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Research article

Approximation of Jakimovski-Leviatan-Beta type integral operators via q-calculus

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Abstract: We construct Jakimovski-Leviatan-Beta type $q-$integral operators and show that these positive linear operators are uniformly convergent to a continuous functions. We obtain the Korovkin type results, the rate of convergence as well as some direct theorems.

Keywords: Appell polynomials; q-Appell polynomials; Jakimovski-Levitian operators; Jakimovski-Leviatan-Beta operators; Korovkin's theorem; modulus of continuity
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1. Preliminaries and introduction

In 1880, Appell investigated the polynomials named as Appell polynomials (see [1]). Jakimovski and Leviatan introduced and modified the Appell polynomials [2] in 1969 by the identity defined bellow

$$P(u)e^{uy} = \sum_{k=0}^{\infty} \beta_k(y)u^k, \tag{1.1}$$

where $\beta_k(y) = \sum_{i=0}^{k} \alpha_i \frac{y^{n-i}}{(n-i)!}$ $(n \in \mathbb{N})$ and $P(u) = \sum_{k=0}^{\infty} \alpha_k u^k$, $P(1) \neq 0$. Let $E[0,\infty)$ denote the set of functions defined on $[0,\infty)$ such that $|f(x)| \leq \kappa e^{\gamma x}$, where $\kappa,\ \gamma$ are positive constants.

We recall some basic notations on q-calculus (see [3–5]). For each non-negative integer j, the

q-integer is defined as

$$[j]_q = \begin{cases} \frac{1-q^j}{1-q}, & q \neq 1 \\ j, & q = 1 \end{cases} \text{ for } j \in \mathbb{N} \text{ and } [0]_q = 0,$$

For $|q| < 1$, the q-factorial $[j]_q!$ is defined by

$$[j]_q! = \begin{cases} 1 & (j = 0) \\ \prod_{k=1}^{j} [k]_q & (j \in \mathbb{N}). \end{cases} \tag{1.2}$$

In the standard approach the exponential functions for q-calculus:

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!}. \tag{1.3}$$

The improper integral of function f is defined by:

$$\int_0^{\infty/A} f(x) d_q x = (1-q) \sum_{n \in \mathbb{N}} f(\frac{q^n}{A}) \frac{q^n}{A},\ A \in \mathbb{R} - \{0\}. \tag{1.4}$$

Al-Salam (see [6, 7]) introduced the family of q-Appell polynomials through the generating functions $P_q(t) = \sum_{n=0}^{\infty} P_{n,q} \frac{t^n}{[n]_q!},\ P_q(1) \neq 0$. We have

$$P_{n,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q A_{n-k,q} x^k,\ (n \in \mathbb{N})$$

and q-differential, $D_{q,x}(P_{n,q}(x)) = [n]_q P_{n-1,q}(x),\ n = 1, 2, \ldots$, where $P_{0,q}(x)$ is a non zero constant let say $P_{0,q}$ and $D_{q,x}(P_{1,q}(x)) = [1]_q P_{0,q}(x) = P_{0,q}$. Also $P_q(t) e_q(tx) = \sum_{n=0}^{\infty} P_{n,q}(x) \frac{t^n}{[n]_q!}$, $\quad 0 < q < 1$.

Recently in [8], authors studied the q-analogue of Jakimovski-Levitian operators involving q-Appell polynomials; and in [9], Stacu type Jakimovski-Levitian-Durmeyer operators have been studied. Recently such q-analogues operators have also been studied in [10–13]. Our aim is to construct Jakimovski-Leviatan-Beta type $q-$integral operators and show that these positive linear operators are uniformly convergent to a continuous functions. Further, we obtain the Korovkin type results, the rate of convergence as well as some direct theorems.

Let $x \in [0, \infty),\ P_{r,q}(x) \geq 0$ and $P_q(1) \neq 0$. For every $f \in C_\zeta[0, \infty) = \{f \in C[0, \infty) : f(t) = O(t^\zeta),\ t \to \infty\}$, and $m \in \mathbb{N},\ \ 0 < q < 1$, we define

$$\mathcal{J}^*_{m,q}(f; x) = \frac{e_q(-[m]_q x)}{P_q(1)} \sum_{r=0}^{\infty} \frac{P_{r,q}([m]_q x)}{[r]_q!} \frac{\mathcal{K}(A, r+1)}{B_q(r+1, m)} \int_0^{\infty/A} \frac{t^r}{(1+t)_q^{r+m+1}} f(q^r t)\, d_q t, \tag{1.5}$$

where $\zeta > m$ and

$$B_q(r, m) = \mathcal{K}(A, r) \int_0^{\infty/A} \frac{x^{r-1}}{(1+x)_q^{r+m}} d_q x = \frac{[r-1]_q}{[m]_q} B_q(r-1, m+1), \quad r > 1,\ m > 0,$$

