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Research article

Several expressions of truncated Bernoulli-Carlitz and truncated Cauchy-Carlitz numbers

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Abstract: The truncated Bernoulli-Carlitz numbers and the truncated Cauchy-Carlitz numbers are defined as analogues of hypergeometric Bernoulli numbers and hypergeometric Cauchy numbers, and as extensions of Bernoulli-Carlitz numbers and the Cauchy-Carlitz numbers. These numbers can be expressed explicitly in terms of incomplete Stirling-Carlitz numbers. In this paper, we give several expressions of truncated Bernoulli-Carlitz numbers and truncated Cauchy-Carlitz numbers as natural extensions. One kind of expressions is in continued fractions. Another is in determinants originated in Glaisher, giving several interesting determinant expressions of numbers, including Bernoulli and Cauchy numbers.

Keywords: Bernoulli-Carlitz numbers; hypergeometric Bernoulli numbers; truncated Bernoulli-Carlitz numbers; Cauchy-Carlitz numbers; hypergeometric Cauchy numbers; truncated Cauchy-Carlitz numbers; determinants; recurrence relations; function fields; continued fractions
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1. Introduction

L. Carlitz ([1]) introduced analogues of Bernoulli numbers for the rational function (finite) field $K = \mathbb{F}_r(T)$, which are called Bernoulli-Carlitz numbers now. Bernoulli-Carlitz numbers have been studied since then (e.g., see [2–6]). According to the notations by Goss [7], Bernoulli-Carlitz numbers BC_n are defined by

$$\frac{x}{e_C(x)} = \sum_{n=0}^{\infty} \frac{BC_n}{\Pi(n)} x^n . \tag{1.1}$$

Here, $e_C(x)$ is the Carlitz exponential defined by

$$e_C(x) = \sum_{i=0}^{\infty} \frac{x^{r^i}}{D_i}, \tag{1.2}$$

where $D_i = [i][i-1]^r \cdots [1]^{r^{i-1}}$ $(i \geq 1)$ with $D_0 = 1$, and $[i] = T^{r^i} - T$. The Carlitz factorial $\Pi(i)$ is defined by

$$\Pi(i) = \prod_{j=0}^{m} D_j^{c_j} \tag{1.3}$$

for a non-negative integer i with r-ary expansion:

$$i = \sum_{j=0}^{m} c_j r^j \quad (0 \leq c_j < r). \tag{1.4}$$

As analogues of the classical Cauchy numbers c_n, Cauchy-Carlitz numbers CC_n ([8]) are introduced as

$$\frac{x}{\log_C(x)} = \sum_{n=0}^{\infty} \frac{CC_n}{\Pi(n)} x^n. \tag{1.5}$$

Here, $\log_C(x)$ is the Carlitz logarithm defined by

$$\log_C(x) = \sum_{i=0}^{\infty} (-1)^i \frac{x^{r^i}}{L_i}, \tag{1.6}$$

where $L_i = [i][i-1] \cdots [1]$ $(i \geq 1)$ with $L_0 = 1$.

In [8], Bernoulli-Carlitz numbers and Cauchy-Carlitz numbers are expressed explicitly by using the Stirling-Carlitz numbers of the second kind and of the first kind, respectively. These properties are the extensions that Bernoulli numbers and Cauchy numbers are expressed explicitly by using the Stirling numbers of the second kind and of the first kind, respectively.

On the other hand, for $N \geq 1$, hypergeometric Bernoulli numbers $B_{N,n}$ ([9–12]) are defined by the generating function

$$\frac{1}{{}_1F_1(1; N+1; x)} = \frac{x^N/N!}{e^x - \sum_{n=0}^{N-1} x^n/n!} = \sum_{n=0}^{\infty} B_{N,n} \frac{x^n}{n!}, \tag{1.7}$$

where

$${}_1F_1(a; b; z) = \sum_{n=0}^{\infty} \frac{(a)^{(n)}}{(b)^{(n)}} \frac{z^n}{n!}$$

is the confluent hypergeometric function with $(x)^{(n)} = x(x+1) \cdots (x+n-1)$ $(n \geq 1)$ and $(x)^{(0)} = 1$. When $N = 1$, $B_n = B_{1,n}$ are classical Bernoulli numbers defined by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}.$$

In addition, hypergeometric Cauchy numbers $c_{N,n}$ (see [13]) are defined by

$$\frac{1}{{}_2F_1(1,N;N+1;-x)} = \frac{(-1)^{N-1}x^N/N}{\log(1+t) - \sum_{n=1}^{N-1}(-1)^{n-1}x^n/n} = \sum_{n=0}^{\infty} c_{N,n}\frac{x^n}{n!}\,, \tag{1.8}$$

where

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)^{(n)}(b)^{(n)}}{(c)^{(n)}}\frac{z^n}{n!}$$

is the Gauss hypergeometric function. When $N = 1$, $c_n = c_{1,n}$ are classical Cauchy numbers defined by

$$\frac{x}{\log(1+x)} = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!}\,.$$

