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Research article

On the Tame automorphisms of differential polynomial algebras

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Abstract: Let $R\{x, y\}$ be the differential polynomial algebra in two differential indeterminates x, y over a differential domain R with a derivation operator δ. In this paper, we study on automorphisms of the differential polynomial algebra $R\{x, y\}$ with one derivation operator. Using a method in group theory, we prove that the Tame subgroup of automorphism of $R\{x, y\}$ is the amalgamated free product of the Triangular and the Affine subgroups over their intersection.

Keywords: differential algebras; differential polynomial algebras; derivations; Tame automorphisms; amalgamated product
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1. Introduction

P. Cohn [3] proved that the automorphisms of a free Lie algebra with a finite set of generators are tame. The tameness of the automorphisms of polynomial algebras and free associative algebras in two variables is well known [2,6,9,10]. It was proved [13,15] that polynomial algebras and free associative algebras in three variables in the case of characteristic zero have wild automorphisms. For example the Nagata automorphism [11,13] is a wild automorphism of a free Poisson algebra in three variables. Let $\Delta = \{\delta_1, \dots, \delta_m\}$ be a basic set of derivation operators and $R\{x_1, x_2, \dots, x_n\}$ be the polynomial algebra in the variables $x_1, x_2, \dots, x_n$ over a differential ring R. The basic concepts of differential algebras can be found in [1,5,7,8,12,14]. The tameness of automorphisms of differential polynomial algebras is studied by B. A. Duisengaliyeva, A. S. Naurazbekova and U. U. Umirbaev [4]. They have proved that the tame automorphism group of a differential polynomial algebra over a field of characteristic 0 in two variables with m commuting derivations is a free product with amalgamation. In this paper we give some important subgroups of the group of differential automorphisms of $R\{x, y\}$ with one derivation operator. Furthermore using the method in Essen [16], we prove that the Tame subgroup of automorphism of $R\{x, y\}$ is the amalgamated free product of the Triangular and the Affine subgroups over their intersection.

2. Differential polynomial algebras

Let R be any commutative ring with unity. A mapping $d : R \to R$ is called a *derivation* if

$$d(s+t) = d(s) + d(t)$$

$$d(st) = d(s)t + sd(t)$$

holds for all $s, t \in R$.

Let $\Delta = \{\delta_1, \ldots, \delta_m\}$ be a basic set of derivation operators.

A ring R is said to be a *differential ring* or Δ*-ring* if all elements of Δ act on R as a commuting set of derivations, i.e., the derivations $\delta_i : R \to R$ are defined for all i and $\delta_i\delta_j = \delta_j\delta_i$ for all i, j. If $\Delta = \{\delta\}$ (that is, if Δ consists of only one derivation), then R is called an *ordinary differential ring* and will be denoted as δ*-ring*, unless R is called a *partial differential ring*. In this work, we study on ordinary differential rings.

Let Θ be the free commutative monoid generated by a derivation operator δ. For $n = 0, 1, 2, ...$, the elements

$$\theta = (\underbrace{\delta\delta\ldots\delta}_{(n-1)-times})\delta = \delta^n,$$

of the monoid Θ are called *derivative operators*. The *order* of θ is defined as $|\theta| = n$. If R is a δ*-ring* and R is a field (resp. integral domain), then R is called a *differential field (resp. integral domain)*. Let x be a differential indeterminate and let $\Theta x = \{\theta x | \theta \in \Theta\}$ be the set of symbols enumerated by the elements of Θ. Consider the polynomial algebra $R[\Theta x]$ over a δ-ring R generated by the set of (algebraically) independent indeterminates Θx. It is easy to check that the derivation δ can be uniquely extended to a derivation of $R[\Theta x]$ by $\delta(\theta x) = (\delta\theta)x$. Denote this differential ring by $R\{x\}$; it is called the *ring of differential polynomials* in x over R.

