Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 4(1): 134–146.
DOI:10.3934/Math.2019.1.134
Received: 25 November 2018
Accepted: 11 February 2019
Published: 14 February 2019

Research article

The dynamics of Zika virus with Caputo fractional derivative

Muhammad Altaf Khan[1,*], **Saif Ullah**[2] **and Muhammad Farhan**[3]

[1] Department of mathematics, City university of Science and Information Technology, Peshawar, KP, Pakistan

[2] Department of mathematics, University of Peshawar, KP, Pakistan

[3] Department of mathematics, Abdul wali khan university, Mardan, KP, Pakistan

* **Correspondence:** Email: altafdir@gmail.com, makhan@cusit.edu.pk; Tel: +923469247983.

Abstract: In the present paper, we investigate a fractional model in Caputo sense to explore the dynamics of the Zika virus. The basic results of the fractional Zika model are presented. The local and global stability analysis of the proposed model is obtained when the basic reproduction reproduction number is less or greater than 1. To show the global stability of the fractional Zika model, we use the Lyapunov function theory in fractional environment. Further, we simulate the fractional Zika model to present the graphical results for different values of fractional order and model parameters.

Keywords: Zika virus model; stability analysis; generalized mean value theorem; Lyapunov function; Caputo derivative; numerical results
Mathematics Subject Classification: 37C75, 34A08

1. Introduction

Zika infection is a kind of vector-borne disease caused and spread by the bite infected Aedes mosquitos. The Zika infection was first discovered in Uganda in 1947. In 2007, the first case of Zika virus was reported occurred in the Island of Yap (Federated States of Micronesia). After that, it spread very quickly in Asia, Africa and USA [1]. The Aides mosquitoes is the main source from which the Zika virus is spread and is also responsible for dengue infection. The transmission of virus of Zika infection to humans occurred by the bites of infected female mosquitoes from the Aedes genus. This infection can also be transmitted having unprotected sexual relations, if one partner is suffering from Zika virus. People who have infected with Zika will have mild symptom due to which they feel mild illness and get severe ailment. Zika infected people main symptoms are skin rashes, headache, mild fever, conjunctivitis, and muscle pains. Usually the symptoms last for 2–7 days but sometimes the infected individuals due to Zika virus de not developed symptoms. This infection can also affect a pregnant women to her developing fetus [2, 3]. If this happened then most probably the newly born babies have abnormal brain and small head development along with muscle weakness which effects nervous system.

Epidemic models are used as powerful tool to predict the dynamics and control of various communicable diseases. These models usually consist of nonlinear differential equations describing the dynamics of the concern disease. A number of transmission models and effective possible controlling strategies have been developed in literature to explore the effective strategies for controlling of Zika infection in different regions around the globe. Kucharsk et al. [4] proposed a mathematical model and provided a detail analysis of French Polynesia Zika outbreak appeared in 2013-14. Kucharsk et al. used the total Zika infected cases between October 2013 till April 2014 which are reported in six main places of French Polynesia for model parameters estimation. Bonyah and Okosun [5] used optimal control theory to derived three different controlling strategies to reduce the spreed of this infection. The impact of bednets, used of insecticides spry and possible treatment was studied in detail in [6]. However, these models are based integer-order classical differential systems. The classical integer-order derivatives have some limitations as they are local in nature and do not posses the memory effects which are appear in most of biological systems. Secondly, classical derivative are unable to provides information about the rate of changes between two points not necessarily same. To overcome such limitations of local derivatives, various concepts on new derivatives with non-integer or fractional order were developed in recent years and can e found in [7, 9, 10]. The classical Caputo fractional operator [7] has been used to model many complex phenomena in different fields. For example in [11], a numerical scheme was proposed for of the diffusive fractional HBV model in Caputo sense. A numerical scheme for Caputo fractional reaction-diffusion equation and its stability analysis can be found in [12]. Also a detail stability analysis and simulations of Caputo sub-diffusion equation has been developed in [13]. The real world application of non-local and non-singular fractional operator [9] can be found in [14]. A comparative analysis Sturm-Liouville fractional problems has been carried out in [15]. Other applications of singular and non-singular fractional order operators in modeling various phenomena can be found in [16–20]. There is no rich literate on the modeling of Zika virus in fractional order. Only few models with fractional order has been presented in literature for Zika infection [21, 22]. Keeping the above discussion in view and applicability of fractional order derivatives, in the preset investigation, a mathematical transmission model is considered in the Caputo sense in order to explore the dynamics of the Zika virus. We simulate the proposed Zika model for different values of relevant parameters and for several values of arbitrary fractional order α.

The structure of the paper is follows is as: groundwork of the fractional derivative is given in Section 2. The basic model formulation is given in Section 3. Sections 4 is devoted to explore the basic properties of the model. Sections 5 and 6 are concern to obtain the stability results of the model equilibria. Graphical analysis are given in Section 7. The whole work is summarized with a brief conclusion in Section 8.

2. Preliminaries

The basic definitions regarding the fractional derivative in Caputo sense are as follows [7, 8]:

Definition 2.1. The Caputo fractional derivative of order $\alpha \in (n-1, n)$ with $n \in \mathbb{N}$ for a function $h \in C^n$

is stated as follow:

$$^{C}D_t^{\alpha}(h(t)) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{h^{(n)}\xi}{(t-\xi)^{\alpha-n+1}} d\xi.$$

Clearly $^{C}D_t^{\alpha}(h(t))$ approaches to $h^{'}(t)$ as $\alpha \to 1$.

