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Research article

Darboux helices in three dimensional Lie groups

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Abstract: In this paper, we introduce Darboux helices in a three dimensional Lie group G with a bi-invariant metric and give some characterizations of Darboux helices. Besides, we give some relations between some special curves (general helices and slant helices) and Darboux helices. Moreover, we prove that all Darboux helices are not a slant helix if G is commutative.

Keywords: curves in Lie groups; slant helix; angular velocity vector; Darboux helix
Mathematics Subject Classification: 22E15, 53A04, 53C40

1. Introduction

In Euclidean space $\mathbb{E}^3$ a regular curve whose tangent vector T make a constant angle with a fixed direction, is called a *general helix* (or curve of the constant slope) ([1]). A classical result stated by M. A. Lancret in 1802 and first proved by B. de Saint Venant in 1845 is: a regular curve α with the first curvature $\kappa \neq 0$ and second curvature τ in $\mathbb{E}^3$ is the general helix if and only if it has constant conical curvature $\frac{\tau}{\kappa}$ (see [1] for details). The *slant helix* is a curve with non-vanishing curvature whose principal normal vector N makes a constant angle with a fixed direction and characterized these curves with

$$\frac{\kappa^2}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\left(\frac{\tau}{\kappa}\right)' \tag{1.1}$$

is a constant function ([2]). There is a nice relation between slant helices and general helices. Namely, slant helices are the successor curves of general helices ([3]). In particular, general helix has constant conical curvature, while the geodesic curvature of the principal normal indicatrix of a slant helix is constant. They can be found on general Hopf cylinders ([4, 5]) and on helix surfaces ([5]). The *Darboux helix* is defined by the property that the Darboux vector ω makes a constant angle with a fixed direction and give some characterizations of these curves [6]. Also, several kinds of helices are introduced and characterized by many researchers in [7–14].

In addition to these, more and more researchers have still paid their attention to these curves. In [15] Çiftçi defined a general helix in a Lie group with a bi-invariant metric as a curve whose tangent vector makes a constant angle with a left-invariant vector field and characterized general helices by a relation between the curvature κ and the torsion τ:

$$\frac{\tau - \tau_G}{\kappa} = constant, \tag{1.2}$$

where $\tau_G = \frac{1}{2}\langle [T, N], B\rangle$ and $\{T, N, B\}$ being the Frenet frame. In [16] Okuyucu et al. defined a slant helix in a Lie group with a bi-invariant metric as a curve whose principal normal vector makes a constant angle with a left-invariant vector field. As a result, the necessary and sufficient condition for a curve to be a slant helix is given by

$$\frac{\kappa\left(H^2 + 1\right)^{\frac{3}{2}}}{H'} = constant, \tag{1.3}$$

where $H = \frac{\tau - \tau_G}{\kappa}$. Then, Yampolsky et al., [17], defined the first, second, and third kind of helices on 3-dimensional Lie groups with left-invariant metric and obtained their description in terms of new geometric invariants of the curve. Also, they generalized corresponding descriptions for helices in three dimensional Lie groups with bi-invariant metric.

In this paper, we introduce Darboux helices in a three dimensional Lie group with a bi-invariant metric. We give some characterizations for such curves and obtain parameter equations of their axes. Besides, we give some relations between some special curves (general helices and slant helices) and Darboux helices. Moreover, we show that all Darboux helices are not a slant helix if G is commutative (i.e. $G = \mathbb{E}^3$).

2. Preliminaries

Let G be a smooth manifold which is also a topological group with multiplication map *mult* : $G \times G \longrightarrow G$ and inverse map *inv* : $G \longrightarrow G$ and view $G \times G$ as the product manifold. Then G is a *Lie group* if mult, inv are smooth maps. By the manifold properties, any two points in G can be connected by a smooth trajectory, and at any point $g \in G$ one can define a differential dg that is tangent to G. The differential at the neutral element $e \in G$ (identity) is particularly important. The tangent space at the identity of G is $\mathfrak{g} = T_eG$, which is called the *Lie algebra* for G. The Lie algebra $\mathfrak{g}$ along with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ called the *Lie br*acke*t*, forms a vector space.