By adjoining more differential indeterminates, we obtain the differential ring $R\{x_1, x_2, \ldots, x_n\}$ of the differential polynomials in $x_1, x_2, \ldots, x_n$ over R. The ring $R\{x_1, x_2, \ldots, x_n\}$ coincides with the polynomial algebra

$$R[\theta x_i : \theta \in \Theta, 1 \leq i \leq n].$$

Consequently, the set of *differential monomials*

$$M = \theta_1 x_{k_1} \cdots \theta_s x_{k_s}$$

where $1 \leq k_i \leq n$, $\theta_i \in \Theta$, $1 \leq i \leq s$, form a linear basis of $R\{x_1, x_2, \ldots, x_n\}$ over R.

The degree $\deg(M)$ for a monomial $M = \prod_{i=1}^{s}(\theta_i x_{k_i})$, $1 \leq k_i \leq n$, $\theta_i \in \Theta$, $1 \leq i \leq s$ is defined as in the algebraic case: $\deg(M)$=s. It is clear that $\deg(\theta x_i) = 1$, for each i, $\theta \in \Theta$. The elements of the ring $R\{x_1, x_2, \ldots, x_n\}$ are called *differential polynomials*. The degree of a differential polynomial f is $\deg(f)=\max_{M\in f}\deg(M)$, where $M \in f$ means that M is a differential monomials occurring in f. If each term of a differential polynomial f has the same degree, then f is a *homogeneous* differential polynomial. By $\deg_{x_i} f$ we denote the degree of f with respect to x_i and its derivatives. We have $\deg_{x_i}(\theta x_j) = \delta_{ij}$ where $1 \leq i \leq n$, $1 \leq j \leq n$, $\theta \in \Theta$ and δ_{ij} is the Kronecker delta function.

On the other hand one defines *the weight* of the monomial M as $\mathrm{wt}(M) = \sum_{i=1}^{s} |\theta_i|$, where $\theta_i \in \Theta$ and the weight of a differential polynomial f as $\mathrm{wt}(f)=\max_{M\in f} \mathrm{wt}(M)$. The terms occurring in f that have the same weight are called *isobaric* component of f.

Let R be a δ-ring, an ideal $\boldsymbol{a} \in R$ is a differential ideal (δ-ideal) if $a \in \boldsymbol{a}$, we have $\delta(a) \in \boldsymbol{a}$. Additionally if $\boldsymbol{a}$ is a radical ideal, then it is called as radical δ-ideal.

A subset $V \subseteq R^2$ is Kolchin-closed if it is the set of all common zeros in R^2 of a radical differential ideal $\boldsymbol{a} \subseteq R\{x, y\}$.

Let R be a δ-field, we say that R is differentially closed if every consistent system of differential polynomial equations with coefficients in R has a solution in R.

3. Differential automorphisms of $R\{x, y\}$

Definition 1. *Let R be a δ-ring with a derivation operator δ. A differential ring homomorphism, or simply a δ-homomorphism, of R is a ring automorphism φ of R such that $\varphi(\delta(a)) = \delta(\varphi(a))$, for all $a \in R$. If the δ-homomorphism φ is a ring automorphism of R, then φ is a δ-automorphism.*

Definition 2. *Let K be a differential ring extension of a δ-ring R. A δ-automorphism φ of K is called an R-δ-automorphism, provided $\varphi(a) = a$, for all $a \in R$. The set of all R-δ-automorphism of K is a group under composition, denoted by $Aut_\delta(K|R)$.*

From now, let R be an integral domain with only derivation operator δ and $R\{x, y\}$ be the differential polynomial ring in x, y over R. Notice that since $R\{x, y\}$ is the free object on the set $\{x, y\}$ in the category of differential R-δ-algebras and hence has the universal mapping property.