In [14], for $N \ge 0$, the truncated Bernoulli-Carlitz numbers $BC_{N,n}$ and the truncated Cauchy-Carlitz numbers $CC_{N,n}$ are defined by

$$\frac{x^{r^N}/D_N}{e_C(x) - \sum_{i=0}^{N-1} x^{r^i}/D_i} = \sum_{n=0}^{\infty} \frac{BC_{N,n}}{\Pi(n)}x^n \tag{1.9}$$

and

$$\frac{(-1)^N x^{r^N}/L_N}{\log_C(x) - \sum_{i=0}^{N-1}(-1)^i x^{r^i}/L_i} = \sum_{n=0}^{\infty} \frac{CC_{N,n}}{\Pi(n)}x^n\,, \tag{1.10}$$

respectively. When $N = 0$, $BC_n = BC_{0,n}$ and $CC_n = CC_{0,n}$ are the original Bernoulli-Carlitz numbers and Cauchy-Carlitz numbers, respectively. These numbers $BC_{N,n}$ and $CC_{N,n}$ in (1.9) and (1.10) in function fields are analogues of hypergeometric Bernoulli numbers in (1.7) and hypergeometric Cauchy numbers in (1.8) in complex numbers, respectively. In [15], the *truncated Euler polynomials* are introduced and studied in complex numbers.

It is known that any real number α can be expressed uniquely as the simple continued fraction expansion:

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}\,, \tag{1.11}$$

where a_0 is an integer and $a_1, a_2, \dots$ are positive integers. Though the expression is not unique, there exist general continued fraction expansions for real or complex numbers, and in general, analytic functions $f(x)$:

$$f(x) = a_0(x) + \cfrac{b_1(x)}{a_1(x) + \cfrac{b_2(x)}{a_2(x) + \cfrac{b_3(x)}{a_3(x) + \ddots}}}\,, \tag{1.12}$$

where $a_0(x), a_1(x), \ldots$ and $b_1(x), b_2(x), \ldots$ are polynomials in x. In [16, 17] several continued fraction expansions for non-exponential Bernoulli numbers are given. For example,

$$\sum_{n=1}^{\infty} B_{2n}(4x)^n = \cfrac{x}{1+\frac{1}{2}+\cfrac{x}{\frac{1}{2}+\frac{1}{3}+\cfrac{x}{\frac{1}{3}+\frac{1}{4}+\cfrac{x}{\ddots}}}}. \tag{1.13}$$

More general continued fractions expansions for analytic functions are recorded, for example, in [18]. In this paper, we shall give expressions for truncated Bernoulli-Carlitz numbers and truncated Cauchy-Carlitz numbers.

In [19], the hypergeometric Bernoulli numbers $B_{N,n}$ ($N \ge 1$, $n \ge 1$) can be expressed as

$$B_{N,n} = (-1)^n n! \begin{vmatrix} \frac{N!}{(N+1)!} & 1 & 0 & & \\ \frac{N!}{(N+2)!} & \frac{N!}{(N+1)!} & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ \frac{N!}{(N+n-1)!} & \frac{N!}{(N+n-2)!} & \cdots & \frac{N!}{(N+1)!} & 1 \\ \frac{N!}{(N+n)!} & \frac{N!}{(N+n-1)!} & \cdots & \frac{N!}{(N+2)!} & \frac{N!}{(N+1)!} \end{vmatrix}.$$

When $N = 1$, we have a determinant expression of Bernoulli numbers ([20, p.53]). In addition, relations between $B_{N,n}$ and $B_{N-1,n}$ are shown in [19].

In [21, 22], the hypergeometric Cauchy numbers $c_{N,n}$ ($N \ge 1$, $n \ge 1$) can be expressed as

$$c_{N,n} = n! \begin{vmatrix} \frac{N}{N+1} & 1 & 0 & & \\ \frac{N}{N+2} & \frac{N}{N+1} & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ \frac{N}{N+n-1} & \frac{N}{N+n-2} & \cdots & \frac{N}{N+1} & 1 \\ \frac{N}{N+n} & \frac{N}{N+n-1} & \cdots & \frac{N}{N+2} & \frac{N}{N+1} \end{vmatrix}.$$

When $N = 1$, we have a determinant expression of Cauchy numbers ([20, p.50]).

