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Research article

Partial sums of generalized q-Mittag-Leffler functions

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Abstract: In the present investigation, our main aim is to give lower bounds for the ratio of some normalized q-Mittag-Leffler function and their sequences of partial sums. We consider various corollaries and consequences of our main results.

Keywords: univalent functions; analytic functions; partial sums; q-derivative (or q-difference) operator; normalized q-Mittag-Leffler function
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1. Introduction

Let $\mathcal{A}$ denote the class of all functions f which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$$

and normalized by the following condition:

$$f(0) = 0 = f'(0) - 1,$$

that is, a function $f \in \mathcal{A}$ has the following Taylor-Maclaurin series representation:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{U}). \tag{1.1}$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all univalent functions in $\mathbb{U}$.

We denote the class of starlike functions by $\mathcal{S}^*$, which is the usual subclass of the normalized univalent function class $\mathcal{S}$. That is, $\mathcal{S}^*$ consists of functions $f \in \mathcal{A}$ that satisfy the following inequality:

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (z \in \mathbb{U}).$$

We now recall some basic definitions and concept details of the q-calculus, which are used in this paper. We suppose throughout the paper that $0 < q < 1$ and that

$$\mathbb{N} = \{1, 2, 3, ...\} = \mathbb{N}_0 \setminus \{0\} \qquad (\mathbb{N}_0 = \{0, 1, 2, ...\}).$$

Definition 1.1. Let $q \in (0, 1)$ and define the q-number $[\lambda]_q$ by

$$[\lambda]_q = \begin{cases} \dfrac{1 - q^\lambda}{1 - q} & (\lambda \in \mathbb{C}) \\ \sum\limits_{k=0}^{n-1} q^k = 1 + q + q^2 + ... + q^{n-1} & (\lambda = n \in \mathbb{N}). \end{cases}$$

Definition 1.2. Let $q \in (0, 1)$ and define the q-factorial $[n]_q!$ by

$$[n]_q! = \begin{cases} 1 & n = 0 \\ \prod\limits_{k=1}^{n} [k]_q & n \in \mathbb{N}. \end{cases}$$

Definition 1.3. Let $q \in (0, 1)$ and define q-generalized Pochhammer symbol by

$$\left([t]_q\right)_n = \begin{cases} 1 & (n = 0) \\ \prod\limits_{k=0}^{n} [t + k]_q & (n \in \mathbb{N}). \end{cases}$$

We note that

$$\left([t]_q\right)_n = [t]_q \left([t + 1]_q\right)_{n-1} \qquad (n \in \mathbb{N}) \tag{1.2}$$

and

$$\left([t]_q\right)_n \geqq \left([t]_q\right)^n \qquad (n \in \mathbb{N}). \tag{1.3}$$

Definition 1.4. For $t > 0$, let the q-gamma function be defined by

$$\Gamma_q(t + 1) = [t]_q \Gamma_q(t) \qquad \text{and} \qquad \Gamma_q(1) = 1.$$

Definition 1.5. (*see* [9] and [10]; see also [1,20] and [27]) The q-derivative (or the q-difference) operator D_q for a function $f \in \mathcal{A}$ in given subset of $\mathbb{C}$ is defined by

$$D_q f(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z} & (z \neq 0) \\ \\ f'(0) & (z = 0), \end{cases} \tag{1.4}$$

provided that $f'(0)$ exists.

We deduce from Definition 1.5 that

$$\lim_{q\to 1^-} \left(D_q f\right)(z) = \lim_{q\to 1-} \left(\frac{f(z) - f(qz)}{(1-q)z}\right) = f'(z)$$

for a differentiable function f in a given subset of $\mathbb{C}$. It can be easily seen from (1.1) and (1.4) that

$$\left(D_q f\right)(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}. \tag{1.5}$$

In geometric function theory, the operator D_q (see Definition 1.5) provides an important tool that has been used in order to investigate various subclasses of the class $\mathcal{S}$ of normalized univalent functions. Historically speaking, Ismail et al. (see [8]) were the first who introduced a q-analogue of the class $\mathcal{S}^*$ of normalized starlike functions in $\mathbb{U}$ (see Definition 1.6 below). However, an important usage of the q-calculus in the context of geometric function theory was actually provided and the basic (or q-) hypergeometric functions were first used in geometric function theory in a book chapter by Srivastava (see, for details, [22, pp. 347 *et seq.*] (see also some more recent works [13, 24].

