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Research article

A new generalized family of distributions: Properties and applications

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Abstract: We come up with a new class called log-logistic tan generalized family which provides sub-models with left skewed, symmetrical, right skewed, unimodal, bimodal and reversed-J densities, and increasing, decreasing, modified bathtub, bathtub, unimodal, reversed-J shaped, and J-shaped hazard rates. Some of its sub-models are provided along with some general structural properties. The parameter estimation has been conducted via maximum likelihood. Moreover, the estimators behavior are assessed using various simulation results. The capability of the log-logistic tan-Weibull model is proved using two real-life data sets. It provides higher quality fit than competing Weibull extensions, among others.

Keywords: bathtub failure rates; maximum likelihood estimation; order statistics; simulation; Weibull distribution
Mathematics Subject Classification: 60E05, 62F10

1. Introduction

The classical probability distributions were generalized through the induction of location, scale and shape parameters. Recently, appreciable attempts have been made in the development of the new probability distributions which have big privilege of more flexibility, fitting specific and several real world sequence of events.

The improvement in the G-Classes revolution began with the fundamental article of Alzaatreh et al. [9] in which they proposed *transformed (T)-transformer (X)* (*T-X*) family.

Consider the random variable (rv) $T \in [c, d]$ for $-\infty \leq c < d \leq \infty$ with probability density function (pdf) $r(t)$, and consider a link function, $W(\cdot) : [0, 1] \to R$, that satisfy the conditions: For any baseline cumulative distribution function (cdf) $G(x)$, $W[G(x)] \in [c, d]$, is monotonically non-decreasing and differentiable, for $x \to -\infty$, $W[G(x)] \to c$, and for $x \to \infty$, $W[G(x)] \to d$. Hence, the cdf of the *T-X* class has the form

$$F(x) = \int_a^{W[G(x)]} r(t)\, dt. \tag{1.1}$$

Many authors constructed extended generalized families by using T-X approach. Some examples of generalized classes are, beta-G [19], Kw-G type-1 [17], log-gamma-G type-2 [10], gamma-X [40], exponentiated T-X [9], Weibull-G [11], exponentiated-Weibull-H [13] and generalized odd Lindley-G [3].

Motivated by the new prospect in term of accuracy and exibility of a new distribution, we propose a new propose a new flexible family called, log-logistic tan generalized (LLT-G) family which provides greater accuracy and flexibility in fitting real-life data. Some general properties of the LLT-G class will be provided here. We provide two applications for one special sub-model of the proposed class, called log-logistic tan-Weibull (LLT-W) distribution which has decreasing, increasing, bathtub and unimodal hazard rate functions.

Let $r(t) = \frac{c}{s}\left(\frac{t}{s}\right)^{c-1}\left[1 + (\frac{t}{s})^C\right]^{-2}$ be the pdf of a rv $0 < t < \infty$ and $W[G(x)] = \tan\left[\frac{\pi}{2}\, G^{\alpha}(x)\right]$ which satisfies the conditions of T-X family. The cdf and pdf of the new LLT-G family take the forms

$$F(x) = \int_0^{tan\left[\frac{\pi}{2} G^{\alpha}(x)\right]} r(t)\, dt = 1 - \left(1 + s^{-c}\left\{\tan\left[\frac{\pi}{2}\, G^{\alpha}(x)\right]\right\}^{c}\right)^{-1}, \qquad x > 0,\ \alpha, c, s > 0 \tag{1.2}$$

and

$$\begin{aligned} f(x) &= \frac{\pi\,\alpha\,c}{2\,s^{c}}\, g(x)\,[G(x)]^{\alpha-1}\left\{\sec^{2}\left[\frac{\pi}{2}\, G^{\alpha}(x)\right]\right\}\left\{\tan\left[\frac{\pi}{2}\, G^{\alpha}(x)\right]\right\}^{c-1} \\ &\times \left(1 + s^{-c}\left\{\tan\left[\frac{\pi}{2}\, G^{\alpha}(x)\right]\right\}^{c}\right)^{-2}, \qquad x > 0,\ \alpha, c, s > 0. \end{aligned} \tag{1.3}$$

The LLT-G hazard rate function (hrf) has the form

$$\begin{aligned} h(x) &= \frac{\pi\,\alpha\,c}{2\,s^{c}}\, g(x)\,[G(x)]^{\alpha-1}\left[\sec^{2}\left(\frac{\pi}{2}\, G^{\alpha}(x)\right)\right]\left\{\tan\left[\frac{\pi}{2}\, G^{\alpha}(x)\right]\right\}^{c-1} \\ &\times \left(1 + s^{-c}\left\{\tan\left[\frac{\pi}{2}\, G^{\alpha}(x)\right]\right\}^{c}\right)^{-1}, \qquad x > 0,\ \alpha, c, s > 0. \end{aligned} \tag{1.4}$$

Henceforth, the rv with PDF (1.3) is denoted by X ~LLT-G(α, c, s).

The LLT-G class has some desirable properties including the following.

(i) The special sub-models of the LLT-G class can provide left skewed, symmetrical, right skewed, unimodal, bimodal and reversed-J densities, and increasing, modified bathtub, decreasing, bathtub, upside-down bathtub, reversed-J shaped, and J-shaped hazard rates. Hence, their special sub-models are capable of fitting different shapes of failure criteria.

(ii) The LLT-G class provides extended versions of baseline distributions with closed forms for the cdf and hrf. Hence the special sub-models of this family can be used in modeling and analyzing censored data.

(iii) The LLT-Weibull as a special sub-model of LLT-G class provides adequate fits than other modified models generated by other existing families under same baseline model.

Recently, several authors have proposed many generalized classes based on the logistic and log logistic distributions. For example, Afify et al. [4], Alizadeh et al. [5], Alizadeh et al. [6], Alizadeh et al. [7], Altun et al. [8], Cordeiro et al. [14], Cordeiro et al. [15], Cordeiro et al. [16], Gleaton and Lynch [20], Mehdi et al. [21], Hassan et al [26], Haghbin et al. [27], Korkmaz et al. [28], Mahadevi et al. [29] and Torabi and Montazeri [40].

This paper is outlined as follows. Five special models of the LLT-G family are provided in Section 2. In Section 3, we derived some mathematical properties of LLT-G family. Estimation of the LLT-G parameters using maximum likelihood is discussed in Section 4. Further, we present simulations for the LLT-W model to address the performance of the proposed estimators in Section 4. In Section 5, we show the importance and applicability of the LLT-G family via two real-life data applications. In Section 6, the paper is summarized.

2. Five special distributions

We study five sub-models of the LLT-G class using the baseline Weibull, normal, Rayleigh, exponential and Burr XII distributions (1.3) and provide some plots for their pdfs and hrfs. Figures 1–5 reveals that the special sub-models of the LLT-G class can provide left skewed, symmetrical, right skewed, unimodal, bimodal and reversed-J densities, and increasing, modified bathtub, decreasing, bathtub, unimodal, reversed-J shaped, and J-shaped failure rates.