This paper always considers $\mathfrak{g}$ endowed with the Euclidean metric $\langle\ ,\ \rangle$ such that the following two identities hold for all $X, Y, Z \in \mathfrak{g}$:

$$\langle X, [Y, Z]\rangle = \langle [X, Y], Z\rangle, \tag{2.1}$$

and

$$D_X Y = \frac{1}{2}[X, Y], \tag{2.2}$$

where D is the Levi-Civita connection of G.

Let $\{X_1, X_2, \ldots, X_n\}$ be an orthonormal basis of $\mathfrak{g}$ and $\alpha : I \subset \mathbb{R} \longrightarrow G$ be a parametrized curve in terms of arc-length s. The covariant derivative of $U = \{u_1, u_2, \ldots, u_n\} \in \mathfrak{g}$ along the curve α with the notation $D_{\alpha'}U$ is given as follows

$$D_{\alpha'}U = \dot{U} + \frac{1}{2}[T, U] \tag{2.3}$$

where $T = \alpha' = \frac{d\alpha}{ds}$ is the tangent vector field of α and $\dot{U} = \sum_{i=1}^{n} \frac{du_i}{dt} X_i$. Note that $\dot{U} = 0$ if U is the left-invariant vector field of α, [15].

Now, let G be a three dimensional Lie group and $\alpha : I \subset \mathbb{R} \longrightarrow G$ be a curve with the Frenet apparatus $\{T, N, B, \kappa, \tau\}$, parameterized by the arc-length parameter s. Then, by using (2.3) the curvatures κ and τ of α are given by

$$\begin{aligned} \kappa &= \|D_T T\| = \left\|\dot{T}\right\|, \\ \tau &= \|D_T B\| - \tau_G, \end{aligned} \tag{2.4}$$

where

$$\tau_G = \frac{1}{2}\langle [T, N], B\rangle, \tag{2.5}$$

or

$$\tau_G = \frac{1}{2\kappa^2\tau}\left[\left\langle \ddot{T}, \left[T, \dot{T}\right]\right\rangle + \frac{1}{2}\left\|\left[T, \dot{T}\right]\right\|^2\right]. \tag{2.6}$$

([15]).

Proposition 1. *Let α be a curve in three dimensional Lie group G with the Frenet apparatus $\{T, N, B, \kappa, \tau\}$, parameterized by the arc-length parameter s. Then the following equalities hold*

$$\begin{aligned} [T, N] &= \langle [T, N], B\rangle = 2\tau_G B, \\ [T, B] &= \langle [T, B], N\rangle = -2\tau_G N, \\ [N, B] &= \langle [N, B], T\rangle = 2\tau_G T. \end{aligned}$$

Proposition 2. *([15, 18])Let G be a three dimensional Lie group with bi-invariant metric.Then, the following statements hold:*
i) if G is a commutative group, then $\tau_G = 0$,
ii) if G is $SO(3)$, then $\tau_G = \frac{1}{2}$,
iii) if G is $S^3 \cong SU(2)$, then $\tau_G = 1$.

3. Darboux helices in three dimensional Lie groups

In this section, we will define Darboux helices in a three dimensional Lie group $\mathbb{G}$ furnished with a bi-invariant metric $\langle , \rangle$ and characterize these curves. Also, we obtain some interesting relations between some special curves (general helices and slant helices) and Darboux helices. Furthermore, we prove that all Darboux helices are not a slant helix if G is commutative (i.e. $G = \mathbb{E}^3$)

When the Frenet frame $\{T, N, B\}$ of a curve α makes an instantaneous helix motion in three dimensional Lie group G, there exists an axis of frame's rotation whose direction is given by the vector

$$\omega(s) = \tau(s) T(s) + \kappa(s) B(s), \tag{3.1}$$

which is called a *Darboux vector* (*centrode*).

