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Research article

New extensions of Chebyshev-Pólya-Szegö type inequalities via conformable integrals

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Abstract: Recently, several papers related to integral inequalities involving various fractional integral operators have been presented. In this work, motivated essentially by the previous works, we prove some new Polya-Szegö inequalities via conformable fractional integral operator and use them to prove some new fractional Chebyshev type inequalities concerning the integral of the product of two functions and the product of two integrals which are improvement of the results in the paper [Ntouyas, S.K., Agarwal, P. and Tariboon, J., *On Polya-Szegö and Chebyshev type inequalities involving the Riemann-Liouville fractional integral operators*, J. Math. Inequal (see [9])].

Keywords: Chebyshev inequality; Polya-Szegö type inequalities; conformable fractional integrals
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1. Introduction and preliminaries

This article is based on the well known Chebyshev functional [1]:

$$T(f,g)=\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx-\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right) \tag{1.1}$$

where f ve g are two integrable functions which are synchronous on $[a,b]$, i.e.

$$(f(x)-f(y))(g(x)-g(y))\geq 0$$

for any $x,y\in[a,b]$, then the Chebyshev inequality states that $T(f,g)\geq 0$.

The functional (1.1) has attracted many researchers attention due to diverse applications in numerical quadrature, transform theory, probability and statistical problems. Among those applications, the functional (1.1) has also been employed to yield a number of integral inequalities. For some recent counterparts, generalizations of Chebyshev inequality, the reader may refer to [2, 3, 9] and [10].

Another important inequality which will be useful to prove our main results is Pólya and Szegö inequality: (see [4])

$$\frac{\int_a^b f^2(x)\,dx \int_a^b g^2(x)\,dx}{\left(\int_a^b f(x)\,g(x)\,dx\right)^2} \leq \frac{1}{4}\left(\sqrt{\frac{MN}{mn}} + \sqrt{\frac{mn}{MN}}\right)^2$$

In [5], Dragomir and Diamond obtained the following Grüss type inequality by using the Pólya-Szegö inequality:

Theorem 1.1. *Let $f, g : [a,b] \to \mathbb{R}_+$ be two integrable functions so that*

$$\begin{aligned} 0 &< m \leq f(x) \leq M < \infty \\ 0 &< n \leq g(x) \leq N < \infty \end{aligned}$$

for $x \in [a,b]$. Then we have

$$|T(f,g;a,b)| = \frac{1}{4}\frac{(M-m)(N-n)}{\sqrt{mnMN}}\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right) \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.2) in the sense it can not be replaced by a smaller constant.

Let us recall some well-known concepts. We note that the beta function $B(\alpha,\beta)$ is defined (see [11])

$$B(\alpha,\beta) = \begin{cases} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt & (\mathbb{R}(\alpha), \mathbb{R}(\beta) > 0) \\ \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} & \left(\alpha,\beta \in \mathbb{C}\setminus\mathbb{Z}_0^-\right) \end{cases}$$

where Γ is the familiar Gamma function. Here and in the following, let $\mathbb{C}, \mathbb{R}, \mathbb{R}^+$ and $\mathbb{Z}_0^-$ be the sets of complex numbers, real numbers, positive real numbers and non-positive integers, respectively and let $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$.

Definition 1.1. *(See [6, 7]) Let $f \in L[a,b]$. The Riemann-Liouville integrals $J_{a^+}^{\alpha} f$ and $J_{b^-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$\left(J_{a^+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,dt \qquad (x > a), \tag{1.3}$$

and

$$\left(J_{b^-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\,dt \qquad (x < b), \tag{1.4}$$

where $\Gamma(\alpha) = \int_0^\infty e^{-x}x^{\alpha-1}dx$ is the Gamma function.

