Research article

On discrete time Beverton-Holt population model with fuzzy environment

Qianhong Zhang[1,*]**, Fubiao Lin**[1] **and Xiaoying Zhong**[2]

[1] School of Mathematics and Statistics, Guizhou University of Finance and Economics, Guiyang, Guizhou 550025, China

[2] Library, Guizhou University of Finance and Economics, Guiyang, Guizhou 550025, China

* **Correspondence:** Email: zqianhong68@163.com.

Abstract: In this work, dynamical behaviors of discrete time Beverton-Holt population model with fuzzy parameters are studied. It provides a flexible model to fit population data. For three different fuzzy parameters and fuzzy initial conditions, according to a generalization of division (g-division) of fuzzy number, it can represent dynamical behaviors including boundedness, global asymptotical stability and persistence of positive solution. Finally, two examples are given to demonstrate the effectiveness of the results obtained.

Keywords: Beverton-Holt population model; fuzzy discrete system; boundedness; persistence; global asymptotic behavior

1. Introduction

Discrete time single-species model is the most appropriate mathematical description of life histories of organism whose reproduction occurs only once a year during a very short season. It is assumed that, in the rest of the year, the population is only subjected to mortality, but not to births. Therefore the between year dynamics is characterized by a first order difference equation $x_{n+1} = f(x_n)$, where x_n denotes population at the nth generation. The production function f is usually density-dependent, and the strength of density dependence is determined by several parameters including growth rate, the probability of surviving the reproductive season, the carrying capacity of surrounding environment, and intraspecific cooperation or competition factors. These models are widely used in fisheries and many organisms [1]. Beverton-Holt model [2] is one of classic population model which has been studied

$$x_{n+1} = \frac{\beta x_n}{1 + \delta x_n}, n = 0, 1, \cdots,$$

where x_n is population at the nth generation, β represents a productivity parameter, and δ controls the level of density dependence. Since then, many results on the model and the generation of the model

have been widely obtained by some researchers [3, 4, 5].

In fact, the identification of the parameters of the model is usually based on statistical method, starting from data experimentally obtained and on the choice of some method adapted to the identification. These models, even the classic deterministic approach, are subjected to inaccuracies (fuzzy uncertainty) that can be caused by the nature of the state variables, by parameters as coefficients of the model and by initial conditions.

In our real life, we have learned to deal with uncertainty. Scientists also accept the fact that uncertainty is very important study in most applications. Modeling the real life problems in such cases, usually involves vagueness or uncertainty in some of the parameters. The concept of fuzzy set and system was introduced by Zadeh [6] and its development has been growing rapidly to various situation of theory and application including fuzzy differential and fuzzy difference equations. It is well known that fuzzy difference equation is a difference equation whose parameters or the initial values are fuzzy numbers, and its solutions are sequences of fuzzy numbers. It has been used to model a dynamical systems under possibility uncertainty [7]. Due to the applicability of fuzzy difference equation for the analysis of phenomena where imprecision is inherent, this class of difference equation is a very important topic from theoretical point of view and also its applications. Recently there has been an increasing interest in the study of fuzzy difference equations (see [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]).

Inspired with the previous, by virtue of the theory of fuzzy difference equation, in this work, we consider the following discrete time Beverton-Holt model with fuzzy uncertainty parameters and initial conditions.

$$x_{n+1} = \frac{A x_n}{\tilde{1} + B x_n}, n = 0, 1, \cdots, \qquad (1.1)$$

where x_n is population at the nth generation, A denotes a productivity parameter, B controls the level of density dependence. Furthermore $A, \tilde{1}, B$ and the initial value x_0 are positive fuzzy numbers.

The main aim of this work is to study the existence of positive solutions of Beverton-Holt population model (1.1). Furthermore, according to a generation of division (g-division) of fuzzy numbers, we derive some conditions so that every positive solution of Beverton-Holt population model (1.1) is bounded and persistent. Finally, under some conditions we prove that Beverton-Holt population model (1.1) has a unique positive equilibrium x and every positive solution tends to x as $n \to \infty$.

2. Preliminary and definitions

Firstly, we give the following definitions.

