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Research article

A degree condition for fractional (g, f, n)-critical covered graphs

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Abstract: A graph G is called a fractional (g, f)-covered graph if for any $e \in E(G)$, G admits a fractional (g, f)-factor covering e. A graph G is called a fractional (g, f, n)-critical covered graph if for any $W \subseteq V(G)$ with $|W| = n$, $G - W$ is a fractional (g, f)-covered graph. In this paper, we demonstrate that a graph G of order p is a fractional (g, f, n)-critical covered graph if $p \geq \frac{(a+b)(a+b+n+1)-(b-m)n+2}{a+m}$, $\delta(G) \geq \frac{(b-m)(b+1)+2}{a+m} + n$ and for every pair of nonadjacent vertices u and v of G, $\max\{d_G(u), d_G(v)\} \geq \frac{(b-m)p+(a+m)n+2}{a+b}$, where g and f are integer-valued functions defined on $V(G)$ satisfying $a \leq g(x) \leq f(x) - m \leq b - m$ for every $x \in V(G)$.

Keywords: graph; degree condition; fractional (g, f)-factor; fractional (g, f)-covered graph; fractional (g, f, n)-critical covered graph
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1. Introduction

All graphs considered here are finite, undirected and simple. Let G be a graph. The vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$, respectively. Let $d_G(x)$ denote the degree of a vertex x in G, and $N_G(x)$ denote the neighborhood of a vertex x in G. Set $N_G[x] = N_G(x) \cup \{x\}$. Let X be a vertex subset of G. We use $G[X]$ to denote the subgraph of G induced by X, and write $G - X = G[V(G) \setminus X]$. If no two vertices in X are adjacent, then we call X an independent set of G.

For two integer-valued functions g and f with $f(x) \geq g(x) \geq 0$ for any $x \in V(G)$, a (g, f)-factor of G is defined as a spanning subgraph F of G such that $g(x) \leq d_F(x) \leq f(x)$ for any $x \in V(G)$. Let $E_x = \{e : e = xy \in E(G)\}$. A fractional (g, f)-indicator function is a function h that assigns each edge of G to a number in $[0, 1]$ so that $g(x) \leq \sum_{e \in E_x} h(e) \leq f(x)$ for every $x \in V(G)$. Let h be a fractional (g, f)-indicator function of G. Write $E_h = \{e : e \in E(G), h(e) \neq 0\}$. If G_h is a spanning subgraph of G with $E(G_h) = E_h$, then G_h is called a fractional (g, f)-factor of G. If $h(e) \in \{0, 1\}$ for any $e \in E(G)$, then G_h is just a (g, f)-factor of G. A graph G is said to be a fractional (g, f)-covered graph if for any

$e \in E(G)$, there exists a fractional (g, f)-factor G_h satisfying $h(e) = 1$. If $g(x) = a$ and $f(x) = b$ for any $x \in V(G)$, then a fractional (g, f)-covered graph is called a fractional $[a, b]$-covered graph. A fractional $[k, k]$-covered graph is simply called a fractional k-covered graph. A graph G is said to be a fractional (g, f, n)-critical covered graph if $G - W$ is a fractional (g, f)-covered graph for any $W \subseteq V(G)$ with $|W| = n$. If $g(x) = a$ and $f(x) = b$ for any $x \in V(G)$, then a fractional (g, f, n)-critical covered graph is called a fractional (a, b, n)-critical covered graph. A fractional (k, k, n)-critical covered graph is simply called a fractional (k, n)-critical covered graph.

In recent years, the problems related to factors and fractional factors of graphs have raised attention in computer networks and graph theory. Correa and Matamala [1] gave some results about factors of graphs. Li [2] studied $[a, b]$-factors of $K_{1,t}$-free graphs. Zhou, Sun and Xu [3] obtained a result on the existence of edge-disjoint factors in digraphs. Akbari and Kano [4] discussed the existence of factors in r-regular graphs. Li and Cai [5] derived a degree condition for graphs to have $[a, b]$-factors. Zhou et al [6–11] gained some results on factors of graphs. Egawa and Kano [12] posed some sufficient conditions for graphs to admit (g, f)-factors. Ota and Tokuda [13] considered the existence of regular factors in $K_{1,n}$-free graphs. Liu and Zhang [14] investigated the existence of fractional factors in graphs. Jiang [15, 16] discussed fractional factors of graphs. Zhou et al. [17–20] verified some results on fractional factors of graphs. Yuan and Hao [21] showed a degree condition for a graph to be a fractional $[a, b]$-covered graph. Zhou, Xu and Sun [22] improved and extended the result, and presented a degree condition for a graph to be a fractional (a, b, n)-critical covered graph.