Definition 1.2. *(See [8]) Let $\alpha \in (n, n+1]$, $n = 0, 1, 2, \ldots$ and set $\beta = \alpha - n$. Then the left conformable fractional integral of order $\alpha > 0$ is defined by*

$$I_\alpha^a f(t) = \frac{1}{n!}\int_a^t (t-x)^n (x-a)^{\beta-1} f(x)\,dx, \quad t > a, \tag{1.5}$$

the right conformable fractional integral of order $\alpha > 0$ is defined by

$$^{b}I_{\alpha}f(x) = \frac{1}{n!}\int_{t}^{b}(x-t)^{n}(b-x)^{\beta-1}f(x)\,dx, \quad t<b. \tag{1.6}$$

Notice that if $\alpha = n+1$ then $\beta = \alpha - n = n+1-n = 1$ and hence $(I_{\alpha}^{a}f)(t) = \left(J_{n+1}^{a}f\right)(t)$.

Several recent results related to different kinds of fractional integral operators can be found in [12–24].

The main aim of this present paper is to prove certain new Pólya-Szegö and Chebyshev types integral inequalities involving conformable fractional integral operator. We also give some special cases of our results.

2. Main results

In this section, we establish certain Pólya-Szegö type integral inequalities for positive integral functions involving conformable fractional integral operator.

Lemma 2.1. *Let f and g be two positive integrable functions on $[0,\infty)$. Also let $\alpha \in (n, n+1]$, $n = 0, 1, 2, ...$, set $\beta = \alpha - n$. Assume that there exist four positive integrable functions v_1, v_2, w_1 and w_2 such that:*

$$0 < v_1(\tau) \le f(\tau) \le v_2(\tau),\ 0 < w_1(\tau) \le g(\tau) \le w_2(\tau) \quad (\tau \in [0,x],\ x>0) \tag{2.1}$$

Then the following inequality holds:

$$\frac{I_{\alpha}^{0}\left\{w_1w_2f^2\right\}(x)\, I_{\alpha}^{0}\left\{v_1v_2g^2\right\}(x)}{\left(I_{\alpha}^{0}\left\{(v_1w_1+v_2w_2)fg\right\}(x)\right)^2} \le \frac{1}{4}. \tag{2.2}$$

Proof. From (2.1), for $\tau \in [0,x]$, $x>0$, we can write

$$\left(\frac{v_2(\tau)}{w_1(\tau)} - \frac{f(\tau)}{g(\tau)}\right) \ge 0 \tag{2.3}$$

and

$$\left(\frac{f(\tau)}{g(\tau)} - \frac{v_1(\tau)}{w_2(\tau)}\right) \ge 0 \tag{2.4}$$

multiplying (2.3) and (2.4), we get

$$\left(\frac{v_2(\tau)}{w_1(\tau)} - \frac{f(\tau)}{g(\tau)}\right)\left(\frac{f(\tau)}{g(\tau)} - \frac{v_1(\tau)}{w_2(\tau)}\right) \ge 0.$$

From the above inequality, we can write

$$\begin{aligned} &(v_1(\tau)w_1(\tau) + v_2(\tau)w_2(\tau))f(\tau)g(\tau) \\ \ge\ & w_1(\tau)w_2(\tau)f^2(\tau) + v_1(\tau)v_2(\tau)g^2(\tau). \end{aligned} \tag{2.5}$$

Multiplying both sides of (2.5) by $\frac{1}{n!}(x-\tau)^n \tau^{\alpha-n-1}$ and integrating the resulting inequality with respect to τ over $(0,x)$, we get

$$I_\alpha^0 \{(v_1w_1+v_2w_2) fg\}(x) \geq I_\alpha^0 \left\{w_1w_2f^2\right\}(x) + I_\alpha^0 \left\{v_1v_2g^2\right\}(x) \tag{2.6}$$

applying the AM-GM inequality, i.e. ($a+b \geq 2\sqrt{ab}$, $a,b \in \mathbb{R}^+$), we have

$$I_\alpha^0 \{(v_1w_1+v_2w_2) fg\}(x) \geq 2\sqrt{I_\alpha^0 \{w_1w_2f^2\}(x) + I_\alpha^0 \{v_1v_2g^2\}(x)}$$

which implies that

$$I_\alpha^0 \left\{w_1w_2f^2\right\}(x) + I_\alpha^0 \left\{v_1v_2g^2\right\}(x) \leq \frac{1}{4}\left(I_\alpha^0 \{(v_1w_1+v_2w_2) fg\}(x)\right)^2.$$