Definition 1. *Let α be a curve in three dimensional Lie group G with the Frenet apparatus $\{T, N, B, \kappa, \tau\}$, parameterized by the arc-length parameter s. The curve α is called a* Darboux helix *if its unit Darboux vector makes a constant angle with a left-invariant vector field $U \in g$.*

The vector field U express the *axis* of helix. We will exclude the case when the Darboux vector ω is constant (i.e. $\|\omega\| = \tau^2 + \kappa^2 \neq constant$), since it trivially makes constant angle with any fixed direction. Also, let $\mathbb{R}_0$ denote $\mathbb{R}\backslash \{0\}$.

Now, let α be a Darboux helix in three dimensional Lie group G parameterized by the arc-length function s. Then, from the Definition 1 there exists a unit left-invariant vector field $U \in g$ such that the following relation is satisfied

$$\langle \omega, U \rangle = c, \tag{3.2}$$

and with respect to the Frenet frame $\{T, N, B\}$ the axis U can be decomposed as

$$U = u_1 T + u_2 N + u_3 B, \tag{3.3}$$

where $u_1 = \langle T, U \rangle$, $u_2 = \langle N, U \rangle$ and $u_3 = \langle B, U \rangle$ are some differentiable functions of the arclength parameter s and $c \in \mathbb{R}$. Then, from (3.2) and (3.3) we get,

$$u_1 \tau + u_3 \kappa = c. \tag{3.4}$$

Also, differentiating the Eq. (3.3), we obtain

$$D_T U = u_1' T + u_1 D_T T + u_2' N + u_2 D_T N + u_3' B + u_3 D_T B, \tag{3.5}$$

where

$$D_T T = \kappa N, \quad D_T N = -\kappa T + \tau B, \quad D_T B = -\tau N. \tag{3.6}$$

Since $U \in \mathfrak{g}$ is the left-invariant vector field of α, i.e. $\dot{U} = 0$, we have

$$D_T U = \frac{1}{2} [T, U],$$

and from the Proposition 1

$$D_T U = -u_3 \tau_G N + u_2 \tau_G B. \tag{3.7}$$

Combining the relations (3.5)–(3.7) we have

$$-u_3 \tau_G N + u_2 \tau_G B = (u_1' - \kappa u_2) T + (u_2' + u_1 \kappa - \tau u_3) N + (u_3' + \tau u_2) B,$$

and then we obtain the following system of differential equations

$$\begin{cases} u_1' - \kappa u_2 = 0, \\ u_2' + \kappa u_1 - (\tau - \tau_G) u_3 = 0, \\ u_3' + (\tau - \tau_G) u_2 = 0. \end{cases} \tag{3.8}$$

In what follows, we derive the functions $u_1(s)$, $u_2(s)$ and $u_3(s)$ satisfying (3.4) and (3.8) by considering two cases: **(A)** $u_2(s) = constant$; **(B)** $u_2(s) \neq constant$. In each of these cases, we will distinguish the subcase when ω is orthogonal to U ($c = 0$) and when ω is not orthogonal to U ($c \neq 0$).

(A) $u_2(s) = constant$. Then from the relation (3.8) we have,

$$\begin{cases} u_1' - \kappa u_2 = 0, \\ \kappa u_1 - (\tau - \tau_G) u_3 = 0, \\ u_3' + (\tau - \tau_G) u_2 = 0. \end{cases} \tag{3.9}$$

(A1) If ω is not orthogonal to U $(c \neq 0)$, then from the relation (3.4) and the second equation of (3.9), we have,

$$\begin{cases} u_1 = c\frac{\tau - \tau_G}{\kappa^2 + \tau(\tau - \tau_G)}, \\ u_3 = c\frac{\kappa}{\kappa^2 + \tau(\tau - \tau_G)}. \end{cases} \tag{3.10}$$

Substituting (3.10) in the first equation of (3.9), we obtain,

$$\frac{\left(\frac{\tau - \tau_G}{\kappa}\right)'}{\left[\kappa^2 + \tau(\tau - \tau_G)\right]\left(1 + \left(\frac{\tau - \tau_G}{\kappa}\right)^2\right)} = constant. \tag{3.11}$$