Definition 2.2. The corresponding fractional integral having order $\alpha > 0$ of the function $h : R^+ \to R$ is described by the following expression

$$I_t^{\alpha}(h(t)) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} h(\xi) d\xi,$$

where Γ represent the Gamma function.

Definition 2.3. The constant point x^* is said to be an equilibrium point of the following Caputo fractional dynamic system:

$$^{C}D_t^{\alpha} x(t) = h(t, x(t)), \qquad \alpha \in (0, 1), \tag{2.1}$$

if and only if it observed that $h(t, x^*) = 0$.

To present the stability analysis of nonlinear fractional systems in the Caputo sense via Lyapunov method we first recall the following necessary results from [23, 24].

Theorem 2.4. *Suppose x^* be an equilibrium point for the above system (2.1) and $\Omega \in R^n$ be a domain containing x^*. Let $L : [0, \infty) \times \Omega$ into to R be a continuously differentiable function such that*

$$W_1(x) \leq L(t, x(t)) \leq W_2(x),$$

and

$$^{C}D_t^{\alpha} L(t, x(t)) \leq -W_3(x),$$

$\forall\ \alpha \in (0, 1)$ *and $x \in \Omega$. Whereas $W_1(x)$, $W_2(x)$ and $W_3(x)$ are continuously positive definite functions on Ω. Then x^* is uniformly asymptotically stable equilibrium point for the model (2.1).*

Next we recall the following lemma from [24], which we will use in presenting the global stability via Lyapunov function.

Lemma 2.5. *For a continuous and derivable function $z(t) \in R_+$ and $\alpha \in (0, 1)$, then for any time $t \geq t_0$ we have*

$$^{C}D_t^{\alpha}\left\{z(t) - z^* - z^* \ln \frac{z(t)}{z^*}\right\} \leq \left(1 - \frac{z^*}{z(t)}\right) {}^{C}D_t^{\alpha} z(t), \quad z^* \in R_+.$$

3. Mathematical model and discussion

To formulate the model, we divide the human population into two sub-classes, susceptible individuals and infected individuals. The total human population is represented by $x_h(t) = x_1(t) + x_2(t)$, where x_1 represent susceptible and x_2 represent numbers of infected human individuals. Similarly, x_m is the total number of mosquitos which are further divided into susceptible mosquitos x_3, and infected mosquitos x_4, so that $x_m(t) = x_3(t) + x_4(t)$. The compartmental mathematical model is given by the following system of four ordinary differential equations to describe the mechanism of the transmission of Zika virus.

$$\begin{cases} {}^C D_t^\alpha x_1 = \Lambda_h - \beta_1\gamma_1 x_1(t)x_4(t) - d_1 x_1(t), \\ {}^C D_t^\alpha x_2 = \beta_1\gamma_1 x_1(t)x_4(t) - d_1 x_2(t), \\ {}^C D_t^\alpha x_3 = \Lambda_m - \beta_2\gamma_2 x_2(t)x_3(t) - d_2 x_3(t), \\ {}^C D_t^\alpha x_4 = \beta_2\gamma_2 x_2(t)x_3(t) - d_2 x_4(t), \end{cases} \tag{3.1}$$

with the initial conditions

$$x_1(0) = x_{10} \geq 0,\ x_2(0) = x_{20} \geq 0,\ x_3(0) = x_{30} \geq 0,\ x_4(0) = x_{40} \geq 0.$$

In the above proposed model Λ_h and Λ_m respectively represent the recruitment rate of human and mosquito populations. The natural death rate of the human and mosquitos are d_1 and d_2 respectively. The contact rate of suspectable human and infected mosquitos is β_1, while β_2 is the contact rate OF susceptible mosquitos and infected humans. The parameters γ_1 and γ_2 shows the transmission probabilities of humans and mosquitos.

4. Existence and positivity of the solution

In order to present the non-negativity of the system solution, let

$$R_+^4 = \{y \in R^4 \mid y \geq 0\} \ \ and \ \ y(t) = \Big(x_1(t), x_2(t), x_3(t), x_4(t)\Big)^T.$$

To proceeds further, first we recall the generalized mean values theorem [25].

Lemma 4.1. *Let suppose that $h(t) \in C[a,b]$ and ${}^C D_t^\alpha h(t) \in (a,b]$, then*

$$h(t) = h(a) + \frac{1}{\Gamma(\alpha)}({}^C D_t^\alpha h)(\zeta)(t-a)^\alpha,$$

with $a \leq \zeta \leq t$, $\forall\ t \in (a,b]$.

Corollary 4.2. *Suppose that $h(t) \in C[a,b]$ and ${}^C D_t^\alpha h(t) \in (a,b]$, where $\alpha \in (0,1]$. Then if*

$$(i)\ \ {}^C D_t^\alpha h(t) \geq 0, \forall\ \ t \in (a,b),\ \ then\ \ h(t)\ \ is\ \ non-decreasing.$$

$$(ii)\ \ {}^C D_t^\alpha h(t) \leq 0, \forall\ \ t \in (a,b),\ \ then\ \ h(t)\ \ is\ \ non-increasing.$$

We are now able to give the following result.