2.1. LLT-Weibull (LLT-W) distribution

Using the Weibull pdf, $g(x) = b\,x^{b-1}\,\mathrm{e}^{-x^b}$, $x > 0,\ b > 0$, we obtain the cdf of the LLT-W distribution

$$F(x) = 1 - \left(1 + s^{-c}\left\{\tan\left[\frac{\pi}{2}\left(1 - \mathrm{e}^{-x^b}\right)^{\alpha}\right]\right\}^{c}\right)^{-1}. \tag{2.1}$$

The pdf associated with (2.1) reduces to

$$\begin{aligned} f(x) &= \frac{\pi\,\alpha b\,c}{2\,s^c}\,x^{b-1}\,\mathrm{e}^{-x^b}\left(1 - \mathrm{e}^{-x^b}\right)^{\alpha-1}\left\{\sec^2\left[\frac{\pi}{2}\left(1 - \mathrm{e}^{-x^b}\right)^{\alpha}\right]\right\} \\ &\times \left\{\tan\left[\frac{\pi}{2}\left(1 - \mathrm{e}^{-x^b}\right)^{\alpha}\right]\right\}^{c-1}\left(1 + s^{-c}\left\{\tan\left[\frac{\pi}{2}\left(1 - \mathrm{e}^{-x^b}\right)^{\alpha}\right]\right\}^{c}\right)^{-2}. \end{aligned} \tag{2.2}$$

The plots in Figure 1 depict pdf and hrf shapes of the LLT-W distribution for different parametric values.

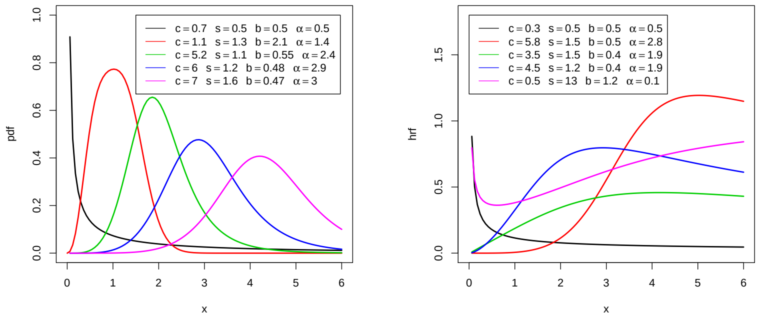


Figure 1. The LLT-W density (left) and hazard rate (right) plots for different parametric values.

2.2. *LLT-normal (LLT-N) distribution*

The pdf of normal distribution is $g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-x^2}{2\sigma^2}}$, $x \in \Re$, $\sigma > 0$. The cdf of the LLT-N reduces to

$$F(x) = 1 - \left\{1 + s^{-c}\left[\tan\left(\frac{\pi}{2}\left\{\frac{1}{2}\left[1 + erf\left(\frac{x}{\sigma\sqrt{2}}\right)\right]\right\}^{\alpha}\right)\right]^{c}\right\}^{-1}. \tag{2.3}$$

The LLT-N pdf takes the form

$$\begin{aligned} f(x) &= \frac{\pi\,\alpha\,c}{2\,s^c\sigma\sqrt{2\pi}} e^{\frac{-x^2}{2\sigma^2}} \left\{\frac{1}{2}\left[1 + erf\left(\frac{x}{\sigma\sqrt{2}}\right)\right]\right\}^{\alpha-1} \left[\sec^2\left(\frac{\pi}{2}\left\{\frac{1}{2}\left[1 + erf\left(\frac{x}{\sigma\sqrt{2}}\right)\right]\right\}^{\alpha}\right)\right] \\ &\times \left[\tan\left(\frac{\pi}{2}\left\{\frac{1}{2}\left[1 + erf\left(\frac{x}{\sigma\sqrt{2}}\right)\right]\right\}^{\alpha}\right)\right]^{c-1} \left\{1 + s^{-c}\left[\tan\left(\frac{\pi}{2}\left\{\frac{1}{2}\left[1 + erf\left(\frac{x}{\sigma\sqrt{2}}\right)\right]\right\}^{\alpha}\right)\right]^{c}\right\}^{-2} \end{aligned} \tag{2.4}$$

Some plots of the density and hazard rate functions of the LLT-N distribution for several parametric values are shown in Figure 2.

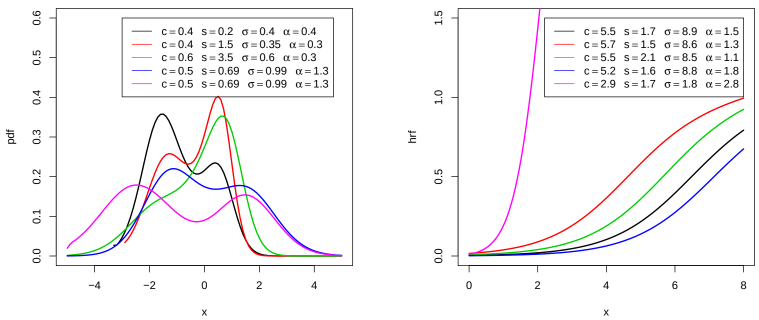


Figure 2. The LLT-N density (left) and hazard rate (right) plots for different parametric values.

2.3. LLT-Rayleigh (LLT-R) distribution

Using the Rayleigh *rv* with pdf, $g(x) = 2\,a\,x\,\mathrm{e}^{-a\,x^2}$, $x > 0$, $a > 0$, the cdf of the LLT-R distribution becomes

$$F(x) = 1 - \left(1 + s^{-c}\left\{\tan\left[\frac{\pi}{2}\left(1 - \mathrm{e}^{-a\,x^2}\right)^{\alpha}\right]\right\}^{c}\right)^{-1}. \tag{2.5}$$

The pdf of the LLT-R reduces to

$$\begin{aligned} f(x) &= \frac{\pi\,\alpha\,c}{a\,s^c}\,x\,\mathrm{e}^{-a\,x^2}\left(1 - \mathrm{e}^{-a\,x^2}\right)^{\alpha-1}\left\{\sec^2\left[\frac{\pi}{2}\left(1 - \mathrm{e}^{-a\,x^2}\right)^{\alpha}\right]\right\} \\ &\times \left\{\tan\left[\frac{\pi}{2}\left(1 - \mathrm{e}^{-a\,x^2}\right)^{\alpha}\right]\right\}^{c-1}\left(1 + s^{-c}\left\{\tan\left[\frac{\pi}{2}\left(1 - \mathrm{e}^{-a\,x^2}\right)^{\alpha}\right]\right\}^{c}\right)^{-2}. \end{aligned} \tag{2.6}$$

Figure 3 depicts possible plots of the LLT-R pdf and hrf for various parametric values.

