Materials Science

AIMS Materials Science, 3(4): 1365-1381.
DOI: 10.3934/matersci.2016.4.1365
Received: 18 August 2016
Accepted: 08 October 2016
Published: 14 October 2016

http://www.aimspress.com/journal/Materials

Research article

Moving row of antiplane shear cracks within one-dimensional piezoelectric quasicrystals

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Abstract: Closed-form expressions are deduced and discussed, using an extended form of the classical dislocation layer method, for the phonon and phason stress and electric displacement components and intensity factors generated in one-dimensional piezoelectric quasicrystals by a collinear row of moving shear cracks. Representative numerical results are presented graphically. Additionally, this analysis yields the fields of a single crack moving in a finite piezoelectric quasicrystalline plate and also of a moving edge crack in a plate.

Keywords: moving antiplane shear cracks; piezoelectric quasicrystals; dislocation layers

1. Introduction

Since the announcement in 1984 by Shechtman et al. [1] of the exciting discovery of the new class of novel alloys that are now known as quasicrystals, there has been an enormous concerted effort by especially the exponents of materials science, solid mechanics and solid state physics to investigate their behaviour both theoretically and experimentally. They have desirable properties that are now widely exploited in, for example, engine surface coatings in the aerospace industry and nuclear fuel containers.

However, with the increasing maturity of the technological applications of quasicrystals, it is crucial that their behaviour should continue to be fully researched and particularly that their intrinsic piezoelectric coupling effects are incorporated. Piezoelectric materials are utilized extensively in signal processing in, for example, transducers, sensors and attenuators and piezoelectric quasicrystals can play a vital role in the design of intelligent systems and structures. It is vital that fundamental solutions of all types of boundary value problems and particularly those involving various configurations of inclusions and cracks are made available, as experiments have demonstrated that quasicrystals are quite brittle and thus can experience premature failure.

Fan [2,3] and Ding et al. [4], for example, have most conveniently comprehensively referenced and reviewed the growing literature of mathematical theories and analyses of physical problems in quasicrystals. But encouragingly, more recently relevant techniques have begun to be developed for studying the corresponding situations within piezoelectric quasicrystals.

The fundamental equations that govern the behaviour of three-dimensional piezoelectric quasicrystals were developed by Altay and Dökmeci [5] in both differential and variational invariant forms. Using the theory of group representation, Li and Liu [6] derived the matrix forms of the thermal-expansion coefficients and piezoelectric coefficient tensors for all 31 point-groups of one-dimensional quasicrystals. These provided the starting point for Wang and Pan [7] to produce elementary expressions for the induced fields of a uniformly moving screw dislocation within a one-dimensional hexagonal piezoelectric quasicrystal. Thereafter, the generalized formalism of Stroh was used by Yang et al. [8] to give the piezoelectric fields created in a one-dimensional piezoelectric quasicrystal by a motionless straight dislocation that is parallel to its axis of periodicity.

By introducing two functions of displacement and rigorously applying the theory of operators, a set of general solutions of static problems in three-dimensions for hexagonal piezoelectric quasicrystals was obtained by Li et al. [9]. Yu et al. [10] proposed a method of complex variables for an elliptical anti-plane cavity to derive explicit solutions for the elastic-electric fields in a one-dimensional hexagonal piezoelectric quasicrystal. Further, Yu et al. [11] used operator and complex variable function methods to rigorously deduce general solutions of piezoelasticity plane problems for quasicrystals and considered an antiplane, constantly-loaded, stationary crack in a hexagonal piezoelastic quasicrystal by the semi-inverse method. Zhang et al. [12] introduced four potential functions in terms of which solutions of the equations governing plane problems in one-dimensional orthorhombic quasicrystals having piezoelectric effects are expressed.

Most recently, in 2016, methods of complex variable functions and techniques of conformal mappings assisted Yang and Li [13] in studying a circular hole with a straight crack and Guo et al. [14] in considering an elliptical inclusion embedded in composites of one-dimensional hexagonal piezoelectric quasicrystals. Fan et al. [15] presented basic solutions of three-dimensional cracks in such media using Green's functions, Hankel transforms and methods of boundary integral equations and Tupholme [16] derived closed-form expressions for the stress and electric fields created by a moving uniformly-loaded shear crack in one-dimensional hexagonal piezoelectric quasicrystals.

The continuing wider interest in quasicrystals is further demonstrated by, for example, the proposal and analysis by Guo and Pan [17] of a three-phase model of one-dimensional hexagonal piezoelectric quasicrystal composites and a study of two-dimensional thermoelastic deformations of a conductive elliptical hole embedded within a two-dimensional decagonal quasicrystal by Guo et al. [18], and the references therein.

However, there has been no analysis whatsoever presented previously using any technique of a row of moving, anti-plane cracks in quasicrystals with piezoelectric effects. The focus of the current investigation is to demonstrate that the continuous dislocation layers method, that was devised originally for purely elastic isotropic solids, can be extended most conveniently to yield expressions for the components of the fields induced by such cracks.

In Section 2, a summary is provided of the underlying basic three-dimensional equations which govern the deformation of piezoelectric quasicrystals and the fundamental constitutive equations of one-dimensional hexagonal piezoelectric quasicrystals with point group 6 mm are stated. A formulation is then given of the problem being studied here. The required components of the phonon and phason displacement and stress field and the electric potential of a moving screw dislocation are presented in Section 3, before an extension of the classical dislocation layer method is described and adapted in Section 4 to yield closed-form representations for the fields around the cracks. Illustrative numerical results are displayed graphically for the variation of the stress component ahead of a crack tip for a range of values of the geometric parameters. The results for the situation that has not been presented previously of a stationary row of cracks are deduced. As observed in the concluding Section 5, the solutions are also provided by this analysis to the problems of a moving single central crack within a finite plate and a finite plate having a moving edge crack in a piezoelectric quasicrystal.

2. Fundamental Equations of Piezoelectric Quasicrystals and Problem Formulation

The deformation fields within the linear theory of piezoelectric quasicrystals have components that are governed by the general three-dimensional equations which Altay and Dökmeci [5] have presented in both differential and variational invariant forms. In the absence of body forces and an electric charge density, the quasistatic equilibrium equations and the constitutive equations can be written compactly, relative to a fixed system of rectangular Cartesian coordinates (x_1, x_2, x_3), using a suffix notation where *i, j, k, l* = 1, 2, 3 with the adoption of the repeated suffices summation convention, as

$$\sigma_{ij,i} = 0, \quad H_{ij,i} = 0, \quad D_{ij,i} = 0 , \tag{1}$$

$$\sigma_{ij} = c_{ijkl}(u_{k,l} + u_{l,k})/2 + R_{ijkl}w_{k,l} - e_{kij}E_k , \tag{2}$$

$$H_{ij} = R_{klij}(u_{k,l} + u_{l,k})/2 + K_{ijkl}w_{k,l} - e'_{kij}E_k , \tag{3}$$

$$D_i = e_{kij}(u_{j,k} + u_{k,j})/2 + e'_{kij}w_{j,k} - \varepsilon_{ij}E_j , \tag{4}$$

with a comma followed by p denoting partial differentiation with respect to x_p for $p = i, j, k, l$.

The components of the phonon stress and displacement, the phason stress and displacement and the electric displacement and field are denoted by σ_{ij}, u_i, H_{ij}, w_i, D_i and E_i, respectively, and c_{ijkl}, R_{ijkl}, K_{ijkl}, e_{ijk}, e'_{ijk} and ε_{ij} are the phonon elastic constants, the phonon-phason coupling constants, the phason elastic constants, the phonon and phason piezoelectric constants and the dielectric constants, respectively.