Definition 2.1. [24] $u : R \to [0, 1]$ *is said to be a fuzzy number if it satisfies conditions (i)-(iv) written below:*

(i) u is normal, i. e., there exists an $x \in R$ such that $u(x) = 1$;

(ii) u is fuzzy convex, i. e., for all $t \in [0, 1]$ and $x_1, x_2 \in R$ such that

$$u(tx_1 + (1 - t)x_2) \geq \min\{u(x_1), u(x_2)\};$$

(iii) u is upper semicontinuous;

(iv) The support of u, $\text{supp}u = \overline{\bigcup_{\alpha \in (0,1]}[u]^\alpha} = \overline{\{x : u(x) > 0\}}$ is compact.

For $\alpha \in (0, 1]$, the α−cuts of fuzzy number u is $[u]_\alpha = \{x \in R : u(x) \geq \alpha\}$ and for $\alpha = 0$, the support of u is defined as $\text{supp}u = [u]^0 = \overline{\{x \in R | u(x) > 0\}}$.

Definition 2.2. Fuzzy Number (Parametric form): [24] *A fuzzy number u in a parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r), \overline{u}(r), 0 \le r \le 1$, which satisfies the following requirements:*
(1) $\underline{u}(r)$ is a bounded monotonic increasing left continuous function,
(2) $\overline{u}(r)$ is a bounded monotonic decreasing left continuous function,
(3) $\underline{u}(r) \le \overline{u}(r), 0 \le r \le 1$.

A crisp (real) number x is simply represented by $(\underline{u}(r), \overline{u}(r)) = (x, x), 0 \le r \le 1$. The fuzzy number space $\{(\underline{u}(r), \overline{u}(r))\}$ becomes a convex cone E^1 which could be embedded isomorphically and isometrically into a Banach space [24].

Definition 2.3. [24] *The distance between two arbitrary fuzzy numbers u and v is defined as follows:*

$$D(u, v) = \sup_{\alpha \in [0,1]} \max\{|u_{l,\alpha} - v_{l,\alpha}|, |u_{r,\alpha} - v_{r,\alpha}|\}. \tag{2.1}$$

It is clear that (E^1, D) is a complete metric space.

Definition 2.4. [24] *Let $u = (\underline{u}(r), \overline{u}(r)), v = (\underline{v}(r), \overline{v}(r)) \in E^1, 0 \le r \le 1$, and arbitrary $k \in R$. Then*
(i) $u = v$ iff $\underline{u}(r) = \underline{v}(r), \overline{u}(r) = \overline{v}(r)$,
(ii) $u + v = (\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r))$,
(iii) $u - v = (\underline{u}(r) - \overline{v}(r), \overline{u}(r) - \underline{v}(r))$,
(iv) $ku = \begin{cases} (k\underline{u}(r), k\overline{u}(r)), & k \ge 0; \\ (k\overline{u}(r), k\underline{u}(r)), & k < 0, \end{cases}$
(v) $uv = (\min\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\overline{v}(r), \overline{u}(r)\underline{v}(r), \overline{u}(r)\overline{v}(r)\}, \max\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\overline{v}(r), \overline{u}(r)\underline{v}(r), \overline{u}(r)\overline{v}(r)\}).$

Definition 2.5. Triangular Fuzzy Number. [24] *A triangular fuzzy number (TFN) denoted by A is defined as (a, b, c) where the membership function*

$$A(x) = \begin{cases} 0, & x \le a; \\ \frac{x-a}{b-a}, & a \le x \le b; \\ 1, & x = b; \\ \frac{c-x}{c-b}, & b \le x \le c; \\ 0, & x \ge c. \end{cases}$$

The $\alpha-$cuts of $A = (a, b, c)$ are described by $[A]^\alpha = \{x \in R : A(x) \ge \alpha\} = [a + \alpha(b - a), c - \alpha(c - b)] = [A_{l,\alpha}, A_{r,\alpha}], \alpha \in [0, 1]$, it is clear that the $[A]^\alpha$ are closed interval. A fuzzy number is positive if supp$A \subset (0, \infty)$.

The following proposition is fundamental since it characterizes a fuzzy set through the α-levels.