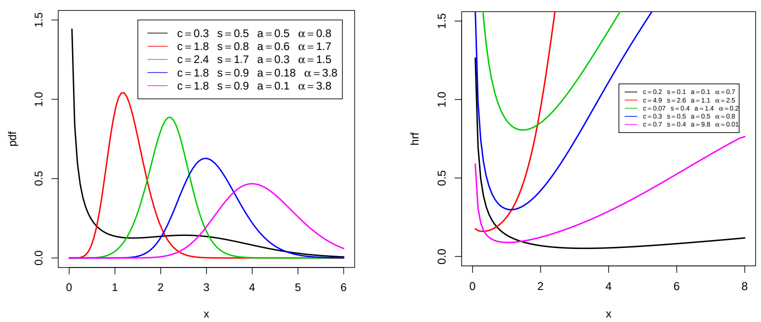


Figure 3. The LLT-R density (left) and hrf (right) plots for different parametric values.

2.4. LLT-exponential (LLT-E) distribution

Consider the exponential distribution with pdf $g(x) = b\,\mathrm{e}^{-b\,x}$, $x > 0$, $b > 0$. Hence, the cdf and pdf of LLT-E distribution are

$$F(x) = 1 - \left(1 + s^{-c}\left\{\tan\left[\frac{\pi}{2}\left(1 - \mathrm{e}^{-b\,x}\right)^{\alpha}\right]\right\}^{c}\right)^{-1} \tag{2.7}$$

and

$$\begin{aligned} f(x) &= \frac{\pi\,\alpha\,bc}{2\,s^c}\,\mathrm{e}^{-b\,x}\left(1 - \mathrm{e}^{-b\,x}\right)^{\alpha-1}\left\{\sec^2\left[\frac{\pi}{2}\left(1 - \mathrm{e}^{-b\,x}\right)^{\alpha}\right]\right\} \\ &\times \left\{\tan\left[\frac{\pi}{2}(1 - \mathrm{e}^{-b\,x})^{\alpha}\right]\right\}^{c-1}\left(1 + s^{-c}\left\{\tan\left[\frac{\pi}{2}\left(1 - \mathrm{e}^{-b\,x}\right)^{\alpha}\right]\right\}^{c}\right)^{-2}. \end{aligned} \tag{2.8}$$

Figure 4 depicts pdf and hrf plots of the LLT-E distribution for different parametric values.

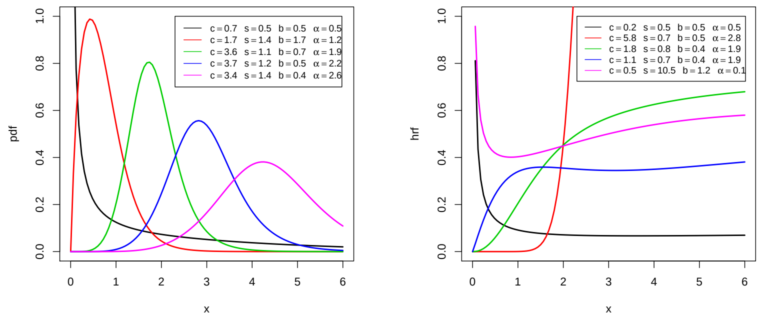


Figure 4. The LLT-E density (left) and hazard rate (right) plots for different parametric values.

2.5. *LLT-Burr XII (LLT-BXII) distribution*

Let X be a Burr XII *rv* with pdf $g(x; a, b) = a\,b\,x^{a-1}\,(1 + x^{a})^{-b-1}$, $x > 0$, $a, b > 0$. The cdf of the LLT-BXII distribution is

$$F(x) = 1 - \left[1 + s^{-c}\left(\tan\left\{\frac{\pi}{2}\left[1 - (1 + x^{a})^{-b}\right]^{\alpha}\right\}\right)^{c}\right]^{-1}. \tag{2.9}$$

The pdf of the LLT-BXII reduces to

$$\begin{aligned} f(x) &= \frac{\pi\,\alpha\,abc}{2\,s^{c}}\,x^{a-1}\,(1 + x^{a})^{-b-1}\left[1 - (1 + x^{a})^{-b}\right]^{\alpha-1}\left(\sec^{2}\left\{\frac{\pi}{2}\left[1 - (1 + x^{a})^{-b}\right]^{\alpha}\right\}\right) \\ &\times \left(\tan\left\{\frac{\pi}{2}\left[1 - (1 + x^{a})^{-b}\right]^{\alpha}\right\}\right)^{c-1}\left[1 + s^{-c}\left(\tan\left\{\frac{\pi}{2}\left[1 - (1 + x^{a})^{-b}\right]^{\alpha}\right\}\right)^{c}\right]^{-2}. \end{aligned} \tag{2.10}$$

Figure 5 displays density and hazard rate plots of the LLT-BXII distribution.

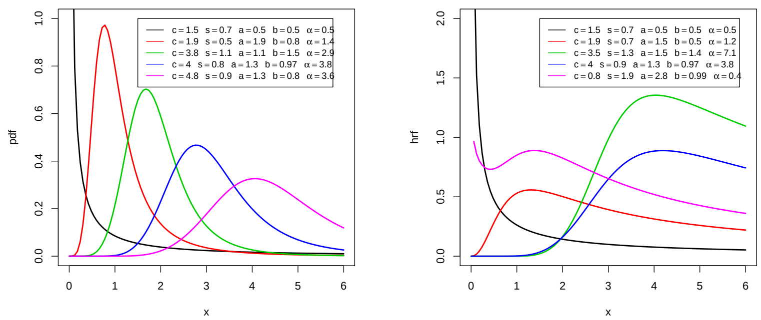


Figure 5. The LLT-BXII density (left) and hazard rate (right) plots for different parametric values.

3. Properties

In this section, the LLT-G properties such as quantile function (qf), useful expansion, moments, generating function, and order statistics are derived. The expressions derived for the LLT-G family can be handled using symbolic computation software, such as, Maple, Matlab, Mathematica, Mathcad, and R because of their ability to deal with complex and formidable size mathematical expressions. Established explicit formulae to evaluate statistical and mathematical measures can be more efficient than computing them directly by numerical integration. It is noted that the infinity limit in the sums of these expressions can be substituted by a large positive integer such as 40 or 50 for most practical purposes.

3.1. Quantile function

The qf of X is calculated directly by inverting (2.1) as

$$Q(u) = G^{-1}\left\{\frac{2}{\pi}\,\tan^{-1}\left[s\left(\frac{u}{1-u}\right)^{\frac{1}{c}}\right]\right\}^{\frac{1}{\alpha}}, \tag{3.1}$$

Eq (3.1) can be used to simulate any baseline model and to obtain the median=Q(1/2), Bowley's skewness and Moors kurtosis.

3.2. Useful expansions

The formation of the pdf and cdf in power series is an important concept.

Some essential properties of the exponentiated-G (exp-G) distributions are studied by [23–25, 31–35].

