AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 5(5): 4874–4888.
DOI: 10.3934/math.2020311
Received: 01 March 2020
Accepted: 22 May 2020
Published: 03 June 2020

Research article

New inequalities of Wilker's type for circular functions

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Abstract: In the article, we establish three new Wilker type inequalities involving tangent and sine functions by use of a double inequality for the ratio of two consecutive non-zero Bernoulli numbers.

Keywords: Wilker type inequality; circular function; tangent function; sine function; Bernoulli number
Mathematics Subject Classification: 33B10, 26D05

1. Introduction

Inequalities are ubiquitous in all branches of pure and applied mathematics, while the trigonometric and hyperbolic functions inequalities are the indispensable parts in the whole theory of inequality, there are numerous researchers devoted to the study of mathematical inequalities. Recently, there are many novel inequalities in different areas have been discovered, for example, the Hermite-Hadamard type inequalities [1–6], bivariate means inequalities [7–11], gamma function inequalities [12], complete elliptic integrals inequalities [13–15], Bessel functions inequalities [16], Jensen type inequalities [17–19], Ostrowski type inequalities [20], Cauchy-Schwarz inequality [21], reverse Minkowski inequality [22], Petrović-type inequalities [23], Pólya-Szegö and Ćebyšev type inequalities [24], delay dynamic double integral inequalities [25], integral majorization type inequalities [26], generalized convex functions inequalities [27], generalized proportional fractional integral operators inequalities [28], generalized trigonometric and hyperbolic functions inequalities [29], exponentially convex inequalities [30] and so on.

In 1989, Montgomery et al. [31] proposed two open problems as follows:

(1) Does the inequality

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 \tag{1.1}$$

holds for all $0 < x < \pi/2$?

(2) Is there a largest constant c such that the inequality

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 + cx^3 \tan x \tag{1.2}$$

holds for all $0 < x < \pi/2$?

Wilker et al. [32] gave positive answers to the inequalities (1.1) and (1.2), and proved that the double inequality

$$\frac{16}{\pi^4} x^3 \tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2 < \frac{8}{45} x^3 \tan x \tag{1.3}$$

holds for all $x \in (0, \pi/2)$ with the best constants $16/\pi^4$ and $8/45$.

For different proofs of inequalities (1.1) and (1.3), we recommend the literature [33,34] to the interested readers.

In [35], Zhu proved that the Wilker type inequality

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} - 2 > \frac{8}{45} x^3 \tanh x \tag{1.4}$$

for hyperbolic functions holds whenever $x \neq 0$, and the constant $8/45$ in the right of inequality (1.4) can not be replaced by any larger number.

Sun and Zhu [36] provided a new version of the Wilker type inequality for hyperbolic functions as follows:

$$\left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x} - 2 < \frac{2}{45} x^3 \sinh x \tag{1.5}$$

for all $x \neq 0$, and the constant $2/45$ in the right of inequality (1.5) can not be replaced by any smaller number.

Very recently, Zhu [37] established the following novel results (Propositions 1.1–1.3) involving the Wilker type inequality for hyperbolic functions by use of the power series formulas and monotonicity criteria of the quotient of two power series [38].

Proposition 1.1. The inequality

$$\left(\frac{\sinh x}{x}\right)^2 + \frac{\tanh x}{x} - 2 > \frac{8}{45} x^4 \left(\frac{\tanh x}{x}\right)^{6/7} \tag{1.6}$$

takes place for all $x \neq 0$ with the best possible constant $8/45$ in the right of inequality (1.6).

Proposition 1.2. $x_0 = 1.54471\cdots$ is the unique point on the interval $(0, \infty)$ such that the function

$$G(x) = \frac{\left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x} - 2}{x^3 \tanh x}$$

attain its maximum

$$\theta_0 = G(x_0) = \max_{x \in (0,\infty)} G(x) = 0.050244\cdots,$$

and θ_0 is the best possible constant such that the following inequality (1.7)

$$\left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x} - 2 < \theta_0 x^3 \tanh x \tag{1.7}$$

holds for all $x \in (0, \infty)$.