Proposition 2.1 [24] *If $\{A^\alpha : \alpha \in [0, 1]\}$ is a compact, convex and not empty subset family of R^n such that*
(i) $\overline{\bigcup A^\alpha} \subset A^0$.
(ii) $A^{\alpha_2} \subset A^{\alpha_1}$ if $\alpha_1 \le \alpha_2$.
(iii) $A^\alpha = \bigcap_{k \ge 1} A^{\alpha_k}$ if $\alpha_k \uparrow \alpha > 0$.

Then there is $u \in E^n(E^n$ denotes n dimensional fuzzy number space) such that $[u]^\alpha = A^\alpha$ for all $\alpha \in (0, 1]$ and $[u]^0 = \overline{\bigcup_{0 < \alpha \le 1} A^\alpha} \subset A^0$.

Definition 2.6. [25] *Suppose that $A, B \in E^1$ have α-cuts $[A]^\alpha = [A_{l,\alpha}, A_{r,\alpha}], [B]^\alpha = [B_{l,\alpha}, B_{r,\alpha}]$, with $0 \notin [B]^\alpha, \forall \alpha \in [0, 1]$. The g-division $\div_g$ is the operation that calculates the fuzzy number $C = A \div_g B$*

having level cuts $[C]^\alpha = [C_{l,\alpha}, C_{r,\alpha}]$(*here* $[A]^{\alpha-1} = [1/A_{r,\alpha}, 1/A_{l,\alpha}]$) *defined by*

$$[C]^\alpha = [A]^\alpha \div_g [B]^\alpha \iff \begin{cases} (i) & [A]^\alpha = [B]^\alpha [C]^\alpha, \\ or & \\ (ii) & [B]^\alpha = [A]^\alpha [C]^{\alpha-1} \end{cases} \qquad (2.2)$$

provided that C is a proper fuzzy number ($C_{l,\alpha}$ *is nondecreasing,* $C_{r,\alpha}$ *is nondecreasing,* $C_{l,1} \leq C_{r,1}$).

Remark 2.1. *According to [25], in this paper the fuzzy number is positive, if* $A \div_g B = C \in E^1$ *exists, then the following two cases are possible*
Case I. if $A_{l,\alpha} B_{r,\alpha} \leq A_{r,\alpha} B_{l,\alpha}, \forall \alpha \in [0, 1]$, *then* $C_{l,\alpha} = \frac{A_{l,\alpha}}{B_{l,\alpha}}, C_{r,\alpha} = \frac{A_{r,\alpha}}{B_{r,\alpha}}$,
Case II. if $A_{l,\alpha} B_{r,\alpha} \geq A_{r,\alpha} B_{l,\alpha}, \forall \alpha \in [0, 1]$, *then* $C_{l,\alpha} = \frac{A_{r,\alpha}}{B_{r,\alpha}}, C_{r,\alpha} = \frac{A_{l,\alpha}}{B_{l,\alpha}}$.
    The fuzzy analog of the boundedness and persistence (see [9, 10]) is as follows:

Definition 2.7. *A sequence of positive fuzzy numbers* (x_n) *is persistence (resp. bounded) if there exists a positive real number M (resp. N) such that*

$$supp\ x_n \subset [M, \infty)(resp.\ supp\ x_n \subset (0, N]), n = 1, 2, \cdots,$$

A sequence of positive fuzzy numbers (x_n) *is bounded and persistence if there exist positive real numbers* $M, N > 0$ *such that*
$$supp\ x_n \subset [M, N], n = 1, 2, \cdots.$$

    A sequence of positive fuzzy numbers $(x_n), n = 1, 2, \cdots$, is an unbounded if the norm $\|x_n\|, n = 1, 2, \cdots$, is an unbounded sequence.

Definition 2.8. x_n *is a positive solution of (1.1) if* (x_n) *is a sequence of positive fuzzy numbers which satisfies (1.1). A positive fuzzy number x is called a positive equilibrium of (1.1) if*

$$x = \frac{Ax}{\tilde{1} + Bx}.$$

    Let (x_n) be a sequence of positive fuzzy numbers and x is a positive fuzzy number, $x_n \to x$ as $n \to \infty$ if $\lim_{n\to\infty} D(x_n, x) = 0$.