Here an infinite homogeneous one-dimensional hexagonal piezoelectric quasicrystal of point group 6 mm is considered that has a constant density, ρ, and is initially in an undisturbed reference state which is free of stress and at rest everywhere. It is oriented, with reference to a system of fixed rectangular Cartesian coordinates (x, y, z), so that its periodic plane is the x-y plane and its quasiperiodic direction is that of the positive z-axis.

A problem of mode III fracture is addressed in which a periodic infinite array of collinear, plane Griffith-type, moving, strip cracks of equal constant width $2c$ is embedded in such a material that is subjected to remote uniform phonon, phason and electrical loads.

Within the piezoelectric quasicrystal, the created components σ_{XY}, ε_{XY} and u_X of the phonon stress and strain tensors and displacement vector, H_{zX}, w_{zX} and w_X of the phason stress and strain tensors and displacement vector, and D_X and E_X of the electric displacement and field vectors, for X and $Y = x, y$ or z, are linked by the matrix constitutive equations

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \\ H_{zz} \\ H_{zx} \\ H_{zy} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 & R_1 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 & R_1 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 & R_2 & 0 & 0 \\ 0 & 0 & 0 & 2c_{44} & 0 & 0 & 0 & 0 & R_3 \\ 0 & 0 & 0 & 0 & 2c_{44} & 0 & 0 & R_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{11}-c_{12} & 0 & 0 & 0 \\ R_1 & R_1 & R_2 & 0 & 0 & 0 & K_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2R_3 & 0 & 0 & K_2 & 0 \\ 0 & 0 & 0 & 2R_3 & 0 & 0 & 0 & 0 & K_2 \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy} \\ w_{zz} \\ w_{zx} \\ w_{zy} \end{bmatrix}$$

$$- \begin{bmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{31} \\ 0 & 0 & e_{33} \\ 0 & e_{15} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e'_{33} \\ e'_{15} & 0 & 0 \\ 0 & e'_{15} & 0 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}, \tag{5}$$

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 2e_{15} & 0 & 0 & e'_{15} & 0 \\ 0 & 0 & 0 & 2e_{15} & 0 & 0 & 0 & 0 & e'_{15} \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 & e'_{33} & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy} \\ w_{zz} \\ w_{zx} \\ w_{zy} \end{bmatrix}$$

$$+\begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{bmatrix}\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}. \tag{6}$$

with

$$\varepsilon_{XY} = \frac{1}{2}\left(\frac{\partial u_X}{\partial Y} + \frac{\partial u_Y}{\partial X}\right), \qquad w_{zX} = \frac{\partial w_z}{\partial X}. \tag{7}$$

The constants involved are the phonon elastic moduli, c_{ij}, the phason elastic moduli, K_i, the phonon-phason coupling elastic moduli, R_i, the piezoelectric moduli, e_{ij} and e'_{ij}, and the dielectric moduli, ε_{ij}, where *i* and *j* take integer values and the customary Voigt's contracted notation is being used.

At time *t,* it is assumed that the cracks are moving uniformly parallel to their axes in their own planes with a speed *v* and occupy the regions R_t of the $y = 0$ plane, where

$$R_t = \{\, (x, y, z):\ vt - c + 2nh < x < vt + c + 2nh, \quad y = 0, \quad -\infty < z < \infty \,\}, \tag{8}$$

for $n = 0, \pm 1, \pm 2, \ldots$, with their centres at $x = vt,\ vt \pm 2h,\ vt \pm 4h, \ldots$, as illustrated in Figure 1.

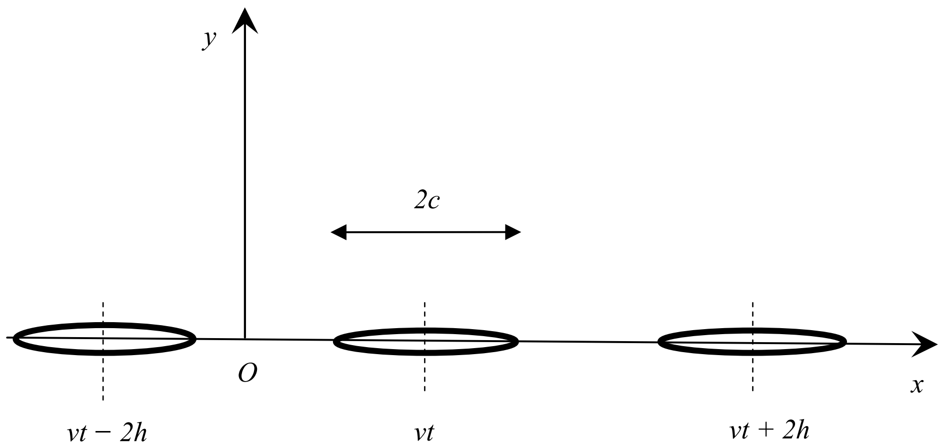

Figure 1. Row of strip cracks moving in the *x*-direction with constant speed *v*.

It is appropriate to utilize a moving coordinate, ξ, defined by

$$\xi \quad = \quad x - vt \tag{9}$$

and an electric potential, ϕ, which is related to the electric field vector, $\boldsymbol{E}$, by

$$\boldsymbol{E} \quad = \quad -\nabla\phi \tag{10}$$

The field variables in the antiplane deformation under consideration here are independent of *z*. Thus the required relations between the phonon and phason displacements' components, u_z and w_z, and the strain components involved become

$$\varepsilon_{xz} = \frac{1}{2}\frac{\partial u_z}{\partial x}, \quad \varepsilon_{yz} = \frac{1}{2}\frac{\partial u_z}{\partial y}, \tag{11}$$

$$w_{zx} = \frac{\partial w_z}{\partial x}, \quad w_{zy} = \frac{\partial w_z}{\partial y}, \tag{12}$$

and the corresponding components of the phonon and phason stresses and electric displacement are given by Eqs. (5) and (6) in the forms.

$$\sigma_{yz} = 2c_{44}\,\varepsilon_{yz} + R\,w_{zy} - e_{15}E_y, \quad \sigma_{xz} = 2c_{44}\,\varepsilon_{xz} + R\,w_{zx} - e_{15}E_x, \tag{13}$$

$$H_{zy} = 2R\,\varepsilon_{yz} + K\,w_{zy} - e'_{15}\,E_y, \quad H_{zx} = 2R\,\varepsilon_{xz} + K\,w_{zx} - e'_{15}\,E_x, \tag{14}$$

$$D_y = 2e_{15}\,\varepsilon_{yz} + e'_{15}\,w_{zy} + \varepsilon_{11}E_y, \quad D_x = 2e_{15}\,\varepsilon_{xz} + e'_{15}\,w_{zx} + \varepsilon_{11}E_x. \tag{15}$$

Here, and subsequently, R and K are used as abbreviations for the constants R_3 and K_2 for the sake of brevity of presentation.

With the solid subjected to phonon, phason and electrical loading such that

$$\sigma_{yz} \to \mathcal{T}, \quad H_{zy} \to \mathcal{H}, \quad D_y \to \mathcal{D} \quad \text{at infinity}, \tag{16}$$

where the constants $\mathcal{T}$, $\mathcal{H}$ and $\mathcal{D}$ are specified, an antiplane mode III deformation is induced.

Similarly, an interested reader could develop the corresponding analyses to that given below if analogously any desired combination of three of the phonon, phason and electrical components σ_{yz}, ε_{yz}, H_{zy}, w_{zy}, D_y, or E_y are prescribed instead.