Definition 1.6. (see [8] and [27]) A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{S}_q^*$ if

$$f(0) = 0 = f'(0) - 1 \tag{1.6}$$

and

$$\left|\frac{z}{f(z)}\left(D_q f\right)(z) - \frac{1}{1-q}\right| \leqq \frac{1}{1-q} \qquad (z \in \mathbb{U}). \tag{1.7}$$

It is readily observed that, as $q \to 1-$, the closed disk

$$\left|w - \frac{1}{1-q}\right| \leqq \frac{1}{1-q}$$

becomes the right-half complex plane and the class $\mathcal{S}_q^*$ reduces to the above-mentioned well-known class $\mathcal{S}^*$ of normalized starlike functions in $\mathbb{U}$.

We note that the notation $\mathcal{S}_q^*$ was first used by Sahoo and Sharma (see [19]).

We now recall the familiar Mittag-Leffler function $E_\alpha(z)$ (see [14]) and its two-parameter extension $E_{\alpha,\beta}(z)$ having similar properties to those of the Mittag-Leffler function $E_\alpha(z)$ (see [28] and [29]), which are defined (as usual) by means of the following series:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \qquad (z \in \mathbb{C};\ \alpha > 0) \tag{1.8}$$

and

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)} \qquad (z \in \mathbb{C};\ \alpha > 0;\ \beta > 0)\,, \tag{1.9}$$

respectively. For a detailed account of the properties, generalizations and applications of the functions in (1.8) and (1.9), one may refer to [6, 7, 17, 25].

The above-defined Mittag-Leffler functions $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$ can be normalized as follows:

$$\mathbf{E}_{\alpha,\beta}(z) = z\Gamma(\beta)\, E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha n+\beta)} z^{n+1} \qquad (z \in \mathbb{U};\ \alpha > 0;\ \beta > 0)\,.$$

We note that

$$\begin{cases} \left(\mathbf{E}_{\alpha,\beta}\right)_0(z) = z \\ \left(\mathbf{E}_{\alpha,\beta}\right)_j(z) = z + \sum\limits_{n=1}^{j} \omega_n z^{n+1} \quad (j \in \mathbb{N}), \end{cases} \tag{1.10}$$

where

$$\omega_n = \frac{\Gamma(\beta)}{\Gamma(\alpha n+\beta)} \qquad (\alpha > 0;\ \beta > 0\ n \in \mathbb{N}).$$

Geometric properties including starlikeness, convexity and close-to-convexity for the Mittag-Leffler function $E_{\alpha,\beta}(z)$ were investigated by Bansal and Prajapat in [3] and, more recently, by Srivastava and Bansal (see [24]). In fact, the generalized Mittag-Leffler function $E_{\alpha,\beta}(z)$ and its extensions and generalizations continue to be used in many different contexts in geometric function theory (see, for details, [23]).

The q-Mittag-Leffler function $\mathfrak{M}_{\alpha,\beta}(z;q)$ is normalized as follows (see, for example, [21]):

$$\mathfrak{M}_{\alpha,\beta}(z;q) = z\Gamma_q(\beta)\, E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{\Gamma_q(\beta)}{\Gamma_q(\alpha n+\beta)} z^{n+1}, \tag{1.11}$$
$$(z \in \mathbb{C};\ \alpha > 0;\ \beta \in \mathbb{C} \setminus \{0, -1, -2, ...\})\,.$$

Some special cases of the normalized q-Mittag-Leffler function $\mathfrak{M}_{\alpha,\beta}(z;q)$ are listed below:

$$\begin{cases} \mathfrak{M}_{0,\beta}(z;q) = \frac{z}{1-z} \\ \mathfrak{M}_{1,1}(z;q) = ze_q^z \\ \mathfrak{M}_{1,2}(z;q) = e_q^z - 1 \\ \mathfrak{M}_{1,3}(z;q) = \dfrac{\left(e_q^z - z - 1\right)(1+q)}{z} \\ \mathfrak{M}_{1,4}(z;q) = \dfrac{(1+q)\left(1+q+q^2\right)}{z^2}\left(e_q^z - z - 1 - \frac{z^2}{1+q}\right), \end{cases} \tag{1.12}$$

where e_q^z is one of the q-analogues of the exponential function e^z, which is given by (see [25, p. 488, Eq. 6.3 (7)])

$$e_q^z = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_q(n+1)}. \tag{1.13}$$

Recently, several results were given such as those related to partial sums of special functions, such as the Struve function [30], meromorphic functions (see [11] and [2]), the Bessel function [15], the Lommel function [4] and the Wright functions [5]. Several other works dealing with partial sums of various subclasses of the analytic function class $\mathcal{A}$, the interested reader may refer (for example) to [12, 16] and [26].