The cdf of the LLT-G family can be written as

$$F(x) = 1 - \left(1 + s^{-c}\left\{\tan\left[\frac{\pi}{2}\,G^{\alpha}(x)\right]\right\}^{c}\right)^{-1}, \qquad x > 0,\ \alpha, c, s > 0. \tag{3.2}$$

Using the binomial expansion

$$(1+x)^{-1} = \sum_{i=1}^{\infty}(-1)^{i}\,x^{i}, \qquad x \le 1$$

and the convergent series of tan function which can be calculated using the `Mathematica` software.

$$\left[\tan\left(\frac{\pi G(x)}{2}\right)\right]^{ic} = \sum_{j=0}^{\infty}\left[b_j(ic)\right]\left(\frac{\pi G(x)}{2}\right)^{\alpha(2j+ic)}, \qquad G(x) \le 1, x > 0.$$

Applying the last two equations to (3.2), the cdf of LLT-G reduces to

$$F(x) \;=\; \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{(i+1)}}{s^{ic}}\left[b_j(ic)\right]\left(\frac{\pi}{2}\right)^{(2j+ic)}\left[G(x)\right]^{\alpha(2j+ic)}, \quad x > 0. \tag{3.3}$$

We can rewrite the Eq (3.2) as

$$F(x) \;=\; \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} W_{i,j}\; H_{\alpha(2j+ic)}(x), \tag{3.4}$$

where

$$W_{i,j} = \frac{(-1)^{(i+1)}}{s^{ic}}\ b_j\ (ic)\ \left(\frac{\pi}{2}\right)^{(2j+ic)}. \tag{3.5}$$

The below power series can be calculated using the `Mathematica` software

$$\left[\tan\left(\frac{\pi G(x)}{2}\right)\right]^{ic} = \sum_{j=0}^{\infty} b_j(ic)\ \left(\frac{\pi G(x)}{2}\right)^{\alpha(2j+ic)}, \tag{3.6}$$

where $b_0(ic) = 1,\ b_1\ (ic) = \frac{ic}{3},\ b_2\ (ic) = \frac{ic(5\,i+_{3}7)}{90}$, etc. and $H_{\alpha(2j+ic)}(x) = [G(x)]^{\alpha(2j+ic)}$ denotes the cdf of exp-G model with parameter $\alpha(2j + ic)$. So, the LLT-G pdf reduces to

$$f(x) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} w_{i,j}\ h_{\alpha(2j+ic)}(x), \tag{3.7}$$

where $h_{\alpha(2j+ic)}(x)$ denotes the exp-G pdf with parameter $\alpha(2j + ic)$. Hereafter, a *rv* having the pdf $h_{\alpha(2j+ic)}(x)$ is denoted by $Y_{\alpha(2j+i)} \sim$ exp-G$\alpha(2j + ic)$. Eq (3.7) delivers up that the LLT-G pdf is a linear combination of exp-G densities. Thus, a few properties of LLT-G class can be calculated someplace precisely from exp-G properties.

3.3. Moments

Assuming that Z is a *rv* with a baseline $G(x)$. The moments of X can be derived from the (r, k)th probability weighted moment (PWM) of Z defined by [22] as

$$\tau_{r,k} = \mathrm{E}[G(Z)^k\ Z^r] = \int_{-\infty}^{\infty} x^r\ g(x)\ G(x)^k\ dx. \tag{3.8}$$

Using Eq (3.7), one can write

$$\mu'_r = \mathrm{E}(X^r) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty} w_{i,j}\ h_{\alpha(2j+ic)}(x)\ \tau_{r,\alpha(2j+ic)}, \tag{3.9}$$

where $\tau_{r,\alpha(2j+ic)} = \int_0^1 Q_G(u)^r\ u^{\alpha(2j+ic)}du$ which is computed numerically for any baseline qf.

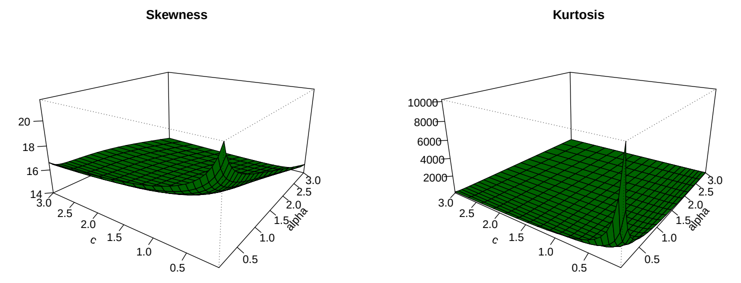
The measures of skewness, β_1, and kurtosis, β_2, can be defined by the following two formulae

$$\beta_1 = \frac{\mu'_3 + 2\ {\mu'_1}^3 - 3\,\mu'_2\ \mu'_1}{\left(\mu'_2 - {\mu'_1}^2\right)^{\frac{3}{2}}} \tag{3.10}$$

and

$$\beta_2 = \frac{\mu'_4 - 4\ \mu'_1\ \mu'_3 + 6\,{\mu'_1}^2\,\mu'_3 - 3\ {\mu'_1}^4}{\mu'_2 - {\mu'_1}^2} \tag{3.11}$$

The plots of skewness and kurtosis for the LLT-W distribution are visualized in Figure 6.



(a) (b)

Figure 6. Skewness and kurtosis Plots for the LLT-W model.

3.4. Moment generating function

We present two formulae for the mgf, $M(s) = E(e^{sX})$, of the *rv* X.

The first one comes from Eq (3.7) as

$$M(s) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} w_{i,j}\, M_{\alpha(2j+ic)}(s), \tag{3.12}$$

where $M_{\alpha(2j+ic)}(s)$ is the exp-G mgf with power parameter $\alpha(2j + ic)$.

Further, Eq (3.12) can also take the form

$$M(s) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \alpha(2j + ic)\; w_{i,j}\, \rho_{\alpha(2j+ic)}(s), \tag{3.13}$$

where the quantity $\rho_{2j+i\beta}(s) = \int_0^1 \exp\left[s\, Q_G(u)\right] u^{\alpha(2j+ic)} du$ is calculated numerically.

3.5. Incomplete moments

Incomplete moments are beneficial in calculating some inequality measures and mean deviations. The nth incomplete moments of the LLT-G family has the form

$$m_n(y) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} w_{i,j}\; \alpha(2j + ic) \int_0^{G(y;\xi)} Q_G(u)^n\; u^{\alpha(2j+ic)}\; du. \tag{3.14}$$

For most baseline distributions, the above (3.14) can be obtained numerically.