Theorem 1 ([22]). Let a, b and n be integers with $n \geq 0$, $a \geq 1$ and $b \geq \max\{2, a\}$, and let G be a graph of order p with $p \geq \frac{(a+b)(a+b-1)+bn+3}{b}$. If $\delta(G) \geq a + n + 1$ and

$$\max\{d_G(u), d_G(v)\} \geq \frac{ap + bn + 2}{a + b}$$

for every pair of nonadjacent vertices u and v of G, then G is a fractional (a, b, n)-critical covered graph.

In this paper, we extend Theorem 1 to fractional (g, f, n)-critical covered graph, and derive the following result.

Theorem 2. Let a, b, m and n be integers satisfying $m \geq 0$, $n \geq 0$, $a \geq 1$ and $b \geq a + m$, let G be a graph of order p with $p \geq \frac{(a+b)(a+b+n+1)-(b-m)n+2}{a+m}$, and let g and f be integer-valued functions defined on $V(G)$ satisfying $a \leq g(x) \leq f(x) - m \leq b - m$ for every $x \in V(G)$. If $\delta(G) \geq \frac{(b-m)(b+1)+2}{a+m} + n$ and for every pair of nonadjacent vertices u and v of G,

$$\max\{d_G(u), d_G(v)\} \geq \frac{(b - m)p + (a + m)n + 2}{a + b},$$

then G is a fractional (g, f, n)-critical covered graph.

The following result holds if setting $m = 0$ in Theorem 2.

Corollary 1. Let a, b and n be integers satisfying $n \geq 0$ and $b \geq a \geq 1$, let G be a graph of order p with $p \geq \frac{(a+b)(a+b+n+1)-bn+2}{a}$, and let g and f be integer-valued functions defined on $V(G)$ satisfying $a \leq g(x) \leq f(x) \leq b$ for every $x \in V(G)$. If $\delta(G) \geq \frac{b(b+1)+2}{a} + n$ and for every pair of nonadjacent vertices u and v of G,

$$\max\{d_G(u), d_G(v)\} \geq \frac{bp + an + 2}{a + b},$$

then G is a fractional (g, f, n)-critical covered graph.

The following result holds if setting $n = 0$ in Theorem 2.

Corollary 2. Let a, b and m be integers satisfying $m \geq 0$, $a \geq 1$ and $b \geq a + m$, let G be a graph of order p with $p \geq \frac{(a+b)(a+b+1)+2}{a+m}$, and let g and f be integer-valued functions defined on $V(G)$ satisfying $a \leq g(x) \leq f(x) - m \leq b - m$ for every $x \in V(G)$. If $\delta(G) \geq \frac{(b-m)(b+1)+2}{a+m}$ and for every pair of nonadjacent vertices u and v of G,

$$\max\{d_G(u), d_G(v)\} \geq \frac{(b-m)p+2}{a+b},$$

then G is a fractional (g, f)-covered graph.

2. Proof of Theorem 2

The following theorem derived by Li, Yan and Zhang [23] is essential to the proof of Theorem 2.

Theorem 3 ([23]). Let G be a graph, and let g and f be integer-valued functions defined on $V(G)$ satisfying $0 \leq g(x) \leq f(x)$ for any $x \in V(G)$. Then G is a fractional (g, f)-covered graph if and only if

$$\delta_G(S, T) = f(S) + d_{G-S}(T) - g(T) \geq \varepsilon(S)$$

for each $S \subseteq V(G)$, where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq g(x)\}$ and $\varepsilon(S)$ is defined by

$$\varepsilon(S) = \begin{cases} 2, & if\ S\ is\ not\ independent, \\ 1, & if\ S\ is\ independent\ and\ there\ is\ an\ edge\ joining \\ & S\ and\ V(G) \setminus (S \cup T),\ or\ there\ is\ an\ edge\ e = uv \\ & joining\ S\ and\ T\ such\ that\ d_{G-S}(v) = g(v)\ for \\ & v \in T, \\ 0, & otherwise. \end{cases}$$

We now verify Theorem 2. Let $H = G - W$ for any $W \subseteq V(G)$ with $|W| = n$. In order to justify Theorem 2, it suffices to show that H is a fractional (g, f)-covered graph. Suppose that H is not a fractional (g, f)-covered graph. Then by Theorem 3, there exists some subset S of $V(H)$ such that

$$\delta_H(S, T) = f(S) + d_{H-S}(T) - g(T) \leq \varepsilon(S) - 1, \tag{2.1}$$

where $T = \{x : x \in V(H) \setminus S, d_{H-S}(x) \leq g(x)\}$.