3. Moving Screw Dislocation in a Piezoelectric Quasicrystal

The stimulus is provided for the current study by the fundamental properties of the fields of a moving "piezoelectric quasicrystal screw dislocation". This has a generalized Burgers vector which is extended from that of a traditional purely elastic screw dislocation. Across its slip plane, such a dislocation has discontinuities of finite magnitudes b in the component u_z of the phonon displacement, d in the component w_z of the phason displacement and b_4 in ϕ, the electric potential.

Expressions have been derived explicitly by Wang and Pan [7] for the components of the fields created around a dislocation at the origin of this type, which is parallel to the z-axis and moves in a one-dimensional hexagonal piezoelectric quasicrystal with point group 6 mm with a speed v along the x-axis. These can be written, after appropriate rearrangement and some renaming of the terms and moduli of the medium as

$$u_z^{\mathrm{III}}(\xi, y) = \frac{1}{2\pi(\alpha^2 + \overline{R}^2)}\left[b\left\{\alpha^2 \tan^{-1}\left(\frac{\beta_1 y}{\xi}\right) + \overline{R}^2 \tan^{-1}\left(\frac{\beta_2 y}{\xi}\right)\right\} + d\alpha\overline{R}\left\{\tan^{-1}\left(\frac{\beta_1 y}{\xi}\right) - \tan^{-1}\left(\frac{\beta_2 y}{\xi}\right)\right\}\right] \tag{17}$$

$$w_z^{\mathrm{III}}(\xi, y) = \frac{1}{2\pi(\alpha^2 + \overline{R}^2)}\left[b\alpha\overline{R}\left\{\tan^{-1}\left(\frac{\beta_1 y}{\xi}\right) - \tan^{-1}\left(\frac{\beta_2 y}{\xi}\right)\right\} + d\left\{\overline{R}^2 \tan^{-1}\left(\frac{\beta_1 y}{\xi}\right) + \alpha^2 \tan^{-1}\left(\frac{\beta_2 y}{\xi}\right)\right\}\right] \tag{18}$$

$$\phi^{\mathrm{III}}(\xi, y) = \frac{1}{2\pi}\left\{ b\left[\frac{1}{\varepsilon_{11}(\alpha^2 + \overline{R}^2)}\left\{ \alpha(e_{15}\alpha + e'_{15}\overline{R})\tan^{-1}\left(\frac{\beta_1 y}{\xi}\right) - \overline{R}(e'_{15}\alpha - e_{15}\overline{R})\tan^{-1}\left(\frac{\beta_2 y}{\xi}\right)\right\}\right.\right.$$

$$\left. - \frac{e_{15}}{\varepsilon_{11}}\tan^{-1}\left(\frac{y}{\xi}\right)\right] + d\left[\frac{1}{\varepsilon_{11}(\alpha^2 + \overline{R}^2)}\left\{ \overline{R}(e_{15}\alpha + e'_{15}\overline{R})\tan^{-1}\left(\frac{\beta_1 y}{\xi}\right) + \alpha(e'_{15}\alpha - e_{15}\overline{R})\tan^{-1}\left(\frac{\beta_2 y}{\xi}\right)\right\}\right.$$

$$\left.\left. - \frac{e'_{15}}{\varepsilon_{11}}\tan^{-1}\left(\frac{y}{\xi}\right)\right] + b_4\tan^{-1}\left(\frac{y}{\xi}\right)\right\}, \tag{19}$$

where, here and henceforth, the superscript III indicates that the terms are related to the antiplane mode III deformation.

It is convenient to put

$$\alpha = \left\{ \bar{c}_{44} - \overline{K} + \sqrt{(\bar{c}_{44} - \overline{K})^2 + 4\overline{R}^2} \right\}/2, \tag{20}$$

where the piezoelectrically stiffened elastic constants, $\bar{c}_{44}$ and $\overline{K}$, in the phonon and phason fields, and the piezoelectrically stiffened phonon-phason coupling elastic constant, $\overline{R}$, are defined by

$$\bar{c}_{44} = c_{44} + \frac{e_{15}^{\ 2}}{\varepsilon_{11}}, \quad \overline{K} = K + \frac{e'^{\ 2}_{15}}{\varepsilon_{11}}, \quad \overline{R} = R + \frac{e_{15}\, e'_{15}}{\varepsilon_{11}}, \tag{21}$$

and

$$\beta_i = \sqrt{1 - \frac{v^2}{s_i^{\ 2}}} \qquad \text{for } i = 1 \text{ and } 2. \tag{22}$$

Under antiplane shear conditions, the piezoelectrically stiffened wave speeds, s_1 and s_2, are given by

$$s_i = \sqrt{\varepsilon_i / \rho}, \tag{23}$$

where

$$\varepsilon_1 = \left\{ \bar{c}_{44} + \overline{K} + \sqrt{(\bar{c}_{44} - \overline{K})^2 + 4\overline{R}^2} \right\}/2, \quad \varepsilon_2 = \left\{ \bar{c}_{44} + \overline{K} - \sqrt{(\bar{c}_{44} - \overline{K})^2 + 4\overline{R}^2} \right\}/2. \tag{24}$$

Then the phonon and phason stresses and electric displacement have non-zero components which can be deduced, using the constitutive equations (5) and (6), from Eqs. (17)–(19) in the forms.

$$\sigma_{xz}^{\mathrm{III}}(\xi, y) = -\frac{y}{2\pi}\left\{ b\left[\frac{1}{(\alpha^2 + \overline{R}^2)}\left\{ \frac{\beta_1\alpha(\bar{c}_{44}\alpha + \overline{R}^2)}{\xi^2 + \beta_1^{\ 2} y^2} + \frac{\beta_2\overline{R}^2(\bar{c}_{44} - \alpha)}{\xi^2 + \beta_2^{\ 2} y^2}\right\} - \frac{e_{15}^{\ 2}}{\varepsilon_{11}(\xi^2 + y^2)}\right]\right.$$

$$+d\left[\frac{\overline{R}}{(\alpha^2+\overline{R}^2)}\left\{\frac{\beta_1(\bar{c}_{44}\alpha+\overline{R}^2)}{\xi^2+\beta_1^2y^2}-\frac{\beta_2\alpha(\bar{c}_{44}-\alpha)}{\xi^2+\beta_2^2y^2}\right\}-\frac{e_{15}\,e'_{15}}{\varepsilon_{11}(\xi^2+y^2)}\right]+b_4\frac{e_{15}}{\xi^2+y^2}\Bigg\}, \tag{25}$$

$$\sigma_{yz}^{\mathrm{III}}(\xi,y)=\frac{\xi}{2\pi}\Bigg\{b\left[\frac{1}{(\alpha^2+\overline{R}^2)}\left\{\frac{\beta_1\alpha(\bar{c}_{44}\alpha+\overline{R}^2)}{\xi^2+\beta_1^2y^2}+\frac{\beta_2\overline{R}^2(\bar{c}_{44}-\alpha)}{\xi^2+\beta_2^2y^2}\right\}-\frac{e_{15}^{\ 2}}{\varepsilon_{11}(\xi^2+y^2)}\right]$$

$$+d\left[\frac{\overline{R}}{(\alpha^2+\overline{R}^2)}\left\{\frac{\beta_1(\bar{c}_{44}\alpha+\overline{R}^2)}{\xi^2+\beta_1^2y^2}-\frac{\beta_2\alpha(\bar{c}_{44}-\alpha)}{\xi^2+\beta_2^2y^2}\right\}-\frac{e_{15}\,e'_{15}}{\varepsilon_{11}(\xi^2+y^2)}\right]+b_4\frac{e_{15}}{\xi^2+y^2}\Bigg\}, \tag{26}$$