The first incomplete moment, $m_1(z)$, can be calculated from Eq (3.7) as

$$m_1(z) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} w_{i,j}\; J_{\alpha(2j+ic)}(z), \tag{3.15}$$

where

$$J_{\alpha(2j+ic)}(z) = \int_{-\infty}^{z} x\ h_{\alpha(2j+ic)}(x)dx. \tag{3.16}$$

Eq (3.15) is the primary quantity to obtain the mean deviations. We apply (3.15) to the LLT-W model. The LLT-W pdf with power parameter $\alpha(2j + ic)$, reduces to

$$h_{\alpha(2j+ic)}(x) \quad = \quad \alpha(2j + ic)\ g(x)\ [G(x)]^{\alpha(2j+ic)-1}, \tag{3.17}$$

and then

$$J_{\alpha(2j+ic)}(x) \quad = \quad \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{(i+1)}}{s^{ic}}\ b_j(ic)\ \left(\frac{\pi}{2}\right)^{\alpha(2j+ic)} \int_0^z x\ \alpha(2j + ic)\ g(x)\ [G(x)]^{\alpha(2j+ic)-1} dx. \tag{3.18}$$

Another formula for $m_1(z)$ follows, Eq (3.7) with $u = G(x)$, as

$$m_1(z) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \alpha(2j + ic)\ w_{i,j}\ T_{\alpha(2j+ic)}(z), \tag{3.19}$$

where $T_{\alpha(2j+ic)}(z) = \int_0^{G(z)} Q_G(u)\ u^{\alpha(2j+ic)} du.$

3.6. Order statistics

Consider a random sample from the LLT-G class, $X_1, \ldots, X_n$. The pdf of the ith order statistic, $X_{i:n}$, has the form

$$f_{i:n}(x) = D\ \sum_{p=0}^{n-i} (-1)^p \binom{n-i}{p}\ F(x)^{p+i-1}\ f(x), \tag{3.20}$$

where $D = n!/[(i - 1)!\,(n - i)!]$.

After some algebra, the pdf of $X_{i:n}$ takes the form

$$f_{i:n}(x) = \sum_{m,r,t=0}^{\infty} d_{m,r,t}\ h_{\alpha(2(r+t)+c(m+1))}(x), \tag{3.21}$$

where the exp-G density, $h_{\alpha(2(r+t)+c(m+1))}(x)$, has a power parameter $\alpha(2(r + t) + c(m + 1))$

$$d_{m,r,t} \quad = \quad \sum_{j,l=0}^{\infty} k\, \frac{(-1)^{j+l}}{s^{c(m+1)}} \binom{n-i}{j} \binom{i+j-1}{l} \binom{-(l+2)}{m} b_r[c(m + 1) - 1]\, c_t(2) \left(\frac{\pi}{2}\right)^{2(r+t)+c(m+1)}$$

where $b_r[c(m + 1) - 1]$ and $c_t(2)$ are the coefficients of the power series expansion of tan and sec functions by using `Mathematica`.

Eq (3.21) can be used to calculate some quantities of the LLT-G order statistics from the exp-G quantities.

4. Estimation and simulation

We will discuss the estimation of the LLT-G parameters using the maximum likelihood (MLE) and study the performance of these estimators via simulations.

4.1. Maximum likelihood estimation

Consider a random sample from the LLT-G class, $x_1, \ldots, x_n$, with $\Theta = (\alpha, s, c, \boldsymbol{\xi})^\top$. Then, the log-likelihood of Θ reduces to

$$\begin{aligned}
\ell_n(\Theta) &= n \log(\frac{\pi}{2}) + n \log(\alpha c) - n \log(s) + \sum_{i=1}^{n} \log \ G(x_i; \boldsymbol{\xi}) + \sum_{i=1}^{n} \log\left[\sec^2 \ (A_i)\right] \\
&+ (\alpha - 1) \sum_{i=1}^{n} \log[G(x_i; \boldsymbol{\xi})] + (c-1) \sum_{i=1}^{n} \log\left[\frac{1}{s} \tan(A_i)\right] \\
&- 2 \sum_{i=1}^{n} \log\left\{1 + \left[\frac{1}{s} \tan(A_i)\right]^c\right\}, \qquad (4.1)
\end{aligned}$$

where $A_i = \frac{\pi}{2} G^\alpha(x_i; \boldsymbol{\xi})$.

The above log-likelihood (4.1) is maximized simply by statistical programs namely, `R`, `Mathcad`, `SAS`, `Mathematica` and `Ox` programs. It also is maximized by solving the following likelihood equations.

The score vector elements,
$U_n(\Theta) = (\partial\ell_n/\partial\alpha, \partial\ell_n/\partial s, \partial\ell_n/\partial c, \partial\ell_n/\partial\boldsymbol{\xi})^\top$, take the following formulae

$$\begin{aligned}
\frac{\partial \ell_n}{\partial \alpha} &= \sum_{i=1}^{\infty} \log[G(x_i)] - \frac{\pi c}{s} \sum_{i=1}^{\infty} \frac{\left[\sec^2(A_i)\right]\left[\frac{1}{s}\tan(A_i)\right]^{c-1} \{G^\alpha(x_i) \ \log \ [G(x_i)]\}}{\left\{\left[\frac{1}{s}\tan(A_i)\right]^c + 1\right\}} \\
&+ (c-1) \sum_{i=1}^{\infty} \frac{\pi}{2} [\csc(A_i)] [\sec(A_i)] \{G^\alpha(x_i) \log \ [G(x_i)]\} \\
&+ \sum_{i=1}^{\infty} \pi \left[\sec^2(A_i)\right] [\tan(A_i)] \{G^\alpha(x_i) \log \ [G(x_i)]\}, \qquad (4.2)
\end{aligned}$$

$$\frac{\partial \ell_n}{\partial s} = (1-c) \sum_{i=1}^{\infty} \frac{1}{s} - \frac{n}{s} - \frac{2c}{s^2} \sum_{i=1}^{\infty} - \frac{[\tan(A_i)]\left[\frac{1}{s}\tan(A_i)\right]^{c-1}}{\left\{\left[\frac{1}{s}\tan(A_i)\right]^c + 1\right\}}, \qquad (4.3)$$

$$\frac{\partial \ell_n}{\partial c} = \frac{n}{c} + \sum_{i=1}^{\infty} \log\left[\frac{1}{s}\tan(A_i)\right] - 2 \sum_{i=1}^{\infty} \frac{\left[\frac{1}{s}\tan(A_i)\right]^c \log\left[\frac{1}{s}\tan(A_i)\right]}{\left\{\left[\frac{1}{s}\tan(A_i)\right]^c + 1\right\}} \qquad (4.4)$$

and

$$\frac{\partial \ell_n}{\partial \boldsymbol{\xi}} = \sum_{i=1}^{n} \frac{g(x_i; \ \boldsymbol{\xi})^{(\xi)}}{g(x_i; \ \boldsymbol{\xi})} + (\alpha - 1) \sum_{i=1}^{n} \frac{G(x_i; \ \boldsymbol{\xi})^{(\xi)}}{G(x_i; \ \boldsymbol{\xi})} + \pi \sum_{i=1}^{n} \left[\frac{1}{s}\tan(A_i)\right] G(x_i; \ \boldsymbol{\xi})^{(\alpha\,\xi)}$$

$$- \frac{\pi}{s} \sum_{i=1}^{\infty} \frac{\left[\frac{c}{s}\tan(A_i)\right]^{c-1}\left[\sec^2(A_i)\right]}{\left\{\left[\frac{1}{s}\tan(A_i)\right]^{c}+1\right\}} G(x_i;\,\boldsymbol{\xi})^{(\alpha\,\xi)}$$

$$+ \frac{\pi}{2}(c-1)\sum_{i=1}^{\infty}\frac{\sec^2(A_i)}{\tan(A_i)} G(x_i;\,\boldsymbol{\xi})^{(\alpha\,\xi)}, \tag{4.5}$$

where $g^{(\xi)}(\cdot) = \partial g / \partial \boldsymbol{\xi}$.