Conversely, we assume that the relation (3.11) holds. Consider the vector U given by

$$U = c\left(\frac{\tau - \tau_G}{\kappa^2 + \tau(\tau - \tau_G)}\right)T + cu_2 N + c\left(\frac{\kappa}{\kappa^2 + \tau(\tau - \tau_G)}\right)B, \tag{3.12}$$

where $c \in \mathbb{R}_0$ and u_2 is non-zero constant given by

$$u_2 = \frac{\left(\frac{\tau - \tau_G}{\kappa}\right)'}{\left[\kappa^2 + \tau(\tau - \tau_G)\right]\left(1 + \left(\frac{\tau - \tau_G}{\kappa}\right)^2\right)}. \tag{3.13}$$

Differentiating the Eq. (3.12) and using the Eqs. (2.2) and (2.3), we find $\dot{U} = 0$. Hence U is a left-invariant vector field. It can be easily checked that $\langle \omega, U \rangle = c$. According to the Definition 1, α is a Darboux helix whose axis is spanned by U. This proves the next theorem.

Theorem 1. *Let α be a unit speed curve in three dimensional Lie group G with the non-zero curvatures κ and τ. Then, α is a Darboux helix if and only if*

$$\frac{\left(\frac{\tau - \tau_G}{\kappa}\right)'}{\left[\kappa^2 + \tau(\tau - \tau_G)\right]\left(1 + \left(\frac{\tau - \tau_G}{\kappa}\right)^2\right)} = constant \tag{3.14}$$

and the axis of α is given by (3.12).

Corollary 1. *Every general helix in three dimensional Lie group G is a Darboux helix with the axis given by (3.12).*

Corollary 2. *A Darboux helix with the axis given by (3.12) in three dimensional Lie group G is a slant helix.*

(A2) If ω is orthogonal to U ($c = 0$), then from the relation (3.4), we obtain

$$u_3 = -\frac{\tau}{\kappa}u_1, \tag{3.15}$$

and from the second equation of (3.9), we have

$$\frac{\tau - \tau_G}{\kappa} = -\frac{\kappa}{\tau}, \tag{3.16}$$

where $\tau_G \neq 0$ for all s since the Darboux vector ω is not constant.
Substituting (3.15) in the third equation of (3.9) we obtain

$$-\left(\frac{\tau}{\kappa}\right)' u_1 - \left(\frac{\tau}{\kappa}\right) u_1' + (\tau - \tau_G)\, u_2 = 0, \tag{3.17}$$

and from the first equation of (3.9)

$$\left(\frac{\tau}{\kappa}\right)' u_1 - \tau_G u_2 = 0. \tag{3.18}$$

Then we have the following subcases:
(A2.1) if $\frac{\tau}{\kappa} = constant$, then $u_2 = 0$ and from the first equation of (3.9) and the relation (3.15) we get

$$\begin{cases} u_1 = u_0 = constant \neq 0, \\ u_3 = -\frac{\tau}{\kappa}u_0. \end{cases} \tag{3.19}$$

Substituting (3.19) in the second equation of (3.9), we have

$$\frac{\tau - \tau_G}{\kappa} = constant. \tag{3.20}$$

Therefore, the axis U is given by

$$U = u_0 T - \frac{\tau}{\kappa}u_0 B, \tag{3.21}$$

where $u_0 \in \mathbb{R}_0$.
(A2.2) if $\frac{\tau}{\kappa} \neq constant$, then $u_2 \neq 0$ and from the relation (3.18) we get,

$$u_1 = \frac{\tau_G}{\left(\frac{\tau}{\kappa}\right)'}u_2, \tag{3.22}$$

where $\tau_G \neq 0$. Hence, the axis is given by

$$U = -\frac{\tau_G}{\left(\frac{\tau}{\kappa}\right)'}u_2 T + u_2 N + u_2\left(\frac{\tau}{\kappa}\frac{\tau_G}{\left(\frac{\tau}{\kappa}\right)'}\right)B, \tag{3.23}$$

where $u_2 \in \mathbb{R}_0$ for all s.
Thus, we can give the next theorem.