Definition 3. *Let $F_1 = F_1(x, y)$ and $F_2 = F_2(x, y)$ be two differential polynomials in $R\{x, y\}$. A tuple $F = (F_1, F_2)$ in $R\{x, y\}^2$ defines uniquely a R-δ-homomorphism $\sigma_F : R\{x, y\} \to R\{x, y\}$ such that $\sigma_F(x) = F_1$, $\sigma_F(y) = F_2$. For any $P \in R\{x, y\}$, $\sigma_F(P) = P(F_1, F_2)$ (that is, P acts like a bivariate differential operator on pair (F_1, F_2)). Conversely, every R-δ-homomorphism σ of $R\{x, y\}$(in particular, the inverse of an R-δ-automorphism $\sigma \in Aut_\delta(R\{x, y\}|R)$) is of form σ_F for some tuple $F = (F_1, F_2) \in R\{x, y\}^2$.*

Definition 4. *A differential polynomial map is a polynomial map $\varphi_F : R^2 \to R^2$ defined by a tuple $F = (F_1, F_2)$ in $R\{x, y\}^2$ such that $\varphi_F(a, b) = (F_1(a, b)\ , F_2(a, b))$ for any $(a, b) \in R^2$.*

Definition 5. *A differential polynomial map φ_F said to be invertible if there exists $G = (G_1(x, y), G_2(x, y)) \in R\{x, y\}^2$ such that $\varphi_F \circ \varphi_G = \iota$, where ι is the identity map on the set R^2.*

The following theorem is a simplified version of a theorem of Kolchin [8, Theorem 4, p. 105] and is analogous to the algebraic fact that in characteristic zero, any two transcendence bases of a field extension K over k have the same cardinality. Kolchin proved the analog of for arbitrary characteristic in terms of differential inseparability basis of a partial differential field extension K over a partial differential field k.

Theorem 1. *Let K be a differential field extension of an ordinary differential field k of characteristic zero. Then every set $\Sigma \subset K$ that (differentially) generates K over k (that is, $K = k\langle\Sigma\rangle$) contains a differential transcendence basis K of over k and any two differential transcendence bases of K over k have the same cardinality (called the differential dimension of K over k).*

The next theorem is known as differential analogue of Hilbert's Nullstellensatz, but is a much deeper result as it involves the notion of a differential closed field.

Theorem 2. *Let R be differentially closed field of characteristic zero. The correspondence between the set of radical differential ideals $\boldsymbol{a}$ of $R\{x, y\}$ and the set of Kolchin-closed subset $V \subseteq R^2$, given by*

$$\boldsymbol{a} \mapsto V(\boldsymbol{a}) = \{(a, b) \in R^2 | P(a, b) = 0 \textit{ for all } P \in \boldsymbol{a}\}$$

is bijective with the inverse given by

$$V \subseteq R^2 \mapsto \boldsymbol{a}(V) = \{P \in R\{x, y\} | P(a, b) = 0 \textit{ for all } (a, b) \in V\}.$$

Remark 1. It follows that $V((0)) = R^2$ and $\boldsymbol{a}(R^2) = (0)$. In particular, we have this property : "For any $P \in R\{x, y\}$, if P vanishes for every $(a, b) \in R^2$, then $P = 0$".

From now, for simplicity, we will use P instead of $P(x, y)$ for a differential polynomial $P(x, y)$ of $R\{x, y\}$.

In the next theorem, we show the relation between differential polynomial maps and differential automorphisms. Here when we say that "for some G" we mean "for some $G = (G_1, G_2) \in R\{x, y\}^2$".

Theorem 3. *Let R be a differential integral domain. Let $F = (F_1, F_2) \in R\{x, y\}^2$.*

I. *The following are equivalent:*
(a) σ_F is a differential automorphism with an inverse σ_G for some G.
(b) σ_F is surjective (equivalently, $R\{x, y\} = R\{F_1, F_2\}$).
(c) $x = G_1(F_1, F_2)$ and $y = G_2(F_1, F_2)$ for some G.

II. *If F satisfies any (and all) of the conditions in I., then*
(d) φ_F is a invertible differential polynomial map (that is, $\varphi_F \circ \varphi_G = \iota$ for some G).
(e) $x = F_1(G_1, G_2)$ and $y = F_2(G_1, G_2)$ for some G.
(f) φ_F has an inverse φ_G for some G.