4.2. Simulations

This section deals with checking the MLEs performance in estimating the LLT-W parameters using a simulation study. For $n = 50, 100, 300$ and 500, we generated 1,000 samples from the LLT-W model for various parametric values using the inversion method via the qf of the LLT-W given by

$$Q(u) = \left[-\log\left(1 - \left\{\frac{2}{\pi}\tan^{-1}\left[s\left(\frac{u}{1-u}\right)^{\frac{1}{c}}\right]\right\}^{\frac{1}{\alpha}}\right)\right]^{\frac{1}{b}}. \tag{4.6}$$

The numerical results of the mean square error (MSE), bias, coverage probability (C.P) and average width (AW) of the MLEs of the model parameters are obtained using R software. Further, the graphical representation of bias, MSE and C.P on several parametric values are also provided. The numerical results for bias, MSE, C.P and AW are listed in Tables 1 and 2. The bias, MSE and C.P are depicted in Figures 7–9. The values in Tables 1 and 2, and shapes of Figures 7–9 show quite stable estimates for the considered sample sizes. Moreover, the MSEs decrease as n increases, proving the consistency of maximum likelihood estimators.

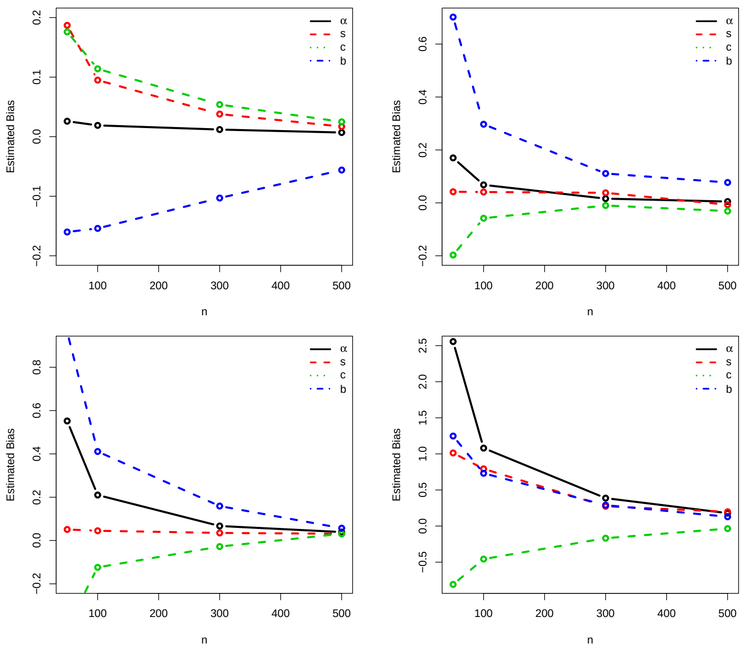


Figure 7. Biases plots of the LLT-W for different parametric values.

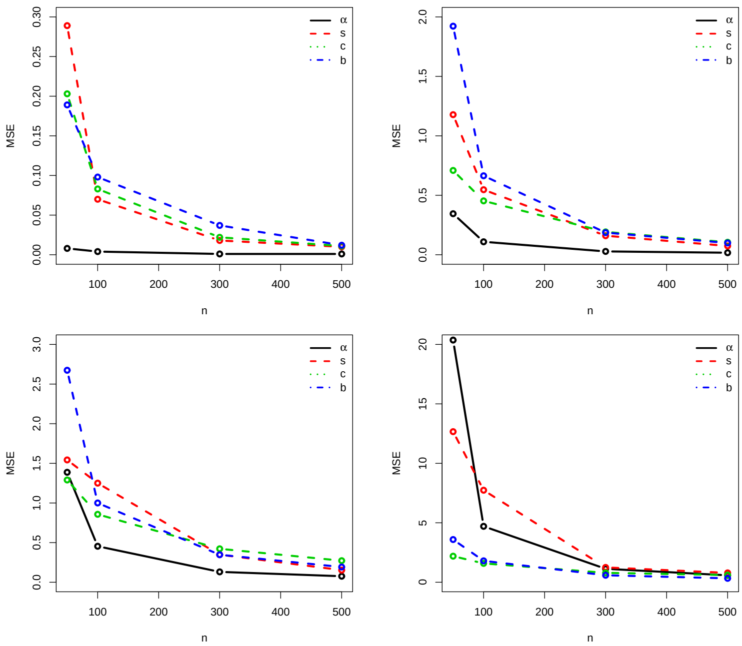


Figure 8. MSEs plots of the LLT-W for different parametric values.

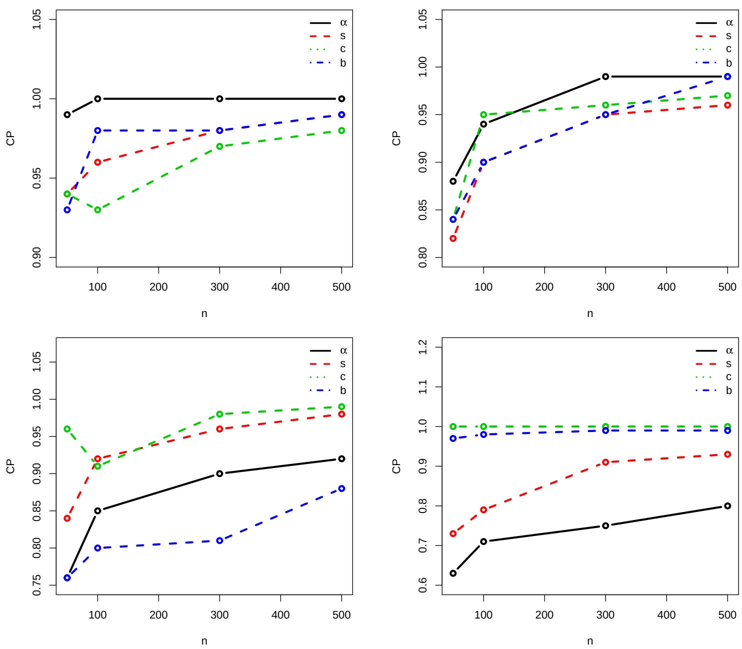


Figure 9. C.P plots of the LLT-W for different parametric values.

Table 1. Estimated biases, MSE, C.P, and AW for MLEs of the LLT-W parameters.