$$H_{zx}^{\mathrm{III}}(\xi,y)=-\frac{y}{2\pi}\Bigg\{b\left[\frac{\overline{R}}{(\alpha^2+\overline{R}^2)}\left\{\frac{\beta_1\alpha(\alpha+\overline{K})}{\xi^2+\beta_1^2y^2}-\frac{\beta_2(\alpha\overline{K}-\overline{R}^2)}{\xi^2+\beta_2^2y^2}\right\}-\frac{e_{15}\,e'_{15}}{\varepsilon_{11}(\xi^2+y^2)}\right]$$

$$+d\left[\frac{1}{(\alpha^2+\overline{R}^2)}\left\{\frac{\beta_1\overline{R}^2(\alpha+\overline{K})}{\xi^2+\beta_1^2y^2}+\frac{\beta_2\alpha(\alpha\overline{K}-\overline{R}^2)}{\xi^2+\beta_2^2y^2}\right\}-\frac{e'^{\,2}_{15}}{\varepsilon_{11}(\xi^2+y^2)}\right]+b_4\frac{e'_{15}}{\xi^2+y^2}\Bigg\}, \tag{27}$$

$$H_{zy}^{\mathrm{III}}(\xi,y)=\frac{\xi}{2\pi}\Bigg\{b\left[\frac{\overline{R}}{(\alpha^2+\overline{R}^2)}\left\{\frac{\beta_1\alpha(\alpha+\overline{K})}{\xi^2+\beta_1^2y^2}-\frac{\beta_2(\alpha\overline{K}-\overline{R}^2)}{\xi^2+\beta_2^2y^2}\right\}-\frac{e_{15}\,e'_{15}}{\varepsilon_{11}(\xi^2+y^2)}\right]$$

$$+d\left[\frac{1}{(\alpha^2+\overline{R}^2)}\left\{\frac{\beta_1\overline{R}^2(\alpha+\overline{K})}{\xi^2+\beta_1^2y^2}+\frac{\beta_2\alpha(\alpha\overline{K}-\overline{R}^2)}{\xi^2+\beta_2^2y^2}\right\}-\frac{e'^{\,2}_{15}}{\varepsilon_{11}(\xi^2+y^2)}\right]+b_4\frac{e'_{15}}{\xi^2+y^2}\Bigg\}, \tag{28}$$

$$D_x^{\mathrm{III}}(\xi,y)=-\frac{y}{2\pi}\frac{be_{15}+de'_{15}-b_4\varepsilon_{11}}{\xi^2+y^2}, \tag{29}$$

$$D_y^{\mathrm{III}}(\xi,y)=\frac{\xi}{2\pi}\frac{be_{15}+de'_{15}-b_4\varepsilon_{11}}{\xi^2+y^2}. \tag{30}$$

If required, the corresponding components of the phonon and phason strain and electric field can be deduced analogously using Eqs. (17)–(19) together with Eqs. (10)–(12).

4. Solution by Extending the Technique of Dislocation Layers

The realization that an equivalent continuous planar array of elastic dislocations usefully models a strip crack in classical elastic materials inspired the development of the so-called "dislocation layer method", as conveniently summarized, for example, by Bilby and Eshelby [19] and Lardner [20]. This fundamental method is suitably extended here to represent the moving row of shear cracks in piezoelectric quasicrystals by appropriately spreading a distribution of moving piezoelectric quasicrystal screw dislocations of the kind introduced in Section 3 above.

To the right of each crack the screws are positive and they are negative to the left. The distributions of the components of the phonon and phason displacements and the electric potential therefore have discontinuities with densities $f(\xi)$, $g(\xi)$ and $f_4(\xi)$, respectively, which are stipulated as being odd functions of ξ.

The corresponding phonon and phason stresses and electric displacement then have relevant components at a point on the ξ-axis given by Eqs. (26), (28) and (30) as

$$\sigma_{yz}^{\mathrm{III}}(\xi,0)=\frac{b}{2\pi}\left[\left\{\frac{\beta_1\alpha(\bar{c}_{44}\alpha+\bar{R}^2)+\beta_2\bar{R}^2(\bar{c}_{44}-\alpha)}{\alpha^2+\bar{R}^2}\right\}-\frac{e_{15}^{\;2}}{\varepsilon_{11}}\right]\int_{-\infty}^{\infty}\frac{f(\xi')}{\xi-\xi'}d\xi'$$

$$+\frac{d}{2\pi}\left[\bar{R}\left\{\frac{\beta_1(\bar{c}_{44}\alpha+\bar{R}^2)-\beta_2\alpha(\bar{c}_{44}-\alpha)}{\alpha^2+\bar{R}^2}\right\}-\frac{e_{15}\,e'_{15}}{\varepsilon_{11}}\right]\int_{-\infty}^{\infty}\frac{g(\xi')}{\xi-\xi'}d\xi'+\frac{b_4e_{15}}{2\pi}\int_{-\infty}^{\infty}\frac{f_4(\xi')}{\xi-\xi'}d\xi', \tag{31}$$

$$H_{zy}^{\mathrm{III}}(\xi,0)=\frac{b}{2\pi}\left[\bar{R}\left\{\frac{\beta_1\alpha(\alpha+\bar{K})-\beta_2(\alpha\bar{K}-\bar{R}^2)}{\alpha^2+\bar{R}^2}\right\}-\frac{e_{15}\,e'_{15}}{\varepsilon_{11}}\right]\int_{-\infty}^{\infty}\frac{f(\xi')}{\xi-\xi'}d\xi'$$

$$+\frac{d}{2\pi}\left[\left\{\frac{\beta_1\bar{R}^2(\alpha+\bar{K})+\beta_2\alpha(\alpha\bar{K}-\bar{R}^2)}{\alpha^2+\bar{R}^2}\right\}-\frac{e'^{\;2}_{15}}{\varepsilon_{11}}\right]\int_{-\infty}^{\infty}\frac{g(\xi')}{\xi-\xi'}d\xi'+\frac{b_4e'_{15}}{2\pi}\int_{-\infty}^{\infty}\frac{f_4(\xi')}{\xi-\xi'}d\xi', \tag{32}$$

$$D_{y}^{\mathrm{III}}(\xi,0)=\frac{be_{15}}{2\pi}\int_{-\infty}^{\infty}\frac{f(\xi')}{\xi-\xi'}d\xi'+\frac{de'_{15}}{2\pi}\int_{-\infty}^{\infty}\frac{g(\xi')}{\xi-\xi'}d\xi'-\frac{b_4\varepsilon_{11}}{2\pi}\int_{-\infty}^{\infty}\frac{f_4(\xi')}{\xi-\xi'}d\xi'. \tag{33}$$

The Plemelj formulae indicate that a Cauchy principal value interpretation must be assigned to the improper integrals in Eqs. (31)–(33).