If $T = \emptyset$, then using (2.1) and $\varepsilon(S) \leq |S|$ we derive $\varepsilon(S) - 1 \geq \delta_H(S, T) = f(S) \geq (a + m)|S| \geq |S| \geq \varepsilon(S)$, a contradiction. Therefore, we admit $T \neq \emptyset$. Next, we define

$$d_1 = \min\{d_{H-S}(x) : x \in T\}$$

and select $x_1 \in T$ with $d_{H-S}(x_1) = d_1$. Note that $d_1 \leq d_{H-S}(x) \leq g(x) \leq b - m$ holds for any $x \in T$. We shall discuss two cases.

Case 1. $T = N_{H[T]}[x_1]$.

It follows from $0 \leq d_1 \leq b - m$, $|S| + d_1 = |S| + d_{H-S}(x_1) \geq d_H(x_1) = d_{G-W}(x_1) \geq d_G(x_1) - |W| \geq \delta(G) - n \geq \frac{(b-m)(b+1)+2}{a+m}$, $|T| = |N_{H[T]}[x_1]| \leq d_{H-S}(x_1) + 1 = d_1 + 1 \leq b - m + 1$ and $\varepsilon(S) \leq 2$ that

$$\delta_H(S, T) \;=\; f(S) + d_{H-S}(T) - g(T)$$

$$
\begin{aligned}
&\geq (a+m)|S| + d_{H-S}(T) - (b-m)|T| \\
&= (a+m)|S| + d_1|T| - (b-m)|T| \\
&\geq (a+m)\Big(\frac{(b-m)(b+1)+2}{a+m} - d_1\Big) - (b-m-d_1)(b-m+1) \\
&= (b-m-d_1)m + 2 + (b-a-m+1)d_1 \\
&\geq 2 \geq \varepsilon(S),
\end{aligned}
$$

which contradicts (2.1).

Case 2. $T \neq N_{H[T]}[x_1]$.

Obviously, $T \setminus N_{H[T]}[x_1] \neq \emptyset$. We may define

$$
d_2 = \min\{d_{H-S}(x) : x \in T \setminus N_{H[T]}[x_1]\}
$$

and select $x_2 \in T \setminus N_{H[T]}[x_1]$ with $d_{H-S}(x_2) = d_2$. It is clear that $0 \leq d_1 \leq d_2 \leq b-m$ holds.

Note that $x_1x_2 \notin E(H)$. Thus, we easily see that $x_1x_2 \notin E(G)$. According to the hypothesis of Theorem 2 and $H = G - W$, the following inequalities hold:

$$
\begin{aligned}
\frac{(b-m)p + (a+m)n + 2}{a+b} &\leq \max\{d_G(x_1), d_G(x_2)\} \\
&= \max\{d_{H+W}(x_1), d_{H+W}(x_2)\} \\
&\leq \max\{d_H(x_1) + n, d_H(x_2) + n\} \\
&= \max\{d_H(x_1), d_H(x_2)\} + n \\
&\leq \max\{d_{H-S}(x_1) + |S|, d_{H-S}(x_2) + |S|\} + n \\
&= \max\{d_{H-S}(x_1), d_{H-S}(x_2)\} + |S| + n \\
&= \max\{d_1, d_2\} + |S| + n \\
&= d_2 + |S| + n,
\end{aligned}
$$

namely,

$$
|S| \geq \frac{(b-m)p - (b-m)n + 2}{a+b} - d_2. \tag{2.2}
$$

Note that $p-n-|S|-|T| \geq 0$ and $b-m-d_2 \geq 0$. Thus, we derive $(p-n-|S|-|T|)(b-m-d_2) \geq 0$. Combining this inequality with (2.1) and $\varepsilon(S) \leq 2$, we obtain

