$$

4: Compute

$$\xi_h^k = \frac{1}{\rho \int_{\Omega_T} \varphi_h dxdt} \max\Big\{ \int_{\Omega_T} \hat{y}_h^{k+1} dxdt - \delta, 0 \Big\}.$$

and $p_h^k = \hat{p}_h^k + \xi_h^k \psi_h$.

5: Compute $u_h^{k+1} = u_h^k - \rho(\gamma u_h^k + p_h^k)$. and solve the following equation:

$$({}_0D_t^\alpha y_h^{k+1}, v_h)_{\Omega_T} + (\nabla y_h^{k+1}, \nabla v_h)_{\Omega_T} = (f + u_h^{k+1}, v_h)_{\Omega_T}, \forall v_h \in V_h.$$

6: If the stop condition is reached, end the loop. Otherwise, update $k = k + 1$ and go to step (2).

5.2. Numerical Examples

Example 5.1. *In this example, we consider the optimal control problem with* $\Omega = [0, 1], T = 1$. *The exact solution are given by*

$$\begin{aligned}
y &= t^s sin(2\pi x), \\
p &= (1 - t)^s sin(\pi x), \\
u &= -(1 - t)^s sin(\pi x).
\end{aligned}$$

Table 1. $L^2(0,T;L^2(\Omega))$ errors and convergence rates for space variable with $\alpha = 1/3$ and $s = 2$.

M	J	$\|\|y-y_h\|\|_{L^2(0,T;L^2(\Omega))}$	order	$\|\|u-u_h\|\|_{L^2(0,T;L^2(\Omega))}$	order	$\|\xi-\xi_h\|$	order
2^3	2^3	0.06827755	–	0.07066092	–	0.45135857	–
2^4	2^5	0.01675417	2.0269	0.01712133	2.0451	0.12283095	1.8776
2^5	2^7	0.00417107	2.0060	0.00419205	2.0301	0.03229044	1.9275
2^6	2^9	0.00104207	2.0010	0.00103673	2.0156	0.00827863	1.9636
2^7	2^{11}	2.6052e-04	2.0000	2.5779e-04	2.0077	0.00209525	1.9823

Table 2. $L^2(0,T;L^2(\Omega))$ errors and convergence rates for time variable with $\alpha = 1/3$ and $s = 2$.

M	J	$\|\|y-y_h\|\|_{L^2(0,T;L^2(\Omega))}$	order	$\|\|u-u_h\|\|_{L^2(0,T;L^2(\Omega))}$	order	$\|\xi-\xi_h\|$	order
2^3	2^3	0.06827755	–	0.07066092	–	0.45135857	–
2^4	2^4	0.02986199	1.1931	0.03384772	1.0619	0.25331048	0.8334
2^5	2^5	0.01384412	1.1090	0.01663089	1.0252	0.13259718	0.9339
2^6	2^6	0.00665006	1.0578	0.00826286	1.0092	0.06748221	0.9745
2^7	2^7	0.00325723	1.0297	0.00412156	1.0035	0.03399226	0.9893

Table 3. $L^2(0,T;L^2(\Omega))$ errors and convergence rates for space variable with $\alpha = 2/3$ and $s = 2$.

M	J	$\|\|y-y_h\|\|_{L^2(0,T;L^2(\Omega))}$	order	$\|\|u-u_h\|\|_{L^2(0,T;L^2(\Omega))}$	order	$\|\xi-\xi_h\|$	order
2^3	2^3	0.06768648	–	0.06859494	–	0.44263663	–
2^4	2^5	0.01662731	2.0253	0.01660560	2.0464	0.12156339	1.8644
2^5	2^7	0.00414497	2.0041	0.00406390	2.0307	0.03221745	1.9158
2^6	2^9	0.00103661	1.9995	0.00100463	2.0162	0.00831098	1.9548
2^7	2^{11}	2.5934e-04	1.9990	2.4975e-04	2.0081	0.00211248	1.9761

Table 4. $L^2(0,T;L^2(\Omega))$ errors and convergence rates for time variable with $\alpha = 2/3$ and $s = 2$.

M	J	$\|\|y-y_h\|\|_{L^2(0,T;L^2(\Omega))}$	order	$\|\|u-u_h\|\|_{L^2(0,T;L^2(\Omega))}$	order	$\|\xi-\xi_h\|$	order
2^3	2^3	0.06768648	–	0.06859494	–	0.44263663	–
2^4	2^4	0.02962587	1.1920	0.03279230	1.0647	0.25001564	0.8241
2^5	2^5	0.01375176	1.1072	0.01610694	1.0257	0.13154320	0.9265
2^6	2^6	0.00661364	1.0561	0.00800377	1.0089	0.06725091	0.9679
2^7	2^7	0.00324272	1.0282	0.00399304	1.0032	0.03401301	0.9835

Table 5. $L^2(0,T;L^2(\Omega))$ errors and convergence rates for space variable with $\alpha = 0.4$ and $s = 0.51$.