To satisfy the imposed conditions (16), the dislocation array must fulfil the equilibrium equations that

$$\sigma_{yz}^{\mathrm{III}}(\xi,0)\;=\;-\mathcal{T},\qquad H_{zy}^{\mathrm{III}}(\xi,0)\;=\;-\mathcal{H},\quad D_{y}^{\mathrm{III}}(\xi,0)\;=\;-\mathcal{D}\quad\text{in}\;\;R_t. \tag{34}$$

The system of three simultaneous equations which follows by substituting the representations (31)–(33) into Eqs. (34) can be solved to yield three singular integral equations for the densities $f(\xi)$, $g(\xi)$ and $f_4(\xi)$ that, after considerable, intricate manipulation and algebraic simplification, can be written concisely as

$$\int_{-\infty}^{\infty}\frac{f(\xi')}{\xi-\xi'}d\xi'=-\frac{2\pi}{b\beta_1\beta_2\varepsilon_{11}(\bar{c}_{44}\bar{K}-\bar{R}^2)(\alpha^2+\bar{R}^2)}\times$$

$$\Bigg[\;\varepsilon_{11}\left\{\beta_1\bar{R}^2(\alpha+\bar{K})+\beta_2\alpha(\alpha\bar{K}-\bar{R}^2)\right\}\mathcal{T}-\varepsilon_{11}\bar{R}\left\{\beta_1(\bar{c}_{44}\alpha+\bar{R}^2)-\beta_2\alpha(\bar{c}_{44}-\alpha)\right\}\mathcal{H}$$

$$-\left[\;e'_{15}\bar{R}\left\{\beta_1(\bar{c}_{44}\alpha+\bar{R}^2)-\beta_2\alpha(\bar{c}_{44}-\alpha)\right\}-e_{15}\left\{\beta_1\bar{R}^2(\alpha+\bar{K})+\beta_2\alpha(\alpha\bar{K}-\bar{R}^2)\right\}\;\right]\mathcal{D}\;\Bigg], \tag{35}$$

$$\int_{-\infty}^{\infty}\frac{g(\xi')}{\xi-\xi'}d\xi'=-\frac{2\pi}{d\beta_1\beta_2\varepsilon_{11}(\bar{c}_{44}\bar{K}-\bar{R}^2)(\alpha^2+\bar{R}^2)}\times$$

$$\left[-\varepsilon_{11}\overline{R}\left\{\beta_1\alpha(\alpha+\overline{K})-\beta_2(\alpha\overline{K}-\overline{R}^2)\right\}\mathcal{T}+\varepsilon_{11}\left\{\beta_1\alpha(\overline{c}_{44}\alpha+\overline{R}^2)+\beta_2\overline{R}^2(\overline{c}_{44}-\alpha)\right\}\mathcal{H}\right.$$

$$\left. +\left[e'_{15}\left\{\beta_1\alpha(\overline{c}_{44}\alpha+\overline{R}^2)+\beta_2\overline{R}^2(\overline{c}_{44}-\alpha)\right\}-e_{15}\overline{R}\left\{\beta_1\alpha(\alpha+\overline{K})-\beta_2(\alpha\overline{K}-\overline{R}^2)\right\}\right]\mathcal{D}\right], \quad (36)$$

$$\int_{-\infty}^{\infty}\frac{f_4(\xi')}{\xi-\xi'}d\xi' = -\frac{2\pi}{b_4\beta_1\beta_2\varepsilon_{11}(\overline{c}_{44}\overline{K}-\overline{R}^2)(\alpha^2+\overline{R}^2)}\times$$

$$\left\{ -\left[e'_{15}\overline{R}\left\{\beta_1\alpha(\alpha+\overline{K})-\beta_2(\alpha\overline{K}-\overline{R}^2)\right\}-e_{15}\left\{\beta_1\overline{R}^2(\alpha+\overline{K})+\beta_2\alpha(\alpha\overline{K}-\overline{R}^2)\right\}\right]\mathcal{T}\right.$$

$$+\left[e'_{15}\left\{\beta_1\alpha(\overline{c}_{44}\alpha+\overline{R}^2)+\beta_2\overline{R}^2(\overline{c}_{44}-\alpha)\right\}-e_{15}\overline{R}\left\{\beta_1(\overline{c}_{44}\alpha+\overline{R}^2)-\beta_2\alpha(\overline{c}_{44}-\alpha)\right\}\right]\mathcal{H}$$

$$-\left[\beta_1\beta_2(\overline{c}_{44}\overline{K}-\overline{R}^2)(\alpha^2+\overline{R}^2)-\frac{e_{15}^2}{\varepsilon_{11}}\left\{\beta_1\overline{R}^2(\alpha+\overline{K})+\beta_2\alpha(\alpha\overline{K}-\overline{R}^2)\right\}\right.$$

$$-\frac{e'^2_{15}}{\varepsilon_{11}}\left\{\beta_1\alpha(\overline{c}_{44}\alpha+\overline{R}^2)+\beta_2\overline{R}^2(\overline{c}_{44}-\alpha)\right\}+\frac{e_{15}\,e'_{15}}{\varepsilon_{11}}\overline{R}\left(\beta_1\left\{\alpha(\alpha+\overline{K})+(\overline{c}_{44}\alpha+\overline{R}^2)\right\}\right.$$

$$\left.\left.\left.-\beta_2\left\{\alpha(\overline{c}_{44}-\alpha)+(\alpha\overline{K}-\overline{R}^2)\right\}\right)\right]\mathcal{D}\right\}. \quad (37)$$

However, the infinite integrals arising in Eqs. (35)–(37) can be rewritten as the summation of the separate contributions from the individual cracks.

For example, by recalling the stipulation that the density functions must be odd, so that $f(\xi)=-f(-\xi)$, and observing that $f(\xi+2nh)=f(\xi)$, it follows using the relationship

$$\sum_{n=1}^{\infty}\frac{1}{z^2-n^2} = -\frac{1}{2z^2}+\frac{\pi}{2z}\cot\pi z, \quad (38)$$

together with partial fractions, that the left-hand side of Eq. (35) can be expressed alternatively (cf. Leibfried [21]) as

$$\frac{\pi}{2h}\int_{-c}^{c}\frac{\cos(\pi\xi'/2h)}{\sin(\pi\xi/2h)-\sin(\pi\xi'/2h)}f(\xi')\,d\xi'. \quad (39)$$

Further, this can be more concisely rewritten in terms of new variables given by

$$\xi_1 = \sin(\pi\xi/2h), \qquad c_1 = \sin(\pi c/2h), \qquad \xi'_1 = \sin(\pi\xi'/2h), \quad (40)$$

so that finally the integral equation (35) for $f(\xi)$ is converted into the convenient form

$$\int_{-c_1}^{c_1} \frac{f_1(\xi_1')}{\xi_1 - \xi_1'} d\xi_1' = -\frac{2\pi}{b\beta_1\beta_2\varepsilon_{11}(\bar{c}_{44}\bar{K} - \bar{R}^2)(\alpha^2 + \bar{R}^2)} \times$$

$$\Big[\varepsilon_{11}\Big\{ \beta_1\bar{R}^2(\alpha + \bar{K}) + \beta_2\alpha(\alpha\bar{K} - \bar{R}^2)\Big\}\mathcal{T} - \varepsilon_{11}\bar{R}\Big\{ \beta_1(\bar{c}_{44}\alpha + \bar{R}^2) - \beta_2\alpha(\bar{c}_{44} - \alpha)\Big\}\mathcal{H}$$

$$- \Big[e'_{15}\bar{R}\Big\{ \beta_1(\bar{c}_{44}\alpha + \bar{R}^2) - \beta_2\alpha(\bar{c}_{44} - \alpha)\Big\} - e_{15}\Big\{ \beta_1\bar{R}^2(\alpha + \bar{K}) + \beta_2\alpha(\alpha\bar{K} - \bar{R}^2)\Big\}\Big]\mathcal{D}\Big],$$