with

$$\mathcal{K}(A, r+1) = q^r \mathcal{K}(A, r),$$

$$\mathcal{K}(A, r) = q^{\frac{r(r-1)}{2}},\ \mathcal{K}(A, 0) = 1$$

2. Moments of new operators

Lemma 1. Take $e_i = t^{i-1}$ for $i = 1, 2, 3, 4, 5$. Then

$$\begin{aligned}
(1)\quad \mathcal{J}^*_{m,q}(e_1; x) &= 1;\\
(2)\quad \mathcal{J}^*_{m,q}(e_2; x) &= \frac{1}{q[m-1]_q} + \frac{1}{[m-1]_q}\Big([m]_q x + \frac{P'_q(1)}{P_q(1)}\Big);\\
(3)\quad \mathcal{J}^*_{m,q}(e_3; x) &= \frac{(1+q)}{q^3[m-1]_q[m-2]_q} + \frac{(1+2q)}{q^2[m-1]_q[m-2]_q}\Big(([m]_q x + \frac{P'_q(1)}{P_q(1)}\Big)\\
&+ \frac{1}{[m-1]_q[m-2]_q}\Big([m]_q^2 x^2 + \frac{2[m]_q P'_q(1)}{P_q(1)} x + \frac{P''_q(1)}{P_q(1)}\Big);\\
(4)\quad \mathcal{J}^*_{m,q}(e_4; x) &= \frac{1+2q+2q^2+q^3}{q^4[m-1]_q[m-2]_q[m-3]_q} + \frac{(1+3q+4q^2+3q^3}{q^3[m-1]_q[m-2]_q[m-3]_q}\Big(([m]_q x + \frac{P'_q(1)}{P_q(1)}\Big)\\
&+ \frac{(1+2q+3q^2)}{q[m-1]_q[m-2]_q[m-3]_q}\Big([m]_q^2 x^2 + \frac{2[m]_q P'_q(1)}{P_q(1)} x + \frac{P''_q(1)}{P_q(1)}\Big)\\
&+ \frac{q^2}{[m-1]_q[m-2]_q[m-3]_q}\Big([m]_q^3 x^3 + 3[m]_q^2 \frac{P'_q(1)}{P_q(1)} x^2 + 3[m]_q \frac{P''_q(1)}{P_q(1)} x + \frac{P'''_q(1)}{P_q(1)}\Big);\\
(5)\quad \mathcal{J}^*_{m,q}(e_5; x) &= \frac{1+3q+5q^2+6q^3+5q^4+3q^5+q^6}{q^5[m-1]_q[m-2]_q[m-3]_q[m-4]_q}\\
&+ \frac{(1+5q+10q^2+13q^3+12q^4+7q^5+2q^6)}{q^3[m-1]_q[m-2]_q[m-3]_q[m-4]_q}\Big([n]_q x + \frac{P'_q(1)}{P_q(1)}\Big)\\
&+ \frac{(1+3q+7q^2+9q^3+9q^4+6q^5)}{q[m-1]_q[m-2]_q[m-3]_q[m-4]_q}\Big([m]_q^2 x^2 + \frac{2[m]_q P'_q(1)}{P_q(1)} x + \frac{P''_q(1)}{P_q(1)}\Big)\\
&+ \frac{(q^2+2q^3+2q^4+2q^5+q^6+2q^7)}{[m-1]_q[m-2]_q[m-3]_q[m-4]_q}\Big([m]_q^3 x^3 + 3[m]_q^2 \frac{P'_q(1)}{P_q(1)} x^2 + 3[m]_q \frac{P''_q(1)}{P_q(1)} x + \frac{P'''_q(1)}{P_q(1)}\Big)\\
&+ \frac{q^6}{[m-1]_q[m-2]_q[m-3]_q[m-4]_q}\\
&\Big([m]_q^4 x^4 + 4[m]_q^3 \frac{P'_q(1)}{P_q(1)} x^3 + 6[m]_q^2 \frac{P''_q(1)}{P_q(1)} x^2 + 4[m]_q \frac{P'''_q(1)}{P_q(1)} x + \frac{P^{(4)}_q(1)}{P_q(1)}\Big).
\end{aligned}$$

Proof of Lemma 1.

$$\begin{aligned}
\sum_{r=0}^{\infty} \frac{P_{r,q}([m]_q x)}{[r]_q!} &= P_q(1) e_q([m]_q x),\\
\sum_{r=0}^{\infty} r \frac{P_{r,q}([m]_q x)}{[r]_q!} &= \Big[[m]_q P_q(1) x + P'_q(1)\Big] e_q([m]_q x),
\end{aligned}$$

$$\sum_{r=0}^{\infty} r^2 \frac{P_{r,q}([m]_q x)}{[r]_q!} = \Big[[m]_q^2 P_q(1)x^2 + 2[m]_q P_q'(1)x + P_q''(1)\Big] e_q([m]_q x),$$

$$\sum_{r=0}^{\infty} r^3 \frac{P_{r,q}([m]_q x)}{[r]_q!} = \Big[[m]_q^3 P_q(1)x^3 + 3[m]_q^2 P_q'(1)x^2 + 3[m]_q P_q''(1)x + P_q'''(1)\Big] e_q([m]_q x),$$

$$\sum_{r=0}^{\infty} r^4 \frac{P_{r,q}([m]_q x)}{[r]_q!}$$

$$= \Big[[m]_q^4 P_q(1)x^4 + 4[m]_q^3 P_q'(1)x^3 + 6[m]_q^2 P_q''(1)x^2 + 4[m]_q P_q'''(1)x + P_q^{(4)}(1)\Big] e_q([m]_q x).$$

Take $f(t) = e_1$, then

$$\begin{aligned}
\mathcal{J}_{m,q}^*(e_1; x) &= \frac{e_q(-[m]_q x)}{P_q(1)} \sum_{r=0}^{\infty} \frac{P_{r,q}([m]_q x)}{[r]_q!} \frac{\mathcal{K}(A, r+1)}{B_q(r+1, m)} \int_0^{\infty/A} \frac{t^r}{(1+t)^{r+m+1}} d_q t \\
&= \frac{e_q(-[m]_q x)}{P_q(1)} \sum_{r=0}^{\infty} \frac{P_{r,q}([m]_q x)}{[r]_q!} \frac{B_q(r+1, m)}{B_q(r+1, m)} \\
&= 1.
\end{aligned}$$

Take $f(t) = e_2$, then

$$\begin{aligned}
\mathcal{J}_{m,q}^*(e_2; x) &= \frac{e_q(-[m]_q x)}{P_q(1)} \sum_{r=0}^{\infty} \frac{P_{r,q}([n]_q x)}{[r]_q!} q^r \frac{\mathcal{K}(A, r+1)}{B_q(r+1, m)} \int_0^{\infty/A} \frac{t^{r+1}}{(1+t)^{r+m+1}} d_q t \\
&= \frac{e_q(-[m]_q x)}{P_q(1)} \sum_{r=0}^{\infty} \frac{P_{r,q}([m]_q x)}{[r]_q!} q^r \frac{\mathcal{K}(A, r+1)}{B_q(r+1, m)} \frac{B_q(r+2, m-1)}{\mathcal{K}(A, r+2)} \\
&= \frac{1}{q[m-1]_q} \frac{e_q(-[m]_q x)}{P_q(1)} \sum_{r=0}^{\infty} \frac{P_{r,q}([m]_q x)}{[r]_q!} [r+1]_q.
\end{aligned}$$

By applying,

$$[r+1]_q = 1 + q[r]_q, \tag{2.1}$$

$$\begin{aligned}
\mathcal{J}_{m,q}^*(e_2; x) &= \frac{1}{q[m-1]_q} \frac{e_q(-[m]_q x)}{P_q(1)} \sum_{r=0}^{\infty} \frac{P_{r,q}([m]_q x)}{[r]_q!} \\
&+ \frac{1}{[m-1]_q} \frac{e_q(-[m]_q x)}{P_q(1)} \sum_{r=0}^{\infty} \frac{P_{r,q}([m]_q x)}{[r]_q!} [r]_q \\
&= \frac{1}{q[m-1]_q} + \frac{1}{[m-1]_q} \Big([m]_q x + \frac{P_q'(1)}{P_q(1)} \Big)
\end{aligned}$$