$$
\begin{aligned}
&(p-n-|S|-|T|)(b-m-d_2) \geq 0 \geq \varepsilon(S) - 2 \geq \delta_H(S,T) - 1 \\
= \;& f(S) + d_{H-S}(T) - g(T) - 1 \\
\geq \;& (a+m)|S| + d_1|N_{H[T]}[x_1]| + d_2(|T| - |N_{H[T]}[x_1]|) - (b-m)|T| - 1 \\
= \;& (a+m)|S| + (d_1 - d_2)|N_{H[T]}[x_1]| - (b-m-d_2)|T| - 1 \\
\geq \;& (a+m)|S| + (d_1 - d_2)(d_1+1) - (b-m-d_2)|T| - 1,
\end{aligned}
$$

where $|T| \geq |N_{H[T]}[x_1]| + 1$, $d_1 - d_2 \leq 0$ and $|N_{H[T]}[x_1]| \leq d_1 + 1$. Then from the above inequality we get

$$
-1 \leq (p-n)(b-m-d_2) - (a+b-d_2)|S| - (d_1-d_2)(d_1+1). \tag{2.3}
$$

It follows from (2.2), (2.3), $0 \leq d_1 \leq d_2 \leq b-m$ and $p \geq \frac{(a+b)(a+b+n+1)-(b-m)n+2}{a+m}$ that

$$
-1 \leq (p-n)(b-m-d_2) - (a+b-d_2)|S| - (d_1-d_2)(d_1+1)
$$

$$
\begin{aligned}
&\leq (p-n)(b-m-d_2)-(a+b-d_2)\Big(\frac{(b-m)p-(b-m)n+2}{a+b}-d_2\Big)\\
&\quad -(d_1-d_2)(d_1+1)\\
&= -\frac{(a+m)p+(b-m)n-2}{a+b}d_2+(a+b+n+1)d_2-d_1(d_1+1)\\
&\quad +d_2(d_1-d_2)-2\\
&\leq -\frac{(a+m)p+(b-m)n-2}{a+b}d_2+(a+b+n+1)d_2-2\\
&\leq -\frac{(a+b)(a+b+n+1)-(b-m)n+2+(b-m)n-2}{a+b}d_2\\
&\quad +(a+b+n+1)d_2-2\\
&= -2,
\end{aligned}
$$

which is a contradiction. Theorem 2 is proved. □

3. Remark

Let us explain that $\max\{d_G(u), d_G(v)\} \geq \frac{(b-m)p+(a+m)n+2}{a+b}$ in Theorem 2 is best possible, namely, it can not be replaced by $\max\{d_G(u), d_G(v)\} \geq \frac{(b-m)p+(a+m)n+2}{a+b} - 1$. Let $b = a + m$, $g(x) \equiv b - m$ and $f(x) \equiv a + m$. We construct a graph $G = K_{(b-m)t+n} \vee ((a + m)tK_1)$ with order p, where $\vee$ means "join". Then $p = (a + b)t + n$ and

$$
\begin{aligned}
\frac{(b-m)p+(a+m)n+2}{a+b}-1 &\leq \max\{d_G(u), d_G(v)\}\\
&= (b-m)t+n\\
&= \frac{(b-m)p+(a+m)n}{a+b}\\
&< \frac{(b-m)p+(a+m)n+2}{a+b}
\end{aligned}
$$

for every pair of nonadjacent vertices u and v of G. Let $W = V(K_n) \subseteq V(K_{(b-m)t+n})$ and $H = G - W = K_{(b-m)t} \vee ((a + m)tK_1)$. Select $S = V(K_{(b-m)t})$ and $T = V((a + m)tK_1)$, and $\varepsilon(S) = 2$. Thus, we derive

$$
\begin{aligned}
\delta_H(S,T) &= f(S)+d_{H-S}(T)-g(T)\\
&= (a+m)|S|-(b-m)|T|\\
&= (a+m)(b-m)t-(b-m)(a+m)t\\
&= 0 < 2 = \varepsilon(S).
\end{aligned}
$$

In light of Theorem 3, H is not a fractional (g, f)-covered graph, and so G is not a fractional (g, f)-critical covered graph.

4. Conclusions

In this paper, we investigate the relationship between degree conditions and the existence of fractional (g, f, n)-critical covered graphs. A sufficient condition for a graph being a fractional

(g, f, n)-critical covered graph is derived. Furthermore, the sharpness of the main result in this paper is illustrated by constructing a special graph class. In addition, some other graph parameter conditions for graphs being fractional (g, f, n)-critical covered graphs can be studied further.