		MLE($\alpha = 0.20,\ s = 0.50,\ c = 1.30,\ b = 1.50$)			
n		Bias	MSE	C.P	AW
50	α	0.02632	0.00861	0.99023	2.36578
	s	0.18709	0.28943	0.94067	2.19272
	c	0.17656	0.20302	0.94821	0.44550
	b	-0.16017	0.18987	0.93041	1.56365
100	α	0.01978	0.00444	1.00022	1.25564
	s	0.09531	0.07086	0.96764	1.40323
	c	0.11487	0.08334	0.93042	0.29029
	b	-0.15492	0.09865	0.98562	1.49110
300	α	0.01298	0.00175	1.00276	0.65864
	s	0.03895	0.01812	0.98054	0.758178
	c	0.05497	0.02252	0.97097	0.15874
	b	-0.10343	0.03767	0.9819	1.47684
500	α	0.00787	0.00156	1.00876	0.48943
	s	0.01798	0.01028	0.99098	0.56132
	c	0.02509	0.01161	0.98987	0.11954
	b	-0.05684	0.01249	0.99076	1.50318
		MLE($\alpha = 0.80,\ s = 1.10,\ c = 1.60,\ b = 1.80$)			
n		Bias	MSE	C.P	AW
50	α	0.17012	0.34589	0.88320	5.73426
	s	0.04296	1.17838	0.82209	3.51921
	c	-0.19793	0.70917	0.84190	2.01749
	b	0.70237	1.92221	0.84084	3.51065
100	α	0.06832	0.10920	0.94890	3.29742
	s	0.04132	0.54720	0.90908	2.76689
	c	-0.05829	0.45318	0.9593	1.25046
	b	0.29772	0.66458	0.9032	2.72295
300	α	0.01620	0.02809	0.99870	1.46536
	s	0.03836	0.16018	0.95430	1.62928
	c	-0.01011	0.19239	0.9673	0.65048
	b	0.11101	0.18638	0.95650	2.23627
500	α	0.00503	0.01737	0.99201	1.02895
	s	-0.00776	0.07429	0.96807	1.24003
	c	-0.03110	0.10385	0.97956	0.49509
	b	0.07738	0.09929	0.9982	2.12406

Table 2. Estimated biases, MSE, C.P, and AW for MLEs of the LLT-W parameters.

		MLE(α = 1.20, s = 1.50, c = 2.00, b = 2.50)			
n		Bias	MSE	C.P	AW
50	α	0.55202	1.38710	0.76780	7.32437
	s	0.05138	1.54372	0.8452	6.13638
	c	-0.40001	1.29038	0.96980	4.15307
	b	0.96352	2.67448	0.76980	5.539872
100	α	0.21052	0.45489	0.85098	5.69928
	s	0.04592	1.25029	0.92290	4.14301
	c	-0.12428	0.85789	0.91428	2.57578
	b	0.41103	1.00012	0.80098	4.19802
300	α	0.06776	0.13128	0.90097	2.44256
	s	0.03576	0.34829	0.96980	2.50462
	c	-0.02872	0.42232	0.98098	1.40965
	b	0.15902	0.34752	0.81980	3.36362
500	α	0.03962	0.07652	0.92620	1.48882
	s	0.03172	0.15372	0.98095	1.99142
	c	0.02030	0.27289	0.99980	1.07363
	b	0.05729	0.19258	0.88765	3.09498
		MLE(α = 1.80, s = 2.20, c = 2.60, b = 3.20)			
n		Bias	MSE	C.P	AW
50	α	2.55609	20.36652	0.63065	21.15316
	s	1.01329	12.664i2	0.73098	7.87152
	c	-0.80862	2.19272	1.00098	13.64625
	b	1.24726	3.59262	0.97980	11.27043
100	α	1.08072	4.70272	0.71980	16.88462
	s	0.79362	7.37992	0.79809	6.31162
	c	-0.45762	1.58666	1.00450	7.46862
	b	0.73178	1.80152	0.98609	7.66562
300	α	0.38762	1.13352	0.75780	5.98709
	s	0.27457	1.24709	0.91550	3.92154
	c	-0.16804	0.79128	1.00010	3.93678
	b	0.29061	0.58889	0.99987	5.45865
500	α	0.17910	0.58267	0.80087	4.02328
	s	0.19884	0.78817	0.93709	3.28665
	c	-0.03543	0.60569	1.00120	2.99141
	b	0.12976	0.33093	0.99120	4.82464

5. Modeling real-life data

This section deals with checking the flexibility of the LLT-W distribution using two real-life data applications.

The discrimination measures namely, the Kolmogrov-Smirnov (K-S) (with its p-value), Anderson-Darling (A^*) and Cramér–von Mises (W^*) are calculated to compare the competing models selected, odd log-logistic modified-Weibull (OLLMW) [37], odd Dagum-Weibull (ODW) [1], odd log logistic exponentiated-Weibull (OLLEW) [2], Kumaraswamy-Weibull (KW) [18], Kumaraswamy-exponentiated Burr XII (KEBXII) [30], Weibull-Lomax (WL) [39], and beta-Burr XII (BBXII) [36] distributions.

The first real data contain 63 observations about strengths of 1.5 cm glass fibers which are reported in Smith and Naylor [38]. The second data set from Choulakian and Stephens (2001) [12], is the exceedances of flood peaks (in m3/s) of Wheaton River, Yukon Territory, Canada. The data consists of 72 exceedances for the years 1958–1984, rounded to one decimal place.

Tables 3 and 4 report the MLEs (with their associated (standard errors)) of the parameters of competing models, and the statistics K-S, p-value, A^* and W^*.

The LLT-W model is compared with the ODW, OLLEW, KW, WBXII, KEBXII, BBXII, WL distributions in Tables 3 and 4. We note that the LLT-W model gives the lowest values for all discrimination measures and largest p-value among all fitted models. Hence,the LLT-W model could be chosen as a good alternative to explain glass fibers and Wheaton river data. The results in Tables 3 and 4 indicate that LLT-W provides better fits for glass fibers and Wheaton river data sets as compared to other competing models. More visual comparison of the four best competing distributions are provided in Figures 10 and 11. The fitted densities of the four best models are shown in Figure 10 for glass fibers and Wheaton river data, whereas the estimated distribution functions for four best models are depicted in Figure 11. Further, the hrf plots of the LLT-W distribution for glass fibers and Wheaton river data are illustrated in Figure 12. Based on visual comparison, we can conclude that the LLT-W distribution provides a close fit for glass fibers and Wheaton river data and it can be utilized in fitting data with increasing and modified bathtub hazard rates.

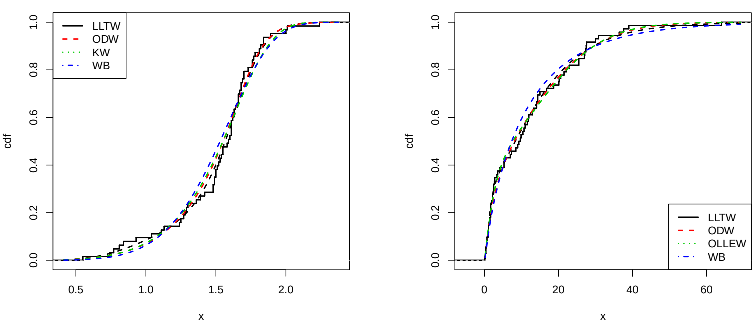


Figure 10. Estimated cdfs for the LLT-W model and other competing models (left) for glass fibers and (right) for Wheaton river data.