So, we get the desired result. □

Corollary 2.1. *If $v_1 = m$, $v_2 = M$, $w_1 = n$ and $w_2 = N$, then we have*

$$\frac{\left(I_\alpha^0 f^2\right)(x)\left(I_\alpha^0 g^2\right)(x)}{\left(\left(I_\alpha^0 fg\right)(x)\right)^2} \leq \frac{1}{4}\left(\sqrt{\frac{mn}{MN}} + \sqrt{\frac{MN}{mn}}\right)^2.$$

Lemma 2.2. *Let f and g be two positive integrable functions on $[0,\infty)$. Also let $\alpha \in (n,n+1]$, $\theta \in (k,k+1]$, $n,k = 0,1,2,....$ Assume that there exist four positive integrable functions v_1, v_2, w_1 and w_2 satisfying condition (2.1). Then the following inequality holds:*

$$\begin{aligned} & I_\alpha^0 \{v_1v_2\}(x)\, I_\theta^0 \{w_1w_2\}(x) \times I_\alpha^0 \left\{f^2\right\}(x)\, I_\theta^0 \left\{g^2\right\}(x) \\ \leq\ & \frac{1}{4}\left(I_\alpha^0 \{v_1f\}(x)\, I_\theta^0 \{w_1g\}(x) + I_\alpha^0 \{v_2f\}(x)\, I_\theta^0 \{w_2g\}(x)\right)^2 \end{aligned} \tag{2.7}$$

Proof. From (2.1), we get

$$\left(\frac{v_2(\tau)}{w_1(\xi)} - \frac{f(\tau)}{g(\xi)}\right) \geq 0$$

and

$$\left(\frac{f(\tau)}{g(\xi)} - \frac{v_1(\tau)}{w_2(\xi)}\right) \geq 0$$

which leads to

$$\left(\frac{v_1(\tau)}{w_2(\xi)} + \frac{v_2(\tau)}{w_1(\xi)}\right)\frac{f(\tau)}{g(\xi)} \geq \frac{f^2(\tau)}{g^2(\xi)} + \frac{v_1(\tau)\,v_2(\tau)}{w_1(\xi)\,w_2(\xi)}. \tag{2.8}$$

Multiplying both sides of (2.8) by $w_1(\xi)\,w_2(\xi)\,g^2(\xi)$, we have

$$\begin{aligned} & v_1(\tau)\, f(\tau)\, w_1(\xi)\, g(\xi) + v_2(\tau)\, f(\tau)\, w_2(\xi)\, g(\xi) \\ \geq\ & w_1(\xi)\, w_2(\xi)\, f^2(\tau) + v_1(\tau)\, v_2(\tau)\, g^2(\xi). \end{aligned} \tag{2.9}$$

Multiplying both sides (2.9) by $\left(\frac{1}{n!}\right)\left(\frac{1}{k!}\right)(x-\tau)^n (x-\xi)^k \tau^{\alpha-n-1}\xi^{\theta-k-1}$and integrating the resulting inequality with respect to τ and ξ over $(0,x)^2$, we get

$$I_\alpha^0 \{v_1f\}(x)\, I_\theta^0 \{w_1g\}(x) + I_\alpha^0 \{v_2f\}(x)\, I_\theta^0 \{w_2g\}(x)$$

$$\geq \quad I_{\alpha}^{0}\left\{f^{2}\right\}(x)\, I_{\theta}^{0}\{w_1 w_2\}(x)+I_{\alpha}^{0}\{v_1 v_2\}(x)\, I_{\theta}^{0}\left\{g^{2}\right\}(x).$$

Applying the AM-GM inequality, we obtain

$$\begin{aligned} & I_{\alpha}^{0}\{v_1 f\}(x)\, I_{\theta}^{0}\{w_1 g\}(x)+I_{\alpha}^{0}\{v_2 f\}(x)\, I_{\theta}^{0}\{w_2 g\}(x) \\ \geq\ & 2\sqrt{I_{\alpha}^{0}\{f^{2}\}(x)\, I_{\theta}^{0}\{w_1 w_2\}(x)\times I_{\alpha}^{0}\{v_1 v_2\}(x)\, I_{\theta}^{0}\{g^{2}\}(x)} \end{aligned}$$