$$\text{for} \quad |\xi_1| < c_1, \tag{41}$$

with $f_1(\xi_1) = f_1(\sin(\pi\xi/2h)) \equiv f(\xi)$. The relevant solution of this is readily deducible by the classical methods of Muskhelishvili [22], and Gakhov [23], for example, to be

$$f_1(\xi_1) = -\frac{2}{\pi b\beta_1\beta_2\varepsilon_{11}(\bar{c}_{44}\bar{K} - \bar{R}^2)(\alpha^2 + \bar{R}^2)} \times$$

$$\Big[\varepsilon_{11}\Big\{ \beta_1\bar{R}^2(\alpha + \bar{K}) + \beta_2\alpha(\alpha\bar{K} - \bar{R}^2)\Big\}\mathcal{T} - \varepsilon_{11}\bar{R}\Big\{ \beta_1(\bar{c}_{44}\alpha + \bar{R}^2) - \beta_2\alpha(\bar{c}_{44} - \alpha)\Big\}\mathcal{H}$$

$$- \Big[e'_{15}\bar{R}\Big\{ \beta_1(\bar{c}_{44}\alpha + \bar{R}^2) - \beta_2\alpha(\bar{c}_{44} - \alpha)\Big\} - e_{15}\Big\{ \beta_1\bar{R}^2(\alpha + \bar{K}) + \beta_2\alpha(\alpha\bar{K} - \bar{R}^2)\Big\}\Big]\mathcal{D}\Big] \times$$

$$\frac{1}{\left(c_1^2 - \xi_1^2\right)^{1/2}} \int_{-c_1}^{c_1} \frac{\left(c_1^2 - \xi_1'^2\right)^{1/2}}{\xi_1' - \xi_1} d\xi_1' \tag{42}$$

$$= \frac{2}{b\beta_1\beta_2\varepsilon_{11}(\bar{c}_{44}\bar{K} - \bar{R}^2)(\alpha^2 + \bar{R}^2)} \times$$

$$\Big[\varepsilon_{11}\Big\{ \beta_1\bar{R}^2(\alpha + \bar{K}) + \beta_2\alpha(\alpha\bar{K} - \bar{R}^2)\Big\}\mathcal{T} - \varepsilon_{11}\bar{R}\Big\{ \beta_1(\bar{c}_{44}\alpha + \bar{R}^2) - \beta_2\alpha(\bar{c}_{44} - \alpha)\Big\}\mathcal{H}$$

$$- \Big[e'_{15}\bar{R}\Big\{ \beta_1(\bar{c}_{44}\alpha + \bar{R}^2) - \beta_2\alpha(\bar{c}_{44} - \alpha)\Big\} - e_{15}\Big\{ \beta_1\bar{R}^2(\alpha + \bar{K}) + \beta_2\alpha(\alpha\bar{K} - \bar{R}^2)\Big\}\Big]\mathcal{D}\Big] \times$$

$$\frac{\xi_1}{\left(c_1^2 - \xi_1^2\right)^{1/2}}, \tag{43}$$

which, by reverting to the original variables using Eqs. (40), becomes

$$f(\xi) = \frac{2}{b\beta_1\beta_2\varepsilon_{11}(\bar{c}_{44}\bar{K} - \bar{R}^2)(\alpha^2 + \bar{R}^2)} \times$$

$$\Big[\varepsilon_{11}\Big\{ \beta_1\bar{R}^2(\alpha + \bar{K}) + \beta_2\alpha(\alpha\bar{K} - \bar{R}^2)\Big\}\mathcal{T} - \varepsilon_{11}\bar{R}\Big\{ \beta_1(\bar{c}_{44}\alpha + \bar{R}^2) - \beta_2\alpha(\bar{c}_{44} - \alpha)\Big\}\mathcal{H}$$

$$-\left[\ e'_{15}\,\bar{R}\left\{\beta_1(\bar{c}_{44}\alpha+\bar{R}^2)-\beta_2\alpha(\bar{c}_{44}-\alpha)\right\}-e_{15}\left\{\beta_1\bar{R}^2(\alpha+\bar{K})+\beta_2\alpha(\alpha\bar{K}-\bar{R}^2)\right\}\ \right]\mathcal{D}\Bigg]\times$$

$$\frac{\sin(\pi\xi/2h)}{\left[\sin^2(\pi c/2h)-\sin^2(\pi\xi/2h)\right]^{1/2}}. \tag{44}$$

The phason and electric densities satisfying the Eqs. (36) and (37) can be deduced analogously to be

$$g(\xi)\ =\ \frac{2}{d\beta_1\beta_2\varepsilon_{11}(\bar{c}_{44}\bar{K}-\bar{R}^2)(\alpha^2+\bar{R}^2)}\ \times$$

$$\Bigg[-\varepsilon_{11}\bar{R}\left\{\beta_1\alpha(\alpha+\bar{K})-\beta_2(\alpha\bar{K}-\bar{R}^2)\right\}\mathcal{T}+\varepsilon_{11}\left\{\beta_1\alpha(\bar{c}_{44}\alpha+\bar{R}^2)+\beta_2\bar{R}^2(\bar{c}_{44}-\alpha)\right\}\mathcal{H}$$

$$+\left[\ e'_{15}\left\{\beta_1\alpha(\bar{c}_{44}\alpha+\bar{R}^2)+\beta_2\bar{R}^2(\bar{c}_{44}-\alpha)\right\}-e_{15}\bar{R}\left\{\beta_1\alpha(\alpha+\bar{K})-\beta_2(\alpha\bar{K}-\bar{R}^2)\right\}\ \right]\mathcal{D}\Bigg]\times$$

$$\frac{\sin(\pi\xi/2h)}{\left[\sin^2(\pi c/2h)-\sin^2(\pi\xi/2h)\right]^{1/2}} \tag{45}$$

and

$$f_4(\xi)=\frac{2}{b_4\beta_1\beta_2\varepsilon_{11}(\bar{c}_{44}\bar{K}-\bar{R}^2)(\alpha^2+\bar{R}^2)}\ \times$$

$$\Bigg\{-\left[e'_{15}\,\bar{R}\left\{\beta_1\alpha(\alpha+\bar{K})-\beta_2(\alpha\bar{K}-\bar{R}^2)\right\}-e_{15}\left\{\beta_1\bar{R}^2(\alpha+\bar{K})+\beta_2\alpha(\alpha\bar{K}-\bar{R}^2)\right\}\ \right]\mathcal{T}$$

$$+\left[e'_{15}\left\{\beta_1\alpha(\bar{c}_{44}\alpha+\bar{R}^2)+\beta_2\bar{R}^2(\bar{c}_{44}-\alpha)\right\}-e_{15}\bar{R}\left\{\beta_1(\bar{c}_{44}\alpha+\bar{R}^2)-\beta_2\alpha(\bar{c}_{44}-\alpha)\right\}\ \right]\mathcal{H}$$

$$-\Bigg[\beta_1\beta_2(\bar{c}_{44}\bar{K}-\bar{R}^2)(\alpha^2+\bar{R}^2)-\frac{e_{15}^{\,2}}{\varepsilon_{11}}\left\{\beta_1\bar{R}^2(\alpha+\bar{K})+\beta_2\alpha(\alpha\bar{K}-\bar{R}^2)\right\}$$

$$-\frac{e'^{\,2}_{15}}{\varepsilon_{11}}\left\{\beta_1\alpha(\bar{c}_{44}\alpha+\bar{R}^2)+\beta_2\bar{R}^2(\bar{c}_{44}-\alpha)\right\}+\frac{e_{15}\,e'_{15}}{\varepsilon_{11}}\bar{R}\,\Big(\beta_1\left\{\alpha(\alpha+\bar{K})+(\bar{c}_{44}\alpha+\bar{R}^2)\right\}$$

$$-\beta_2\left\{\alpha(\bar{c}_{44}-\alpha)+(\alpha\bar{K}-\bar{R}^2)\right\}\Big)\Bigg]\mathcal{D}\Bigg\}\ \frac{\sin(\pi\xi/2h)}{\left[\sin^2(\pi c/2h)-\sin^2(\pi\xi/2h)\right]^{1/2}}. \tag{46}$$

With the necessary densities $f(\xi)$, $g(\xi)$ and $f_4(\xi)$ now determined, explicit expressions for any of the components of the phonon and phason stress and electric fields which are required can be deduced from Eqs. (25)–(30) and (44)–(46). These generally depend not only upon the values of the loads $\mathcal{T}$, $\mathcal{H}$ and $\mathcal{D}$ which are applied and the geometric constants *c* and *h*, but also upon the piezoelectric quasicrystal material moduli and the speed *v* of the cracks.