Take $f(t) = e_3$, then

$$\mathcal{J}_{m,q}^*(e_3; x) = \frac{e_q(-[m]_q x)}{P_q(1)} \sum_{r=0}^{\infty} \frac{P_{r,q}([m]_q x)}{[r]_q!} q^{2r} \frac{\mathcal{K}(A, r+1)}{B_q(r+1, m)} \int_0^{\infty/A} \frac{t^{r+2}}{(1+t)^{r+m+1}} d_q t$$

$$= \frac{e_q(-[m]_q x)}{P_q(1)} \sum_{r=0}^{\infty} \frac{P_{r,q}([m]_q x)}{[r]_q!} q^{2r} \frac{\mathcal{K}(A, r+1)}{\mathcal{K}(A, r+3)} \frac{B_q(r+3, m-2)}{B_q(r+1, m)}$$

$$= \frac{1}{q^3[m-1]_q[m-2]_q} \frac{e_q(-[m]_q x)}{P_q(1)} \sum_{r=0}^{\infty} \frac{P_{r,q}([m]_q x)}{[r]_q!} [r+2]_q[r+1]_q.$$

By applying (2.1) and $[r+2]_q = 1 + q + q^2[r]_q$,

$$\begin{aligned} \mathcal{J}^*_{m,q}(e_3; x) &= \frac{1}{q^3[m-1]_q[m-2]_q} \frac{e_q(-[m]_q x)}{P_q(1)} \sum_{r=0}^{\infty} \frac{P_{r,q}([m]_q x)}{[r]_q!} \\ &\times \Big((1+q) + q(1+2q)[r]_q + q^3[r]_q^2\Big) \\ &= \frac{(1+q)}{q^3[m-1]_q[m-2]_q} + \frac{(1+2q)}{q^2[m-1]_q[m-2]_q} \left(([m]_q x + \frac{P'_q(1)}{P_q(1)}\right) \\ &+ \frac{1}{[m-1]_q[m-2]_q} \left([m]_q^2 x^2 + \frac{2[m]_q P'_q(1)}{P_q(1)} x + \frac{P''_q(1)}{P_q(1)}\right) \end{aligned}$$

Take $f(t) = e_4$, then

$$\mathcal{J}^*_{m,q}(e_4; x) = \frac{1}{q^4[m-1]_q[m-2]_q[m-3]_q} \frac{e_q(-[m]_q x)}{P_q(1)} \sum_{r=0}^{\infty} \frac{P_{r,q}([m]_q x)}{[r]_q!} [r+3]_q[r+2]_q[r+1]_q$$

By a simple calculation we have

$$\begin{aligned} &[r+3]_q[r+2]_q[r+1]_q \\ &= (1+q)(1+q+q^2) + \Big\{q(1+2q)(1+q+q^2) + q^3(1+q)\Big\} [r]_q \\ &+ \Big\{q^3(1+q+q^2) + q^4(1+2q)\Big\} [r]_q^2 + q^6[r]_q^3. \end{aligned}$$

If $f(t) = e_5$, then

$$\begin{aligned} \mathcal{J}^*_{m,q}(e_5; x) &= \frac{1}{q^5[m-1]_q[m-2]_q[m-3]_q[m-4]_q} \\ &\times \frac{e_q(-[m]_q x)}{P_q(1)} \sum_{r=0}^{\infty} \frac{P_{r,q}([m]_q x)}{[r]_q!} [r+4]_q[r+3]_q[r+2]_q[r+1]_q \end{aligned}$$

A simple calculation leads to
$[r+4]_q[r+3]_q[r+2]_q[r+1]_q$

$$\begin{aligned} &= (1+q)(1+2q+3q^2+3q^3+2q^4+q^5) + \Big\{q(1+2q)(1+2q+3q^2+3q^3+2q^4+q^5) \\ &+ q^3(1+q)(1+2q+2q^2+2q^3)\Big\}[r]_q \\ &+ \Big\{q^3(1+2q+3q^2+3q^3+2q^4+q^5) + q^4(1+2q)(1+2q+2q^2+2q^3) + q^7(1+q)\Big\} [r]_q^2 \\ &+ \Big\{q^6(1+2q+2q^2+2q^3) + q^8(1+2q)\Big\} [r]_q^3 + q^{10}[r]_q^4. \end{aligned}$$

□

Lemma 2.
Let $\mu_j = (e_2 - x)^j$ for $j = 1, 2$. For all $x \in [0, \infty)$, $\quad 0 < q < 1, \quad P_{r,q}(x) \geq 0$ and $P_q(1) \neq 0$, we have:

$$\left(\delta^*_{m,q}\right)^2 = \mathcal{J}^*_{m,q}(\mu_2; x) \text{ for } j = 2,\ m > 2; \text{and } \left(\delta^*_{m,q}\right)^2 = \mathcal{J}^*_{m,q}(\mu_1; x) \text{ for } j = 1,\ m > 1. \tag{2.2}$$

That is,

$$\left(\delta^*_{m,q}\right)^2 = \begin{cases} \left(\dfrac{[m]_q^2}{[m-1]_q[m-2]_q} + 1 - \dfrac{2[m]_q}{[m-1]_q}\right)x^2 \\ + \dfrac{1}{[m-1]_q}\left(\dfrac{(1+2q)[m]_q}{q^2[m-2]_q} + \dfrac{2[m]_q}{[m-2]_q}\dfrac{P'_q(1)}{P_q(1)} - \dfrac{2P'_q(1)}{P_q(1)} - \dfrac{2}{q}\right)x \\ + \dfrac{1}{q^2[m-1]_q[m-2]_q}\left(\dfrac{(1+q)}{q} + (1+2q)\dfrac{P'_q(1)}{P_q(1)}\right) \\ \quad \text{for } \ j = 2,\ m > 2 \\ \left(\dfrac{[m]_q}{[m-1]_q} - 1\right)x + \dfrac{1}{[m-1]_q}\left(\dfrac{1}{q} + \dfrac{P'_q(1)}{P_q(1)}\right) \\ \quad \text{for } \ j = 1,\ m > 1. \end{cases}$$

3. Korovkin type approximation

We write $C_B(\mathbb{R}^+)$ for the set of all bounded and continuous functions with

$$\| f \|_{C_B} = \sup_{x \geq 0} | f(x) | .$$

Let

$$E := \{f : x \in [0, \infty), \frac{f(x)}{1 + x^2} \quad \text{is} \quad \text{convergent} \quad \text{as} \quad x \to \infty\}.$$

We choose $q = q_m$ where $0 < q_m < 1$ such that

$$\lim_m q_m \to 1, \qquad \lim_\nu q_m^m \to \alpha \tag{3.1}$$

Theorem 1. For any function $f \in C[0, \infty) \cap E$, we have

$$\lim_{\nu \to \infty} \mathcal{J}^*_{m,q_m}(f; x) \to f(x)$$

uniformly on each compact subset of $[0, \infty)$.