Recently, in ([23]) the truncated Euler-Carlitz numbers $EC_{N,n}$ ($N \ge 0$), introduced as

$$\frac{x^{q^{2N}}/D_{2N}}{\mathrm{Cosh}_C(x) - \sum_{i=0}^{N-1} x^{q^{2i}}/D_{2i}} = \sum_{n=0}^{\infty} \frac{EC_{N,n}}{\Pi(n)} x^n,$$

are shown to have some determinant expressions. When $N = 0$, $EC_n = EC_{0,n}$ are the Euler-Carlitz numbers, denoted by

$$\frac{x}{\mathrm{Cosh}_C(x)} = \sum_{n=0}^{\infty} \frac{EC_n}{\Pi(n)} x^n,$$

where

$$\mathrm{Cosh}_C(x) = \sum_{i=0}^{\infty} \frac{x^{q^{2i}}}{D_{2i}}$$

is the Carlitz hyperbolic cosine. This reminds us that the hypergeometric Euler numbers $E_{N,n}$ ([24]), defined by

$$\frac{t^{2N}/(2N)!}{\cosh t - \sum_{n=0}^{N-1} t^{2n}/(2n)!} = \sum_{n=0}^{\infty} E_{N,n}\frac{x^n}{n!}\,,$$

have a determinant expression [25, Theorem 2.3] for $N \ge 0$ and $n \ge 1$,

$$E_{N,2n} = (-1)^n (2n)! \begin{vmatrix} \frac{(2N)!}{(2N+2)!} & 1 & 0 & \\ \frac{(2N)!}{(2N+4)!} & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1 \\ \frac{(2N)!}{(2N+2n)!} & \cdots & \frac{(2N)!}{(2N+4)!} & \frac{(2N)!}{(2N+2)!} \end{vmatrix}.$$

When $N = 0$, we have a determinant expression of Euler numbers (*cf.* [20, p.52]). More general cases are studied in [26].

In this paper, we also give similar determinant expressions of truncated Bernoulli-Carlitz numbers and truncated Cauchy-Carlitz numbers as natural extensions of those of hypergeometric numbers.

2. Continued fraction expansions of truncated Bernoulli-Carlitz and Cauchy-Carlitz numbers

Let the n-th convergent of the continued fraction expansion of (1.12) be

$$\frac{P_n(x)}{Q_n(x)} = a_0(x) + \cfrac{b_1(x)}{a_1(x) + \cfrac{b_2(x)}{a_2(x) + \ddots + \cfrac{b_n(x)}{a_n(x)}}}\,. \tag{2.1}$$

There exist the fundamental recurrence formulas:

$$\begin{aligned} P_n(x) &= a_n(x)P_{n-1}(x) + b_n(x)P_{n-2}(x)\ (n \ge 1),\\ Q_n(x) &= a_n(x)Q_{n-1}(x) + b_n(x)Q_{n-2}(x)\ (n \ge 1), \end{aligned} \tag{2.2}$$

with $P_{-1}(x) = 1$, $Q_{-1}(x) = 0$, $P_0(x) = a_0(x)$ and $Q_0(x) = 1$.

From the definition in (1.9), truncated Bernoulli-Carlitz numbers satisfy the relation

$$\left(D_N \sum_{i=0}^{\infty} \frac{x^{r^{N+i}-r^N}}{D_{N+i}}\right)\left(\sum_{n=0}^{\infty} \frac{BC_{N,n}}{\Pi(n)}x^n\right) = 1\,.$$

Thus,

$$P_m(x) = \frac{D_{N+m}}{D_N}, \quad Q_m(x) = D_{N+m}\sum_{i=0}^{m} \frac{x^{r^{N+i}-r^N}}{D_{N+i}}$$

yield that

$$Q_m(x)\sum_{n=0}^{\infty} \frac{BC_{N,n}}{\Pi(n)}x^n \sim P_m(x) \quad (m \to \infty)\,.$$

Notice that the n-th convergent p_n/q_n of the simple continued fraction (1.11) of a real number α yields the approximation property

$$|q_n\alpha - p_n| < \frac{1}{q_{n+1}}\,.$$

Now,

$$\frac{P_0(x)}{Q_0(x)} = 1 = \frac{1}{1}, \quad \frac{P_1(x)}{Q_1(x)} = 1 - \frac{x^{r^{N+1}-r^N}}{D_{N+1}/D_N + x^{r^{N+1}-r^N}}$$

and $P_n(x)$ and $Q_n(x)$ ($n \ge 2$) satisfy the recurrence relations

$$P_n(x) = \left(\frac{D_{N+n}}{D_{N+n-1}} + x^{r^{N+n}-r^{N+n-1}}\right)P_{n-1}(x) - \frac{D_{N+n-1}}{D_{N+n-2}}x^{r^{N+n}-r^{N+n-1}}P_{n-2}(x)$$
$$Q_n(x) = \left(\frac{D_{N+n}}{D_{N+n-1}} + x^{r^{N+n}-r^{N+n-1}}\right)Q_{n-1}(x) - \frac{D_{N+n-1}}{D_{N+n-2}}x^{r^{N+n}-r^{N+n-1}}Q_{n-2}(x)$$