which leads to the desired inequality in (2.7). The proof is completed. □

Corollary 2.2. *If $v_1 = m$, $v_2 = M$, $w_1 = n$ and $w_2 = N$, then we have*

$$x^{\alpha+\theta}\frac{\Gamma(\alpha-n)\,\Gamma(\theta-k)}{\Gamma(\alpha+1)\,\Gamma(\theta+1)}\times\frac{\left(I_{\alpha}^{0}f^{2}\right)(x)\left(I_{\theta}^{0}g^{2}\right)(x)}{\left(\left(I_{\alpha}^{0}f\right)(x)\left(I_{\theta}^{0}g\right)(x)\right)^{2}}\leq\frac{1}{4}\left(\sqrt{\frac{mn}{MN}}+\sqrt{\frac{MN}{mn}}\right)^{2}$$

Lemma 2.3. *Let f and g be two positive integrable functions on $[0,\infty)$. Also let $\alpha\in(n,n+1]$, $\theta\in(k,k+1]$, $n,k=0,1,2,\ldots$. Assume that there exist four positive integrable functions v_1, v_2, w_1 and w_2 satisfying condition (2.1). Then the following inequality holds:*

$$I_{\alpha}^{0}\left\{f^{2}\right\}(x)\, I_{\theta}^{0}\left\{g^{2}\right\}(x)\leq I_{\alpha}^{0}\left\{\frac{v_2 f g}{w_1}\right\}(x)\, I_{\theta}^{0}\left\{\frac{w_2 f g}{w_1}\right\}(x). \tag{2.10}$$

Proof. Using the condition (2.1), we get

$$f^{2}(\tau)\leq\frac{v_2(\tau)}{w_1(\tau)}f(\tau)\,g(\tau). \tag{2.11}$$

Multiplying both sides of (2.11) by $\left(\frac{1}{n!}\right)(x-\tau)^{n}\tau^{\alpha-n-1}$and integrating the resulting inequality with respect to τ over $(0,x)$, we obtain

$$\frac{1}{n!}\int_{0}^{x}(x-\tau)^{n}\tau^{\alpha-n-1}f^{2}(\tau)\,d\tau\leq\frac{1}{n!}\int_{0}^{x}(x-\tau)^{n}\tau^{\alpha-n-1}\frac{v_2(\tau)}{w_1(\tau)}f(\tau)\,g(\tau)\,d\tau$$

which leads to

$$I_{\alpha}^{0}\left\{f^{2}\right\}(x)\leq I_{\alpha}^{0}\left\{\frac{v_2 f g}{w_1}\right\}(x). \tag{2.12}$$

Similarly, we can write

$$g^{2}(\xi)\leq\frac{w_2(\xi)}{v_1(\xi)}f(\xi)\,g(\xi).$$

By a similar argument, we have

$$\frac{1}{k!}\int_{0}^{x}(x-\xi)^{n}\xi^{\theta-n-1}g^{2}(\xi)\,d\xi\leq\frac{1}{k!}\int_{0}^{x}(x-\xi)^{n}\xi^{\theta-n-1}\frac{w_2(\xi)}{v_1(\xi)}f(\xi)\,g(\xi)\,d\xi$$

which implies

$$I_{\theta}^{0}\left\{g^{2}\right\}(x)\leq I_{\theta}^{0}\left\{\frac{w_2 f g}{v_1}\right\}(x). \tag{2.13}$$

Multiplying (2.12) and (2.13), we get the (2.10). The proof is completed. □

Corollary 2.3. *If $v_1 = m$, $v_2 = M$, $w_1 = n$ and $w_2 = N$, then we have*

$$\frac{\left(I_\alpha^0 f^2\right)(x)\left(I_\theta^0 g^2\right)(x)}{\left(\left(I_\alpha^0 fg\right)(x)\left(I_\theta^0 fg\right)(x)\right)^2} \leq \frac{MN}{mn}$$