III. *If R is a differentially closed field of characteristic zero, then (a)–(f) are all equivalent, and the tuple $G \in R\{x, y\}^2$ involved are unique and the same.*

Proof. **I.** The implications $(a) \Rightarrow (b) \Rightarrow (c)$ are trivial. We now show the converses: $(c) \Rightarrow (b) \Rightarrow (a)$. From (c), we have $x, y \in R\{F_1, F_2\}$, hence $\sigma_F(R\{x, y\}) = R\{F_1, F_2\} = R\{x, y\}$, which is (b). From (b), let K be the quotient field of R (R is a domain). Then K is a differential field and the two quotient fields $K\langle x, y\rangle$ of $R\{x, y\}$ and $K\langle F_1, F_2\rangle$ of $R\{F_1, F_2\}$ coincide. By Theorem 1, the set $\{F_1, F_2\}$ forms a differential transcendence basis of $K\langle x, y\rangle = K\langle F_1, F_2\rangle$ over K. So F_1, F_2 are differentially algebraic independent over K and a fortiori over R, which means σ_F is injective and hence (a).

II. We now assume F satisfies (a), (b), and (c) and prove the second part by proving several implications. In this part of the proof, let G be as in (c).

(c)⇒(d):
By (c), for all $(a, b) \in R^2$,

$$\begin{aligned} \varphi_G \circ \varphi_F(a, b) &= \varphi_G(F_1(a, b), F_2(a, b)) \\ &= (G_1(F_1(a, b), F_2(a, b)), G_2(F_1(a, b), F_2(a, b))) \\ &= (a, b). \end{aligned} \tag{3.1}$$

Hence $\varphi_G \circ \varphi_F = \iota$, the identity map on R^2 and (d) holds.

$(c) \Rightarrow (e)$:
From (c), we have $\sigma_F(F_1(G_1, G_2)) = F_1(G_1(F_1, F_2), G_2(F_1, F_2)) = F_1(x, y) = \sigma_F(x)$. Since $(c) \Rightarrow (a)$, σ_F is injective and so $x = F_1(G_1, G_2)$. Similarly, $y = F_2(G_1, G_2)$ and (e) holds.

$(c) \Rightarrow (f)$:
By (c), $\sigma_F \circ \sigma_G = I$,where I is the identity automorphism of $R\{x, y\}$ and from equation (3.1) in $(c) \Rightarrow (d)$, $\varphi_G \circ \varphi_F = \iota$. Since $(c) \Rightarrow (e)$, $\sigma_G \circ \sigma_F = I$ and $\varphi_F \circ \varphi_G = \iota$. Hence (f). In fact, we also prove σ_G is the inverse of σ_F.

III. Let R be a differential closed field of characteristic zero. In **I.** we have proved that the G in (c) works for (a) and vice versa. In **II.** we have proved that the G from (c) works for (d), (e), and (f). We now prove the converses and that the G from any of (d), (e), or (f) works for (c), too and hence G is unique (because G defines the inverse of σ_F) and all G in all the equivalent conditions are the same.

$(d) \Rightarrow (c)$:
Suppose there exists $G = (G_1, G_2) \in R\{x, y\}^2$ such that $\varphi_G \circ \varphi_F = \iota$. Then from equation (3.1), the differential polynomials $P = G_1(F_1, F_2) - x$ and $Q = G_2(F_1, F_2) - y$ each vanishes at all $(a, b) \in R^2$. By Remark 1, $P = Q = 0$, which proves (c).

$(e) \Rightarrow (c)$:
Let $G = (G_1, G_2)$ be as given in (e). We now apply the already proven first two parts of this theorem to G (taking the place of F). If we temporarily ornate the corresponding item labels in the theorem for G with †, then $(c)^\dagger$ holds using the $F = (F_1, F_2)$ from (e). Since we have proved $(c)^\dagger \Rightarrow (e)^\dagger$, using the F of $(c)^\dagger$, it follows that $(e) \Rightarrow (c)$, using the G from (e).