Theorem 4.3. *A unique solution $y(t)$ of the model (3.1) exists and will remain in R^4_+. Further more, the solution is positive.*

Proof. The exitance of the Caputo fractional Zika model can be shown with the help of theorem 3.1 from [26,27], while the uniqueness of the solution can be easily obtained by making use of the Remark 3.2 in [26] for all positive values of t. In order to explore the solution positivity, it is necessary to show that on each hyperplane bounding the positive orthant, the vector field points to R^4_+. Form the system (3.1), we deduced that

$$ {}^{C}D_t^{\alpha} x_1 \mid_{x_1=0} = \Lambda_h \geq 0, \ \ {}^{C}D_t^{\alpha} x_2 \mid_{x_2=0} = \beta_1\gamma_1 x_1(t)x_4(t) \geq 0, $$

$$ {}^{C}D_t^{\alpha} x_3 \mid_{x_3=0} = \Lambda_m \geq 0, \ \ {}^{C}D_t^{\alpha} x_4 \mid_{x_4=0} = \beta_2\gamma_2 x_2(t)x_3(t) \geq 0. $$

Hence, using the above corollary (4.2), we obtain the desired target i.e. the solution will remain in R^4_+ and hence biologically feasible region is constructed as:

$$ \Phi = \Big\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4_+ : x_1, x_2, x_3, x_4 \geq 0 \Big\}. $$

□

Next we explore the equilibria and basic threshold quantity $\mathcal{R}_0$ of the model (3.1) in the following subsection.

4.1. Model equilibria and basic reproduction number

The equilibria of our proposed system (3.1) are obtained by solving the system below

$$ {}^{C}D_t^{\alpha} x_1 = {}^{C}D_t^{\alpha} x_2 = {}^{C}D_t^{\alpha} x_3 = {}^{C}D_t^{\alpha} x_4 = 0. $$

Hence we deduced that the proposed model exhibit two type of equilibrium points. The disease free equilibrium (DFE) calculated as

$$ \mathcal{E}^0 = \left(x_1^0, x_2^0, x_3^0, x_4^0\right) = \left(\frac{\Lambda_h}{d_1}, 0, \frac{\Lambda_m}{d_2}, 0\right), $$

and the endemic equilibrium (EE) is as evaluated as follows $\mathcal{E}_1 = \left(x_1^*, x_2^*, x_3^*, x_4^*\right)$, where

$$ x_1^* = \frac{\Lambda_h}{d_1 + x_4^*\beta_1\gamma_1}, x_2^* = \frac{\Lambda_h x_4^*\beta_1\gamma_1}{d_1\left(d_1 + x_4^*\beta_1\gamma_1\right)}, x_3^* = \frac{d_1\Lambda_m(d_1 + \beta_1\gamma_1 x_4^*)}{\beta_1\gamma_1 x_4^*(d_1 d_2 + \beta_2\gamma_2\Lambda_h) + d_2 d_1^2}. \tag{4.1} $$

The EE $\mathcal{E}_1$, exist only if $\mathcal{R}_0 > 1$. The threshold quantity known as the basic reproduction number for the fractional Zika model is obtained by using the well known approach discussed in [28]. The basic reproduction number is biologically very important and determine the global dynamics of the model. The corresponding matrices F and V are given by

$$ F = \begin{pmatrix} 0 & \beta_1\gamma_1\frac{\Lambda_h}{d_1} \\ \beta_2\gamma_2\frac{\Lambda_m}{d_2} & 0 \end{pmatrix}, \ \ V = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}. $$

Further, the inverse of V is

$$V^{-1} = \begin{pmatrix} \frac{1}{d_1} & 0 \\ 0 & \frac{1}{d_2} \end{pmatrix}, \ FV^{-1} = \begin{pmatrix} 0 & \frac{\beta_1\gamma_1\Lambda_h}{d_1 d_2} \\ \frac{\beta_2\gamma_2\Lambda_m}{d_1 d_2} & 0 \end{pmatrix}$$

The spectral radius $\rho(FV^{-1})$ is the basic reproduction number of the model and after some simplification the reproduction number is

$$\mathcal{R}_0 = \sqrt{\frac{\Lambda_h\Lambda_m\beta_2\gamma_2\beta_1\gamma_1}{d_1^2 d_2^2}}.$$

5. Stability analysis of DFE

In this section we proceed to confirm the stability results in both local and global case. The Jacobian of linearization matrix of model (3.1).

$$J_{\mathcal{E}^0} = \begin{pmatrix} -d_1 & 0 & 0 & -\frac{\beta_1\gamma_1\Lambda_h}{d_1} \\ 0 & -d_1 & 0 & \frac{\beta_1\gamma_1\Lambda_h}{d_1} \\ 0 & -\frac{\beta_2\gamma_2\Lambda_m}{d_2} & -d_2 & 0 \\ 0 & \frac{\beta_2\gamma_2\Lambda_m}{d_2} & 0 & -d_2 \end{pmatrix}.$$