(They are proved by induction). Since by (2.2) for $n \ge 2$

$$a_n(x) = \frac{D_{N+n}}{D_{N+n-1}} + x^{r^{N+n}-r^{N+n-1}} \quad \text{and} \quad b_n(x) = -\frac{D_{N+n-1}}{D_{N+n-2}}x^{r^{N+n}-r^{N+n-1}}\,,$$

we have the following continued fraction expansion.

Theorem 1.

$$\sum_{n=0}^{\infty}\frac{BC_{N,n}}{\Pi(n)}x^n = 1 - \cfrac{x^{r^{N+1}-r^N}}{\frac{D_{N+1}}{D_N} + x^{r^{N+1}-r^N} - \cfrac{\frac{D_{N+1}}{D_N}x^{r^{N+2}-r^{N+1}}}{\frac{D_{N+2}}{D_{N+1}} + x^{r^{N+2}-r^{N+1}} - \cfrac{\frac{D_{N+2}}{D_{N+1}}x^{r^{N+3}-r^{N+2}}}{\frac{D_{N+3}}{D_{N+2}} + x^{r^{N+3}-r^{N+2}} - \ddots}}}\,.$$

Put $N = 0$ in Theorem 1 to illustrate a simpler case. Then, we have a continued fraction expansion concerning the original Bernoulli-Carlitz numbers.

Corollary 1.

$$\sum_{n=0}^{\infty}\frac{BC_n}{\Pi(n)}x^n = 1 - \cfrac{x^{r-1}}{D_1 + x^{r-1} - \cfrac{D_1 x^{r^2-r}}{\frac{D_2}{D_1} + x^{r^2-r} - \cfrac{\frac{D_2}{D_1}x^{r^3-r^2}}{\frac{D_3}{D_2} + x^{r^3-r^2} - \ddots}}}\,.$$

From the definition in (1.10), truncated Cauchy-Carlitz numbers satisfy the relation

$$\left(L_N\sum_{i=0}^{\infty}\frac{(-1)^i x^{r^{N+i}-r^N}}{L_{N+i}}\right)\left(\sum_{n=0}^{\infty}\frac{CC_{N,n}}{\Pi(n)}x^n\right) = 1\,.$$

Thus,

$$P_m(x) = \frac{L_{N+m}}{L_N}, \quad Q_m(x) = L_{N+m} \sum_{i=0}^{m} \frac{(-1)^i x^{r^{N+i}-r^N}}{L_{N+i}}$$

yield that

$$Q_m(x) \sum_{n=0}^{\infty} \frac{CC_{N,n}}{\Pi(n)} x^n \sim P_m(x) \quad (m \to \infty)\,.$$

Now,

$$\frac{P_0(x)}{Q_0(x)} = 1 = \frac{1}{1}, \quad \frac{P_1(x)}{Q_1(x)} = 1 + \frac{x^{r^{N+1}-r^N}}{L_{N+1}/L_N - x^{r^{N+1}-r^N}}$$

and $P_n(x)$ and $Q_n(x)$ ($n \ge 2$) satisfy the recurrence relations

$$P_n(x) = \left(\frac{L_{N+n}}{L_{N+n-1}} - x^{r^{N+n}-r^{N+n-1}}\right) P_{n-1}(x) + \frac{L_{N+n-1}}{L_{N+n-2}} x^{r^{N+n}-r^{N+n-1}} P_{n-2}(x)$$
$$Q_n(x) = \left(\frac{L_{N+n}}{L_{N+n-1}} - x^{r^{N+n}-r^{N+n-1}}\right) Q_{n-1}(x) + \frac{L_{N+n-1}}{L_{N+n-2}} x^{r^{N+n}-r^{N+n-1}} Q_{n-2}(x)\,.$$

Since by (2.2) for $n \ge 2$

$$a_n(x) = \frac{L_{N+n}}{L_{N+n-1}} - x^{r^{N+n}+r^{N+n-1}} \quad \text{and} \quad b_n(x) = \frac{L_{N+n-1}}{L_{N+n-2}} x^{r^{N+n}-r^{N+n-1}}\,,$$

we have the following continued fraction expansion.