Theorem 2. *Let α be a Darboux helix whose axis is orthogonal to ω in three dimensional Lie group G with the non-zero curvatures κ and τ. Then,*
(i) if $\frac{\tau}{\kappa}$ = constant, then

$$\frac{\tau-\tau_G}{\kappa} = constant, \tag{3.24}$$

and its axis lies in the planes spanned by $\{T, B\}$ and is given by

$$U = u_0 T - \frac{\tau}{\kappa} u_0 B, \tag{3.25}$$

where $u_0 \in \mathbb{R}_0$.
(ii) if $\frac{\tau}{\kappa} \neq$ constant, then

$$\frac{\tau-\tau_G}{\kappa} = -\frac{\kappa}{\tau}, \tag{3.26}$$

and its axis is given by

$$U = -\frac{\tau_G}{\left(\frac{\tau}{\kappa}\right)'} u_2 T + u_2 N + u_2 \left(\frac{\tau}{\kappa} \frac{\tau_G}{\left(\frac{\tau}{\kappa}\right)'}\right) B, \tag{3.27}$$

where $\tau_G \neq 0$ and $u_2 \in \mathbb{R}_0$.

Corollary 3. *Let α be a Darboux helix whose axis is orthogonal to ω in three dimensional Lie group G with the non-zero curvatures κ and τ. Then,*
(i) if $\frac{\tau}{\kappa}$ = constant, then the Darboux helix α is both a general helix and a slant helix;
(ii) if $\frac{\tau}{\kappa} \neq$ constant, then the Darboux helix α is not a general helix, but a slant helix.

Using the Proposition 2, we can give the following corollary.

Corollary 4. *Let α be a Darboux helix whose axis is orthogonal to ω in three dimensional Lie group G with the non-zero curvatures κ and τ. If $\frac{\tau}{\kappa} \neq$ constant for all s, then*
(i) there is not defined Darboux helix α in the commutative Lie group G;
(ii) every Darboux helix α in the Lie groups $SU(2)$ and $SO(3)$ is a slant helix with the axis given by (3.27).

(B) $u_2(s) \neq$ *constant.*
(B1) If ω is orthogonal to U ($c = 0$), then from the relation (3.4), we obtain

$$u_3 = -\frac{\tau}{\kappa} u_1. \tag{3.28}$$

Substituting (3.28) in the third equation of (3.8) we obtain,

$$-\left(\frac{\tau}{\kappa}\right)' u_1 - \left(\frac{\tau}{\kappa}\right) u_1' + (\tau - \tau_G) u_2 = 0, \tag{3.29}$$

and from the first equation of (3.8)

$$-\left(\frac{\tau}{\kappa}\right)' u_1 - \tau_G u_2 = 0. \tag{3.30}$$

There are the following two subcases:
(B1.1) if $\frac{\tau}{\kappa} \neq$ *constant* for all s, then $\tau_G \neq 0$ and we have

$$\begin{cases} u_1 = -\frac{\tau_G}{\left(\frac{\tau}{\kappa}\right)'} u_2, \\ u_3 = \frac{\tau}{\kappa} \frac{\tau_G}{\left(\frac{\tau}{\kappa}\right)'} u_2. \end{cases} \tag{3.31}$$

Moreover, from the relation (3.3) we get

$$\langle U, U\rangle = u_1^2 + u_2^2 + u_3^2 = constant. \tag{3.32}$$

Combining the relations (3.31) and (3.32), let $r \in \mathbb{R}^+$ be the constant given by

$$\left(1 + \left[1 + \left(\frac{\tau}{\kappa}\right)^2\right]\left(\frac{\tau_G}{\left(\frac{\tau}{\kappa}\right)'}\right)^2\right)u_2^2 = r^2, \tag{3.33}$$

and we obtain

$$u_2 = \pm\frac{r}{\sqrt{1 + \left[1 + \left(\frac{\tau}{\kappa}\right)^2\right]\left(\frac{\tau_G}{\left(\frac{\tau}{\kappa}\right)'}\right)^2}}. \tag{3.34}$$