3. Main results

3.1. Existence of solution of Beverton-Holt population model (1.1)

    Firstly we study the existence of positive solutions of Beverton-Holt population model (1.1). We need the following lemma.

Lemma 3.1. [24] Let $f : R^+ \times R^+ \times R^+ \to R^+$ be continuous, A, B, C are fuzzy numbers. Then

$$[f(A, B, C)]^\alpha = f([A]^\alpha, [B]^\alpha, [C]^\alpha), \quad \alpha \in (0, 1] \qquad (3.1)$$

Theorem 3.1. *Consider Beverton-Holt population model (1.1) with fuzzy uncertainty parameters and initial condition. Then, for any positive fuzzy number* x_0, *there exists a unique positive solution* x_n *of*

Beverton-Holt population model (1.1) with initial conditions x_0.

Proof. The proof is similar to those of Proposition 2.1 [9]. Suppose that there exists a sequence of fuzzy numbers (x_n) satisfying (1.1) with initial condition x_0. Consider the α−cuts, $\alpha \in (0, 1]$,

$$[x_n]^\alpha = [L_{n,\alpha}, R_{n,\alpha}], n = 0, 1, 2, \cdots, \quad [A]^\alpha = [A_{l,\alpha}, A_{r,\alpha}], [B]^\alpha = [B_{l,\alpha}, B_{r,\alpha}], [\tilde{1}]^\alpha = [\tilde{1}_{l,\alpha}, \tilde{1}_{r,\alpha}], \quad (3.2)$$

It follows from (1.1), (3.2) and Lemma 3.1 that

$$\begin{aligned}
[x_{n+1}]^\alpha &= [L_{n+1,\alpha}, R_{n+1,\alpha}] = \left[\frac{Ax_n}{\tilde{1} + Bx_n}\right]^\alpha = \frac{[A]^\alpha \times [x_n]^\alpha}{[\tilde{1}]^\alpha + [B]^\alpha \times [x_n]^\alpha} \\
&= \frac{[A_{l,\alpha}L_{n,\alpha}, A_{r,\alpha}R_{n,\alpha}]}{[\tilde{1}_{l,\alpha} + B_{l,\alpha}L_{n,\alpha}, \tilde{1}_{r,\alpha} + B_{r,\alpha}R_{n,\alpha}]}
\end{aligned}$$

Noting Remark 2.1, one of the following two cases holds

Case I

$$[x_{n+1}]^\alpha = [L_{n+1,\alpha}, R_{n+1,\alpha}] = \left[\frac{A_{l,\alpha}L_{n,\alpha}}{\tilde{1}_{l,\alpha} + B_{l,\alpha}L_{n,\alpha}}, \frac{A_{r,\alpha}R_{n,\alpha}}{\tilde{1}_{r,\alpha} + B_{r,\alpha}R_{n,\alpha}}\right] \quad (3.3)$$

Case II

$$[x_{n+1}]^\alpha = [L_{n+1,\alpha}, R_{n+1,\alpha}] = \left[\frac{A_{r,\alpha}R_{n,\alpha}}{\tilde{1}_{r,\alpha} + B_{r,\alpha}R_{n,\alpha}}, \frac{A_{l,\alpha}L_{n,\alpha}}{\tilde{1}_{l,\alpha} + B_{l,\alpha}L_{n,\alpha}}\right] \quad (3.4)$$

If Case I holds true, it follows that for $n \in \{0, 1, 2, \cdots\}, \alpha \in (0, 1]$

$$L_{n+1,\alpha} = \frac{A_{l,\alpha}L_{n,\alpha}}{\tilde{1}_{l,\alpha} + B_{l,\alpha}L_{n,\alpha}}, \quad R_{n+1,\alpha} = \frac{A_{r,\alpha}R_{n,\alpha}}{\tilde{1}_{r,\alpha} + B_{r,\alpha}R_{n,\alpha}} \quad (3.5)$$