Theorem 5.1. *For positive integers r_1 and r_2 such that $gcd(r_1, r_2) = 1$. Let $\alpha = (\frac{r_1}{r_2})$ and define $N = r_2$, then the model DFE denoted by $\mathcal{E}^0$ is stable locally asymptotically provided that $|arg(\lambda)| > \frac{\pi}{2N}$, where λ denotes the possible roots of the characteristic equation of the matrix $J_{\mathcal{E}^0}$ given below.*

$$det(diag[\lambda^{p_1}\lambda^{p_1}\lambda^{p_1}\lambda^{p_1}] - J_{\mathcal{E}^0}) = 0. \tag{5.1}$$

Proof. By expansion of Eq. (5.1), we get the below equation in term of λ.

$$(\lambda^{r_1} + d_1)(\lambda^{r_1} + d_2)(\lambda^{2r_1} + a_1\lambda^{r_1} + a_2) = 0, \tag{5.2}$$

where the coefficients are given below:

$$\begin{aligned} a_1 &= d_1 + d_2, \\ a_2 &= d_1 d_1(1 - \mathcal{R}_0). \end{aligned}$$

The arguments of the roots of the equation $\lambda^{p_1} + d_1 = 0$ are as follow:

$$arg(\lambda_k) = \frac{\pi}{r_1} + k\frac{2\pi}{r_1} > \frac{\pi}{N} > \frac{\pi}{2N}, \ \ where \ \ k = 0, 1 \cdots, (r_1 - 1). \tag{5.3}$$

In similar pattern, it can be shown that argument of the roots of $\lambda^{p_1} + d_2 = 0$ are also greater than $\frac{\pi}{2M}$. Further, if $\mathcal{R}_0 < 1$, then the desired condition ($|arg(\lambda)| > \frac{\pi}{2N}$) is satisfied for all roots of polynomial (5.2). For $\mathcal{R}_0 > 1$, then with the help of Descartes rule of signs, there exits exactly one root of characteristic equation with $|arg(\lambda)| < \frac{\pi}{2N}$. Thus the DFE is locally asymptotically stable for $\mathcal{R}_0 < 1$, otherwise unstable. □

For global stability result we prove the following theorem. This subsection provide the global analysis of the model for the DF and endemic case. We have the following results.

Theorem 5.2. *For arbitrary fractional order α in the interval (0,1], and $\mathcal{R}_0 < 1$, the DFE of the proposed model is stable globally asymptotically and unstable otherwise.*

Proof. To prove our result we define consider the following Lyapunov function

$$V(t) = W_1\Big(x_1 - x_1^0 - x_1^0 ln\frac{x_1}{x_1^0}\Big) + W_2x_2 + W_3\Big(x_3 - x_3^0 - x_3^0 ln\frac{x_3}{x_3^0}\Big) + W_4x_4. \tag{5.4}$$

Where W_i, $i = 1, 2\cdots 4$, are arbitrary positive constants to be chosen latter. Using lemma (5.1), the time derivative of (5.4), along the solution of (3.1) is given by

$$\begin{aligned}
{}^CD_t^\alpha V(t) &= W_1\Big(\tfrac{x_1-x_1^0}{x_1}\Big)\,{}^CD_t^\alpha x_1 + W_2\,{}^CD_t^\alpha x_2 + W_3\Big(\tfrac{x_3-x_3^0}{x_3}\Big)\,{}^CD_t^\alpha x_3 + W_4\,{}^CD_t^\alpha x_4 \\
&= W_1\Big(\tfrac{x_1-x_1^0}{x_1}\Big)\Big[\Lambda_h - d_1x_1 - \beta_1\gamma_1x_4x_1\Big] + W_2\Big[\beta_1\gamma_1x_4x_1 - d_1x_2\Big] \\
&\quad +W_3\Big(\tfrac{x_3-x_3^0}{x_3}\Big)\Big[\Lambda_m - d_2x_3 - \beta_2\gamma_2x_3x_2\Big] + W_4\Big[\beta_2\gamma_2x_3x_2 - d_2x_4\Big] \\
&= (W_2 - W_1)\Big[\beta_1\gamma_1x_4x_1\Big] + (W_4 - W_3)\Big[\beta_2\gamma_2x_3x_2\Big] \\
&\quad +x_4(W_1\beta_1\gamma_1x_1^0 - W_4d_2) + x_2(W_3\beta_2\gamma_2x_3^0 - W_2d_1).
\end{aligned}$$

Using $x_1^0 = \frac{\Lambda_h}{d_1}$ and $x_3^0 = \frac{\Lambda_m}{d_2}$, we get

$$\begin{aligned}
{}^CD_t^\alpha V &= (W_2 - W_1)\Big[\beta_1\gamma_1x_4x_1\Big] + (W_4 - W_3)\Big[\beta_2\gamma_2x_3x_2\Big] \\
&\quad +x_4(W_1\beta_1\gamma_1\tfrac{\Lambda_h}{d_1} - W_4d_2) + x_2(W_3\beta_1\gamma_1\tfrac{\Lambda_m}{d_2} - W_2d_1).
\end{aligned}$$

Choosing the constants $W_1 = W_2 = \beta_2\gamma_2\frac{\Lambda_m}{d_2}$ and $W_3 = W_4 = d_1$ and after simplification, we get

$${}^CD_t^\alpha V = x_4d_1d_2\Big(\mathcal{R}_0 - 1\Big).$$

${}^CD_t^\alpha V(t)$ is negative for $\mathcal{R}_0 < 1$. Therefore, by theorem (2.4) [23, 24], the DFE $\mathcal{E}^0$, is globally asymptotically stable in Φ. □