Taking the derivative in the first equation of (3.31) and by using the first and third equation of (3.8), we get:

$$\kappa u_2 = \frac{\kappa\tau_G}{\left(\frac{\tau}{\kappa}\right)'}u_1 - \left(\frac{\tau_G}{\left(\frac{\tau}{\kappa}\right)'}\right)' u_2 - \frac{(\tau - \tau_G)\,\tau_G}{\left(\frac{\tau}{\kappa}\right)'}u_3. \tag{3.35}$$

Substituting (3.31) in (3.35) we obtain

$$\frac{\tau}{\kappa} = \int \tau_G\left[s + \int \frac{\tau\,(\tau - \tau_G)}{\kappa^2}ds\right]ds, \tag{3.36}$$

where $\tau_G \neq 0$ for all s.

Conversely, we assume that the relation (3.36) holds. Consider the vector U given by

$$U = -\frac{\tau_G}{\left(\frac{\tau}{\kappa}\right)'}u_2T + u_2N + \frac{\tau}{\kappa}\frac{\tau_G}{\left(\frac{\tau}{\kappa}\right)'}u_2B, \tag{3.37}$$

where $u_2(s)$ is given by

$$u_2\,(s) = \frac{1}{\sqrt{1 + \left[1 + \left(\frac{\tau}{\kappa}\right)^2\right]\left(\frac{\tau_G}{\left(\frac{\tau}{\kappa}\right)'}\right)^2}} \tag{3.38}$$

is not constant. Differentiating the Eq. (3.37) and using the Eqs. (2.2) and (2.3), we find $\dot{U} = 0$. Hence U is a left-invariant vector field. It can be easily checked that $\langle \omega, U\rangle = 0$. According to the Definition 1, α is a Darboux helix whose axis is spanned by U.
Therefore, we can give the following theorem and corollary.

Theorem 3. *Let α be a unit speed curve in three dimensional Lie group G with the non-zero curvatures κ and τ. Then, α is a Darboux helix whose axis is orthogonal to ω if and only if*

$$\frac{\tau}{\kappa} = \int \tau_G\left[s + \int \frac{\tau\,(\tau - \tau_G)}{\kappa^2}ds\right]ds, \tag{3.39}$$

is not constant everywhere τ_G doesn't vanish.

Corollary 5. *The axis of Darboux helix α, which is orthogonal to ω, is given by (3.37).*

(B1.2) if $\frac{\tau}{\kappa} = constant$ for all s, then $\tau_G = 0$ and so, the relation (3.28) and the second equation of (3.8) yield

$$u_2' + \left(\frac{\kappa^2 + \tau^2}{\kappa}\right) u_1 = 0, \tag{3.40}$$

and from the first equation of (3.8) we get,

$$\frac{d}{ds}\left(\frac{1}{\kappa}\frac{du_1}{ds}\right) + \left[1 + \left(\frac{\tau}{\kappa}\right)^2\right] \kappa u_1 = 0. \tag{3.41}$$

Putting $p(s) = \frac{1}{\kappa(s)}$, the Eq. (3.41) can be written as

$$\frac{d}{ds}\left(p(s)\frac{du_1}{ds}\right) + \left[1 + \left(\frac{\tau}{\kappa}\right)^2\right] \frac{u_1(s)}{p(s)} = 0. \tag{3.42}$$

By changing the variables in the above equation with $t(s) = \int \frac{1}{p(s)} ds$, we find

$$\frac{d^2 u_1}{dt^2} + \left[1 + \left(\frac{\tau}{\kappa}\right)^2\right] u_1 = 0. \tag{3.43}$$