Then it is obvious that, for any initial condition $(L_{0,\alpha}, R_{0,\alpha}), \alpha \in (0, 1]$, there is a unique solution $(L_{n,\alpha}, R_{n,\alpha})$. Now we prove that $[L_{n,\alpha}, R_{n,\alpha}], \alpha \in (0, 1]$, where $(L_{n,\alpha}, R_{n,\alpha})$ is the solution of system (3.5) with initial conditions $(L_{0,\alpha}, R_{0,\alpha})$, determines the solution x_n of (1.1) with initial conditions x_0 such that

$$[x_n]^\alpha = [L_{n,\alpha}, R_{n,\alpha}], \ \alpha \in (0, 1], \ n = 0, 1, 2, \cdots. \quad (3.6)$$

For $n = 1$, since $A, B, \tilde{1}$ and x_0 are positive fuzzy numbers, it is easy to see that $[L_{1,\alpha}, R_{1,\alpha}]$ is the α-cuts of $x_1 = \frac{Ax_0}{\tilde{1}+Bx_0}$, for any $\alpha \in (0, 1]$, we have

$$[L_{1,\alpha}, R_{1,\alpha}] = \left[\frac{A_{l,\alpha}L_{0,\alpha}}{\tilde{1}_{l,\alpha} + B_{l,\alpha}L_{0,\alpha}}, \frac{A_{r,\alpha}R_{0,\alpha}}{\tilde{1}_{r,\alpha} + B_{r,\alpha}R_{0,\alpha}}\right] = \frac{[A]^\alpha[x_0]^\alpha}{[\tilde{1}]^\alpha + [B]^\alpha[x_0]^\alpha}$$

Working inductively, let $[L_{k,\alpha}, R_{k,\alpha}], k \geq 1$, be the α-cuts of fuzzy number x_k as $[x_k]^\alpha = [L_{k,\alpha}, R_{k,\alpha}]$, we show that $[L_{k+1,\alpha}, R_{k+1,\alpha}]$ determines the α-cuts of fuzzy number $x_{k+1} = \frac{Ax_k}{\tilde{1}+Bx_k}$.

According to (3.5), for $\alpha \in (0, 1]$, we have

$$\begin{aligned}
[L_{k+1,\alpha}, R_{k+1,\alpha}] &= \left[\frac{A_{l,\alpha}L_{k,\alpha}}{\tilde{1}_{l,\alpha} + B_{l,\alpha}L_{k,\alpha}}, \frac{A_{r,\alpha}R_{k,\alpha}}{\tilde{1}_{r,\alpha} + B_{r,\alpha}R_{k,\alpha}}\right] = \frac{[A_{l,\alpha}L_{k,\alpha}, A_{r,\alpha}R_{k,\alpha}]}{[\tilde{1}_{l,\alpha} + B_{l,\alpha}L_{k,\alpha}, \tilde{1}_{r,\alpha} + B_{r,\alpha}R_{k,\alpha}]} \\
&= \frac{[A]^\alpha[x_k]^\alpha}{[\tilde{1}]^\alpha + [B]^\alpha[x_k]^\alpha}. \quad (3.7)
\end{aligned}$$

Then $[L_{k+1,\alpha}, R_{k+1,\alpha}]$ determines the α-cuts of fuzzy number $x_{k+1} = \frac{Ax_k}{1+Bx_k}$. Then for each n, $[L_{n,\alpha}, R_{n,\alpha}], \alpha \in (0, 1]$ determines the α-cuts of fuzzy number x_n satisfying with (3.6).

Next we prove the uniqueness of the solution. Suppose that there exists another solution $\overline{x}_n$ of (1.1) with the initial condition x_0. Then arguing as above, it is easy to get

$$[\overline{x}_n]^\alpha = [L_{n,\alpha}, R_{n,\alpha}], \ \alpha \in (0, 1], \ n = 0, 1, \cdots . \tag{3.8}$$

Then from (3.6) and (3.8) we have $[x_n]^\alpha = [\overline{x}_n]^\alpha, \alpha \in (0, 1], n = 0, 1, 2, \cdots$, from which it follows that $x_n = \overline{x}_n, n = 0, 1, \cdots$. Thus the proof is completed.

If Case II holds true, the proof is similar to those of Case I. Thus the proof of Theorem 3.1 is completed.