6. Stability of endemic equilibrium

Here, we present the global stability of the system (3.1) at $\mathcal{E}_1$. At the steady state the model (3.1) at $\mathcal{E}_1$ we obtained

$$\begin{cases}
\Lambda_h = \beta_1\gamma_1x_4^*x_1^* + d_1x_1^*, \\
d_1x_2^* = \beta_1\gamma_1x_4^*x_1^*, \\
\Lambda_m = \beta_2\gamma_2x_3^*x_2^* + d_2x_3^*, \\
d_2x_4^* = \beta_2\gamma_2x_3^*x_2^*.
\end{cases} \tag{6.1}$$

Theorem 6.1. *If $\mathcal{R}_0 > 1$, the EE $\mathcal{E}_1$ of the system (3.1) is stable globally asymptotically.*

Proof. We consider the following Lyapunov function:

$$\begin{aligned} L(t) &= \Big(x_1 - x_1^* - x_1^* log\frac{x_1}{x_1^*}\Big) + \Big(x_2 - x_2^* - x_2^* log\frac{x_2}{x_2^*}\Big) + \Big(x_3 - x_3^* - x_3^* log\frac{x_3}{x_3^*}\Big) \\ &\quad + \Big(x_4 - x_4^* - x_4^* log\frac{x_4}{x_4^*}\Big). \end{aligned}$$

Using lemma (5.1), the derivative of $L(t)$ along the solutions of (3.1) is as follows

$$ {}^C D_t^\alpha L = \Big(1 - \frac{x_1^*}{x_1}\Big)\, {}^C D_t^\alpha x_1 + \Big(1 - \frac{x_2^*}{x_2}\Big)\, {}^C D_t^\alpha x_2 + \Big(1 - \frac{x_3^*}{x_3}\Big)\, {}^C D_t^\alpha x_3 + \Big(1 - \frac{x_4^*}{x_4}\Big)\, {}^C D_t^\alpha x_4. $$

By direct calculations, we have that:

$$\begin{aligned} (1 - \frac{x_1^*}{x_1})\, {}^C D_t^\alpha x_1 &= (1 - \frac{x_1^*}{x_1})\Big(\Lambda_h - d_1 x_1 - \beta_1\gamma_1 x_4 x_1\Big) \\ (1 - \frac{x_2^*}{x_2})\, {}^C D_t^\alpha x_2 &= (1 - \frac{x_2^*}{x_2})\Big(\beta_1\gamma_1 x_4 x_1 - d_1 x_2\Big) \\ (1 - \frac{x_3^*}{x_3})\, {}^C D_t^\alpha x_3 &= (1 - \frac{x_3^*}{x_3})\Big(\Lambda_m - d_2 x_3 - \beta_2\gamma_2 x_3 x_2\Big) \\ (1 - \frac{x_4^*}{x_4})\, {}^C D_t^\alpha x_2 &= (1 - \frac{x_4^*}{x_4})\Big(\beta_2\gamma_2 x_3 x_2 - d_2 x_4\Big). \end{aligned} \tag{6.2}$$

$$\begin{aligned} (1 - \frac{x_1^*}{x_1})\, {}^C D_t^\alpha x_1 &= (1 - \frac{x_1^*}{x_1})\Big(\Lambda_h - d_1 x_1 - \beta_1\gamma_1 x_4 x_1\Big) \\ &= (1 - \frac{x_1^*}{x_1})\Big(d_2 x_1^* + \beta_1\gamma_1 x_4^* x_1^* - d_2 x_1 - \beta_1\gamma_1 x_4 x_1\Big) \\ &= d_2 x_1^*(1 - \frac{x_1^*}{x_1})(1 - \frac{x_1}{x_1^*}) + (1 - \frac{x_1^*}{x_1})\Big(\beta_1\gamma_1 x_4^* x_1^* - \beta_1\gamma_1 x_4 x_1\Big) \\ &= d_2 x_1^*\Big(2 - \frac{x_1^*}{x_1} - \frac{x_1}{x_1^*}\Big) + \beta_1\gamma_1 x_4^* x_1^* - \beta_1\gamma_1 x_4 x_1 \\ &\quad -\beta_1\gamma_1 x_4^* x_1^* \frac{x_1^*}{x_1} + \beta_1\gamma_1 x_4 x_1^*. \end{aligned} \tag{6.3}$$

$$\begin{aligned} (1 - \frac{x_2^*}{x_2})\, {}^C D_t^\alpha x_2 &= (1 - \frac{x_2^*}{x_2})\Big(\beta_1\gamma_1 x_4 x_1 - d_1 x_2\Big) \\ &= \beta_1\gamma_1 x_4 x_1 - d_1 x_2 - \beta_1\gamma_1 x_4 x_1 \frac{x_2^*}{x_2} + d_1 x_2^* \\ &= \beta_1\gamma_1 x_4 x_1 - \beta_1\gamma_1 x_4^* x_1^* \frac{x_2}{x_2^*} - \beta_1\gamma_1 x_4 x_1 \frac{x_2^*}{x_2} + \beta_1\gamma_1 x_4^* x_1^*. \end{aligned} \tag{6.4}$$