M	J	$\|y - y_h\|_{L^2(0,T;L^2(\Omega))}$	order	$\|u - u_h\|_{L^2(0,T;L^2(\Omega))}$	order	$\vert\xi - \xi_h\vert$	order
2^3	2^3	0.06128791	–	0.07655843	–	0.58083469	–
2^4	2^5	0.01672425	1.8737	0.02362219	1.6964	0.13448958	2.1106
2^5	2^7	0.00447242	1.9028	0.00681702	1.7929	0.03242069	2.0525
2^6	2^9	0.00118203	1.9198	0.00190351	1.8405	0.00795783	2.0265
2^7	2^{11}	3.0990e-04	1.9314	5.2116e-04	1.8689	0.00197047	2.0138

Table 6. $L^2(0,T;L^2(\Omega))$ errors and convergence rates for time variable with $\alpha = 0.4$ and $s = 0.51$.

M	J	$\|y - y_h\|_{L^2(0,T;L^2(\Omega))}$	order	$\|u - u_h\|_{L^2(0,T;L^2(\Omega))}$	order	$\vert\xi - \xi_h\vert$	order
2^3	2^3	0.06128791	–	0.07655843	–	0.58083469	–
2^4	2^4	0.02666199	1.2008	0.04268519	0.8428	0.28928895	1.0056
2^5	2^5	0.01288063	1.0496	0.02344623	0.8644	0.14138401	1.0329
2^6	2^6	0.00654646	0.9764	0.01269053	0.8856	0.06895288	1.0359
2^7	2^7	0.00339134	0.9489	0.00679131	0.9020	0.03375646	1.0304

In this example we consider a uniform partition for space with $h = \frac{1}{M}$. We first consider the case with smooth solution. Tables 1–4 present the errors and convergence orders of state, control and Lagrange multiplier for $\alpha = \frac{1}{3}$ and $\frac{2}{3}$ with $s = 2$. Tables 5 and 6 present the errors and convergence orders of state, control and Lagrange multiplier for $\alpha = 0.4$ with $s = 0.51$. We can observe that the convergence rates with respect to time and space variable are in agreement with the theoretical findings predicted in Remark 4.6, i.e., first order convergence in time and second order convergence in space.

Secondly we consider the nonsmooth case about time variable with $s = \alpha - 0.49$. In this case the right hand term f and y_d with respect to time belong to $L^2(0,T)$. The errors and convergence rates of state, control and Lagrange multiplier are listed in Tables 7 and 8 with $\alpha = 2/3$. In this case we can observe that the convergence rate for space variable and the convergence rate for time variable approach to 2 and α, which are in agreement with the theoretical findings given in Theorem 4.5.

Table 7. $L^2(0,T;L^2(\Omega))$ errors and convergence rates for space variable with $\alpha = 2/3$ and $s = \alpha - 0.49$.

M	J	$\|y - y_h\|_{L^2(0,T;L^2(\Omega))}$	order	$\|u - u_h\|_{L^2(0,T;L^2(\Omega))}$	order	$\vert\xi - \xi_h\vert$	order
2^3	10	0.05186062	–	0.02626828	–	0.08453145	–
2^4	10^2	0.01142312	2.1827	0.00623473	2.0749	0.02266773	1.8988
2^5	10^3	0.00263417	2.1165	0.00157102	1.9886	0.00597317	1.9241
2^6	10^4	6.3914e-04	2.0431	4.0675e-04	1.9495	0.00158371	1.9152

Table 8. $L^2(0,T;L^2(\Omega))$ errors and convergence rates for time variable with $\alpha = 2/3$ and $s = \alpha - 0.49$.

M	J	$\|y-y_h\|_{L^2(0,T;L^2(\Omega))}$	order	$\|u-u_h\|_{L^2(0,T;L^2(\Omega))}$	order	$\|\xi-\xi_h\|$	order
2^7	2^7	0.00498051	–	0.00522381	–	0.00402921	–
2^8	2^8	0.00314271	0.6643	0.00339663	0.6210	0.00246073	0.7114
2^9	2^9	0.00200124	0.6511	0.00221703	0.6155	0.00156024	0.6573
2^{10}	2^{10}	0.00128416	0.6401	0.00145172	0.6109	0.00101600	0.6189

6. Conclusion

In this paper we discussed a space-time finite element approximation of time fractional optimal control problem with integral state constraint. A priori error estimate for the discrete scheme is derived. Numerical examples are presented to illustrate the theoretical findings.

In our future work we are going to investigate the finite element approximation of time fractional optimal control problem with integral state constraint taking the form $\int_\Omega y(x,t) \le \delta, t \in [0,T]$.

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Conflict of interest

The authors declare there is no conflict of interests.

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