Remark 3.1. From a theoretical point of view, the existence of solution for fuzzy difference equation is very important and meaningful with initial condition. Therefore, in that sense, the existence of positive fuzzy solution for Beverton-Holt population model is of vital importance and practical significance. In fact, it can see that the positive solution of Beverton-Holt population model with fuzzy environment is a sequence of positive fuzzy numbers describing the fuzzy uncertainty.

3.2. Dynamics of Beverton-Holt population model (1.1)

To study the dynamical behavior of the positive solutions of Beverton-Holt population model (1.1), according to Definition 2.3, we consider two cases.

Firstly, if Case I holds true, we need the following lemma.

Lemma 3.2 *Consider the system of difference equations*

$$y_{n+1} = \frac{ay_n}{p + cy_n}, \ z_{n+1} = \frac{bz_n}{q + dz_n}, \ n = 0, 1, \cdots , \tag{3.9}$$

where $p \in (0, 1), q \in (1, +\infty), a, b, c, d \in (0, +\infty), y_0, z_0 \in (0, +\infty)$, if $a > p, b > q$, then the following statements are true:
(i) The system exists unique positive equilibrium $\left(\frac{a-p}{c}, \frac{b-q}{d}\right)$ which is globally asymptotically stable.
(ii) y_n and z_n are bounded and persistent.

Proof. (i) Let $(\overline{y}, \overline{z})$ be equilibrium point of (3.9). It is easy to get positive equilibrium $(\overline{y}, \overline{z}) = \left(\frac{a-p}{c}, \frac{b-q}{d}\right)$. The linearized equation associated with (3.9) about equilibrium $(\overline{y}, \overline{z})$ is

$$y_{n+1} = \frac{p}{a}y_n, \ z_{n+1} = \frac{q}{b}z_n.$$

Since $a > p, b > q$, it follows that the system is locally asymptotically stable.

On the other hand, set $f(y) = \frac{ay}{p+cy}, g(z) = \frac{bz}{q+dz}$, then

$$f'(y) = \frac{ap}{(p + cy)^2} > 0, \quad g'(z) = \frac{bq}{(q + dz)^2} > 0. \tag{3.10}$$

Namely, the sequences (y_n) and (z_n) are increasing and

$$y_n = \frac{ay_{n-1}}{p + cy_{n-1}} < \frac{a}{c}, \quad z_n = \frac{bz_{n-1}}{q + dz_{n-1}} < \frac{b}{d}. \tag{3.11}$$

Therefore, from (3.10) and (3.11), it follows that the limitation of $(y_n), (z_n)$ exist. Set $\lim_{n\to\infty} y_n = y, \lim_{n\to\infty} z_n = z$, integrating with (3.9), it can follows that

$$\lim_{n\to\infty} y_n = \frac{a-p}{c}, \quad \lim_{n\to\infty} z_n = \frac{b-q}{d}. \tag{3.12}$$

Therefore, it follows that the positive equilibrium $\left(\frac{a-p}{c}, \frac{b-q}{d}\right)$ is globally asymptotically stable.

(ii) Set $Y_n = \frac{1}{y_n}$, $Z_n = \frac{1}{z_n}$, then (3.9) can be transformed into

$$Y_{n+1} = \frac{c}{a} + \frac{p}{a}Y_n, \quad Z_{n+1} = \frac{d}{b} + \frac{q}{b}Z_n, \quad n = 0, 1, \cdots, \tag{3.13}$$

It follows from (3.13) that

$$\begin{aligned}
Y_n &= \frac{c}{a} + \frac{pc}{a^2} + \frac{p^2}{a^2}Y_{n-2} = \frac{c}{a} + \frac{pc}{a^2} + \cdots + \frac{p^{n-1}c}{a^n} + \frac{p^n}{a^n}Y_0 \\
&= \frac{\frac{c}{a}\left[1 - \left(\frac{p}{a}\right)^{n-1}\right]}{1 - \frac{p}{a}} + \frac{p^n}{a^n}Y_0 \leq \frac{c}{a-p} + Y_0 := \frac{1}{\delta}
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
Z_n &= \frac{d}{b} + \frac{qd}{b^2} + \frac{q^2}{b^2}Z_{n-2} = \frac{d}{b} + \frac{qd}{b^2} + \cdots + \frac{q^{n-1}d}{b^n} + \frac{q^n}{b^n}Z_0 \\
&= \frac{\frac{d}{b}\left[1 - \left(\frac{q}{b}\right)^{n-1}\right]}{1 - \frac{q}{b}} + \frac{q^n}{b^n}Z_0 \leq \frac{d}{b-q} + Z_0 := \frac{1}{\gamma}
\end{aligned} \tag{3.15}$$