Proposition 1.3. $2/45$ is the best possible constant such that the inequality (1.8)

$$\left(\frac{x}{\sinh x}\right)^2 + \frac{x}{\tanh x} - 2 < \frac{2}{45}x^4\left(\frac{\tanh x}{x}\right)^{4/7} \tag{1.8}$$

holds for all $x \neq 0$.

This main purpose of the article is to find new Wilker type inequalities for trigonometric functions. Our main results are the following Theorems 1.1–1.3.

Theorem 1.1. The inequality

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2 > \frac{8}{45}x^4\left(\frac{\tan x}{x}\right)^{6/7} \tag{1.9}$$

is valid for all $x \in (0, \pi/2)$ with the best possible constant $8/45$ in the right of inequality (1.9).

Theorem 1.2. $\alpha = 0$ and $\beta = 2/45$ are the best possible constants such that the two-sided inequality (1.10)

$$\alpha x^3 \tan x < \left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} - 2 < \beta x^3 \tan x \tag{1.10}$$

takes place for all $x \in (0, \pi/2)$.

Theorem 1.3. Let $0 < x < \pi/2$. Then the double inequality

$$\alpha x^4\left(\frac{\tan x}{x}\right)^{4/7} < \left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} - 2 < \beta x^4\left(\frac{\tan x}{x}\right)^{4/7} \tag{1.11}$$

holds with the best constants $\alpha = 0$ and $\beta = 2/45$ in the left and right sides of inequality (1.11), respectively.

2. Lemmas

In order to prove our main results, we need four lemmas which we present in this section.

Lemma 2.1. Let $|x| < \pi/2$. Then we have the following power series formulas

$$\tan x = \sum_{n=1}^{\infty} \frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}|B_{2n}|x^{2n-1}, \tag{2.1}$$

$$\tan^2 x = \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)(2n-1)}{(n)}|B_{2n}|x^{2n-2},$$

$$\tan^3 x = \sum_{n=3}^{\infty} \frac{2^{2n}(2^{2n}-1)(2n-1)(2n-2)}{2(2n)!}|B_{2n}|x^{2n-3} - \sum_{n=2}^{\infty} \frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}|B_{2n}|x^{2n-1},$$

$$\tan^4 x = \sum_{n=4}^{\infty} \frac{2^{2n}(2^{2n}-1)(2n-1)(2n-2)(2n-3)|B_{2n}|}{6(2n)!} x^{2n-4} - \sum_{n=3}^{\infty} \frac{2^{2n+2}\left(2^{2n}-1\right)(2n-1)|B_{2n}|}{3(2n)!} x^{2n-2}$$

and

$$\sec^2 x \tan x = \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)(2n-1)(2n-2)}{2(2n)!} |B_{2n}| x^{2n-3},$$

$$\sec^2 x \tan^2 x = \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)(2n-1)(2n-2)(2n-3)}{6(2n)!} |B_{2n}| x^{2n-4} - \sum_{n=1}^{\infty} \frac{2^{2n}\left(2^{2n}-1\right)(2n-1)}{3(2n)!} |B_{2n}| x^{2n-2},$$

$$\sec^2 x \tan^3 x = \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)(2n-1)(2n-2)(2n-3)(2n-4)|B_{2n}|}{24(2n)!} x^{2n-5} - \sum_{n=2}^{\infty} \frac{2^{2n+2}\left(2^{2n}-1\right)(2n-1)(2n-2)}{12(2n)!} |B_{2n}| x^{2n-3},$$

where B_n is the Bernoulli number.

Proof. The power series formula (2.1) can be found in the literature [39, 1.3.1.4(2)]. Next, we prove the remain power series formulas.