$$
\begin{aligned}
(1-\frac{x_3^*}{x_3})\,{}^{C}D_t^{\alpha}x_3 &= (1-\frac{x_3^*}{x_3})\Big(\Lambda_m - d_2x_3 - \beta_2\gamma_2x_3x_2\Big)\\
&= (1-\frac{x_3^*}{x_3})\Big(d_2x_3^* + \beta_2\gamma_2x_3^*x_2^* - d_2x_3 - \beta_2\gamma_2x_3x_2\Big)\\
&= d_2x_3^*(1-\frac{x_3^*}{x_3})(1-\frac{x_3}{x_3^*}) + (1-\frac{x_3^*}{x_3})\Big(\beta_2\gamma_2x_3^*x_2^* - \beta_2\gamma_2x_3x_2\Big)\\
&= d_2x_3^*\Big(2-\frac{x_3^*}{x_3}-\frac{x_3}{x_3^*}\Big) + \beta_2\gamma_2x_3^*x_2^{**} - \beta_2\gamma_2x_3x_2\\
&\quad -\beta_2\gamma_2x_3^*x_3^{**}\frac{x_3^*}{x_3} + \beta_2\gamma_2x_3x_2^*. \qquad (6.5)
\end{aligned}
$$

$$
\begin{aligned}
(1-\frac{x_4^*}{x_4})\,{}^{C}D_t^{\alpha}x_4 &= (1-\frac{x_4^*}{x_4})\Big(\beta_2\gamma_2x_3x_2 - d_2x_4\Big)\\
&= \beta_2\gamma_2x_3x_2 - d_2x_4 - \beta_2\gamma_2x_3x_2\frac{x_4^*}{x_4} + d_2x_4^*\\
&= \beta_2\gamma_2x_3x_2 - \beta_2\gamma_2x_3^*x_x^*2\frac{x_4}{x_4^*} - \beta_2\gamma_2x_3x_2\frac{x_4^*}{x_4} + \beta_2\gamma_2x_3^*x_2^*. \qquad (6.6)
\end{aligned}
$$

It follows from (6.3-6.6)

$$
\begin{aligned}
{}^{C}D_t^{\alpha}L &= d_1x_1^*\Big(2-\frac{x_1^*}{x_1}-\frac{x_1}{x_1^*}\Big) + \beta_1\gamma_1x_4^*x_1^*\Big(2-\frac{x_1^*}{x_1}-\frac{x_2}{x_2^*}-\frac{x_4}{x_4^*}\Big(\frac{x_1x_2^*}{x_1^*x_2}-1\Big)\Big)\\
&\quad + d_2x_3^*\Big(2-\frac{x_3^*}{x_3}-\frac{x_3}{x_3^*}\Big) + \beta_2\gamma_2x_3^*x_2^*\Big(2-\frac{x_3^*}{x_3}-\frac{x_4}{x_4^*}-\frac{x_2}{x_2^*}\Big(\frac{x_3x_4^*}{x_3^*x_4}-1\Big)\Big). \qquad (6.7)
\end{aligned}
$$

Make use of arithmetical-geometrical inequality we have in equation (6.7)

$$d_1x_1^*\Big(2-\frac{x_1^*}{x_1}-\frac{x_1}{x_1^*}\Big) \le 0,$$

$$d_2x_3^*\Big(2-\frac{x_3^*}{x_3}-\frac{x_3}{x_3^*}\Big) \le 0,$$

$$\beta_1\gamma_1x_4^*x_1^*\Big(2-\frac{x_1^*}{x_1}-\frac{x_x2}{x_2^*}-\frac{x_3}{x_3^*}\Big(\frac{x_1x_2^*}{x_1^*x_2}-1\Big)\Big) \le 0,$$

$$\beta_2\gamma_2x_3^*x_2^*\Big(2-\frac{x_3^*}{x_3}-\frac{x_4}{x_4^*}-\frac{x_2}{x_2^*}\Big(\frac{x_3x_4^*}{x_3^*x_4}-1\Big)\Big) \le 0.$$

Therefore, ${}^{C}D_t^{\alpha}L \le 0$ and hence by using theorem (2.4) the EE $\mathcal{E}_1$ of the proposed model is globally asymptotically stable whenever $\mathcal{R}_0 > 1$. □

7. Numerical results

The present section is devoted to obtain the numerical results of the proposed Zika fractional order model (3.1). The Adams-type predictor-corrector method is applied to obtained the approximate solution of the model. The numerical values used in the simulations are $\Lambda_h = 1.2$, $\Lambda_m = 0.3$, $\beta_1 = 0.0004, \beta_2 = 0.005, d_1 = 0.004, d_2 = 0.0014, \gamma_1 = 0.02$. $\gamma_2 = 0.0003$. The graphical results using different five values of fractional order $\alpha = 1, 0.95, 0.9, 0.85, 0.8$ are presented in the Figures 1–4. From these figures we can see than the susceptible human and mosquitoes are increasing when we decreases the fractional order α. While there is a significant decrease in the graphs of infected humans and mosquitoes with decrease in α. Hence, the fractional order α has an important role in the model.