The solution of the previous differential equation is given by

$$u_1 = C_2 \sin\left(\sqrt{1 + \left(\frac{\tau}{\kappa}\right)^2} t\right) + C_1 \cos\left(\sqrt{1 + \left(\frac{\tau}{\kappa}\right)^2} t\right), \tag{3.44}$$

where $C_1, C_2 \in \mathbb{R}$. Finally, since $t(s) = \int \kappa(s) ds$, we obtain

$$u_1(s) = C_2 \sin\left(\sqrt{1 + \left(\frac{\tau}{\kappa}\right)^2} \int \kappa(s) ds\right) + C_1 \cos\left(\sqrt{1 + \left(\frac{\tau}{\kappa}\right)^2} \int \kappa(s) ds\right). \tag{3.45}$$

Conversely, we assume that $\frac{\tau}{\kappa} = constant$ and $\tau_G = 0$ for all s holds. Consider the vector U given by

$$U = u_1(s) T + u_2(s) N + u_3(s) B, \tag{3.46}$$

where

$$\begin{aligned}
u_1(s) &= C_2 \sin\left(\sqrt{1 + \left(\tfrac{\tau}{\kappa}\right)^2} \int \kappa(s) ds\right) + C_1 \cos\left(\sqrt{1 + \left(\tfrac{\tau}{\kappa}\right)^2} \int \kappa(s) ds\right), \\
u_2(s) &= \sqrt{1 + \left(\tfrac{\tau}{\kappa}\right)^2}\left(C_2 \cos\left(\sqrt{1 + \left(\tfrac{\tau}{\kappa}\right)^2} \int \kappa(s) ds\right) - C_1 \sin\left(\sqrt{1 + \left(\tfrac{\tau}{\kappa}\right)^2} \int \kappa(s) ds\right)\right), \\
u_3(s) &= -\tfrac{\tau}{\kappa}\left(C_2 \sin\left(\sqrt{1 + \left(\tfrac{\tau}{\kappa}\right)^2} \int \kappa(s) ds\right) + C_1 \cos\left(\sqrt{1 + \left(\tfrac{\tau}{\kappa}\right)^2} \int \kappa(s) ds\right)\right),
\end{aligned} \tag{3.47}$$

and $C_1, C_2 \in \mathbb{R}$. Differentiating the Eq. (3.46) and using the Eqs. (2.2) and (2.3), we find $\dot{U} = 0$. Hence U is a left-invariant vector field. It can be easily checked that $\langle \omega, U \rangle = 0$. According to the Definition 1, α is a Darboux helix whose axis is spanned by U.

Therefore, we can give the following theorem.

Theorem 4. *Let α be a unit speed curve in the commutative Lie group G with the non-zero curvatures κ and τ. Then, α is a Darboux helix if and only if*

$$\frac{\tau}{\kappa} = constant, \tag{3.48}$$

and the axis of α is given by (3.46).

Corollary 6. *There is not any curve both a Darboux helix and a slant helix with the axis given by (3.46) in the commutative Lie group G.*

(B2) If ω is not orthogonal to U ($c \neq 0$), then from the relation (3.4), we obtain

$$u_3 = -\frac{c}{\kappa} - \frac{\tau}{\kappa}u_1, \tag{3.49}$$

where $\kappa \neq 0$ for all s.
Substituting (3.28) in the third equation of (3.8) we obtain,

$$-\left(\frac{\tau}{\kappa}\right)' u_1 - \left(\frac{\tau}{\kappa}\right) u_1' + (\tau - \tau_G)\, u_2 = 0, \tag{3.50}$$

and from the first equation of (3.8)

$$u_1 = c\frac{\left(\frac{1}{\kappa}\right)'}{\left(\frac{\tau}{\kappa}\right)'} - \frac{\tau_G}{\left(\frac{\tau}{\kappa}\right)'}u_2. \tag{3.51}$$

Taking the derivative the relation (3.4) and by using the first and third equation of (3.8), we get

$$u_1\tau' + \kappa\tau_G u_2 + u_3\kappa' = 0. \tag{3.52}$$

Substituting (3.49) and (3.51) in (3.52) we obtain,

$$\left(\kappa^2 - \kappa\right)\left(\tau_G u_2 - c\left(\frac{1}{\kappa}\right)'\right) = 0. \tag{3.53}$$