However, in considering practical criteria for the fracture of materials, traditionally there is especial interest in the components' magnitudes directly in front of the tip of a crack.

The component of the phonon shear stress on the y = 0 plane, for example, is obtained as the sum of the contributions from all the image cracks. Using a summation of finite integrals as a replacement for the resulting infinite integral and recalling Eqs. (38) and (39) gives

$$\begin{aligned}
\sigma_{yz}(\xi,0) \;&=\; \mathcal{T}+\frac{b}{2\pi}\left[\left\{\frac{\beta_1\alpha(\bar{c}_{44}\alpha+\bar{R}^2)+\beta_2\bar{R}^2(\bar{c}_{44}-\alpha)}{\alpha^2+\bar{R}^2}\right\}-\frac{e_{15}{}^2}{\varepsilon_{11}}\right]\int_{-c_1}^{c_1}\frac{f_1(\xi_1')}{\xi_1-\xi_1'}d\xi_1' \\
&+\frac{d}{2\pi}\left[\bar{R}\left\{\frac{\beta_1(\bar{c}_{44}\alpha+\bar{R}^2)-\beta_2\alpha(\bar{c}_{44}-\alpha)}{\alpha^2+\bar{R}^2}\right\}-\frac{e_{15}\,e'_{15}}{\varepsilon_{11}}\right]\int_{-c_1}^{c_1}\frac{g_1(\xi_1')}{\xi_1-\xi_1'}d\xi' \\
&+\frac{b_4e_{15}}{2\pi}\int_{-c_1}^{c_1}\frac{f_{41}(\xi')}{\xi-\xi'}d\xi',
\end{aligned} \tag{47}$$

where $g_1(\xi_1)=g_1(\sin(\pi\xi/2h))\equiv g(\xi)$, $f_{41}(\xi_1)=f_{41}(\sin(\pi\xi/2h))\equiv f_4(\xi)$. The substitution into this of the expressions (44)–(46) for f_1, g_1 and f_{41} appears to yield extremely unwieldy terms, but extensive rearrangement and manipulation leads to

$$\sigma_{yz}(\xi,0) \;=\; \mathcal{T}+\frac{\mathcal{T}}{\pi}\int_{-c_1}^{c_1}\frac{\xi_1'}{\left(c_1{}^2-\xi_1'^2\right)^{1/2}}\frac{d\xi_1'}{\xi_1-\xi_1'}. \tag{48}$$

Evaluation of the involved integral using the method of contour integration in complex variable analysis and the relationships (40) ultimately gives

$$\sigma_{yz}(\xi,0) \;=\; \mathcal{T}\,\frac{\sin(\pi\xi/2h)}{\left[\sin^2(\pi\xi/2h)-\sin^2(\pi c/2h)\right]^{1/2}}. \tag{49}$$

Analogously, the components of the phason stress and electric displacement on the y = 0 plane are found to be

$$\begin{aligned}
H_{zy}(\xi,0) \;&=\; \mathcal{H}+\frac{b}{2\pi}\left[\bar{R}\left\{\frac{\beta_1\alpha(\alpha+\bar{K})-\beta_2(\alpha\bar{K}-\bar{R}^2)}{\alpha^2+\bar{R}^2}\right\}-\frac{e_{15}\,e'_{15}}{\varepsilon_{11}}\right]\int_{-c_1}^{c_1}\frac{f_1(\xi_1')}{\xi_1-\xi_1'}d\xi' \\
&+\frac{d}{2\pi}\left[\left\{\frac{\beta_1\bar{R}^2(\alpha+\bar{K})+\beta_2\alpha(\alpha\bar{K}-\bar{R}^2)}{\alpha^2+\bar{R}^2}\right\}-\frac{e'_{15}{}^2}{\varepsilon_{11}}\right]\int_{-c_1}^{c_1}\frac{g_1(\xi_1')}{\xi_1-\xi_1'}d\xi'\Bigg] \\
&+\frac{b_4e'_{15}}{2\pi}\int_{-c_1}^{c_1}\frac{f_{41}(\xi_1')}{\xi_1-\xi_1'}d\xi' \\
&=\; \mathcal{H}\,\frac{\sin(\pi\xi/2h)}{\left[\sin^2(\pi\xi/2h)-\sin^2(\pi c/2h)\right]^{1/2}},
\end{aligned} \tag{50}$$

$$D_y(\xi,0) = \mathcal{D} + \frac{be_{15}}{2\pi}\int_{-c_1}^{c_1}\frac{f_1(\xi_1')}{\xi_1-\xi_1'}d\xi' + \frac{de'_{15}}{2\pi}\int_{-c_1}^{c_1}\frac{g_1(\xi_1')}{\xi_1-\xi_1'}d\xi' - \frac{b_4\varepsilon_{11}}{2\pi}\int_{-c_1}^{c_1}\frac{f_{41}(\xi_1')}{\xi_1-\xi_1'}d\xi'$$

$$= \mathcal{D}\,\frac{\sin(\pi\xi/2h)}{\left[\sin^2(\pi\xi/2h)-\sin^2(\pi c/2h)\right]^{1/2}}. \tag{51}$$

It is appropriate to extend the classical notion of a stress intensity factor for a stationary crack within a purely elastic material to moving cracks in piezoelectric quasicrystals. The phonon and phason stress and electric displacement intensity factors, $K_{\mathcal{T}}$, $K_{\mathcal{H}}$ and $K_{\mathcal{D}}$, respectively, can thus be defined and evaluated, from Eqs. (49)–(51), in the forms

$$K_{\mathcal{T}} = \lim_{\xi\to c+}\ (\xi-c)^{1/2}\sigma_{yz}(\xi,0) = \mathcal{T}\left[\frac{h}{\pi}\tan\left(\frac{\pi c}{2h}\right)\right]^{\frac{1}{2}}, \tag{52}$$

$$K_{\mathcal{H}} = \lim_{\xi\to c+}\ (\xi-c)^{1/2}H_{zy}(\xi,0) = \mathcal{H}\left[\frac{h}{\pi}\tan\left(\frac{\pi c}{2h}\right)\right]^{\frac{1}{2}}, \tag{53}$$

$$K_{\mathcal{D}} = \lim_{\xi\to c+}\ (\xi-c)^{1/2}D_y(\xi,0) = \mathcal{D}\left[\frac{h}{\pi}\tan\left(\frac{\pi c}{2h}\right)\right]^{\frac{1}{2}}. \tag{54}$$

As observed above, the density functions given by Eqs. (44)–(46) depend upon the values of the piezoelectric quasicrystal material constants, as well as the loads applied and the parameters *c*, *h* and *v*. However, it is of interest to note that on the otherhand the stress intensity factors, $K_{\mathcal{T}}$, $K_{\mathcal{H}}$ and $K_{\mathcal{D}}$, given by Eqs. (52)–(54), each depend only upon *c* and *h* and the respective applied loads $\mathcal{T}$, $\mathcal{H}$, $\mathcal{D}$.