Motivated by the above-mentioned results, in this paper we investigate the ratio of the normalized q-Mittag-Leffler function $\mathfrak{M}_{\alpha,\beta}(z;q)$ defined by (1.11) to its sequence of partial sums:

$$\begin{cases} \left(\mathfrak{M}_{\alpha,\beta}\right)_0(z;q) = z \\ \left(\mathfrak{M}_{\alpha,\beta}\right)_j(z;q) = z + \sum\limits_{n=1}^{j} K_n z^{n+1} \quad (j \in \mathbb{N}), \end{cases} \tag{1.14}$$

where

$$K_n = \frac{\Gamma_q(\beta)}{\Gamma_q(\alpha n + \beta)} \qquad (\alpha > 0,\ \beta > 0\ n \in \mathbb{N}).$$

We obtain the lower bounds on such ratios as those given below:

$$\Re\left\{\frac{\mathfrak{M}_{\alpha,\beta}(z;q)}{\left(\mathfrak{M}_{\alpha,\beta}\right)_j(z;q)}\right\}, \quad \Re\left\{\frac{\left(\mathfrak{M}_{\alpha,\beta}\right)_j(z;q)}{\mathfrak{M}_{\alpha,\beta}(z;q)}\right\}$$

$$\Re\left\{\frac{D_q\mathfrak{M}_{\alpha,\beta}(z;q)}{D_q\left(\mathfrak{M}_{\alpha,\beta}\right)_j(z;q)}\right\}, \quad \Re\left\{\frac{D_q\left(\mathfrak{M}_{\alpha,\beta}\right)_j(z;q)}{D_q\mathfrak{M}_{\alpha,\beta}(z;q)}\right\}.$$

2. Main results

The following lemma will be required in order to derive our main results.

Lemma 2.1. *Let $q \in (0,1)$, $\alpha \geqq 1$ and $\beta \geqq 1$. Then the function $\mathfrak{M}_{\alpha,\beta}(z;q)$ satisfies the following inequalities*:

$$\left|\mathfrak{M}_{\alpha,\beta}(z;q)\right| \leqq \frac{1 + \left(q^\beta + q - 3\right)q^{\beta+1}}{(1-q^\beta)^2\, q} \tag{2.1}$$

and

$$\left|D_q\mathfrak{M}_{\alpha,\beta}(z;q)\right| \leqq \frac{6 + q^{2\beta} + 3q^{\beta+1} - 5q^\beta + 2q^2 - 7q}{(1-q^\beta)^2}. \tag{2.2}$$

Proof. It is well-known that

$$\Gamma_q(\alpha+\beta) \leqq \Gamma_q(\alpha n + \beta).$$

Therefore, we have

$$\frac{\Gamma_q(\beta)}{\Gamma_q(\alpha n+\beta)} \leqq \frac{\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)} = \left(\frac{1}{[\beta]_q}\right)_n. \tag{2.3}$$

By making use of (1.3), (2.3) and well-known triangle inequality for ($z \in \mathbb{U}$), we find that

$$\begin{aligned}
\left|\mathfrak{M}_{\alpha,\beta}(z;q)\right| &= \left|z+\sum_{n=1}^{\infty}\frac{\Gamma_q(\beta)}{\Gamma_q(\alpha n+\beta)}z^{n+1}\right| < 1+\sum_{n=1}^{\infty}\frac{\Gamma_q(\beta)}{\Gamma_q(\alpha n+\beta)} \\
&< 1+\sum_{n=1}^{\infty}\left(\frac{1}{[\beta]_q}\right)_n \\
&= 1+\frac{1}{[\beta]_q}\sum_{n=1}^{\infty}\left(\frac{1}{[\beta+1]_q}\right)_{n-1} < 1+\frac{1}{[\beta]_q}\sum_{n=1}^{\infty}\left(\frac{1}{[\beta+1]_q}\right)^{n-1} \\
&= 1+\frac{1}{[\beta]_q}\sum_{n=0}^{\infty}\left(\frac{1}{[\beta+1]_q}\right)^{n} = \frac{1+\left(q^{\beta}+q-3\right)q^{\beta+1}}{(1-q^{\beta})^2\,q}.
\end{aligned}$$

Hence, the inequality (2.1) is proved. Similarly, we can prove the inequality (2.2). □