Therefore we have the following three subcases:
(B2.1) if $\kappa = 1$ and $\tau_G \neq 0$, then the relations (3.49) and (3.51) yield

$$\begin{cases} u_1 = -\frac{\tau_G}{\tau'}u_2, \\ u_3 = c + \frac{\tau\tau_G}{\tau'}u_2. \end{cases} \tag{3.54}$$

Substituting (3.54) in the second equation of (3.8) we obtain

$$u_2' - \frac{\left(\tau^2 + 1\right)\tau_G - \tau\tau_G}{\tau'}u_2 = c\,(\tau - \tau_G)\,. \tag{3.55}$$

The solution of the previous differential equation is given by

$$u_2 = ce^{\int\left(\frac{\left(\tau^2+1\right)\tau_G - \tau\tau_G}{\tau'}\right)ds} \int (\tau - \tau_G)\, ds. \tag{3.56}$$

Using the relations (3.54) and (3.56), we have the axis U as

$$U = -u_2\left(\frac{\tau_G}{\tau'}\right)T + u_2 N + \left(c + \frac{\tau\tau_G}{\tau'}u_2\right)B. \tag{3.57}$$

Differentiating (3.57) and using the Eqs. (2.2) and (2.3) gives $\dot{U} = 0$. Hence U is a left-invariant vector field. Thus, α is a Darboux helix. So, we can give the following theorem.

Theorem 5. *Every unit speed curve α with the curvatures $\kappa = 1$ and $\tau \neq 0$ in three dimensional Lie group G is a Darboux helix with the axis given by*

$$U = -u_2\left(\frac{\tau_G}{\tau'}\right)T + u_2N + \left(c + \frac{\tau\tau_G}{\tau'}u_2\right)B, \tag{3.58}$$

where $\tau_G \neq 0$ and $u_2(s) \neq constant$ is given by (3.56).

(B2.2) if $\kappa = constant$ and $\tau_G = 0$, then the relations (3.49) and the first and third equations of (3.8) we obtain

$$-\left(\frac{\tau}{\kappa}\right)' u_1 = 0, \tag{3.59}$$

Here, we have the following cases:
(i) if $u_1 = 0$ for all s, then the first equation of (3.8) $u_2(s) = 0$ which is contradiction;
(ii) if $\frac{\tau}{\kappa} = constant$, then $\tau = constant$ which is a contradiction with the assumption that ω is not a constant vector.
Then, we can give the following corollary.

Corollary 7. *There is not any Darboux helix α with the curvature $\kappa = constant$ in the commutative Lie group G if $\langle U, N\rangle = u_2 \neq constant$.*

(B2.3) if $\kappa \neq constant$ and $\tau_G \neq 0$, then the relations (3.49)–(3.52) yield

$$\begin{cases} u_1 = 0, \\ u_2 = c\frac{\left(\frac{1}{\kappa}\right)'}{\tau_G}, \\ u_3 = \frac{c}{\kappa}. \end{cases} \tag{3.60}$$

Therefore, the axis of α is given by

$$U = c\frac{\left(\frac{1}{\kappa}\right)'}{\tau_G}N + \frac{c}{\kappa}B, \tag{3.61}$$

where $c \in \mathbb{R}_0$. Differentiating (3.61) and using the Eqs. (2.2) and (2.3) gives $\dot{U} = 0$. Hence U is a left-invariant vector field. Thus, α is a Darboux helix. So, we can give the following theorem.

Theorem 6. *Every unit speed curve α with the curvatures $\kappa \neq constant$ and $\tau \neq 0$ in three dimensional Lie group G is a Darboux helix with the axis given by*

$$U = c\frac{\left(\frac{1}{\kappa}\right)'}{\tau_G}N + \frac{c}{\kappa}B, \tag{3.62}$$

where $\tau_G \neq 0$ and $c \in \mathbb{R}_0$.

Corollary 8. *Every Darboux helix α with the curvatures $\kappa \neq constant$ and $\tau \neq 0$ in three dimensional Lie group G is a general helix but not a slant helix with the same axis given by (3.62).*

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Conflict of interest

All authors declare that there is no conflict of interest in this paper.

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