Proof of Theorem 1. From the well-known Korovkin's theorem [14] (see [15,16]), it is enough to show

$$\lim_{m \to \infty} \mathcal{J}^*_{m,q_m}(e_j; x) = x^{j-1}, \quad j = 1, 2, 3$$

uniformly on $[0, 1]$.
Using (3.1), $\frac{1}{[m]_{q_m}} \to 0$ and $\frac{[m]_{q_m}}{[m-1]_{q_m}} \to 1$ $(\nu \to \infty)$, we have

$$\lim_{\nu \to \infty} \mathcal{J}^*_{m,q_m}(e_2; x) = x, \quad \lim_{m \to \infty} \mathcal{J}^*_{m,q_m}(e_3; x) = x^2.$$

Which complete the proof. □

Let $\varrho(x) = 1+\phi^2(x), \lim_{x\to\infty}\varrho(x) = \infty$, where $\phi(x)$ is a continuous and strictly increasing function. Let $B_\varrho(\mathbb{R}^+)$ be a set of functions defined on $\mathbb{R}^+$ such that there is a constant M_f,

$$|f(x)| \leq M_f\varrho(x),$$

Its subset of continuous functions is denoted by $C_\varrho(\mathbb{R}^+)$. Note that

$$\|f\|_\varrho = \sup_{x\geq 0}\frac{|f(x)|}{\varrho(x)}$$

is the usual norm on $B_\varrho(\mathbb{R}^+)$. Let $C^0_\varrho(\mathbb{R}^+)$ be a subset of $C_\varrho(\mathbb{R}^+)$ such that

$$\lim_{x\to\infty}\frac{f(x)}{\varrho(x)} = K_f$$

exists and is finite.

Theorem 2. Let $\{\mathcal{J}^*_{m,q}\}_{m\geq 1}$ be the sequence of linear positive operators (1.5) from $C_\varrho(\mathbb{R}^+)$ into $B_\varrho(\mathbb{R}^+)$ such that

$$\lim_{m\to\infty}\|\mathcal{J}^*_{m,q}(\varphi^{i-1}(t);x) - \varphi^{i-1}(x)\|_\varrho = 0 \ (i = 1, 2, 3).$$

Then for $f \in C^0_\varrho(\mathbb{R}^+)$,

$$\lim_{m\to\infty}\|\mathcal{J}^*_{m,q}(f(t);x) - f\|_\varrho = 0.$$

Proof of Theorem 2. Consider $\varphi(x) = x, \quad \varrho(x) = 1 + x^2$, and

$$\|\mathcal{J}^*_{m,q}(e_i;x) - x^{i-1}\|_\varrho = \sup_{x\geq 0}\frac{|\ \mathcal{J}^*_{m,q}(e_i;x) - x^{i-1}\ |}{1+x^2}.$$

Then for $i = 1, 2, 3$ it is easily proved (by Theorem 1) that

$$\lim_{m\to\infty}\|\mathcal{J}^*_{m,q}(e_i;x) - x^{i-1}\|_\varrho = 0.$$

Using Korovkin's theorem, we get

$$\lim_{m\to\infty}\|\mathcal{J}^*_{m,q}(f(t);x) - f\|_\varrho = 0.$$

□

Theorem 3.

Let $x \in [0,\infty), \quad f \in C^0_\varrho(\mathbb{R}^+)$ with $\varrho(x) = 1 + x^2$. Then for $P_{r,q}(x) \geq 0, \quad P_q(1) \neq 0$, the operators $\mathcal{J}^*_{m,q}(\,\cdot\,;\,\cdot\,)$ defined by (1.5) satisfying

$$\lim_{m\to\infty}\|\mathcal{J}^*_{m,q}(f;x) - f\|_\varrho \to 0.$$

Proof of Theorem 3. Using Theorem 2, consider

$$\|\mathcal{J}^*_{m,q}(e_i;x)-x^{i-1}\|_\varrho=\sup_{x\geq 0}\frac{|\mathcal{J}^*_{m,q}(e_i;x)-x^{i-1}|}{1+x^2},$$

for $i=1,2,3$.
From Lemma 1, for $i=1$, we have $|\mathcal{J}^*_{m,q}(e_1;x)-1|\to 0$, and therefore

$$\lim_{m\to\infty}\|\mathcal{J}^*_{m,q}(e_1;x)-1\|_\varrho=0.$$

For $i=2$

$$\begin{aligned}\sup_{x\geq 0}\frac{|\mathcal{J}^*_{m,q}(e_2;x)-x|}{1+x^2} &\leq \left|\frac{[m]_q}{[m-1]_q}-1\right|\sup_{x\geq 0}\frac{x}{1+x^2}\\ &+ \left|\frac{1}{[m-1]_q}\left(\frac{P'_q(1)}{P_q(1)}+\frac{1}{q}\right)\right|\sup_{x\geq 0}\frac{1}{1+x^2}.\end{aligned}$$

Therefore

$$\lim_{m\to\infty}\|\mathcal{J}^*_{m,q}(e_2;x)-x\|_\varrho=0.$$

For $i=3$

$$\begin{aligned}\sup_{x\geq 0}\frac{|\mathcal{J}^*_{m,q}(e_3;x)-x^2|}{1+x^2} &\leq \left|\frac{[m]_q^2}{[m-2]_q[m-1]_q}-1\right|\sup_{x\geq 0}\frac{x^2}{1+x^2}\\ &+ \left|\frac{[m]_q}{[m-2]_q[m-1]_q}\left(\frac{2P'_q(1)}{P_q(1)}+\frac{1+2q}{q^2}\right)\right|\sup_{x\geq 0}\frac{x}{1+x^2}\\ &+ \left|\frac{1}{[m-2]_q[m-1]_q}\left(\frac{P''_q(1)}{P_q(1)}+\frac{(1+2q)}{q^2}\frac{P'_q(1)}{P_q(1)}+\frac{(1+q)}{q^3}\right)\right|\sup_{x\geq 0}\frac{1}{1+x^2}.\end{aligned}$$