From (3.11), (3.14) and (3.15), This implies

$$\delta \leq y_n \leq \frac{a}{c}, \quad \gamma \leq z_n \leq \frac{b}{d}.$$

This completes the proof of Lemma 3.2.

Theorem 3.2 *Consider Beverton-Holt population model (1.1) with fuzzy uncertainty parameters and initial condition. If*

$$0 < \tilde{I}_{l,\alpha} \leq A_{l,\alpha}, \quad \tilde{I}_{r,\alpha} \leq A_{r,\alpha}, \quad \alpha \in (0, 1], \tag{3.16}$$

then the following statements are true
(i) Every positive solution x_n of Beverton-Holt population model (1.1) is bounded and persistent.
(ii) Every positive solution x_n of Beverton-Holt population model (1.1) tends to the positive equilibrium point x as $n \to \infty$.

Proof. (i) Since $A, \tilde{I}, B$ and the initial value x_0 are positive fuzzy numbers, there exist positive real numbers $M_A, N_A, M_B, N_B, M_0, N_0, P, Q$, such that, for all $\alpha \in (0, 1]$,

$$[A_{l,\alpha}, A_{r,\alpha}] \subset [M_A, N_A], \ [B_{l,\alpha}, B_{r,\alpha}] \subset [M_B, N_B], \ [L_{0,\alpha}, R_{0,\alpha}] \subset [M_0, N_0], [\tilde{I}_{l,\alpha}, \tilde{I}_{r,\alpha}] \subset [P, Q], \tag{3.17}$$

Let x_n be a positive solution of Beverton-Holt population model (1.1), from (3.16), (3.17) and Lemma 3.2, we get

$$L_{n,\alpha} > \frac{(A_{l,\alpha} - \tilde{1}_{l,\alpha})L_{0,\alpha}}{B_{l,\alpha}L_{0,\alpha} + A_{l,\alpha} - \tilde{1}_{l,\alpha}} > \frac{(M_A - Q)M_0}{N_B N_0 + N_A - Q} := M, \ R_{n,\alpha} < \frac{A_{r,\alpha}}{B_{r,\alpha}} < \frac{N_A}{M_B} := N \qquad (3.18)$$

From which, we get for $n \geq 1$, $\bigcup_{\alpha \in (0,1]}[L_{n,\alpha}, R_{n,\alpha}] \subset [M, N]$, and so $\overline{\bigcup_{\alpha \in (0,1]}[L_{n,\alpha}, R_{n,\alpha}]} \subseteq [M, N]$. Thus the positive solution is bounded and persistent.

(ii) Suppose that there exists a fuzzy number x such that

$$x = \frac{Ax}{\tilde{1} + Bx}, \quad [x]^\alpha = [L_\alpha, R_\alpha], \quad \alpha \in (0, 1]. \qquad (3.19)$$

where $L_\alpha, R_\alpha \geq 0$. Then from (3.19) we can prove that

$$L_\alpha = \frac{A_{l,\alpha}L_\alpha}{\tilde{1}_{l,\alpha} + B_{l,\alpha}L_\alpha}, \quad R_\alpha = \frac{A_{r,\alpha}R_\alpha}{\tilde{1}_{r,\alpha} + B_{r,\alpha}R_\alpha}. \qquad (3.20)$$

Hence from (3.20), we have

$$L_\alpha = \frac{A_{l,\alpha} - \tilde{1}_{l,\alpha}}{B_{l,\alpha}}, \ R_\alpha = \frac{A_{r,\alpha} - \tilde{1}_{r,\alpha}}{B_{r,\alpha}}. \qquad (3.21)$$