Theorem 2.1. *Let f and g be two positive integrable functions on $[0, \infty)$. Also let $\alpha \in (n, n+1]$, $\theta \in (k, k+1]$, $n, k = 0, 1, 2, \ldots$. Assume that there exist four positive integrable functions v_1, v_2, w_1 and w_2 satisfying condition (2.1). Then the following inequality holds:*

$$\begin{aligned} &\left|\left(x^\alpha \frac{\Gamma(\alpha - n)}{\Gamma(\alpha + 1)}\right)\left(I_\theta^0 fg\right)(x) + \left(x^\theta \frac{\Gamma(\theta - k)}{\Gamma(\theta + 1)}\right)\left(I_\alpha^0 fg\right)(x)\right. \\ &\left. -\left(I_\alpha^0 f\right)(x)\left(I_\theta^0 g\right)(x) - \left(I_\theta^0 f\right)(x)\left(I_\alpha^0 g\right)(x)\right| \\ \leq\ & \left|A_1(f, v_1, v_2)(x) + A_2(f, v_1, v_2)(x)\right|^{1/2} \\ & \times \left|A_1(g, w_1, w_2)(x) + A_2(f, w_1, w_2)(x)\right|^{1/2} \end{aligned} \tag{2.14}$$

where

$$A_1(u, v, w)(x) = \left(x^\theta \frac{\Gamma(\theta - k)}{\Gamma(\theta + 1)}\right) \times \frac{\left(I_\alpha^0 \{(v + w) u\}(x)\right)^2}{4 I_\alpha^0 \{vw\}(x)} - \left(I_\alpha^0 u\right)(x)\left(I_\theta^0 u\right)(x)$$

and

$$A_2(u, v, w)(x) = \left(x^\alpha \frac{\Gamma(\alpha - n)}{\Gamma(\alpha + 1)}\right) \times \frac{\left(I_\theta^0 \{(v + w) u\}(x)\right)^2}{4 I_\theta^0 \{vw\}(x)} - \left(I_\alpha^0 u\right)(x)\left(I_\theta^0 u\right)(x).$$

Proof. Let f and g be two positive integrable functions on $[0, \infty)$. For $\tau, \xi \in (0, x)$ with $x > 0$, we define $H(\tau, \xi)$ as

$$H(\tau, \xi) = (f(\tau) - f(\xi))(g(\tau) - g(\xi))$$

namely

$$H(\tau, \xi) = f(\tau) g(\tau) + f(\xi) g(\xi) - f(\tau) g(\xi) - f(\xi) g(\tau). \tag{2.15}$$

Multiplying both sides of (2.15) by

$$\left(\frac{1}{n!}\right)\left(\frac{1}{k!}\right)(x - \tau)^n (x - \xi)^k \tau^{\alpha - n - 1} \xi^{\theta - k - 1}$$

and double integrating the resulting inequality with respect to τ and ξ over $(0, x)^2$, we get

$$\begin{aligned} &\left(\frac{1}{n!}\right)\left(\frac{1}{k!}\right) \int_0^x \int_0^x (x - \tau)^n (x - \xi)^k \tau^{\alpha - n - 1} \xi^{\theta - k - 1} H(\tau, \xi)\, d\tau d\xi \\ =\ & \left(x^\theta \frac{\Gamma(\theta - k)}{\Gamma(\theta + 1)}\right)\left(I_\alpha^0 fg\right)(x) + \left(x^\alpha \frac{\Gamma(\alpha - n)}{\Gamma(\alpha + 1)}\right)\left(I_\theta^0 fg\right)(x) \\ & -\left(I_\alpha^0 f\right)(x)\left(I_\theta^0 g\right)(x) - \left(I_\theta^0 f\right)(x)\left(I_\alpha^0 g\right)(x). \end{aligned}$$

Applying the Cauchy-Schwarz inequality for double integrals, we can write

$$\left|\left(\frac{1}{n!}\right)\left(\frac{1}{k!}\right) \int_0^x \int_0^x (x - \tau)^n (x - \xi)^k \tau^{\alpha - n - 1} \xi^{\theta - k - 1} H(\tau, \xi)\, d\tau d\xi\right|$$