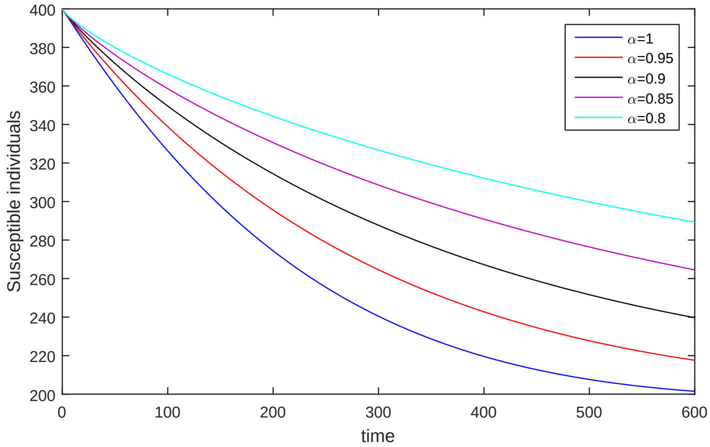

Figure 1. The graph shows the behavior of the susceptible humans for $\alpha = 1, 0.95, 0.90, 0.85, 0.8$.

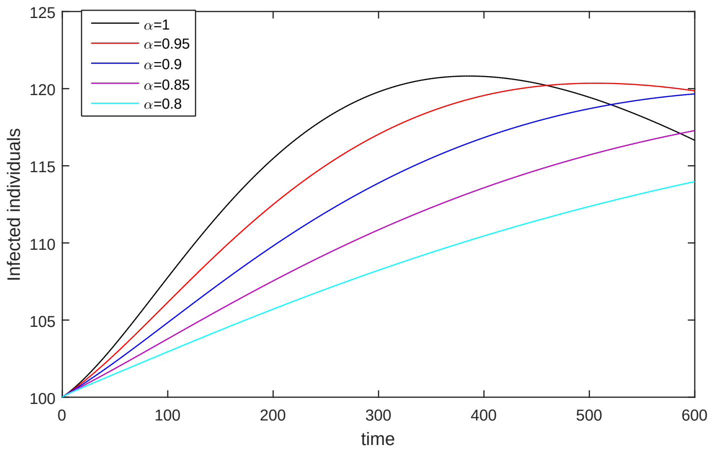

Figure 2. The graph shows the behavior of the infected humans for $\alpha = 1, 0.95, 0.90, 0.85, 0.8$..

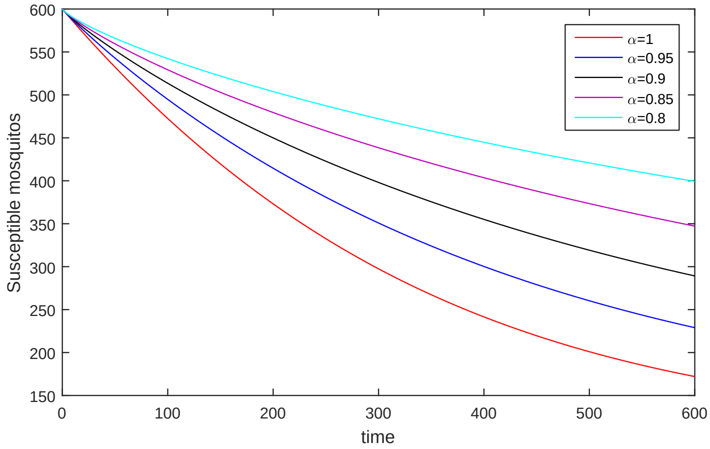

Figure 3. The graph shows the behavior of the susceptible mosquitos for $\alpha = 1, 0.95, 0.90, 0.85, 0.8$.

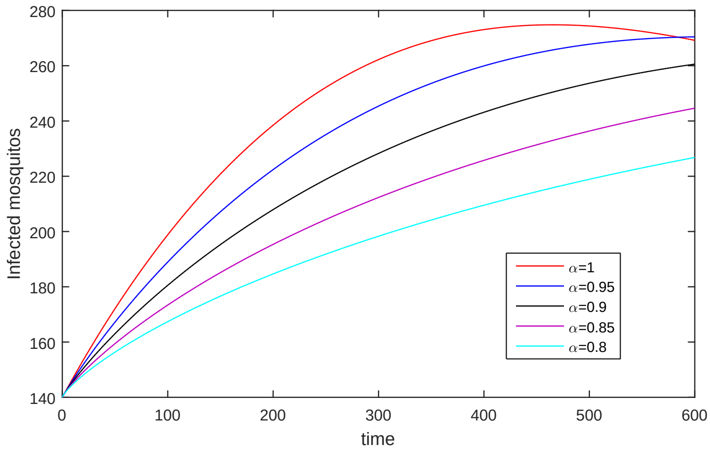

Figure 4. The graph shows the behavior of the infected mosquitos for $\alpha = 1, 0.95, 0.90, 0.85, 0.8$.

8. Conclusion

Zika is a rapidly spreading epidemic and is one of serious health issue, specially for pregnant women. A number of deterministic models have been presented in last few year, for the possible control and eradication of this infection from the community. But, almost all of these models are based on classical or local derivative. In order to better explore the complex behavior of Zika infection, in this paper, a fractional order transmission model in Caputo sense is developed. The detail analysis such as positivity and existence of the solution, basic reproduction numberer and model equilibria of the proposed model are presented. The stability results for both local and global cases are derived in detail in fractional environment. From the numerical results we conclude that the fractional order derivative provides more information about the proposed model which are unable by classical integer-order epidemic models. Also these results ensure that by including the memory effects in the model seems very appropriate for such an investigation. In future, we will explore the proposed model using non-local and non-singular fractional derivatives presented in [9, 10].