Curves are depicted in Figure 2 to representatively illustrate graphically the variation with ξ/c of the non-dimensional, scaled phonon stress component $\sigma_{yz}(\xi,0)/\mathcal{T}$ (and likewise those of $H_{zy}(\xi,0)/\mathcal{H}$ and $D_y(\xi,0)/\mathcal{D}$) for various values of c/h. The curves display the same basic shapes for different values of c/h. But as the ratio c/h increases in magnitude the intensity of $\sigma_{yz}(\xi,0)/\mathcal{T}$ also increases.

It is apparent, from Eqs. (44)–(46), that the analysis above is invalid when $\bar{c}_{44}\bar{K}-\bar{R}^2=0$ or $\beta_1=0$ or $\beta_2=0$. From the definition (22), the value of β_i is zero when the crack speed, *v*, is equal to the wave speed s_i, for *i* = 1, 2, under antiplane shear conditions given by Eq. (23).

The reported data for the values of the material constants of one-dimensional hexagonal piezoelectric quasicrystals are still somewhat variable. However, typically Li et al. [9] report $c_{44}=5.0\times10^{10}\,\mathrm{Nm}^{-2}$, $R=1.2\times10^{9}\,\mathrm{Nm}^{-2}$, $K=3.0\times10^{8}\,\mathrm{Nm}^{-2}$, $e_{15}=-0.138\ \mathrm{Cm}^{-2}$, $e'_{15}=-0.160\ \mathrm{Cm}^{-2}$, $\varepsilon_{11}=82.6\times10^{-12}\ \mathrm{C^2N^{-1}m^{-2}}$, which, with representatively $\rho=5.1\times10^{3}\ \mathrm{kg\,m^{-3}}$, give these wave speeds to be $s_1\approx3139\,\mathrm{ms}^{-1}$ and $s_2\approx333\,\mathrm{ms}^{-1}$. Moreover, it is clear that $\bar{c}_{44}\bar{K}-\bar{R}^2$ does not vanish, since the value of $\bar{c}_{44}\bar{K}$ is much greater than that of $\bar{R}^2$.

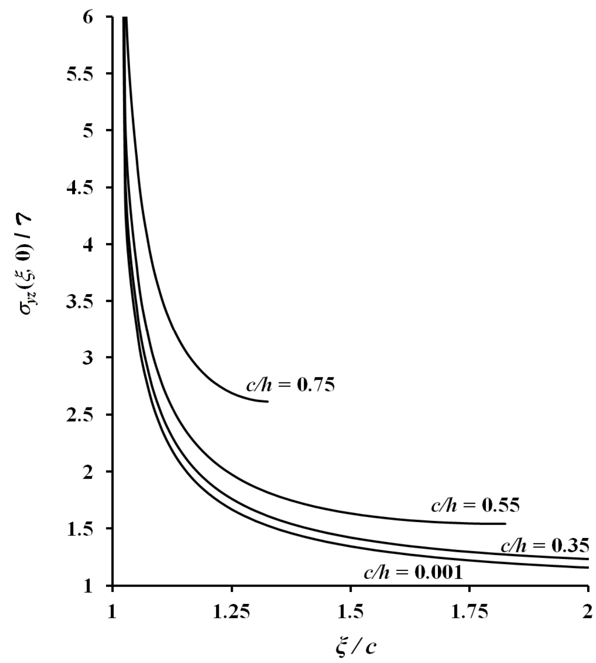

Figure 2. Variation with ξ / c of the scaled stress component $\sigma_{yz}(\xi, 0) / \mathcal{T}$ for a range of values of c / h.

Finally, an analysis of a row of stationary cracks in piezoelectric quasicrystals has not been presented previously and therefore it is worthwhile highlighting briefly the considerably simplified results as a special case of the current analysis. When $v = 0$, from Eq. (22), $\beta_1 = \beta_2 = 1$ and then the densities given by Eqs. (44)–(46) simply become

$$f(x) = -\frac{2}{b\varepsilon_{11}(\bar{c}_{44}\bar{K} - \bar{R}^2)}\{-\varepsilon_{11}\bar{K}\mathcal{T} + \varepsilon_{11}\bar{R}\mathcal{H} + (e'_{15}\bar{R} - e_{15}\bar{K})\mathcal{D}\}\frac{\sin(\pi x/2h)}{\left[\sin^2(\pi c/2h) - \sin^2(\pi x/2h)\right]^{1/2}} \quad (55)$$

$$g(x) = -\frac{2}{d\varepsilon_{11}(\bar{c}_{44}\bar{K} - \bar{R}^2)}\{\varepsilon_{11}\bar{R}\mathcal{T} - \varepsilon_{11}\bar{c}_{44}\mathcal{H} - (e'_{15}\bar{c}_{44} - e_{15}\bar{R})\mathcal{D}\}\frac{\sin(\pi x/2h)}{\left[\sin^2(\pi c/2h) - \sin^2(\pi x/2h)\right]^{1/2}} \quad (56)$$

$$f_4(x) = -\frac{2}{b_4\varepsilon_{11}(\bar{c}_{44}\bar{K} - \bar{R}^2)}\ \{(e'_{15}\bar{R} - e_{15}\bar{K})\mathcal{T} - (e'_{15}\bar{c}_{44} - e_{15}\bar{R})\mathcal{H}$$

$$+ (\bar{c}_{44}\bar{K} - \bar{R}^2 - \frac{e_{15}^{\ 2}}{\varepsilon_{11}}\bar{K} - \frac{e'^{\ 2}_{15}}{\varepsilon_{11}}\bar{c}_{44} + 2\frac{e_{15}e'_{15}}{\varepsilon_{11}}\bar{R})\mathcal{D}\}\ \frac{\sin(\pi x/2h)}{\left[\sin^2(\pi c/2h) - \sin^2(\pi x/2h)\right]^{1/2}}, \quad (57)$$

and these enable any of the components of the phonon, phason and electric fields around the stationary cracks to be derived as desired.

5. Conclusions

The analysis is focussed on adapting and extending the traditional dislocation layer method of elastic media for studying cracks in one-dimensional piezoelectric quasicrystals.

Explicit expressions for the phonon, phason and electric field components of an infinite row of mode III, moving Yoffe-type constant-width cracks in piezoelectric quasicrystals are derived. Illustrative graphs of the variation of the components of the field along the axis ahead of a crack tip are displayed for a range of the geometric parameters.

The phonon, phason and electric densities are distributed antisymmetrically about the planes $x = \pm h$ and thus clearly by an image construction the components of the phonon and phason stress and the electric displacement, σ_{xz}, H_{zx} and D_x vanish on these planes. Therefore the analysis above also describes the piezoelectric quasicrystal fields around a moving single crack originally occupying the region $-c < x < c$ of a finite piezoelectric quasicrystalline plate $-h < x < h$, with load-free surfaces $x = \pm h$.

Further, the solution for an edge crack $0 < x < c$ moving in a finite plate $0 < x < h$ is also provided, since the distributions are odd about $x = 0$.

Conflict of Interest

The author declares that there is no conflict of interest regarding the publication of this paper.

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