$$
\begin{aligned}
\leq\ & \left[\left(\frac{1}{n!}\right)\left(\frac{1}{k!}\right)\int_0^x\int_0^x (x-\tau)^n (x-\xi)^k \tau^{\alpha-n-1}\xi^{\theta-k-1} f^2(\tau)\, d\tau d\xi\right.\\
& +\left(\frac{1}{n!}\right)\left(\frac{1}{k!}\right)\int_0^x\int_0^x (x-\tau)^n (x-\xi)^k \tau^{\alpha-n-1}\xi^{\theta-k-1} f^2(\xi)\, d\tau d\xi\\
& \left.-2\left(\frac{1}{n!}\right)\left(\frac{1}{k!}\right)\int_0^x\int_0^x (x-\tau)^n (x-\xi)^k \tau^{\alpha-n-1}\xi^{\theta-k-1} f(\tau) f(\xi)\, d\tau d\xi\right]^{1/2}\\
& \times\left[\left(\frac{1}{n!}\right)\left(\frac{1}{k!}\right)\int_0^x\int_0^x (x-\tau)^n (x-\xi)^k \tau^{\alpha-n-1}\xi^{\theta-k-1} g^2(\tau)\, d\tau d\xi\right.\\
& +\left(\frac{1}{n!}\right)\left(\frac{1}{k!}\right)\int_0^x\int_0^x (x-\tau)^n (x-\xi)^k \tau^{\alpha-n-1}\xi^{\theta-k-1} g^2(\xi)\, d\tau d\xi\\
& \left.-2\left(\frac{1}{n!}\right)\left(\frac{1}{k!}\right)\int_0^x\int_0^x (x-\tau)^n (x-\xi)^k \tau^{\alpha-n-1}\xi^{\theta-k-1} g(\tau) g(\xi)\, d\tau d\xi\right]^{1/2}
\end{aligned}
$$

As a consequence

$$
\begin{aligned}
& \left|\left(\frac{1}{n!}\right)\left(\frac{1}{k!}\right)\int_0^x\int_0^x (x-\tau)^n (x-\xi)^k \tau^{\alpha-n-1}\xi^{\theta-k-1} H(\tau,\xi)\, d\tau d\xi\right|\\
\leq\ & \left[\left(x^{\theta}\frac{\Gamma(\theta-k)}{\Gamma(\theta+1)}\right)\left(I_{\alpha}^{0} f^2\right)(x)+\left(x^{\alpha}\frac{\Gamma(\alpha-n)}{\Gamma(\alpha+1)}\right)\left(I_{\theta}^{0} f^2\right)(x)\right.\\
& \left.-2\left(I_{\alpha}^{0} f\right)(x)\left(I_{\theta}^{0} f\right)(x)\right]^{1/2}\\
& \times\left[\left(x^{\theta}\frac{\Gamma(\theta-k)}{\Gamma(\theta+1)}\right)\left(I_{\alpha}^{0} g^2\right)(x)+\left(x^{\alpha}\frac{\Gamma(\alpha-n)}{\Gamma(\alpha+1)}\right)\left(I_{\theta}^{0} g^2\right)(x)\right.\\
& \left.-2\left(I_{\alpha}^{0} g\right)(x)\left(I_{\theta}^{0} g\right)(x)\right]^{1/2}
\end{aligned}
$$

Applying Lemma 2.1 with $w_1(\tau)=w_2(\tau)=g(\tau)=1$, we get

$$
\left(x^{\theta}\frac{\Gamma(\theta-k)}{\Gamma(\theta+1)}\right) I_{\alpha}^{0}\left\{f^2\right\}(x) \leq \left(x^{\theta}\frac{\Gamma(\theta-k)}{\Gamma(\theta+1)}\right)\frac{\left(I_{\alpha}^{0}\{(v_1+v_2) f\}(x)\right)^2}{4 I_{\alpha}^{0}\{v_1 v_2\}(x)}.
$$