Hence we have

$$\lim_{m\to\infty}\|\mathcal{J}^*_{m,q}(e_3;x)-x^2\|_\varrho=0.$$

The Korovkin's theorem completes the proof. □

We recall the following spaces:

$$\begin{aligned}P_\sigma(\mathbb{R}^+) &= \left\{f:|f(x)|\leq M_f\sigma(x)\right\},\\ Q_\sigma(\mathbb{R}^+) &= \left\{f:f\in P_\sigma(\mathbb{R}^+)\cap C[0,\infty)\right\},\\ Q^k_\sigma(\mathbb{R}^+) &= \left\{f:f\in Q_\sigma(\mathbb{R}^+)\ \text{ and }\ \lim_{x\to\infty}\frac{f(x)}{\sigma(x)}=k\ \text{(a constant)}\right\},\end{aligned}$$

where $\sigma(x)=1+x^2$. Note that $\|f\|_\sigma=\sup_{x\geq 0}\frac{|f(x)|}{\sigma(x)}$ is the usual norm on $Q_\sigma(\mathbb{R}^+)$.

Theorem 4. Let $0<q_m<1$ and $\mathcal{J}^*_{m,q}(\cdot\,;\cdot)$ be the operators defined by (1.5) with $m>2$. Then for $f\in Q^k_\sigma(\mathbb{R}^+)$, we have

$$\lim_{m\to\infty}\|\mathcal{J}^*_{m,q_m}(f;x)-f\|_\sigma=0.$$

Proof of Theorem 4. Take $j=1$, then Lemma 1, yields the first condition. If we take $j=2,3$, then from Lemma 1, we get

$$\lim_{m\to\infty}\|\mathcal{J}^*_{m,q_m}(e_j;x)-x^{j-1}\|_\sigma=0.$$

□

4. Order of approximation

The modulus of continuity of $f \in C[0,\infty)$ is defined by

$$\varpi(f;\delta) = \sup_{|y_1-y_2|\leq\delta} |\, f(y_1) - f(y_2)\,|, \quad y_1, y_2 \in [0,\infty),\ \delta > 0. \tag{4.1}$$

Note that $\lim_{\delta\to 0+} \varpi(f;\delta) = 0$ for $f \in C[0,\infty)$ and (see ([17], p. 378)

$$|\, f(y_1) - f(y_2)\,| \leq \left(\frac{|\, y_1 - y_2\,|}{\delta} + 1\right)\varpi(f;\delta). \tag{4.2}$$

Theorem 5. For $f \in \tilde{C}[0,\infty)$, $x \geq 0$, $0 < q < 1$ and $P_q(1) \neq 0$, we have

$$|\, \mathcal{J}^*_{m,q}(f;x) - f(x)\,| \leq 2\varpi\left(f;\delta^*_{m,q}\right),$$

where $\tilde{C}[0,\infty)$ is the space of uniformly continuous functions on $\mathbb{R}^+$ and $\delta^*_{m,q}$ is defined in Lemma 2.

Proof of Theorem 5. Using Cauchy-Schwarz inequality and (4.1), (4.2), we get
$|\, \mathcal{J}^*_{m,q}(f;x) - f(x)\,|$

$$\begin{aligned}
&\leq \frac{e_q(-[m]_q x)}{P_q(1)} \sum_{r=0}^{\infty} \frac{P_{r,q}([m]_q x)}{[r]_q!} \frac{\mathcal{K}(A,r+1)}{B_q(r+1,m)} \int_0^{\infty/A} \frac{t^r}{(1+t)_q^{r+m+1}} \,|\, f(t) - f(x)\,|\, d_q t \\
&\leq \frac{e_q(-[m]_q x)}{P_q(1)} \sum_{r=0}^{\infty} \frac{P_{r,q}([m]_q x)}{[r]_q!} \frac{\mathcal{K}(A,r+1)}{B_q(r+1,m)} \\
&\times \int_0^{\infty/A} \frac{t^r}{(1+t)^{r+m+1}} \left(1 + \frac{1}{\delta} \,|\, t-x\,|\right) d_q(t)\varpi(f;\delta) \\
&= \Bigg\{1 + \frac{1}{\delta}\Bigg(\frac{e_q(-[m]_q x)}{P_q(1)} \sum_{r=0}^{\infty} \frac{P_{r,q}([m]_q x)}{[r]_q!} \frac{\mathcal{K}(A,r+1)}{B_q(r+1,m)} \\
&\times \int_0^{\infty/A} \frac{t^r}{(1+t)^{r+m+1}} \,|\, t-x\,|\, d_q t\Bigg)\Bigg\}\varpi(f;\delta) \\
&\leq \Bigg\{1 + \frac{1}{\delta}\Bigg(\frac{e_q(-[m]_q x)}{P_q(1)} \sum_{r=0}^{\infty} \frac{P_{r,q}([m]_q x)}{[r]_q!} \frac{\mathcal{K}(A,r+1)}{B_q(r+1,m)} \\
&\times \int_0^{\infty/A} \frac{t^r}{(1+t)^{r+m+1}} (t-x)^2 d_q(t)\Bigg)^{\frac{1}{2}} \left(\mathcal{J}^*_{m,q}(1;x)\right)^{\frac{1}{2}} \Bigg\}\varpi(f;\delta) \\
&= \left\{1 + \frac{1}{\delta}\left(\mathcal{J}^*_{m,q}(\mu_2;x)\right)^{\frac{1}{2}}\right\}\varpi(f;\delta).
\end{aligned}$$

Choosing $\delta = \delta^*_{m,q} = \sqrt{\mathcal{J}^*_{m,q}((t-x)^2;x)}$, then we get our result. □

5. Rate of convergence

For $f \in C[0,\infty)$, $M > 0$ and $0 < \nu \leq 1$, we define

$$Lip_M(\nu) = \{f :\mid f(\varsigma_1) - f(\varsigma_2) \mid\leq M \mid \varsigma_1 - \varsigma_2 \mid^\nu \quad (\varsigma_1, \varsigma_2 \in [0,\infty))\} \tag{5.1}$$

Theorem 6. For each $f \in Lip_M(\nu)$, $\;M > 0$, $\;0 < \nu \leq 1$ and $0 < q < 1$, we have

$$\mid \mathcal{J}^*_{m,q}(f;x) - f(x) \mid\leq M\left(\lambda_{m,q}(x)\right)^{\frac{\nu}{2}}$$

where $\lambda_{m,q}(x) = \mathcal{J}^*_{m,q}(\mu_2;x)$.