Conflict of Interest

All authors declare no conflict of interest.

References

1. Zika virus, World Health Organization. Available from: `http://www.who.int/mediacentre/factsheets/zika/en/.`
2. G. Calvet, R. S. Aguiar, A. S. O. Melo, et al. *Detection and sequencing of Zika virus from amniotic fluid of fetuses with microcephaly in Brazil: a case study*, Lancet infect dis., **16** (2016), 653–660.
3. T. A. Perkins, A. S. Siraj, C. W. Ruktanonchai, et al. *Model-based projections of Zika virus infections in childbearing women in the Americas*, Nat. Microbiol., **1** (2016), 16126.
4. A. J. Kucharski, S. Funk, R. M. M. Eggo, et al. *Transmission dynamics of Zika virus in island populations: a modelling analysis of the 2013–14 French Polynesia outbreak*, PLoS Neglect. Trop. D., **10** (2016), 38588.
5. E. Bonyah and K. O. Okosun, *Mathematical modeling of Zika virus*, Asian Pacific Journal of Tropical Disease, **6** (2016), 637–679.
6. E. Bonyah, M. A. Khan, K. O. Okosun, et al. *A theoretical model for Zika virus transmission*, Plos one, **12** (2017), 1–26.
7. I. Podlubny, *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Vol. **198**, Elsevier, 1998.
8. S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional integrals and derivatives: theory and applications*, 1993.
9. A. Atangana and D. Baleanu, *New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model*, Therm. Sci., **20** (2016), 763–769.
10. M. Caputo and M. Fabrizio, *A new definition of fractional derivative without singular kernel*, Progr. Fract. Differ. Appl., **1** (2015), 1–13.
11. K. M. Owolabi, *Numerical solution of diffusive HBV model in a fractional medium*, SpringerPlus, **5** (2016), 1643.
12. K. M. Owolabi, *Mathematical modelling and analysis of two-component system with Caputo fractional derivative order*, Chaos Soliton. Fract., **103** (2017), 544–554.
13. K. M. Owolabi, A. Atangana, *Robustness of fractional difference schemes via the Caputo subdiffusion-reaction equations*, Chaos Soliton. Fract., **111** (2018), 119–127.
14. E. Bas, R. Ozarslan, *Real world applications of fractional models by Atangana–Baleanu fractional derivative*, Chaos Soliton. Fract., **116** (2018), 121–125.

15. E. Bas, R. Ozarslan, D. Baleanu, *Comparative simulations for solutions of fractional Sturm-Liouville problems with non-singular operators*, Adv. Differ. Equ-NY, **2018** (2018), 350.

16. E. Bas, *The Inverse Nodal problem for the fractional diffusion equation*, Acta Sci-Technol, **37** (2015), 251–257.

17. E. Bas, F. Metin, *Fractional singular Sturm-Liouville operator for Coulomb potential*, Adv. Differ. Equ-NY, **2013** (2013), 300.

18. S. Ullah, M. A. Khan and M. F. Farooq, *A new fractional model for the dynamics of Hepatitis B virus using Caputo-Fabrizio derivative*, European Physical Journal Plus, **133** (2018), 237.

19. M. A. Khan, S. Ullah and M. F. Farooq, *A new fractional model for tuberculosis with relapse via AtanganaBaleanu derivative*, Chaos Soliton. Fract., **116** (2018), 227–238.

20. M. A. Khan, S. Ullah and M. F. Farooq, *A fractional model for the dynamics of TB virus*, Chaos Soliton. Fract., **116** (2018), 63–71.

21. B. S. T. Alkahtani, A. Atangana, I. Koca, *Novel analysis of the fractional Zika model using the Adams type predictor–corrector rule for non-singular and non-local fractional operators*, The Journal of Nonlinear Sciences and Applications, **10** (2017), 3191–3200.

22. H. Elsaka and E. Ahmed, *A fractional order network model for ZIKA*, BioRxiv, 2016.

23. D. Hadi, D. Baleanu and J. Sadati, *Stability analysis of Caputo fractional-order nonlinear systems revisited*, Nonlinear Dynam., **67** (2012), 2433–2439.

24. V. D. L. Cruz, *Volterra-type Lyapunov functions for fractional-order epidemic systems*, Commun. Nonlinear Sci., **24** (2015), 75–85.

25. Z. M. Odibat and N. T. Shawagfeh, *Generalized Taylors formula*, Appl. Math. Comput., **186** (2007), 286–293.

26. W. Lin, *Global existence theory and chaos control of fractional differential equations*, J. Math. Anal. Appl., **332** (2007), 709–726.

27. H. A. Antosiewicz, J. K. Hale, *Studies in Ordinary Differential Equations*, In: Englewood Cliffs (N. J.) by Mathematical Association of America, 1977.

28. P. van den Driessche and J. Watmough, *Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission*, Bellman Prize in Mathematical Biosciences, **180** (2002), 29–48.

AIMS Press