It follows from (2.1) that

$$\tan^2 x = \sec^2 x - 1 = (\tan x)' - 1 = \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)(2n-1)}{(2n)!} |B_{2n}| x^{2n-2}. \tag{2.2}$$

Making use of (2.1) and (2.2) we get

$$\tan^3 x = \frac{1}{2}\left[\left(\tan^2 x\right)' - 2\tan x\right] = \frac{1}{2}\left(\tan^2 x\right)' - \tan x$$

$$= \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)(2n-1)(2n-2)}{2(2n)!} |B_{2n}| x^{2n-3} - \sum_{n=1}^{\infty} \frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!} |B_{2n}| x^{2n-1}$$

$$= \sum_{n=3}^{\infty} \frac{2^{2n}(2^{2n}-1)(2n-1)(2n-2)}{2(2n)!} |B_{2n}| x^{2n-3} - \sum_{n=2}^{\infty} \frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!} |B_{2n}| x^{2n-1}$$

and

$$\begin{aligned}\tan^4 x &= \frac{1}{3}\left[\left(\tan^3 x\right)' - 3\tan^2 x\right] \\ &= \frac{1}{3}\left(\frac{1}{2}\left(\tan^2 x\right)' - \tan x\right)' - \tan^2 x \\ &= \frac{1}{6}\left(\tan^2 x\right)'' - \frac{1}{3}(\tan x)' - \tan^2 x\end{aligned}$$

$$\begin{aligned}&= \sum_{n=2}^{\infty} \frac{2^{2n}(2^{2n}-1)(2n-1)(2n-2)(2n-3)}{6(2n)!}|B_{2n}|x^{2n-4} \\ &\quad - \sum_{n=1}^{\infty} \frac{\left(2^{2n}-1\right)(2n-1)2^{2n}}{3(2n)!}|B_{2n}|x^{2n-2} \\ &\quad - \sum_{n=2}^{\infty} \frac{(2^{2n}-1)(2n-1)2^{2n}}{(2n)!}|B_{2n}|x^{2n-2} \\ &= \sum_{n=3}^{\infty} \frac{2^{2n}(2^{2n}-1)(2n-1)(2n-2)(2n-3)}{6(2n)!}|B_{2n}|x^{2n-4} \\ &\quad - \sum_{n=2}^{\infty} \frac{2^{2n+2}\left(2^{2n}-1\right)(2n-1)}{3(2n)!}|B_{2n}|x^{2n-2} \\ &= \sum_{n=4}^{\infty} \frac{2^{2n}(2^{2n}-1)(2n-1)(2n-2)(2n-3)}{6(2n)!}|B_{2n}|x^{2n-4} \\ &\quad - \sum_{n=3}^{\infty} \frac{2^{2n+2}\left(2^{2n}-1\right)(2n-1)}{3(2n)!}|B_{2n}|x^{2n-2}.\end{aligned}$$

Note that

$$\sec^2 x \tan^k x = \frac{1}{k+1}\frac{d}{dx}\tan^{k+1} x, k \neq -1.$$

The desired power series formulas for the functions $\sec^2 x \tan x$, $\sec^2 x \tan^2 x$ and $\sec^2 x \tan^3 x$ can be derived easily. □

Lemma 2.2. (See [40]) The power series formula

$$\frac{1}{\sin x} = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2^{2n}-2}{(2n)!}|B_{2n}|x^{2n-1} \tag{2.3}$$

holds for all $x \in (-\pi, 0) \cup (0, \pi)$, where B_n is the Bernoulli number.

Lemma 2.3. (See [41]) The double inequality

$$\frac{2^{2n-1}-1}{2^{2n+1}-1}\frac{(2n+2)(2n+1)}{\pi^2} < \frac{|B_{2n+2}|}{|B_{2n}|} < \frac{2^{2n}-1}{2^{2n+2}-1}\frac{(2n+2)(2n+1)}{\pi^2}. \tag{2.4}$$

holds for $n = 1, 2, 3, \cdots$, where B_n is the Bernoulli number.