Proof of Theorem 6. Hölder inequality and (5.1) imply that

$$\begin{aligned}\mid \mathcal{J}^*_{m,q}(f;x) - f(x) \mid &\leq \mid \mathcal{J}^*_{m,q}(f(t) - f(x);x) \mid \\ &\leq \mathcal{J}^*_{m,q}(\mid f(t) - f(x) \mid; x) \\ &\leq \mid M\mathcal{J}^*_{m,q}(\mid t - x \mid^\nu; x).\end{aligned}$$

So that

$\mid \mathcal{J}^*_{m,q}(f;x) - f(x) \mid$

$$\begin{aligned}&\leq M\frac{e_q(-[m]_qx)}{P_q(1)}\sum_{r=0}^{\infty}\frac{P_{r,q}([m]_qx)}{[r]_q!}\frac{\mathcal{K}(A,r+1)}{B_q(r+1,m)}\int_0^{\infty/A}\frac{t^r}{(1+t)^{r+m+1}}\mid t-x\mid^\nu d_qt \\ &\leq M\frac{e_q(-[m]_qx)}{P_q(1)}\sum_{r=0}^{\infty}\left(\frac{P_{r,q}([m]_qx)}{[r]_q!}\frac{\mathcal{K}(A,r+1)}{B_q(r+1,m)}\right)^{\frac{2-\nu}{2}} \\ &\times\left(\frac{P_{r,q}([m]_qx)}{[r]_q!}\frac{\mathcal{K}(A,r+1)}{B_q(r+1,m)}\right)^{\frac{\nu}{2}}\int_0^{\infty/A}\frac{t^r}{(1+t)^{r+m+1}}\mid t-x\mid^\nu d_qt \\ &\leq M\left(\frac{e_q(-[m]_qx)}{P_q(1)}\sum_{r=0}^{\infty}\frac{P_{r,q}([m]_qx)}{[r]_q!}\frac{\mathcal{K}(A,r+1)}{B_q(r+1,m)}\int_0^{\infty/A}\frac{t^r}{(1+t)^{r+m+1}}d_qt\right)^{\frac{2-\nu}{2}} \\ &\times\left(\frac{e_q(-[n]_qx)}{P_q(1)}\sum_{r=0}^{\infty}\frac{P_{r,q}([m]_qx)}{[r]_q!}\frac{\mathcal{K}(A,r+1)}{B_q(r+1,m)}\int_0^{\infty/A}\frac{t^r}{(1+t)^{r+m+1}}\mid t-x\mid^2 d_qt\right)^{\frac{\nu}{2}} \\ &= M\left(\mathcal{J}^*_{m,q}(\mu_2;x)\right)^{\frac{\nu}{2}}.\end{aligned}$$

Which complete the proof. □

Denote

$$C^2_B(\mathbb{R}^+) = \{\psi \in C_B(\mathbb{R}^+) : \psi', \psi'' \in C_B(\mathbb{R}^+)\}, \tag{5.2}$$

with

$$\parallel \psi \parallel_{C^2_B(\mathbb{R}^+)}=\parallel \psi \parallel_{C_B(\mathbb{R}^+)} + \parallel \psi' \parallel_{C_B(\mathbb{R}^+)} + \parallel \psi'' \parallel_{C_B(\mathbb{R}^+)}, \tag{5.3}$$

also

$$\| \psi \|_{C_B(\mathbb{R}^+)} = \sup_{x\in\mathbb{R}^+} | \psi(x) | . \tag{5.4}$$

Theorem 7. For any $\psi \in C_B^2(\mathbb{R}^+)$, we have

$$| \mathcal{J}^*_{m,q}(\psi; x) - \psi(x) | \leq \left\{ \frac{1}{[m-1]_q} \left(\frac{1}{q} + \frac{P'_q(1)}{P_q(1)} \right) + \left(\frac{[m]_q}{[m-1]_q} - 1 \right) x + \frac{\lambda_{m,q}(x)}{2} \right\} \| \psi \|_{C_B^2(\mathbb{R}^+)}$$

where $\lambda_{m,q}(x)$ is given in Theorem 6.

Proof of Theorem 7. We applying the generalized mean value theorem in the Taylor series expansion, then

$$\psi(t) = \psi(x) + \psi'(x)(t-x) + \psi''(c)\frac{(t-x)^2}{2}, \; x < c < t.$$

By linearity, we have

$$\mathcal{J}^*_{m,q}(\psi; x) - \psi(x) = \psi'(x)\mathcal{J}^*_{m,q}(t-x; x) + \frac{1}{2}\mathcal{J}^*_{m,q}\left(\psi''(c)(t-x)^2; x\right)$$

which gives
$| \mathcal{J}^*_{m,q}(\psi; x) - \psi(x) |$

$$\begin{aligned} &\leq \sup_{x\in\mathbb{R}_+} | \psi'(x) || \mathcal{J}^*_{m,q}(t-x; x) | + \frac{1}{2}\mathcal{J}^*_{m,q}\left(\left(\sup_{c\in\mathbb{R}_+} | \psi''(c) | \right)(t-x)^2; x\right) \\ &= \| \psi' \|_{C_B(\mathbb{R}_+)} | \mathcal{J}^*_{m,q}(t-x; x) | + \frac{1}{2} \| \psi'' \|_{C_B(\mathbb{R}_+)} \mathcal{J}^*_{m,q}((t-x)^2; x). \end{aligned}$$

Hence
$| \mathcal{J}^*_{m,q}(\psi; x) - \psi(x) |$

$$\begin{aligned} &\leq \left\{ \frac{1}{[m-1]_q} \left(\frac{1}{q} + \frac{P'_q(1)}{P_q(1)} \right) + \left(\frac{[m]_q}{[m-1]_q} - 1 \right) x \right\} \| \psi' \|_{C_B(\mathbb{R}^+)} \\ &+ \left\{ \left(\delta^*_{m,q}\right)^2 \right\} \frac{\| \psi'' \|_{C_B(\mathbb{R}^+)}}{2}. \end{aligned}$$