Let $w(z)$ denote an analytic function in $\mathbb{U}$. In the proof of our main results, the following well-known result will be used frequently:

$$\Re\left\{\frac{1+w(z)}{1-w(z)}\right\} > 0$$

if and only if

$$|w(z)| < 1 \quad (z \in \mathbb{U}).$$

Theorem 2.2. *Let* $q \in (0,1)$, $\alpha \leqq 1$ *and* $\beta \leqq \frac{1+\sqrt{5}}{2}$. *Then*

$$\Re\left\{\frac{\mathfrak{M}_{\alpha,\beta}(z;q)}{\left(\mathfrak{M}_{\alpha,\beta}\right)_j(z;q)}\right\} \geqq \frac{q^{\beta+1}\left(q^{\beta}-q-1\right)+2q-1}{(1-q^{\beta})^2\,q} \quad (z \in \mathbb{U}) \tag{2.4}$$

and

$$\Re\left\{\frac{\left(\mathfrak{M}_{\alpha,\beta}\right)_j(z;q)}{\mathfrak{M}_{\alpha,\beta}(z;q)}\right\} \geqq \frac{\left(1-q^{\beta}\right)^2 q}{1+q^{\beta+1}\left(q^{\beta}+q-3\right)} \quad (z \in \mathbb{U}). \tag{2.5}$$

Proof. From the inequality (2.1), we obtain

$$1+\sum_{n=1}^{\infty}K_n \leqq \frac{1+\left(q^{\beta}+q-3\right)q^{\beta+1}}{(1-q^{\beta})^2\,q}, \quad \text{where } K_n = \frac{\Gamma_q(\beta)}{\Gamma_q(\alpha n+\beta)} \quad (n \in \mathbb{N}),$$

which is equivalent to

$$\frac{\left(1-q^{\beta}\right)^2 q}{1-q-q^{\beta+1}+q^{\beta+2}}\sum_{n=1}^{\infty}K_n \leqq 1.$$

In order to prove the inequality (2.4), we set

$$\begin{aligned}
&\frac{\left(1-q^{\beta}\right)^{2} q}{1-q-q^{\beta+1}+q^{\beta+2}}\left[\frac{\mathfrak{M}_{\alpha,\beta}(z;q)}{\left(\mathfrak{M}_{\alpha,\beta}\right)_{j}(z;q)}-\frac{q^{\beta+1}\left(q^{\beta}-q-1\right)+2q-1}{\left(1-q^{\beta}\right)^{2} q}\right] \\
&=\frac{1+\sum_{n=1}^{j} K_{n} z^{n}+\frac{\left(1-q^{\beta}\right)^{2} q}{1-q-q^{\beta+1}+q^{\beta+2}} \sum_{n=j+1}^{\infty} K_{n} z^{n}}{1+\sum_{n=1}^{j} K_{n} z^{n}} \\
&=\frac{1+w(z)}{1-w(z)}, \qquad (2.6)
\end{aligned}$$

where

$$w(z)=\frac{\frac{\left(1-q^{\beta}\right)^{2} q}{1-q-q^{\beta+1}+q^{\beta+2}} \sum_{n=j+1}^{\infty} K_{n} z^{n}}{2+2 \sum_{n=1}^{j} K_{n} z^{n}+\frac{\left(1-q^{\beta}\right)^{2} q}{1-q-q^{\beta+1}+q^{\beta+2}} \sum_{n=j+1}^{\infty} K_{n} z^{n}}$$

and

$$|w(z)|<\frac{\frac{\left(1-q^{\beta}\right)^{2} q}{1-q-q^{\beta+1}+q^{\beta+2}} \sum_{n=j+1}^{\infty} K_{n}}{2-2 \sum_{n=1}^{j} K_{n}-\frac{\left(1-q^{\beta}\right)^{2} q}{1-q-q^{\beta+1}+q^{\beta+2}} \sum_{n=j+1}^{\infty} K_{n}}.$$

The inequality $|w(z)|<1$ holds true if and only if

$$\frac{\left(1-q^{\beta}\right)^{2} q}{1-q-q^{\beta+1}+q^{\beta+2}} \sum_{n=j+1}^{\infty} K_{n} \leqq 1-\sum_{n=1}^{j} K_{n}$$

or, equivalently,

$$\sum_{n=1}^{j} K_{n}+\frac{\left(1-q^{\beta}\right)^{2} q}{1-q-q^{\beta+1}+q^{\beta+2}} \sum_{n=j+1}^{\infty} K_{n} \leqq 1. \qquad (2.7)$$

To prove (2.7), it suffices to show that its left-hand side is bounded above by

$$\frac{\left(1-q^{\beta}\right)^{2} q}{1-q-q^{\beta+1}+q^{\beta+2}} \sum_{n=1}^{\infty} K_{n},$$

which is equivalent to

$$\frac{q^{\beta+1}\left(q^{\beta}-q-1\right)+2q-1}{1-q-q^{\beta+1}+q^{\beta+2}} \sum_{n=1}^{j} K_{n} \geqq 0. \qquad (2.8)$$

We see that the inequality (2.8) holds true for $\beta \geqq \frac{1+\sqrt{5}}{2}$.