Lemma 2.4. (See [38]) Let a_n and b_n ($n = 0, 1, 2, 3, \cdots$) be real numbers, and the power series $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ be convergent for $|x| < R$ ($R \le \infty$). If $b_n > 0$ for all $n \ge 0$ and the sequence $\{a_n/b_n\}_{n=0}^{\infty}$ is increasing (decreasing), then the function $A(x)/B(x)$ is strictly increasing (decreasing) on $(0, R)$.

3. Proof of Theorems 1.1–1.3

Proof of Theorem 1.1. Let $0 < |x| < \pi/2$ and

$$F(x) = 7\ln\left[\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2\right] - \ln\left[\left(\frac{8}{45}\right)^7 x^{22} (\tan x)^6\right].$$

Then elaborated computations lead to

$$F'(x) = \frac{\cos^2 x}{x^3 (\tan x)\left[\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} - 2\right]} f(x),$$

where

$$f(x) = 12x^3 \sec^2 x + x^2 \sec^2 x \tan^3 x + 12x^3 \tan^2 x - 6x \tan^2 x + 14x \tan^2 x$$
$$-29x \sec^2 x \tan^2 x + 45x^2 \sec^2 x \tan x - 36 \tan^3 x - 6x \tan^4 x.$$

By substituting the power series formulas in Lemma 2.1 into $f(x)$, we obtain

$$f(x) = \sum_{n=6}^{\infty} l_n x^{2n-1},$$

where

$$l_n = \frac{6 \cdot 2^{2n}(2^{2n-2} - 1)(2n-3)|B_{2n-2}|}{(2n)!} + \frac{2^{2n}\left(266n^2 - 245n + 164\right)\left(2^{2n} - 1\right)|B_{2n}|}{3(2n)!}$$
$$+ \frac{(2n)\, 2^{2n}(2n-145)(2n+1)(n+1)\left(2^{2n+2} - 1\right)}{3(2n+2)!}|B_{2n+2}|.$$

It follows from (2.4) that

$$\frac{3}{2^{2n}|B_{2n}|} l_n = \frac{18(2^{2n-2} - 1)(2n-3)}{(2n)!}\frac{|B_{2n-2}|}{|B_{2n}|}$$
$$+ \frac{\left(266n^2 - 245n + 164\right)\left(2^{2n} - 1\right)}{(2n)!}$$
$$+ \frac{2n(2n-145)(2n+1)(n+1)\left(2^{2n+2} - 1\right)}{(2n+2)!}\frac{|B_{2n+2}|}{|B_{2n}|}$$
$$> \frac{18(2^{2n-2} - 1)(2n-3)}{(2n)!}\frac{\pi^2(2^{2n} - 1)}{(2n)(2n-1)(2^{2n-2} - 1)}$$

$$+\frac{\left(266n^2-245n+164\right)\left(2^{2n}-1\right)}{(2n)!}$$

$$+\frac{2n\,(2n-145)\,(2n+1)\,(n+1)\left(2^{2n+2}-1\right)}{(2n+2)!}$$

$$\times\frac{(2n+2)(2n+1)(2^{2n-1}-1)}{\pi^2(2^{2n+1}-1)}$$

$$=\frac{18\,(2n-3)}{(2n)!}\,\frac{\pi^2(2^{2n}-1)}{(2n)(2n-1)}+\frac{\left(266n^2-245n+164\right)\left(2^{2n}-1\right)}{(2n)!}$$

$$+\frac{2n\,(2n-145)\,(2n+1)\,(n+1)\left(2^{2n+2}-1\right)}{(2n)!}\,\frac{(2^{2n-1}-1)}{\pi^2(2^{2n+1}-1)},$$

which is equivalent to

$$\frac{3\,(2n)!}{2^{2n}|B_{2n}|}l_n>\frac{18\pi^2\,(2n-3)\,(2^{2n}-1)}{(2n)(2n-1)}+\left(266n^2-245n+164\right)\left(2^{2n}-1\right)$$

$$+\frac{2n\,(2n-145)\,(2n+1)\,(n+1)\left(2^{2n+2}-1\right)(2^{2n-1}-1)}{\pi^2(2^{2n+1}-1)}$$