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Conflict of interest

The author declares no conflict of interest in this paper.

References

1. J. Correa, M. Matamala, *Some results about factors of graphs*, J. Graph Theory, **57** (2008), 265–274.
2. J. Li, *A new degree condition for graphs to have* $[a, b]$*-factor*, Discrete Mathematics, **290** (2005), 99–103.
3. S. Zhou, Z. Sun, Z. Xu, *A result on r-orthogonal factorizations in digraphs*, Eur. J. Combinatorics, **65** (2017), 15–23.
4. S. Akbari, M. Kano, $\{k, r-k\}$*-factors of r-regular graphs*, Graphs and Combinatorics, **30** (2014), 821–826.
5. Y. Li, M. Cai, *A degree condition for a graph to have* $[a, b]$*-factors*, J. Graph Theory, **27** (1998), 1–6.
6. S. Zhou, *Some results about component factors in graphs*, RAIRO-Operations Research, **53** (2019), 723–730.
7. S. Zhou, Z. Sun, *Binding number conditions for* $P_{\geq 2}$*-factor and* $P_{\geq 3}$*-factor uniform graphs*, Discrete Mathematics, **343** (2020), Article 111715, DOI: 10.1016/j.disc.2019.111715.
8. S. Zhou, Z. Sun, H. Liu, *Sun toughness and* $P_{\geq 3}$*-factors in graphs*, Contributions to Discrete Mathematics, **14** (2019), 167–174.
9. S. Zhou, Z. Sun, *Some existence theorems on path factors with given properties in graphs*, Acta Mathematica Sinica, English Series, DOI: 10.1007/s10114-020-9224-5.
10. S. Zhou, T. Zhang, Z. Xu, *Subgraphs with orthogonal factorizations in graphs*, Discrete Applied Mathematics, DOI: 10.1016/j.dam.2019.12.011.
11. S. Zhou, T. Zhang, L. Xu, *A sufficient condition for graphs to have ID-Hamiltonian* $[a, b]$*-factors*, Utilitas Mathematica, **111** (2019), 261–269.
12. Y. Egawa, M. Kano, *Sufficient conditions for graphs to have* (g, f)*-factors*, Discrete Mathematics, **151** (1996), 87–90.

13. K. Ota, T. Tokuda, *A degree condition for the existence of regular factors in $K_{1,n}$-free graphs*, J. Graph Theory, **22** (1996), 59–64.
14. G. Liu, L. Zhang, *Toughness and the existence of fractional k-factors of graphs*, Discrete Mathematics, **308** (2008), 1741–1748.
15. J. Jiang, *A sufficient condition for all fractional $[a,b]$-factors in graphs*, Proceedings of the Romanian Academy, Series A: Mathematics, Physics, Technical Sciences, Information Science, **19** (2018), 315–319.
16. J. Jiang, *Independence number and fractional (g,f)-factors with inclusion and exclusion properties*, Utilitas Mathematica, **111** (2019), 27–33.
17. S. Zhou, Z. Sun, *Neighborhood conditions for fractional ID-k-factor-critical graphs*, Acta Mathematicae Applicatae Sinica, English Series, **34** (2018), 636–644.
18. S. Zhou, Z. Sun, H. Ye, *A toughness condition for fractional (k,m)-deleted graphs*, Information Processing Letters, **113** (2013), 255–259.
19. S. Zhou, L. Xu, Z. Xu, *Remarks on fractional ID-k-factor-critical graphs*, Acta Mathematicae Applicatae Sinica, English Series, **35** (2019), 458–464.
20. S. Zhou, T. Zhang, *Some existence theorems on all fractional (g,f)-factors with prescribed properties*, Acta Mathematicae Applicatae Sinica, English Series, **34** (2018), 344–350.
21. Y. Yuan, R. Hao, *A degree condition for fractional $[a,b]$-covered graphs*, Information Processing Letters, **143** (2019), 20–23.
22. S. Zhou, Y. Xu, Z. Sun, *Degree conditions for fractional (a,b,k)-critical covered graphs*, Information Processing Letters, **152** (2019), Article 105838, DOI: 10.1016/j.ipl.2019.105838.
23. Z. Li, G. Yan, X. Zhang, *On fractional (g,f)-covered graphs*, OR Transactions (China), **6** (2002), 65–68.

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