We next use the same method to prove the inequality (2.5). Consider the function $w(z)$ given by

$$\frac{1+q^{\beta+1}\left(q^{\beta}+q-3\right)}{1-q-q^{\beta+1}+q^{\beta+2}}\left[\frac{\left(\mathfrak{M}_{\alpha,\beta}\right)_j(z;q)}{\mathfrak{M}_{\alpha,\beta}(z;q)}-\frac{\left(1-q^{\beta}\right)^2 q}{1+q^{\beta+1}\left(q^{\beta}+q-3\right)}\right]$$

$$=\frac{1+\sum_{n=1}^{j}K_n z^n-\frac{\left(1-q^{\beta}\right)^2 q}{1-q-q^{\beta+1}+q^{\beta+2}}\sum_{n=j+1}^{\infty}K_n z^n}{1+\sum_{n=1}^{\infty}K_n z^n}$$

$$=\frac{1+w(z)}{1-w(z)}, \tag{2.9}$$

where

$$w(z)=\frac{-\frac{1+q^{\beta+1}\left(q^{\beta}+q-3\right)}{1-q-q^{\beta+1}+q^{\beta+2}}\sum_{n=j+1}^{\infty}K_n z^n}{2+2\sum_{n=1}^{j}K_n z^n-\frac{q^{2\beta+1}-q^{\beta+2}-q^{\beta+1}+2q-1}{1-q-q^{\beta+1}+q^{\beta+2}}\sum_{n=j+1}^{\infty}K_n z^n}$$

and

$$|w(z)|<\frac{\frac{1+q^{\beta+1}\left(q^{\beta}+q-3\right)}{1-q-q^{\beta+1}+q^{\beta+2}}\sum_{n=j+1}^{\infty}K_n}{2-2\sum_{n=1}^{j}K_n-\frac{q^{2\beta+1}-q^{\beta+2}-q^{\beta+1}+2q-1}{1-q-q^{\beta+1}+q^{\beta+2}}\sum_{n=j+1}^{\infty}K_n}.$$

Therefore, we get $|w(z)|<1$ if and only if

$$\frac{\left(1-q^{\beta}\right)^2 q}{1-q-q^{\beta+1}+q^{\beta+2}}\sum_{n=j+1}^{\infty}K_n+\sum_{n=1}^{j}K_n \leqq 1.$$

As the left-hand side of the last inequality is bounded above by

$$\frac{\left(1-q^{\beta}\right)^2 q}{1-q-q^{\beta+1}+q^{\beta+2}}\sum_{n=1}^{\infty}K_n,$$

we are led immediately to the assertion (2.5) of Theorem 2.2. Now we have completed the proof of Theorem 2.2. □

In its special case, if we let $q\to 1-$, Theorem 2.2 yields the following corollary.

Corollary 2.3. (see [18]) *Let* $\alpha \geqq 1$ *and* $\beta \geqq \frac{1+\sqrt{5}}{2}$. *Then*

$$\Re\left\{\frac{E_{\alpha,\beta}(z)}{\left(E_{\alpha,\beta}\right)_j(z)}\right\}\geqq\frac{\beta^2-\beta-1}{\beta^2}\qquad(z\in\mathbb{U})$$

and

$$\Re\left\{\frac{\left(E_{\alpha,\beta}\right)_j(z)}{E_{\alpha,\beta}(z)}\right\}\geqq\frac{\beta^2}{\beta^2+\beta+1}\qquad(z\in\mathbb{U}).$$

We next turn to the ratios involving derivatives.

Theorem 2.4. *Let* $q \in (0,1)$, $\alpha \geqq 1$ *and* $\beta \geqq \frac{3+\sqrt{17}}{2}$. *Then*

$$\Re\left\{\frac{D_q\mathfrak{M}_{\alpha,\beta}(z;q)}{D_q\left(\mathfrak{M}_{\alpha,\beta}\right)_j(z;q)}\right\} \geqq \frac{q^{2\beta}-3q^{\beta+1}+q^{\beta}-2q^2+7q-4}{(1-q^{\beta})^2} \quad (z\in\mathbb{U}) \tag{2.10}$$

and

$$\Re\left\{\frac{D_q\left(\mathfrak{M}_{\alpha,\beta}\right)_j(z;q)}{D_q\mathfrak{M}_{\alpha,\beta}(z;q)}\right\} \geqq \frac{\left(1-q^{\beta}\right)^2 q}{6+q^{2\beta}+3q^{\beta+1}-5q^{\beta}+2q^2-7q} \quad (z\in\mathbb{U}). \tag{2.11}$$