Theorem 2.

$$\sum_{n=0}^{\infty} \frac{CC_{N,n}}{\Pi(n)} x^n = 1 + \cfrac{x^{r^{N+1}-r^N}}{\frac{L_{N+1}}{L_N} - x^{r^{N+1}-r^N} + \cfrac{\frac{L_{N+1}}{L_N} x^{r^{N+2}-r^{N+1}}}{\frac{L_{N+2}}{L_{N+1}} - x^{r^{N+2}-r^{N+1}} + \cfrac{\frac{L_{N+2}}{L_{N+1}} x^{r^{N+3}-r^{N+2}}}{\frac{L_{N+3}}{L_{N+2}} - x^{r^{N+3}-r^{N+2}} + \ddots}}}\,.$$

3. A determinant expression of truncated Cauchy-Carlitz numbers

In [14], some expressions of truncated Cauchy-Carlitz numbers have been shown. One of them is for integers $N \ge 0$ and $n \ge 1$,

$$CC_{N,n} = \Pi(n) \sum_{k=1}^{n} (-L_N)^k \sum_{\substack{i_1,\dots,i_k \ge 1 \\ r^{N+i_1}+\dots+r^{N+i_k}=n+kr^N}} \frac{(-1)^{i_1+\dots+i_k}}{L_{N+i_1} \cdots L_{N+i_k}} \tag{3.1}$$

[14, Theorem 2].

Now, we give a determinant expression of truncated Cauchy-Carlitz numbers.

Theorem 3. *For integers $N \geq 0$ and $n \geq 1$,*

$$CC_{N,n} = \Pi(n) \begin{vmatrix} -a_1 & 1 & 0 & & \\ a_2 & -a_1 & \ddots & & \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & -a_1 & 1 \\ (-1)^n a_n & \cdots & \cdots & a_2 & -a_1 \end{vmatrix},$$

where

$$a_l = \frac{(-1)^i L_N \delta_l^*}{L_{N+i}} \quad (l \geq 1)$$

with

$$\delta_l^* = \begin{cases} 1 & \text{if } l = r^{N+i} - r^N \ (i = 0, 1, \dots); \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

We need the following Lemma in [27] in order to prove Theorem 3.

Lemma 1. *Let $\{\alpha_n\}_{n\geq 0}$ be a sequence with $\alpha_0 = 1$, and $R(j)$ be a function independent of n. Then*

$$\alpha_n = \begin{vmatrix} R(1) & 1 & 0 & & \\ R(2) & R(1) & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ R(n-1) & R(n-2) & \cdots & R(1) & 1 \\ R(n) & R(n-1) & \cdots & R(2) & R(1) \end{vmatrix}. \tag{3.3}$$

if and only if

$$\alpha_n = \sum_{j=1}^{n} (-1)^{j-1} R(j) \alpha_{n-j} \quad (n \geq 1) \tag{3.4}$$

with $\alpha_0 = 1$.

Proof of Theorem 3. By the definition (1.10) with (1.6), we have

$$\begin{aligned} 1 &= \left(\sum_{i=0}^{\infty} (-1)^i \frac{L_N}{L_{N+i}} \right) x^{r^{N+i} - r^N} \left(\sum_{m=0}^{\infty} \frac{CC_m}{\Pi(m)} x^m \right) \\ &= \left(\sum_{l=0}^{\infty} a_l x^l \right) \left(\sum_{m=0}^{\infty} \frac{CC_m}{\Pi(m)} x^m \right) \\ &= \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} a_l \frac{CC_{n-l}}{\Pi(n-l)} x^n . \end{aligned}$$

Thus, for $n \geq 1$, we get

$$\sum_{l=0}^{\infty} a_l \frac{CC_{n-l}}{\Pi(n-l)} = 0 .$$

By Lemma 1, we have

$$\begin{aligned}
\frac{CC_n}{\Pi(n)} &= -\sum_{l=1}^{n} a_l \frac{CC_{n-l}}{\Pi(n-l)} \\
&= \sum_{l=1}^{n} (-1)^{l-1}(-1)^l a_l \frac{CC_{n-l}}{\Pi(n-l)} \\
&= \begin{vmatrix} -a_1 & 1 & 0 & & \\ a_2 & -a_1 & \ddots & & \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & -a_1 & 1 \\ (-1)^n a_n & \cdots & \cdots & a_2 & -a_1 \end{vmatrix}.
\end{aligned}$$

□

Examples. When $n = r^{N+1} - r^N$,

$$\begin{aligned}
\frac{CC_{r^{N+1}-r^N}}{\Pi(r^{N+1}-r^N)} &= \begin{vmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ (-1)^{r^{N+1}-r^N} a_{r^{N+1}-r^N} & 0 & \cdots & 0 \end{vmatrix} \\
&= (-1)^{r^{N+1}-r^N+1}(-1)^{r^{N+1}-r^N} \frac{(-1)^{2N+1} L_N}{L_{N+1}} \\
&= \frac{L_N}{L_{N+1}}.
\end{aligned}$$