From (5.3), we have $\| \psi' \|_{C_B[0,\infty)} \leq \| \psi \|_{C_B^2[0,\infty)}$.
$| \mathcal{J}^*_{m,q}(\psi; x) - \psi(x) |$

$$\begin{aligned} &\leq \left\{ \frac{1}{[m-1]_q} \left(\frac{1}{q} + \frac{P'_q(1)}{P_q(1)} \right) + \left(\frac{[m]_q}{[m-1]_q} - 1 \right) x \right\} \| \psi \|_{C_B^2(\mathbb{R}^+)} \\ &+ \left\{ \left(\delta^*_{m,q}\right)^2 \right\} \frac{\| \psi \|_{C_B^2(\mathbb{R}^+)}}{2}. \end{aligned}$$

This completes the proof. □

6. Direct theorem

The Peetre's K-functional [18] is defined by

$$K_2(f,\delta) = \inf_{C_B^2(\mathbb{R}^+)} \left\{ \left(\| f-\psi \|_{C_B(\mathbb{R}^+)} + \delta \| \psi'' \|_{C_B^2(\mathbb{R}^+)} \right) : \psi \in \mathcal{W}^2 \right\}, \tag{6.1}$$

where

$$\mathcal{W}^2 = \{\psi \in C_B(\mathbb{R}^+) : \psi', \psi'' \in C_B(\mathbb{R}^+)\}. \tag{6.2}$$

Note that $K_2(f,\delta) \leq C\varpi_2(f,\delta^{\frac{1}{2}}),\ \delta > 0, C > 0$, where

$$\varpi_2(f,\delta^{\frac{1}{2}}) = \sup_{0<h<\delta^{\frac{1}{2}}} \sup_{x\in\mathbb{R}^+} | f(x+2h) - 2f(x+h) + f(x) | . \tag{6.3}$$

is the second order modulus of continuity.

Theorem 8. For $f \in C_B(\mathbb{R}^+)$, we have

$$\begin{aligned} | \mathcal{J}^*_{m,q}(f;x) - f(x) | \ &\leq \ 2M\Bigg\{\varpi_2\left(f; \frac{\left\{\frac{1}{[m-1]_q}\left(\frac{1}{q} + \frac{P'_q(1)}{P_q(1)}\right) + \left(\frac{[m]_q}{[m-1]_q} - 1\right)x + \lambda_{m,q}(x)\right\}^{\frac{1}{2}}}{2}\right) \\ &+ \ \min\left(1, \frac{\frac{1}{[m-1]_q}\left(\frac{1}{q} + \frac{P'_q(1)}{P_q(1)}\right) + \left(\frac{[m]_q}{[m-1]_q} - 1\right)x + \lambda_{m,q}(x)}{4}\right) \| f \|_{C_B(\mathbb{R}^+)} \Bigg\}. \end{aligned}$$

Proof of Theorem 8. Applying Theorem 7, we get

$$\begin{aligned} | \mathcal{J}^*_{m,q}(f;x) - f(x) | \ &\leq \ | \mathcal{J}^*_{m,q}(f-\psi;x) | + | \mathcal{J}^*_{m,q}(\psi;x) - \psi(x) | + | f(x) - \psi(x) | \\ &\leq \ 2 \| f-\psi \|_{C_B(\mathbb{R}^+)} + \frac{\lambda_{m,q}(x)}{2} \| \psi \|_{C_B^2(\mathbb{R}^+)} \\ &+ \ \left\{ \frac{1}{[m-1]_q}\left(\frac{1}{q} + \frac{P'_q(1)}{P_q(1)}\right) + \left(\frac{[m]_q}{[m-1]_q} - 1\right)x \right\} \| \psi \|_{C_B(\mathbb{R}^+)} \end{aligned}$$

From (5.3) clearly we have $\| \psi \|_{C_B[0,\infty)} \leq \| \psi \|_{C_B^2[0,\infty)}$.
Therefore
$| \mathcal{J}^*_{m,q}(f;x) - f(x) |$

$$\leq 2\left(\| f-\psi \|_{C_B(\mathbb{R}^+)} + \frac{\frac{1}{[m-1]_q}\left(\frac{1}{q} + \frac{P'_q(1)}{P_q(1)}\right) + \left(\frac{[m]_q}{[m-1]_q} - 1\right)x + \lambda_{m,q}(x)}{4} \| \psi \|_{C_B^2(\mathbb{R}^+)} \right).$$

Taking infimum over all $\psi \in C_B^2(\mathbb{R}^+)$ and using (6.1), we get

$$| \mathcal{J}^*_{m,q}(f;x) - f(x) | \leq 2K_2\left(f; \frac{\frac{1}{[m-1]_q}\left(\frac{1}{q} + \frac{P'_q(1)}{P_q(1)}\right) + \left(\frac{[m]_q}{[m-1]_q} - 1\right)x + \lambda_{m,q}(x)}{4}\right)$$

For an absolute constant $\mathcal{D} > 0$, we use the relation [19]

$$K_2(f;\delta) \leq \mathcal{D}\{\varpi_2(f;\sqrt{\delta}) + \min(1,\delta) \parallel f \parallel\}.$$

This complete the proof. □

For an arbitrary $f \in Q_\sigma^k(\mathbb{R}^+)$, we define (see [20])

$$\Omega(f,\delta) = \sup_{x\in[0,\infty),|h|\leq\delta} \frac{\mid f(x+h) - f(x) \mid}{(1+h^2)(1+x^2)}. \tag{6.4}$$

Note that $\lim_{\delta\to 0} \Omega(f,\delta) \to 0$ and

$$\mid f(t) - f(x) \mid \leq 2\left(1 + \frac{\mid t-x \mid}{\delta}\right)(1+\delta^2)(1+x^2)(1+(t-x)^2)\Omega(f,\delta), \tag{6.5}$$

where $f \in Q_\sigma^k(\mathbb{R}^+)$ and $t, x \in [0,\infty)$.