Proof. From the inequality (2.2), we have

$$1+\sum_{n=1}^{\infty}[n+1]_q K_n \leqq \frac{6+q^{2\beta}+3q^{\beta+1}-5q^{\beta}+2q^2-7q}{(1-q^{\beta})^2}, \tag{2.12}$$

where

$$K_n = \frac{\Gamma_q(\beta)}{\Gamma_q(\alpha n+\beta)} \qquad (n\in\mathbb{N}).$$

Equivalently, we can rewrite the condition in (2.12) as follows:

$$\frac{\left(1-q^{\beta}\right)^2}{5+3q^{\beta+1}-3q^{\beta}+2q^2-7q}\sum_{n=1}^{\infty}[n+1]_q K_n \leqq 1.$$

In order to prove the inequality (2.10), we consider the function $w(z)$ defined by

$$\begin{aligned}
&\frac{\left(1-q^{\beta}\right)^2}{5+3q^{\beta+1}-3q^{\beta}+2q^2-7q}\left[\frac{D_q\mathfrak{M}_{\alpha,\beta}(z;q)}{D_q\left(\mathfrak{M}_{\alpha,\beta}\right)_j(z;q)}\right.\\
&\qquad\left.-\frac{q^{2\beta}-3q^{\beta+1}+q^{\beta}-2q^2+7q-4}{(1-q^{\beta})^2}\right]\\
&=\frac{1+\sum_{n=1}^{j}[n+1]_q K_n z^n+\frac{\left(1-q^{\beta}\right)^2}{5+3q^{\beta+1}-3q^{\beta}+2q^2-7q}\sum_{n=j+1}^{\infty}[n+1]_q K_n z^n}{1+\sum_{n=1}^{j}[n+1]_q K_n z^n}\\
&=\frac{1+w(z)}{1-w(z)}.
\end{aligned} \tag{2.13}$$

From (2.13), we have

$$w(z)=\frac{\frac{\left(1-q^{\beta}\right)^2}{5+3q^{\beta+1}-3q^{\beta}+2q^2-7q}\sum_{n=j+1}^{\infty}[n+1]_q K_n z^n}{2+2\sum_{n=1}^{j}[n+1]_q K_n z^n+\frac{\left(1-q^{\beta}\right)^2}{5+3q^{\beta+1}-3q^{\beta}+2q^2-7q}\sum_{n=j+1}^{\infty}[n+1]_q K_n z^n}$$

or, equivalently

$$w(z)=\frac{\frac{(1-q^{\beta})^2}{5+3q^{\beta+1}-3q^{\beta}+2q^2-7q}\sum_{n=j+1}^{\infty}[n+1]_q K_n}{2-2\sum_{n=1}^{j}[n+1]_q K_n-\frac{(1-q^{\beta})^2}{5+3q^{\beta+1}-3q^{\beta}+2q^2-7q}\sum_{n=j+1}^{\infty}[n+1]_q K_n}.$$

The inequality $|w(z)|<1$ holds true if and only if

$$\sum_{n=1}^{j}[n+1]_q K_n+\frac{(1-q^{\beta})^2}{5+3q^{\beta+1}-3q^{\beta}+2q^2-7q}\sum_{n=j+1}^{\infty}[n+1]_q K_n\leqq 1.$$

The left-hand side of the above inequality is bounded above by

$$\frac{(1-q^{\beta})^2}{5+3q^{\beta+1}-3q^{\beta}+2q^2-7q}\sum_{n=1}^{\infty}[n+1]_q K_n,$$

which is equivalent to

$$\frac{q^{2\beta}-3q^{\beta+1}+q^{\beta}-2q^2+7q-4}{5+3q^{\beta+1}-3q^{\beta}+2q^2-7q}\sum_{n=1}^{j}[n+1]_q K_n\geqq 0. \tag{2.14}$$

The inequality in (2.14) holds true for $\beta\geqq\frac{3+\sqrt{17}}{2}$.

We next use the same method to prove the inequality (2.5). Consider the function $w(z)$ given by

$$\begin{aligned}&\frac{6+q^{2\beta}+3q^{\beta+1}-5q^{\beta}+2q^2-7q}{5+3q^{\beta+1}-3q^{\beta}+2q^2-7q}\left[\frac{D_q\left(\mathfrak{M}_{\alpha,\beta}\right)_j(z;q)}{D_q\mathfrak{M}_{\alpha,\beta}(z;q)}\right.\\&\left.-\frac{(1-q^{\beta})^2}{6+q^{2\beta}+3q^{\beta+1}-5q^{\beta}+2q^2-7q}\right]\\&=\frac{1+\sum_{n=1}^{j}[n+1]_q K_n z^n-\frac{(1-q^{\beta})^2}{5+3q^{\beta+1}-3q^{\beta}+2q^2-7q}\sum_{n=j+1}^{\infty}[n+1]_q K_n z^n}{1+\sum_{n=1}^{\infty}[n+1]_q K_n z^n}\\&=\frac{1+w(z)}{1-w(z)}.\end{aligned} \tag{2.15}$$