$$=\frac{h(n)}{n\,(2^{2n+1}-1)\,(2n-1)\,\pi^2},$$

where

$$h(n)=\left[u_1(n)2^{2n}-v_1(n)\right]2^{2n}+w(n),$$

$$u_1(n)=32n^6-2288n^5+(1064\pi-2328)\,n^4-\left(1512\pi^2-572\right)n^3$$
$$+\left(580+1146\pi^2\right)n^2-\left(328\pi^2-36\pi^4\right)n-54\pi^4,$$

$$v_1(n)=72n^6-5148n^5+\left(1596\pi^2-5238\right)n^4-\left(2268\pi^2-1287\right)n^3$$
$$+\left(1305+1719\pi^2\right)n^2-\left(492\pi^2-54\pi^4\right)n-81\pi^4,$$

$$w(n)=16n^6-1144n^5+\left(532\pi^2-1164\right)n^4-\left(756\pi^2-286\right)n^3$$
$$+\left(573\pi^2+290\right)n^2+\left(18\pi^4-164\pi^2\right)n-27\pi^4>0.$$

It is easy to prove that $2^{2n}>v_1(n)/u_1(n)$ for all $n\geq 6$ by using mathematical induction. Therefore, we deduce that

$$h(n)>0\Rightarrow l_n>0\Rightarrow f(x)>0\Rightarrow F'(x)>0,$$

$F(x)$ is strictly increasing on $(0,\pi/2)$ and $F(x)>F(0^+)=0$. Note that

$$\lim_{x\to0^+}\frac{\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}-2}{x^4\left(\frac{\tan x}{x}\right)^{6/7}}=\frac{8}{45}.$$

Which completes the proofs of Theorem 1.1.

Proof of Theorem 1.2. Let $0 < x < \pi/2$, and

$$A(x) = \frac{x\sin^2 x + x^2\tan x - 2\sin^2 x\tan x}{\sin^2 x}, \quad B(x) = x^3\tan^2 x,$$

$$G(x) = \frac{(\frac{x}{\sin x})^2 + \frac{x}{\tan x} - 2}{x^3\tan x}.$$

Then we clearly see that

$$G(x) = \frac{A(x)}{B(x)}.$$

It follows from Lemma 2.1 and inequality (2.3) that

$$\begin{aligned}
A(x) &= \frac{x\sin^2 x + x^2\tan x - 2\sin^2 x\tan x}{\sin^2 x} \\
&= x - 2\tan x + \frac{x^2}{\sin^2 x}\tan x \\
&= x - 2\tan x + 2x^2\left(\frac{1}{\sin 2x}\right) \\
&= x - 2\sum_{n=1}^{\infty}\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}|B_{2n}|x^{2n-1} \\
&\quad +2x^2\left[\frac{1}{2x} + \sum_{n=1}^{\infty}\frac{2^{2n}\left(2^{2n}-2\right)}{(2n)!}|B_{2n}|x^{2n-1}\right] \\
&= 2\sum_{n=2}^{\infty}\frac{2^{2n}\left(2^{2n}-2\right)}{(2n)!}|B_{2n}|x^{2n+1} - 2\sum_{n=3}^{\infty}\frac{2^{2n}\left(2^{2n}-1\right)}{(2n)!}|B_{2n}|x^{2n-1} \\
&= \sum_{n=2}^{\infty}\frac{2^{2n+1}\left(2^{2n}-2\right)}{(2n)!}|B_{2n}|x^{2n+1} - \sum_{n=2}^{\infty}\frac{2^{2n+3}\left(2^{2n+2}-1\right)}{(2n+2)!}|B_{2n+2}|x^{2n+1} \\
&= \sum_{n=2}^{\infty}\left[\frac{2^{2n+1}\left(2^{2n}-2\right)}{(2n)!}|B_{2n}| - \frac{2^{2n+3}\left(2^{2n+2}-1\right)}{(2n+2)!}|B_{2n+2}|\right]x^{2n+1} =: \sum_{n=2}^{\infty}a_n x^{2n+1}
\end{aligned}$$

and

$$B(x) = \sum_{n=2}^{\infty}\frac{2^{2n}(2^{2n}-1)(2n-1)}{(2n)!}|B_{2n}|x^{2n+1} =: \sum_{n=2}^{\infty}b_n x^{2n+1}.$$