This implies that

$$
\begin{aligned}
& \left(x^{\theta}\frac{\Gamma(\theta-k)}{\Gamma(\theta+1)}\right) I_{\alpha}^{0}\left\{f^2\right\}(x)-\left(I_{\alpha}^{0} f\right)(x)\left(I_{\theta}^{0} f\right)(x)\\
\leq\ & \left(x^{\theta}\frac{\Gamma(\theta-k)}{\Gamma(\theta+1)}\right)\frac{\left(I_{\alpha}^{0}\{(v_1+v_2) f\}(x)\right)^2}{4 I_{\alpha}^{0}\{v_1 v_2\}(x)}-\left(I_{\alpha}^{0} f\right)(x)\left(I_{\theta}^{0} f\right)(x)\\
=\ & A_1(f, v_1, v_2)
\end{aligned}
\tag{2.16}
$$

and

$$
\begin{aligned}
& \left(x^{\alpha}\frac{\Gamma(\alpha-n)}{\Gamma(\alpha+1)}\right) I_{\theta}^{0}\left\{f^2\right\}(x)-\left(I_{\alpha}^{0} f\right)(x)\left(I_{\theta}^{0} f\right)(x)\\
\leq\ & \left(x^{\alpha}\frac{\Gamma(\alpha-n)}{\Gamma(\alpha+1)}\right)\frac{\left(I_{\theta}^{0}\{(v_1+v_2) f\}(x)\right)^2}{4 I_{\theta}^{0}\{v_1 v_2\}(x)}-\left(I_{\alpha}^{0} f\right)(x)\left(I_{\theta}^{0} f\right)(x)
\end{aligned}
$$

$$= \quad A_2(f, v_1, v_2). \tag{2.17}$$

Similarly, applying Lemma 2.1 with $v_1(\tau) = v_2(\tau) = f(\tau) = 1$, we have

$$\begin{aligned} & \left(x^{\theta}\frac{\Gamma(\theta-k)}{\Gamma(\theta+1)}\right) I_{\alpha}^{0}\left\{g^{2}\right\}(x) - \left(I_{\alpha}^{0}g\right)(x)\left(I_{\theta}^{0}g\right)(x) \\ \leq \quad & A_1(g, w_1, w_2) \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} & \left(x^{\alpha}\frac{\Gamma(\alpha-n)}{\Gamma(\alpha+1)}\right) I_{\theta}^{0}\left\{g^{2}\right\}(x) - \left(I_{\alpha}^{0}g\right)(x)\left(I_{\theta}^{0}g\right)(x) \\ \leq \quad & A_2(g, w_1, w_2). \end{aligned} \tag{2.19}$$

Using (2.16)-(2.19), we conclude the result. □

Theorem 2.2. *Let f and g be two positive integrable functions on $[0,\infty)$. Also let $\alpha \in (n, n+1]$, $\theta \in (k, k+1]$, $n, k = 0, 1, 2, \ldots$. Assume that there exist four positive integrable functions v_1, v_2, w_1 and w_2 satisfying condition (2.1). Then the following inequality holds:*

$$\left|\left(x^{\alpha}\frac{\Gamma(\alpha-n)}{\Gamma(\alpha+1)}\right) I_{\alpha}^{0}\{fg\}(x) - \left(I_{\alpha}^{0}f\right)(x)\left(I_{\alpha}^{0}g\right)(x)\right| \leq \left|A(f, v_1, v_2)(x)\, A(g, w_1, w_2)(x)\right|^{1/2} \tag{2.20}$$

where

$$A(u, v, w)(x) = \left(x^{\alpha}\frac{\Gamma(\alpha-n)}{\Gamma(\alpha+1)}\right) \times \frac{\left(I_{\alpha}^{0}\{(v+w)u\}(x)\right)^{2}}{4I_{\alpha}^{0}\{vw\}(x)} - \left(\left(I_{\alpha}^{0}u\right)(x)\right)^{2}.$$

Proof. Setting $\alpha = \theta$ in (2.14), we obtain (2.20). □

Corollary 2.4. *If $v_1 = m$, $v_2 = M$, $w_1 = n$ and $w_2 = N$, then we have*

$$\begin{aligned} & \left|\left(x^{\alpha}\frac{\Gamma(\alpha-n)}{\Gamma(\alpha+1)}\right) I_{\alpha}^{0}\{fg\}(x) - \left(I_{\alpha}^{0}f\right)(x)\left(I_{\alpha}^{0}g\right)(x)\right| \\ \leq \quad & \frac{(M-m)(N-n)}{4\sqrt{MmNn}} \times \left(I_{\alpha}^{0}f\right)(x)\left(I_{\alpha}^{0}g\right)(x). \end{aligned}$$

Conflict of interest

All authors declare no conflicts of interest in this paper.

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