$(f) \Rightarrow (c)$:
Let G be such that φ_G is the inverse of φ_F. Then of course $\varphi_G \circ \varphi_F = \iota$, which is (d), and which implies(c). □

4. The subgroups of the group of differential automorphisms of $R\{x, y\}$

From now let R be a differentially closed field of characteristic zero. Now, for each subset $S \subseteq R\{x, y\}^2$ of interest, we want to find any necessary and sufficient conditions C_S on F that will make σ_F an R-δ-automorphism of $R\{x, y\}$ for every $F \in S$, that is, for σ_F to have an inverse $\sigma_F^{-1} \in Aut_\delta(R\{x, y\}/R)$. Thus C_S is not only the set of necessary and sufficient conditions for σ_F to be both injective and surjective, but also becomes part of defining properties of the set S. Then, it is valid to identify S as a subset $\Sigma(S)$ of $Aut_\delta(R\{x, y\}/R)$ via $F \rightarrow \sigma_F$ and not merely as a set of R-δ-automorphisms. Moreover, we want to identify those $S \subseteq R\{x, y\}^2$ for which $\Sigma(S) = \{\sigma_F | F \in S\}$ are subgroups of $Aut_\delta(R\{x, y\}/R)$.

Notice that there are two properties $\Sigma(S)$ (or S) must satisfy to ensure that $\Sigma(S)$ is a subgroup. The first of these properties is after showing that σ_F^{-1} exists and hence $\sigma_F, \sigma_F^{-1} \in Aut_\delta(R\{x, y\}/R)$, it needs to belong to $\Sigma(S)$, that is, $\sigma_F^{-1} = \sigma_G$ for some $G = (G_1, G_2) \in S$. The second is that for any σ_F and σ_G in $\Sigma(S)$, their composition $\sigma_F \circ \sigma_G$ must also belong to $\Sigma(S)$. That means $\sigma_F \circ \sigma_G = \sigma_H$ for some $H = (H_1, H_2) \in S$.

Some important subgroups of $Aut_\delta(R\{x, y\}/R)$ are defined as follows:

(1). The Affine differential subgroup: We determine it as $Aff_\delta(R\{x, y\}/R)$.
Consider $F_i \in R\{x, y\}$, $deg(F_i) = 1$, $wt(F_i) = 0$, $(i = 1, 2)$, then for $a, b, c, d, e, f \in R$ put

$F_1 = ax + by + c, F_2 = dx + ey + f$ and $det(\begin{bmatrix} a & b \\ d & e \end{bmatrix}) = |JF|$.

After calculating the conditions C_S for $S = Aff_\delta(R\{x, y\}/R)$ and $F = (F_1, F_2) \in S$, we define Affine-δ subgroup as follows:

$Aff_\delta(R\{x, y\}/R) = \{(ax + by + c, dx + ey + f)|a, b, c, d, e, f \in R, |JF| \in R^*\}$.

(2). The Triangular differential subgroup: We determine this subgroup as $J_\delta(R\{x, y\}/R)$ and similarly we calculate it as follows:

$J_\delta(R\{x, y\}/R) = \{(ax + f(y), by + c)|a, b \in R^*, c \in R$ and $f(y) \in R\{y\}\}$.

Here R^* denotes nonzero elements of R.

(3). The Elementary differential subgroup: We determine this subgroup as $E_\delta(R\{x, y\}/R)$ and we calculate it as:

$E_\delta(R\{x, y\}/R) = \{(ax + f(y), y)|a \in R^*$ and $f(y) \in R\{y\}\}$.

Definition 6. *The subgroup of $Aut_\delta(R\{x, y\}|R)$ generated by the Affine R-δ-automorphisms and the Triangular R-δ-automorphisms is called Tame δ-subgroup and is denoted by $T_\delta(R\{x, y\}|R)$.*

Now we will show that $T_\delta(R\{x, y\}|R)$ is the amalgamated free product of $\mathrm{Aff}_\delta(R\{x, y\}|R)$ and $J_\delta(R\{x, y\}|R)$ over their intersection.

To prove the results announced above we need the following lemmas.

Lemma 1. *Let G be a group generated by two subgroups H and K. Then every element g of G can be written as*

$$g = h_0 k_1 h_1 \cdots k_\ell h_\ell$$

for some $\ell \geq 1$, where $h_i \in H \setminus K$ for all $1 \leq i \leq \ell - 1$ and $k_i \in K \setminus H$ for all $1 \leq i \leq \ell$ and $h_0 \in H$.