Let $n = r^{N+2} - r^N$. For simplicity, put

$$\begin{aligned}
\bar{a} &= (-1)^{r^{N+1}-r^N}(-1)^{2N+1} \frac{L_N}{L_{N+1}}, \\
\hat{a} &= (-1)^{r^{N+2}-r^N}(-1)^{2N+2} \frac{L_N}{L_{N+2}}.
\end{aligned}$$

Then by expanding at the first column, we have

$$\frac{CC_{r^{N+1}-r^N}}{\Pi(r^{N+1}-r^N)} = \begin{vmatrix} 0 & 1 & 0 & & & & & \\ \vdots & & \ddots & & & & & \\ 0 & & & & & & & \\ \bar{a} & & & & & & & \\ 0 & \ddots & & & & & & \\ \vdots & & & & & & \ddots & 0 \\ 0 & & \ddots & & & & & 1 \\ \multicolumn{4}{c}{\underbrace{\hat{a} \quad 0 \quad \cdots \quad 0}_{r^{N+2}-r^{N+1}}} & \bar{a} & \multicolumn{3}{c}{\underbrace{0 \quad \cdots \quad 0}_{r^{N+1}-r^N-1}} \end{vmatrix}$$

$$
= (-1)^{r^{N+1}-r^{N}+1}\bar{a}
\begin{vmatrix}
1 &&&&&&& \\
& \ddots &&&&&& \\
&& 1 & 0 & 1 &&& \\
\bar{a} &&&& \ddots &&& \\
& \ddots &&&&&& \\
&& \ddots &&&&& \\
&&& \ddots &&& \ddots & 1 \\
&&&&&& \cdots & 0 \\
\multicolumn{3}{c}{\underbrace{\hphantom{xxxxxxxxxx}}_{r^{N+1}-r^{N}-1}} & \multicolumn{2}{c}{\underbrace{\hphantom{xx}\bar{a}}_{2}} & \multicolumn{3}{c}{\underbrace{\hphantom{xxxxxxxx}}_{r^{N+1}-r^{N}-1}}
\end{vmatrix}
$$

$$
+ (-1)^{r^{N+2}-r^{N}+1}\hat{a}
\begin{vmatrix}
1 & 0 &&&& \\
& \ddots &&&& \\
&&&&& \\
\bar{a} &&&&& \\
& \ddots &&&& \\
&& \ddots && \ddots & 0 \\
&&& \bar{a} && 1
\end{vmatrix}.
$$

The second term is equal to

$$
(-1)^{r^{N+2}-r^{N}+1}(-1)^{r^{N+2}-r^{N}}\frac{L_N}{L_{N+2}} = -\frac{L_N}{L_{N+2}}.
$$

The first term is

$$
(-1)^{r^{N+1}-r^{N}+1}\bar{a}
\begin{vmatrix}
0 & 1 &&&& \\
&&&&& \\
\bar{a} &&&&& \\
& \ddots &&&& \\
&& \ddots &&& \\
&&&&& 1 \\
&&& \bar{a} && 0 \\
\multicolumn{3}{c}{\underbrace{\hphantom{xxxxxxxxxx}}_{r^{N+2}-2r^{N+1}+r^{N}}} & & \multicolumn{2}{c}{\underbrace{\hphantom{xxxxxx}}_{r^{N+1}-r^{N}-1}}
\end{vmatrix}
$$

$$
= (-1)^{2(r^{N+1}-r^{N}+1)}\bar{a}^{2}
\begin{vmatrix}
1 & & & & & & \\
& \ddots & & & & & \\
& & 1 & 0 & 1 & & \\
\bar{a} & & & & \ddots & & \\
& \ddots & & & & & \\
& & \ddots & & & & \\
& & & & & \ddots & 1 \\
& & & \ddots & & \cdots & 0 \\
& & & & \bar{a} & & \\
& \underbrace{\qquad\qquad}_{r^{N+1}-r^{N}-1} & & \underbrace{\qquad\quad}_{2} & & \underbrace{\qquad\quad}_{r^{N+1}-r^{N}-1} &
\end{vmatrix}
$$

$$
= (-1)^{r(r^{N+1}-r^{N}+1)}\bar{a}^{r}
\underbrace{\begin{vmatrix}
0 & 1 & & \\
\vdots & & \ddots & \\
0 & & & 1 \\
\bar{a} & 0 & \cdots & 0
\end{vmatrix}}_{r^{N+1}-r^{N}}
$$