By using Eq. (2.15), we obtain

$$w(z)=\frac{-\frac{6+q^{2\beta}+3q^{\beta+1}-5q^{\beta}+2q^2-7q}{5+3q^{\beta+1}-3q^{\beta}+2q^2-7q}\sum_{n=j+1}^{\infty}[n+1]_q K_n z^n}{2+2\sum_{n=1}^{j}[n+1]_q K_n z^n-\frac{q^{2\beta}-3q^{\beta+1}+q^{\beta}-2q^2+7q-4}{5+3q^{\beta}-3q^{\beta}+2q^2-7q}\sum_{n=j+1}^{\infty}[n+1]_q K_n z^n},$$

which is equivalent to

$$|w(z)| < \frac{\frac{6+q^{2\beta}+3q^{\beta+1}-5q^{\beta}+2q^2-7q}{5+3q^{\beta+1}-3q^{\beta}+2q^2-7q}\sum\limits_{n=j+1}^{\infty}[n+1]_q K_n}{2-2\sum\limits_{n=1}^{j}[n+1]_q K_n - \frac{q^{2\beta}-3q^{\beta+1}+q^{\beta}-2q^2+7q-4}{5+3q^{\beta+1}-3q^{\beta}+2q^2-7q}\sum\limits_{n=j+1}^{\infty}[n+1]_q K_n}.$$

The inequality $|w(z)| < 1$ holds true if and only if

$$\frac{2\left(1-q^{\beta}\right)^2}{5+3q^{\beta+1}-3q^{\beta}+2q^2-7q}\sum_{n=j+1}^{\infty}[n+1]_q K_n \leqq 2-2\sum_{n=1}^{j}[n+1]_q K_n$$

or, equivalently,

$$\sum_{n=1}^{j}[n+1]_q K_n + \frac{\left(1-q^{\beta}\right)^2}{5+3q^{\beta+1}-3q^{\beta}+2q^2-7q}\sum_{n=j+1}^{\infty}[n+1]_q K_n \leqq 1. \tag{2.16}$$

It now suffices to show that the left-hand side of (2.16) is bounded above by

$$\frac{\left(1-q^{\beta}\right)^2}{5+3q^{\beta+1}-3q^{\beta}+2q^2-7q}\sum_{n=j+1}^{\infty}[n+1]_q K_n,$$

which is equivalent to

$$\frac{q^{2\beta}-3q^{\beta+1}+q^{\beta}-2q^2+7q-4}{5+3q^{\beta+1}-3q^{\beta}+2q^2-7q}\sum_{n=j+1}^{\infty}[n+1]_q K_n \geqq 0.$$

This last inequality holds true for $\beta \geqq \frac{3+\sqrt{17}}{2}$. Hence we complete the proof of Theorem 2.4. □

Upon letting $q \to 1-$, Theorem 2.4 yields the following known result.

Corollary 2.5. (see [18]) *Let* $\alpha \geqq 1$ *and* $\beta \geqq \frac{1+\sqrt{5}}{2}$. *Then*

$$\Re\left\{\frac{E'_{\alpha,\beta}(z)}{\left(E_{\alpha,\beta}\right)'_j(z)}\right\} \geqq \frac{\beta^2-3\beta-2}{\beta^2} \quad (z \in \mathbb{U})$$

and

$$\Re\left\{\frac{\left(E_{\alpha,\beta}\right)'_j(z)}{E'_{\alpha,\beta}(z)}\right\} \geqq \frac{\beta^2}{\beta^2+3\beta+2} \quad (z \in \mathbb{U}).$$

Conflicts of interest

The authors declare no conflicts of interest.