Proof. See, [16], Lemma 5.1.1, p.86. □

Let $F = (F_1, F_2), G = (G_1, G_2) \in R\{x, y\}^2$ and let σ_F and σ_F be automorphisms defined by F and G respectively. We define composition of σ_F and σ_G as follows: there exists $H = (H_1, H_2) \in R\{x, y\}^2$ such that

$$\sigma_H(x) = (\sigma_F \circ \sigma_G)(x) = G_1(F_1(x, y), F_2(x, y)) = H_1$$

$$\sigma_H(y) = (\sigma_F \circ \sigma_G)(y) = G_2(F_1(x, y), F_2(x, y)) = H_2.$$

To formulate Lemma 2 and Corollary 1 below, we need following notations: let $F = (F_1, F_2) \in R\{x, y\}^2$. Then

$$\mathrm{deg}F = \max\{\mathrm{deg}(F_1), \mathrm{deg}(F_2)\},$$

$$\mathrm{bideg}F = (\mathrm{deg}(F_1), \mathrm{deg}(F_2)),$$

$$\mathrm{wt}F = \max\{\mathrm{wt}(F_1), \mathrm{wt}(F_2)\},$$

$$\mathrm{biwt}F = (\mathrm{wt}(F_1), \mathrm{wt}(F_2)).$$

Now let $F \in T_\delta(R\{x,y\}|R)$. Then applying Lemma 1 to $G = T_\delta(R\{x,y\}|R)$, $H =\text{Aff}_\delta(R\{x,y\}|R)$, $K = J_\delta(R\{x,y\}|R)$ and $g = F$ we can write

$$F = \lambda_0\tau_1\lambda_1\cdots\tau_\ell\lambda_\ell$$

with $\lambda_i \in\text{Aff}_\delta(R\{x,y\}|R) \setminus J_\delta(R\{x,y\}|R)$ for all $1 \le i \le \ell-1$ and $\tau_i \in J_\delta(R\{x,y\}|R)\setminus \text{Aff}_\delta(R\{x,y\}|R)$ for all $1 \le i \le \ell$. Write

$$\lambda_i = (a_ix + b_iy + c_i, d_ix + e_iy + f_i).$$

Note that for an integer j, if $\lambda_j \in\text{Aff}_\delta(R\{x,y\}|R) \setminus J_\delta(R\{x,y\}|R)$, then $d_j = 0$.

Remark 2. Observe that if $\tau \in J_\delta(R\{x,y\}|R)\setminus \text{Aff}_\delta(R\{x,y\}|R)$ then either $\text{wt}(\tau) \ge 1$ or $\text{wt}(\tau) = 0$ and $\deg(\tau) \ge 2$.

Lemma 2. *For any positive integer ℓ we have*

$$bideg(\lambda_1 \circ \tau_1 \circ \cdots \circ \lambda_\ell \circ \tau_\ell) = (\textstyle\prod_{j=1}^{\ell} deg(\tau_j(x)), \prod_{j=1}^{\ell-1} deg(\tau_j(x)))$$

where for all $1 \le i \le \ell$, $wt(\tau_i) = 0$ and the second product is by definition 1 if $\ell = 1$.

Proof. We prove the lemma by induction on ℓ. The case $\ell = 1$ is obvious. So let us assume that the statement is true for some n and consider $\text{bideg}(\lambda_1 \circ \tau_1 \circ \cdots \circ \lambda_n \circ \tau_n \circ \lambda_{n+1})$. Since $\lambda_{n+1} \notin J_\delta(R\{x,y\}|R)$, we have that $d_{n+1} \neq 0$. Therefore

$$\text{bideg}(\lambda_1 \circ \tau_1 \circ \cdots \circ \lambda_n \circ \tau_n \circ \lambda_{n+1}) = (p_i, \textstyle\prod_{j=1}^{n} \deg(\tau_j(x))),$$

where $p_i \le \prod_{j=1}^{n} \deg(\tau_j(x))$. Finally since $\tau_{n+1} \notin \text{Aff}_\delta(R\{x,y\}|R)$, we have that