$$
= (-1)^{(r+1)(r^{N+1}-r^{N}+1)}\bar{a}^{r+1}
$$

$$
= (-1)^{(r+1)(r^{N+1}-r^{N}+1)}(-1)^{(r^{N+1}-r^{N})(r+1)}(-1)^{r+1}\frac{L_N^{r+1}}{L_{N+1}^{r+1}}
$$

$$
= \frac{L_N^{r+1}}{L_{N+1}^{r+1}}\,.
$$

Therefore,

$$
\frac{CC_{r^{N+1}-r^{N}}}{\Pi(r^{N+1}-r^{N})} = \frac{L_N^{r+1}}{L_{N+1}^{r+1}} - \frac{L_N}{L_{N+2}}\,.
$$

From this procedure, it is also clear that $CC_{N,n} = 0$ if $r^{N+1} - r^{N} \nmid n$, since all the elements of one column (or row) become zero.

4. A determinant expression of truncated Bernoulli-Carlitz numbers

In [14], some expressions of truncated Bernoulli-Carlitz numbers have been shown. One of them is for integers $N \geq 0$ and $n \geq 1$,

$$
BC_{N,n} = \Pi(n)\sum_{k=1}^{n}(-D_N)^{k}\sum_{\substack{i_1,\dots,i_k\geq 1\\ r^{N+i_1}+\cdots+r^{N+i_k}=n+kr^{N}}}\frac{1}{D_{N+i_1}\cdots D_{N+i_k}} \tag{4.1}
$$

[14, Theorem 1].

Now, we give a determinant expression of truncated Bernoulli-Carlitz numbers.

Theorem 4. *For integers $N \geq 0$ and $n \geq 1$,*

$$BC_{N,n} = \Pi(n) \begin{vmatrix} -d_1 & 1 & 0 & & \\ d_2 & -d_1 & \ddots & & \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & -d_1 & 1 \\ (-1)^n d_n & \cdots & \cdots & d_2 & -d_1 \end{vmatrix},$$

where

$$d_l = \frac{D_N \delta_l^*}{D_{N+i}} \quad (l \geq 1)$$

with δ_l^* as in (3.2).

Proof. The proof is similar to that of Theorem 3, using (1.9) and (1.2). □

Example. Let $n = 2(r^{N+1} - r^N)$. For convenience, put

$$\bar{d} = \frac{D_N}{D_{N+1}}.$$

Then, we have

$$\begin{aligned} \frac{BC_{N,2(r^{N+1}-r^N)}}{\Pi(2(r^{N+1}-r^N))} &= \begin{vmatrix} \begin{matrix} 0 & 1 & & \\ \vdots & & & \\ 0 & & & \\ \bar{d} & & & \\ & \ddots & & \\ & & & \bar{d} \end{matrix} & \begin{matrix} & & \\ & & \\ & & \\ & & \\ & & 1 \\ 0 & \cdots & 0 \end{matrix} \\ \underbrace{\qquad\qquad\qquad}_{r^{N+1}-r^N+1} & \underbrace{\qquad\qquad}_{r^{N+1}-r^N-1} \end{vmatrix} \\ &= (-1)^{r^{N+1}-r^N+1} \begin{vmatrix} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \\ \bar{d} & & \\ & \ddots & \\ & & \ddots \end{matrix} & \begin{matrix} \\ \\ \\ 0 \\ \\ \\ \bar{d} \end{matrix} & \begin{matrix} & & \\ & & \\ & & \\ 1 & & \\ & \ddots & \\ & & 1 \\ & & 0 \end{matrix} \\ \underbrace{\qquad\qquad\qquad}_{r^{N+1}-r^N-1} & & \underbrace{\qquad\qquad}_{r^{N+1}-r^N-1} \end{vmatrix} \\ &= (-1)^{r^{N+1}-r^N+1} \bar{d} \underbrace{\begin{vmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ \bar{d} & & & 0 \end{vmatrix}}_{r^{N+1}-r^N} \end{aligned}$$

$$= (-1)^{2(r^{N+1}-r^N+1)}\bar{d}^2 \begin{vmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{vmatrix}$$

$$= \frac{D_N^2}{D_{N+1}^2}.$$

It is also clear that $BC_{N,n} = 0$ if $r^{N+1} - r^N \nmid n$.

5. Applications by the Trudi's formula

We shall use Trudi's formula to obtain different explicit expressions and inversion relations for the numbers $CC_{N,n}$ and $BC_{N,n}$.