Note that

$$\frac{a_n}{b_n} = \frac{\frac{2^{2n+1}\left(2^{2n}-2\right)}{(2n)!}|B_{2n}| - \frac{2^{2n+3}\left(2^{2n+2}-1\right)}{(2n+2)!}|B_{2n+2}|}{\frac{2^{2n}(2^{2n}-1)(2n-1)}{(2n)!}|B_{2n}|}$$

$$= \frac{2\left(2^{2n}-2\right)}{(2^{2n}-1)(2n-1)} - \frac{8\left(2^{2n+2}-1\right)}{(2^{2n}-1)(2n-1)(2n+2)(2n+1)}\frac{|B_{2n+2}|}{|B_{2n}|}.$$

Next, we prove that the sequence $\{a_n/b_n\}$ is decreasing for $n \geq 2$. Indeed, from Lemma 2.3 we clearly see that

$$\frac{a_n}{b_n} > \frac{2\left(2^{2n}-2\right)}{(2^{2n}-1)(2n-1)} - \frac{8\left(2^{2n+2}-1\right)}{(2n-1)(2^{2n+2}-1)\pi^2}$$
$$= \frac{2\left(2^{2n}\pi^2 - 2^{2n+2} - 2\pi^2 + 4\right)}{\pi^2(2n-1)(2^{2n}-1)}$$

and

$$\frac{a_{n+1}}{b_{n+1}} < \frac{2\left(2^{2n+2}-2\right)}{(2^{2n+2}-1)(2n+1)} - \frac{8\left(2^{2n+4}-1\right)\left(2^{2n+1}-1\right)}{(2^{2n+2}-1)(2n+1)(2^{2n+3}-1)\pi^2}$$
$$= \frac{4\left(2^{2n+1}-1\right)\left(2^{2n+3}\pi^2 - 2^{2n+5} - \pi^2 + 2\right)}{\pi^2(2^{2n+3}-1)(2^{2n+2}-1)(2n+1)}.$$

Therefore, the sequence $\{a_n/b_n\}$ is decreasing for $n \geq 2$ follows from the fact that

$$\left(2^{2n}\pi^2 - 2^{2n+2} - 2\pi^2 + 4\right)\left(2^{2n+3}-1\right)\left(2^{2n+2}-1\right)(2n+1)$$
$$-2(2n-1)\left(2^{2n}-1\right)\left(2^{2n+1}-1\right)\left(2^{2n+3}\pi^2 - 2^{2n+5} - \pi^2 + 2\right)$$
$$= \left[\left(64\pi^2 - 256\right)2^{2n} - \left(48n + 48\pi^2 n + 128\pi^2 - 376\right)\right]4^{2n}$$
$$+2^{2n}\left(48n + 6\pi^2 n + 47\pi^2 - 128\right) - \left(4\pi^2 - 8\right) > 0$$

for $n \geq 2$.

Therefore, it follows from Lemma 2.4 that the function $G(x) = A(x)/B(x)$ is strictly increasing on $(0, \pi/2)$, and Theorem 1.2 follows from the limit values

$$\lim_{x\to 0^+} G(x) = \frac{2}{45}, \quad \lim_{x\to\left(\frac{\pi}{2}\right)^-} G(x) = 0.$$

Proof of Theorem 1.3. Let $0 < x < \pi/2$ and

$$H(x) = \ln\left[\left(\frac{2}{45}\right)^7 x^{24}\tan^4 x\right] - 7\ln\left[\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} - 2\right].$$