References

1. M. K. Aouf, T. M. Seoudy, *Convolution properties of certain classes of analytic functions with complex order defined by q-derivative operator*, RACSAM, **113** (2019), 1279–1288.
2. O. Altintas, Ö. Özkan, H. M. Srivastava, *Neighborhoods of a class of analytic functions with negative coefficients*, Appl. Math. Lett., **13** (2000), 63–67.
3. D. Bansal, J. K. Prajapat, *Certain geometric properties of the Mittag-Leffler functions*, Complex Var. Elliptic, **61** (2016), 338–350.
4. M. Çağlar, E. Deniz, *Partial sums of the normalized Lommel functions*, Math. Inequal. Appl., **18** (2015), 1189–1199.
5. M. Din, M. Raza, N. Yağmur, *Partial sums of normalized Wright functions*, arXiv:1606.02750, 2016.
6. R. Gorenflo, F. Mainardi, S. V. Rogosin, *On the generalized Mittag-Leffler type function*, Integr. Transf. Spec. F., **7** (1998), 215–224.
7. I. S. Gupta, L. Debnath, *Some properties of the Mittag-Leffler functions*, Integr. Transf. Spec. F., **18** (2007), 329–336.
8. M. E. H. Ismail, E. Merkes, D. Styer, *A generalization of starlike functions*, Complex Var. Theor. Appl., **14** (1990), 77–84.
9. F. H. Jackson, *On q-definite integrals*, Quart. J. Pure Appl. Math., **41** (1910), 193–203.
10. F. H. Jackson, *q-Difference equations*, Am. J. Math., **32** (1910), 305–314.
11. J. L. Liu, H. M. Srivastava, *Subclasses of meromorphically multivalent functions associated with a certain linear operator*, Math. Comput. Model., **39** (2004), 35–44.
12. S. Maharana, J. K. Prajapat, H. M. Srivastava, *The radius of convexity of partial sums of convex functions in one direction*, P. Natl. A. Sci. India Sect. A Phys. Sci., **87** (2017), 215–219.
13. S. Mahmood, N. Raza, E. S. A. Abujarad, et al. *Geometric properties of certain classes of analytic functions associated with a q-integral operator*, Symmetry, **11** (2019), 719.
14. G. M. Mittag-Leffler, *Sur la nouvelle fonction* $E_\alpha(x)$, CR. Acad. Sci. Paris, **137** (1903), 554–558.
15. H. Orhan, N. Yağmur, *Partial sums of generalized Bessel functions*, J. Math. Inequal., **8** (2014), 863–877.
16. S. Owa, H. M. Srivastava, N. Saito, *Partial sums of certain classes of analytic functions*, Int. J. Comput. Math., **81** (2004), 1239–1256.
17. T. R. Prabhakar, *A singular integral equation with a generalized Mittag-Leffler function in the kernel*, Yokohama Math. J., **19** (1997), 7–15.
18. D. Răducanu, *On partial sums of normalized Mittag-Leffler functions*, An. U. Ovid. Co. Mat., **25** (2017), 123–133.

19. S. K. Sahoo, N. L. Sharma, *On a generalization of close-to-convex functions*, Ann. Polon. Math., **113** (2015), 93–108.

20. T. M. Seoudy, M. K. Aouf, *Coefficient estimates of new class of q-starlike and q-convex functions of complex order*, J. Math. Inequal., **10** (2016), 135–145.

21. S. K. Sharma, R. Jain, *On some properties of generalized q-Mittag Leffler function*, Math. Aeterna, **4** (2014), 613–619.

22. H. M. Srivastava, *Univalent functions, fractional calculus, and associated generalized hypergeometric functions*, In: *Univalent Functions, Fractional Calculus and Their Applications*, New York: Halsted Press, (1989), 329–354.

23. H. M. Srivastava, *Some Fox-Wright generalized hypergeometric functions and associated families of convolution operators*, Appl. Anal. Discrete Math., **1** (2007), 56–71.

24. H. M. Srivastava, D. Bansal, *Close-to-convexity of a certain family of q-Mittag-Leffer functions*, J. Nonlinear Var. Anal., **1** (2017), 61–69.

25. H. M. Srivastava, J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Amsterdam: Elsevier Science Publishers, 2012.

26. H. M. Srivastava, S. Gaboury, F. Ghanim, *Partial sums of certain classes of meromorphic functions related to the Hurwitz-Lerch zeta function*, Moroccan J. Pure Appl. Anal., **1** (2015), 38–50.

27. H. M. Srivastava, A. O. Mostafa, M. K. Aouf, et al. *Basic and fractional q-calculus and associated Fekete-Szego problem for p-valent q-starlike functions and p-valently q-convex functions of complex order*, Miskolc Math. Notes, **20** (2019), 489–509.

28. A. Wiman, *Über den Fundamentalsatz in der Theorie der Funcktionen $E_\alpha(x)$*, Acta Math., **29** (1905), 191–201.

29. A. Wiman, *Über die Nullstellen der Funktionen $E_\alpha(x)$*, Acta Math., **29** (1905), 217–234.

30. N. Yağmur, H. Orhan, *Partial sums of generalized Struve functions*, Miskolc Math. Notes, **17** (2016), 657–670.

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