Lemma 2. *For a positive integer n, we have*

$$\begin{vmatrix} a_1 & a_0 & 0 & \cdots & \\ a_2 & a_1 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ a_{n-1} & & \cdots & a_1 & a_0 \\ a_n & a_{n-1} & \cdots & a_2 & a_1 \end{vmatrix} = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\dots,t_n} (-a_0)^{n-t_1-\cdots-t_n} a_1^{t_1} a_2^{t_2} \cdots a_n^{t_n},$$

where $\binom{t_1+\cdots+t_n}{t_1,\dots,t_n} = \frac{(t_1+\cdots+t_n)!}{t_1!\cdots t_n!}$ *are the multinomial coefficients.*

This relation is known as Trudi's formula [28, Vol.3, p.214], [29] and the case $a_0 = 1$ of this formula is known as Brioschi's formula [30], [28, Vol.3, pp.208–209].

In addition, there exists the following inversion formula (see, e.g. [27]), which is based upon the relation

$$\sum_{k=0}^{n} (-1)^{n-k} \alpha_k D(n-k) = 0 \quad (n \ge 1).$$

Lemma 3. *If* $\{\alpha_n\}_{n\ge 0}$ *is a sequence defined by* $\alpha_0 = 1$ *and*

$$\alpha_n = \begin{vmatrix} D(1) & 1 & 0 & \\ D(2) & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ D(n) & \cdots & D(2) & D(1) \end{vmatrix}, \text{ then } D(n) = \begin{vmatrix} \alpha_1 & 1 & 0 & \\ \alpha_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ \alpha_n & \cdots & \alpha_2 & \alpha_1 \end{vmatrix}.$$

From Trudi's formula, it is possible to give the combinatorial expression

$$\alpha_n = \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\dots,t_n} (-1)^{n-t_1-\cdots-t_n} D(1)^{t_1} D(2)^{t_2} \cdots D(n)^{t_n}.$$

By applying these lemmata to Theorem 3 and Theorem 4, we obtain an explicit expression for the truncated Cauchy-Carlitz numbers and the truncated Bernoulli-Carlitz numbers.

Theorem 5. *For integers $N \geq 0$ and $n \geq 1$, we have*

$$CC_{N,n} = \Pi(n) \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\ldots,t_n} (-1)^{n-t_2-t_4-\cdots-t_{2\lfloor n/2\rfloor}} a_1^{t_1}\cdots a_n^{t_n},$$

where a_n are given in Theorem 3.

Theorem 6. *For integers $N \geq 0$ and $n \geq 1$, we have*

$$BC_{N,n} = \Pi(n) \sum_{t_1+2t_2+\cdots+nt_n=n} \binom{t_1+\cdots+t_n}{t_1,\ldots,t_n} (-1)^{n-t_2-t_4-\cdots-t_{2\lfloor n/2\rfloor}} d_1^{t_1}\cdots d_n^{t_n},$$

where d_n are given in Theorem 4.

By applying the inversion relation in Lemma 3 to Theorem 3 and Theorem 4, we have the following.

Theorem 7. *For integers $N \geq 0$ and $n \geq 1$, we have*

$$a_n = (-1)^n \begin{vmatrix} \frac{CC_{N,1}}{\Pi(1)} & 1 & 0 & & \\ \frac{CC_{N,2}}{\Pi(2)} & \frac{CC_{N,1}}{\Pi(1)} & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ \frac{CC_{N,n-1}}{\Pi(n-1)} & \frac{CC_{N,n-2}}{\Pi(n-2)} & \cdots & \frac{CC_{N,1}}{\Pi(1)} & 1 \\ \frac{CC_{N,n}}{\Pi(n)} & \frac{CC_{N,n-1}}{\Pi(n-1)} & \cdots & \frac{CC_{N,2}}{\Pi(2)} & \frac{CC_{N,1}}{\Pi(1)} \end{vmatrix},$$

where a_n is given in Theorem 3.

Theorem 8. *For integers $N \geq 0$ and $n \geq 1$, we have*

$$d_n = (-1)^n \begin{vmatrix} \frac{BC_{N,1}}{\Pi(1)} & 1 & 0 & & \\ \frac{BC_{N,2}}{\Pi(2)} & \frac{BC_{N,1}}{\Pi(1)} & & & \\ \vdots & \vdots & \ddots & 1 & 0 \\ \frac{BC_{N,n-1}}{\Pi(n-1)} & \frac{BC_{N,n-2}}{\Pi(n-2)} & \cdots & \frac{BC_{N,1}}{\Pi(1)} & 1 \\ \frac{BC_{N,n}}{\Pi(n)} & \frac{BC_{N,n-1}}{\Pi(n-1)} & \cdots & \frac{BC_{N,2}}{\Pi(2)} & \frac{BC_{N,1}}{\Pi(1)} \end{vmatrix},$$

where d_n is given in Theorem 4.

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Conflict of interest

The authors declare no conflict of interest.

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