Then elaborated computations lead to

$$H'(x) = \frac{p(x)}{x(\tan x)^2\left[\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} - 2\right]},$$

where

$$p(x) = 11x^2\tan^2 x - 8x\tan^3 x + 9x\tan x + 11x^2 - 48\tan^2 x + 10x^2\sec^2 x$$

$$+4x^3\sec^2 x\tan x+8\frac{x^3}{\sin 2x}+28\frac{x^3}{\sin 2x}.$$

It follows from Lemmas 2.1 and 2.2 that

$$p(x)=\sum_{n=5}^{\infty}\frac{36\cdot 2^{2n-2}\left(2^{2n-2}-2\right)}{(2n-2)!}|B_{2n-2}|x^{2n}$$

$$+\sum_{n=5}^{\infty}\frac{(2n)\,2^{2n}\left(2^{2n}-1\right)(4n+15)}{(2n)!}|B_{2n}|x^{2n}$$

$$-\sum_{n=5}^{\infty}\frac{2^{2n+5}\,(n+6)\,(2n+1)\left(2^{2n+2}-1\right)}{(2n+2)!}|B_{2n+2}|x^{2n}=:\sum_{n=5}^{\infty}c_nx^{2n},$$

where

$$c_n=\frac{36\cdot 2^{2n-2}\left(2^{2n-2}-2\right)}{(2n-2)!}|B_{2n-2}|+\frac{2n2^{2n}\left(2^{2n}-1\right)(4n+15)}{(2n)!}|B_{2n}|$$

$$-\frac{2^{2n+5}\,(n+6)\,(2n+1)\left(2^{2n+2}-1\right)}{(2n+2)!}|B_{2n+2}|.$$

From Lemma 2.3 we have

$$\frac{c_n}{|B_{2n}|}=\frac{9\cdot 2^{2n}\left(2^{2n-2}-2\right)}{(2n-2)!}\frac{|B_{2n-2}|}{|B_{2n}|}+\frac{(2n)\,2^{2n}\left(2^{2n}-1\right)(4n+15)}{(2n)!}$$

$$-\frac{2^{2n+5}\,(n+6)\,(2n+1)\left(2^{2n+2}-1\right)}{(2n+2)!}\frac{|B_{2n+2}|}{|B_{2n}|}$$

$$>\frac{9\cdot 2^{2n}\left(2^{2n-2}-2\right)}{(2n-2)!}\frac{2^{2n}-1}{2^{2n-2}-1}\frac{\pi^2}{(2n+2)(2n+1)}$$

$$+\frac{(2n)\,2^{2n}\left(2^{2n}-1\right)(4n+15)}{(2n)!}$$

$$-\frac{2^{2n+5}\,(n+6)\,(2n+1)\left(2^{2n+2}-1\right)}{(2n+2)!}\frac{2^{2n}-1}{2^{2n+2}-1}\frac{(2n+2)(2n+1)}{\pi^2}$$

$$=\frac{9\cdot 2^{2n}\left(2^{2n-2}-2\right)}{(2n)!}\frac{2^{2n}-1}{2^{2n-2}-1}\frac{\pi^2\,(2n-1)\,(2n)}{(2n+2)(2n+1)}$$

$$+\frac{(2n)\,2^{2n}\left(2^{2n}-1\right)(4n+15)}{(2n)!}$$

$$-\frac{2^{2n+5}\,(n+6)\,(2n+1)\left(2^{2n+2}-1\right)}{(2n)!}\frac{2^{2n}-1}{2^{2n+2}-1}\frac{1}{\pi^2},$$

that is

$$\frac{(2n)!c_n}{|B_{2n}|} > \frac{9\cdot 2^{2n}\left(2^{2n-2}-2\right)\left(2^{2n}-1\right)\pi^2(2n-1)(2n)}{(2^{2n-2}-1)(2n+2)(2n+1)}$$
$$+(2n)\,2^{2n}\left(2^{2n}-1\right)(4n+15)$$
$$-\frac{32\cdot 2^{2n}(n+6)(2n+1)\left(2^{2n+2}-1\right)\left(2^{2n}-1\right)}{(2^{2n+2}-1)\pi^2}$$