$$\begin{aligned}\text{bideg}(\lambda_1 \circ \tau_1 \circ \cdots \circ \lambda_n \circ \tau_n \circ \lambda_{n+1} \circ \tau_{n+1}) &= (\prod_{j=1}^{n} \deg(\tau_j(x)).\deg(\tau_{n+1}(x)), \prod_{j=1}^{n} \deg(\tau_j(x))) \\ &= (\prod_{j=1}^{n+1} \deg(\tau_j(x)), \prod_{j=1}^{n} \deg(\tau_j(x)))\end{aligned}$$

which completes the proof. □

Corollary 1. *$T_\delta(R\{x,y\}|R)$ is the amalgamated free product of $Aff_\delta(R\{x,y\}|R)$ and $J_\delta(R\{x,y\}|R)$ over their intersection, i.e., $T_\delta(R\{x,y\}|R)$ is generated by these two groups and if $\tau_i \in J_\delta(R\{x,y\}|R) \setminus Aff_\delta(R\{x,y\}|R)$ and $\lambda_i \in Aff_\delta(R\{x,y\}|R) \setminus J_\delta(R\{x,y\}|R)$ then $\tau_1 \circ \lambda_2 \cdots \circ \tau_{n-1} \circ \lambda_n \circ \tau_n$ does not belong to $Aff_\delta(R\{x,y\}|R)$.*

Proof. By the definition, $T_\delta(R\{x,y\}|R)$ is generated by $\text{Aff}_\delta(R\{x,y\}|R)$ and $J_\delta(R\{x,y\}|R)$. So suppose that

$$\tau_1 \circ \lambda_2 \cdots \circ \tau_{n-1} \circ \lambda_n \circ \tau_n = \lambda \in \text{Aff}_\delta(R\{x,y\}|R),$$

with $\tau_i \in J_\delta(R\{x,y\}|R)\setminus\text{Aff}_\delta(R\{x,y\}|R)$ and $\lambda_i \in Aff_\delta(R\{x,y\}|R) \setminus J_\delta(R\{x,y\}|R)$ for all i. Then

$$\lambda^{-1} \circ \tau_1 \circ \lambda_2 \cdots \circ \tau_{n-1} \circ \lambda_n \circ \tau_n(x,y) = (x,y).$$

Therefore we have

$$\text{bideg}(\lambda^{-1} \circ \tau_1 \circ \lambda_2 \cdots \circ \tau_{n-1} \circ \lambda_n \circ \tau_n) = (1, 1) \tag{4.1}$$

$$\text{biwt}(\lambda^{-1} \circ \tau_1 \circ \lambda_2 \cdots \circ \tau_{n-1} \circ \lambda_n \circ \tau_n) = (0, 0) \tag{4.2}$$

Since $\tau_i \in J_\delta(R\{x, y\}|R) \setminus \text{Aff}_\delta(R\{x, y\}|R)$, then

$$\text{wt}(\tau_i) \geq 1 \text{ or } \text{wt}(\tau_i) = 0.$$

Now let us suppose for all i, $\text{wt}(\tau_i) = 0$ so $\deg(\tau_i) \geq 2$. By the Lemma 2, we have

$$\text{bideg}(\lambda^{-1} \circ \tau_1 \circ \lambda_2 \cdots \circ \tau_{n-1} \circ \lambda_n \circ \tau_n) = (\textstyle\prod_{j=1}^{n-1} \deg(\tau_j(x)).\deg(\tau_n(x)), \prod_{j=1}^{n-1} \deg(\tau_j(x)))$$

From the equation (4.1), $\deg(\tau_n) = 1$ which is a contradiction. Let us suppose that for all i, $\text{wt}(\tau_i) \geq 1$, so

$$\text{biwt}(\lambda^{-1} \circ \tau_1 \circ \lambda_2 \cdots \circ \tau_{n-1} \circ \lambda_n \circ \tau_n) > (0, 0)$$

which is a contradiction by (4.2). □

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Conflict of interest

The authors declare no conflict of interest.

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