Table 3. MLEs (SEs) and discrimination measures of competing models for glass fibres data.

Model	α	c	s	b	λ	p-value	K-S	W^*	A^*
LLT-W	2.53601	0.55698	38.13266	3.59346	-	0.70310	0.08880	0.06492	0.38337
	(1.35585)	(0.29409)	(54.65571)	(0.70897)	-				
OLLMW	2.62115	0.30243	0.01341	6.27241	-	0.62310	0.09479	0.08044	0.47130
	(0.78255)	(0.08134)	(0.01179)	(1.09531)	-				
ODW	5.29998	0.29810	9.78229	0.25124	2.32808	0.37470	0.11506	0.09164	0.51433
	(7.86358)	(0.16143)	(55.66654)	(0.30203)	(2.49274)				
OLLEW	0.30232	1.68593	1.99160	8.74967	-	0.22260	0.13196	0.18638	1.03125
	(0.26788)	(0.74479)	(0.29822)	(3.94948)	-				
KW	0.55200	0.22850	0.10530	7.18550	-	0.27860	0.12499	0.15316	0.86834
	(0.14662)	(0.05697)	(0.02394)	(0.01724)	-				
KEBXII	4.27374	658.39551	0.85420	1.31415	4.26494	0.02819	0.18391	0.36750	2.01129
	(36.70618)	(745.20326)	(0.65802)	(1.48982)	(36.63053)				
WL	0.01519	3.39585	7.04854	6.96628	-	0.16120	0.14136	0.18795	1.04676
	(0.02116)	(0.93100)	(12.01524)	(12.82023)	-				
BBXII	103.51855	174.50750	0.55986	0.57767	-	0.00230	0.23172	0.70303	3.84266
	(245.89797)	(401.09856)	(0.69152)	(1.16196)	-				
WBXII	0.01210	2.98508	1.57632	1.46315	-	0.11440	0.15069	0.21760	1.19955
	(0.01215)	(2.60747)	(1.02824)	(0.59959)	-				

Table 4. MLEs (SEs) and discrimination measures of competing models for Wheaton river data.

Model	α	c	s	b	λ	p-value	K-S	W^*	A^*
LLT-W	0.22515	11.81810	0.80678	13.88789	-	0.99580	0.04853	0.02924	0.22831
	(0.07329)	(14.49115)	(0.08631)	(4.14192)	-				
OLLMW	0.00425	0.56057	0.07717	7.29698	-	0.43390	0.10266	0.13676	0.80928
	(0.00289)	(0.07774)	(0.05263)	(4.47235)	-				
ODW	0.31643	13.49715	0.05973	0.03914	1.50574	0.89770	0.06754	0.04044	0.25765
	(0.07961)	(28.57142)	(0.12519)	(0.04984)	(0.36632)				
OLLEW	4.83143	1.38144	4.23678	0.20378	-	0.93890	0.06280	0.03495	0.22261
	(0.00432)	(0.00431)	(0.80929)	(0.02616)	-				
KW	0.54109	1.36525	0.01059	1.39132	-	0.39510	0.10587	0.10785	0.64912
	(0.21553)	(1.19205)	(0.01303)	(0.33498)	-				
KEBXII	-6.33954	63.90982	0.16781	1.72935	-2.82907	0.32580	0.11212	0.19729	1.08811
	(28.76141)	(69.93447)	(0.16987)	(2.40254)	(12.82347)				
WL	1.09893	0.76225	2.66001	46.73481	-	0.44240	0.10198	0.09932	0.60466
	(1.24287)	(0.13144)	(2.91198)	(98.66865)	-				
BBXII	59.58061	9.47	67.54568	0.13740	-	0.11800	0.14020	0.26630	1.49169
	(118.95689)	(139.21525)	(0.14196)	(1.43564)	-				
WBXII	0.12549	0.25801	5.19935	0.63785	-	0.62690	0.08840	0.12470	0.69009
	(0.03716)	(0.22083)	(4.60782)	(0.21590)	-				

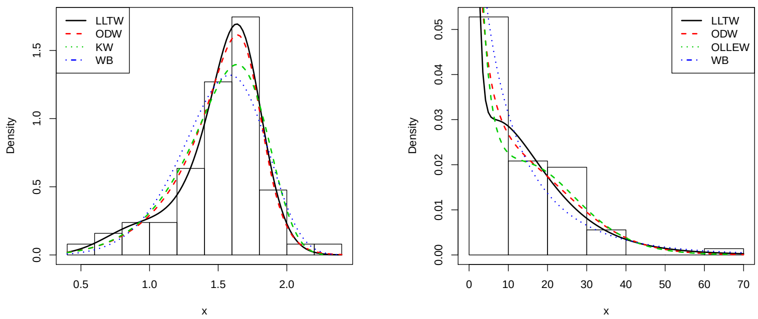


Figure 11. Estimated pdfs for the LLT-W model and other competing models (left) for glass fibers and (right) for Wheaton river data.

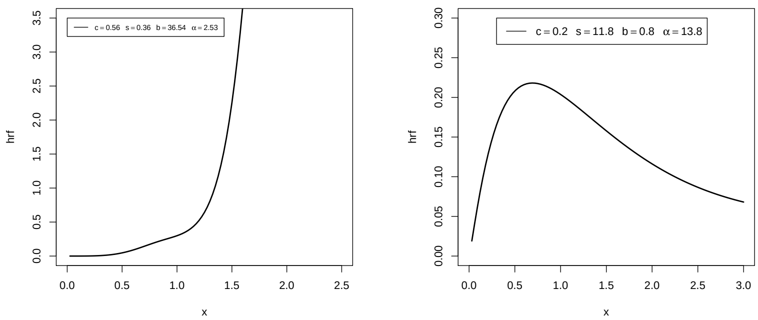


Figure 12. The hrf plots of the LLT-W model (left) for glass fibers and (right) for Wheaton river data.

6. Conclusions

This paper provides a new log-logistic tan generalized (LLT-G) class with three additional shape parameters to capture skewness and kurtosis behavior. Five special models of the LLT-G family are presented by choosing Weibull, normal, Rayleigh, exponential and Burr XII, as baseline distributions in the proposed family, to obtain the LLT-Weibull, LLT-normal, LLT-Rayleigh, LLT-exponential and LLT-Burr XII. The general mathematical properties are obtained for the LLT-G class. The LLT-G parameters estimation is discussed by maximum-likelihood approach and simulation results are obtained to check the performance of these estimators. The importance and flexibility of the LLT-Weibull are checked empirically using two sets of real-life data, proving that it can provide better fit as compared with other competing models, such as odd log-logistic modified-Weibull, odd Dagum-Weibull, odd log logistic exponentiated-Weibull, Weibull-Lomax, Kumaraswamy-Weibull, Kumaraswamy-exponentiated-Burr XII, and beta-Burr XII distributions.

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Conflict of interest

There is no conflict of interest declared by the authors.

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