and

$$\frac{(2n)!c_n}{2^{2n}(2^{2n}-1)|B_{2n}|} > \frac{9\cdot\left(2^{2n-2}-2\right)\pi^2(2n-1)(2n)}{(2^{2n-2}-1)(2n+2)(2n+1)}$$
$$+2n(4n+15)-\frac{32(n+6)(2n+1)}{\pi^2}$$
$$=\frac{q(n)}{\pi^2(n+1)(2n+1)(2^{2n}-4)},$$

where

$$q(n)=u_2(n)2^{2n}-v_2(n),$$
$$u_2(n)=\left(16\pi^2-128\right)n^4-\left(1024n-84\pi^2\right)n^3$$
$$+\left(98\pi^2+18\pi^4-1696\right)n^2-\left(9\pi^4-30\pi^2+992\right)n-192,$$
$$v_2(n)=\left(64\pi^2-512\right)n^4-\left(4096-336\pi^2\right)n^3$$
$$+\left(144\pi^4+392\pi^2-6784\right)n^2-\left(3968-120\pi^2+72\pi^4\right)n-768.$$

It is not difficulty to prove that $q(n) > 0$ for all $n \geq 5$ by use of the mathematical induction. Therefore, $H'(x) > 0$, $F(x)$ is strictly increasing on $(0, \pi/2)$ and $H(x) > H(0^+) = 0$. Note that

$$\lim_{x\to 0^+}\frac{\left(\frac{x}{\sin x}\right)^2+\frac{x}{\tan x}-2}{x^4\left(\frac{\tan x}{x}\right)^{4/7}}=\frac{2}{45}.$$

Which completes the proofs of Theorem 1.3.

4. Remarks

Remark 4.1. The first inequality of (1.3) and our inequality (1.9) are not comparable. Indeed, experiments and numerical simulation show that our inequality (1.9) is better than the first inequality of (1.3) for $x \in (1.0828, \pi/2)$, but the first inequality of (1.3) is stronger than the inequality (1.9) for $x \in (0, 1.0828)$.

Remark 4.2. Let $\alpha = 0$. Then inequality (1.10) or (1.11) leads to

$$\left(\frac{x}{\sin x}\right)^2+\frac{x}{\tan x}>2$$

for $x \in (0, \pi/2)$, which also was proved by Zhu [42]. In addition, the second inequality of (1.11) is stronger than that of (1.10) due to the fact $\tan x > x$ for all $x \in (0, \pi/2)$.

Remark 4.3. It is worth pointing out that it is very meaningful and important to improve the existing inequalities. We recommend the interested readers to read the literature [43–45] where one can found many new inequalities for the circular, hyperbolic, inverse circular, inverse hyperbolic and exponential functions as well as their related special functions.

5. Conclusions

We have established three sharp inequalities of Wilker type for trigonometric functions:

$$\begin{aligned} \beta x^4\left(\frac{\tan x}{x}\right)^{6/7} &< \left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}-2, \\ \alpha x^3\tan x &< \left(\frac{x}{\sin x}\right)^2+\frac{x}{\tan x}-2<\beta x^3\tan x, \\ \alpha x^4\left(\frac{\tan x}{x}\right)^{4/7} &< \left(\frac{x}{\sin x}\right)^2+\frac{x}{\tan x}-2<\beta x^4\left(\frac{\tan x}{x}\right)^{4/7}, \end{aligned}$$

where $0 < x < \pi/2$, $\alpha = 0$, and $\beta = 8/45$. The above three inequalities improve and develop the known famous results.

Acknowledgments

The authors would like to thank the anonymous referees for their valuable comments and suggestions, which led to considerable improvement of the article.

The research is supported by the Natural Science Foundation of China (Grant No. 61772025).

Conflict of interest

The authors declare that they have no competing interests.

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