Theorem 9. Let $q = q_m$, then for $f \in Q_\sigma^k(\mathbb{R}^+)$, we have

$$\sup_{x\in[0,\infty)} \frac{\mid \mathcal{J}^*_{m,q_m}(f;x) - f(x) \mid}{(1+x^2)} \leq (1 + \mathcal{A}_1 + 4\mathcal{A}_1\mathcal{A}_2)\left(1 + \Psi_{m,q_m}(m)\right)\Omega\left(f;\sqrt{\Psi_{m,q_m}(m)}\right),$$

where $\mathcal{A}_1, \mathcal{A}_2 > 0$ and

$$\begin{aligned}\Psi_{m,q_m}(m) &= \max\Bigg\{\frac{[m]^2_{q_m}}{[m-1]_{q_m}[m-2]_{q_m}} + 1 - \frac{2[m]_{q_m}}{[m-1]_{q_m}},\\ &\quad \frac{1}{[m-1]_{q_m}}\left(\frac{(1+2q_m)[m]_{q_m}}{q_m^2[m-2]_{q_m}} + \frac{2[m]_{q_m}}{[m-2]_{q_m}}\frac{P'_{q_m}(1)}{P_{q_m}(1)} - \frac{2P'_{q_m}(1)}{P_{q_m}(1)} - \frac{2}{q_m}\right),\\ &\quad \frac{1}{q_m^2[m-1]_{q_m}[m-2]_{q_m}}\left(\frac{(1+q_m)}{q_m} + (1+2q_m)\frac{P'_{q_m}(1)}{P_{q_m}(1)}\right)\Bigg\}.\end{aligned}$$

Proof of Theorem 9. To prove this theorem our aim is to use the results (6.4), (6.5) and then apply Cauchy-Schwarz inequality, thus we see

$$\begin{aligned}&\left|\mathcal{J}^*_{m,q_m}(f;x) - f(x)\right|\\ &\quad\leq 2(1+\delta^2)(1+x^2)\Omega(f;\delta)\left(1 + \mathcal{J}^*_{m,q_m}((t-x)^2;x) + \mathcal{J}^*_{m,q_m}\left(\left(1+(t-x)^2\right)\frac{\mid t-x\mid}{\delta};x\right)\right)\end{aligned} \tag{6.6}$$

and

$$\begin{aligned}&\mathcal{J}^*_{m,q_m}\left(\left(1+(t-x)^2\right)\frac{\mid t-x\mid}{\delta};y\right)\\ &\quad\leq 2\left(\mathcal{J}^*_{m,q_m}(1+(t-x)^4;x)\right)^{\frac{1}{2}}\left(\mathcal{J}^*_{m,q_m}\left(\frac{(t-x)^2}{\delta^2};x\right)\right)^{\frac{1}{2}}.\end{aligned} \tag{6.7}$$

In the light of Lemma 2, we easily see that

$$\mathcal{J}^*_{m,q_m}((t-x)^2;x) \leq \Psi_{m,q_m}(m)(1+x+x^2), \tag{6.8}$$

where

$$\begin{aligned}\Psi_{m,q_m}(m) = \max\Bigg\{&\frac{[m]^2_{q_m}}{[m-1]_{q_m}[m-2]_{q_m}} + 1 - \frac{2[m]_{q_m}}{[m-1]_{q_m}},\\ &\frac{1}{[m-1]_{q_m}}\left(\frac{(1+2q_m)[m]_{q_m}}{q_m^2[m-2]_{q_m}} + \frac{2[m]_{q_m}}{[m-2]_{q_m}}\frac{P'_{q_m}(1)}{P_{q_m}(1)} - \frac{2P'_{q_m}(1)}{P_{q_m}(1)} - \frac{2}{q_m}\right),\\ &\frac{1}{q_m^2[m-1]_{q_m}[m-2]_{q_m}}\left(\frac{(1+q_m)}{q_m} + (1+2q_m)\frac{P'_{q_m}(1)}{P_{q_m}(1)}\right)\Bigg\},\end{aligned}$$

For a constant $\mathcal{A}_1 > 0$ we have

$$\mathcal{J}^*_{m,q_m}((t-x)^2;x) \leq \mathcal{A}_1(1+x+x^2). \tag{6.9}$$

Similarly a small calculation lead us

$$\mathcal{J}^*_{m,q_m}((t-x)^4;x) = \left(\alpha_{1,q_m}x^2 + \alpha_{2,q_m}x + \alpha_{3,q_m}\right)^2 \leq \varsigma_{m,q_m}(m)\Big(1+x+x^2+x^3+x^4\Big),$$

where

$$\begin{aligned}\varsigma_{m,q_m}(m) \;=\; \max\Big\{&\alpha^2_{1,q_m}(m),\ 2\alpha_{1,q_m}(m)\alpha_{2,q_m}(m),\ \left(2\alpha^2_{1,q_m}(m)\alpha_{3,q_m}(m) + \alpha^2_{2,q_m}(m)\right),\\ &2\alpha_{2,q_m}(m)\alpha_{3,q_m}(m),\ 2\alpha_{3,q_m}(m)\alpha_{2,q_m}(m)\Big\}.\end{aligned}$$

and

$$\begin{aligned}\alpha_{1,q_m}(m) &= \frac{[m]^2_{q_m}}{[m-1]_{q_m}[m-2]_{q_m}} + 1 - \frac{2[m]_{q_m}}{[m-1]_{q_m}},\\ \alpha_{2,q_m}(m) &= \frac{1}{[m-1]_{q_m}}\left(\frac{(1+2q_m)[m]_{q_m}}{q_m^2[m-2]_{q_m}} + \frac{2[m]_{q_m}}{[m-2]_{q_m}}\frac{P'_{q_m}(1)}{P_{q_m}(1)} - \frac{2P'_{q_m}(1)}{P_{q_m}(1)} - \frac{2}{q_m}\right),\\ \alpha_{3,q_m}(m) &= \frac{1}{q_m^2[m-1]_{q_m}[m-2]_{q_m}}\left(\frac{(1+q_m)}{q_m} + (1+2q_m)\frac{P'_{q_m}(1)}{P_{q_m}(1)}\right).\end{aligned}$$

As $\frac{1}{[m-i]_{q_m}} = 0$ for all $i = 1,2$ when $m \to \infty$, imply that for a constant $\mathcal{A}_2 > 0$, easily we have

$$\left(\mathcal{J}^*_{m,q_m}(1+(t-x)^4;x)\right)^{\frac{1}{2}} \leq \mathcal{A}_2\Big(2+x+x^2+x^3+x^4\Big)^{\frac{1}{2}}. \tag{6.10}$$

(6.8), imply that

$$\left(\mathcal{J}^*_{m,q_m}\left(\frac{(t-x)^2}{\delta^2};x\right)\right)^{\frac{1}{2}} \leq \frac{1}{\delta}\left(\Psi_{\kappa,q_m}(m)\right)^{\frac{1}{2}}\left(1+x+x^2\right)^{\frac{1}{2}}. \tag{6.11}$$

Hence, by combining (6.7)–(6.11) and (6.6), and finally if we choosing $\delta = \sqrt{\Psi_{m,q_m}(m)}$ after taking supremum $y \in [0, \Psi_{m,q_m}(m))$, we are denumerable to get the result. □

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Conflict of interest

The authors declare no conflict of interest in this paper.

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