Let x_n be a positive solution of Beverton-Holt population model (1.1). Since (3.16) holds true, we can apply Lemma 3.2 to system (3.5), and so we have

$$\lim_{n\to\infty} L_{n,\alpha} = L_\alpha, \quad \lim_{n\to\infty} R_{n,\alpha} = R_\alpha, \qquad (3.22)$$

Therefore from (3.22) we have

$$\lim_{n\to\infty} D(x_n, x) = \lim_{n\to\infty} \sup_{\alpha \in (0,1]} \{\max\{|L_{n,\alpha} - L_\alpha|, |R_{n,\alpha} - R_\alpha|\}\} = 0.$$

This completes the proof of the Theorem.

Secondly, if Case II holds true, it follows that for $n \in \{0, 1, 2, \cdots\}, \alpha \in (0, 1]$

$$L_{n+1,\alpha} = \frac{A_{r,\alpha}R_{n,\alpha}}{\tilde{1}_{r,\alpha} + B_{r,\alpha}R_{n,\alpha}}, \quad R_{n+1,\alpha} = \frac{A_{l,\alpha}L_{n,\alpha}}{\tilde{1}_{l,\alpha} + B_{l,\alpha}L_{n,\alpha}} \qquad (3.23)$$

We need the following lemmas.

Lemma 3.3. *Consider the system of difference equations*

$$y_{n+1} = \frac{bz_n}{q + dz_n}, \quad z_{n+1} = \frac{ay_n}{p + cy_n}, \quad n = 0, 1, \cdots, \qquad (3.24)$$

where $p \in (0, 1), q \in (1, +\infty), a, b, p, q, c, d, y_0, z_0 \in (0, +\infty)$. If

$$a > p, \quad b > q. \qquad (3.25)$$

Then the following statements are true
(i) The solution of (3.24) is bounded and persistent.
(ii) Furthermore suppose that

$$\frac{a-p}{c} < \frac{b-q}{d}. \tag{3.26}$$

Then system exists unique positive equilibrium point $(\frac{ab-pq}{qc+ad}, \frac{ab-pq}{pd+bc})$ which is globally asymptotically stable.

Proof. (i) It is clear from (3.24) that

$$y_n \le \frac{b}{d}, \quad z_n \le \frac{a}{c}. \tag{3.27}$$

Setting $Y_n = \frac{1}{y_n}$, $Z_n = \frac{1}{z_n}$, then (3.24) can be transformed to

$$Y_n = \frac{d}{b} + \frac{q}{b}Z_{n-1}, \quad Z_n = \frac{c}{a} + \frac{p}{a}Y_{n-1}. \tag{3.28}$$

Working inductively, for $n - 2k \ge 0$, it can conclude that

$$
\begin{aligned}
Y_n &= \frac{d}{b} + \frac{q}{b}\left(\frac{c}{a} + \frac{p}{a}Y_{n-2}\right) = \frac{d}{b} + \frac{qc}{ab} + \frac{pq}{ab}Y_{n-2} \\
&= \frac{d}{b} + \frac{qc}{ab} + \frac{pqd}{ab^2} + \frac{pq^2}{ab^2}Z_{n-3} = \frac{d}{b} + \frac{qc}{ab} + \frac{pqd}{ab^2} + \frac{pq^2c}{a^2b^2} + \frac{p^2q^2}{a^2b^2}Y_{n-4} \\
&= \cdots = \frac{\frac{d}{b}\left[1 - \left(\frac{pq}{ab}\right)^k\right]}{1 - \frac{pq}{ab}} + \frac{\frac{qc}{ab}\left[1 - \left(\frac{pq}{ab}\right)^k\right]}{1 - \frac{pq}{ab}} + \left(\frac{pq}{ab}\right)^k Y_{n-2k} \\
&\le \frac{\frac{d}{b} + \frac{qc}{ab}}{1 - \frac{pq}{ab}} + Y_0 = \frac{bc + pd}{ab - pq} + Y_0 := \frac{1}{\xi}
\end{aligned} \tag{3.29}
$$

$$
\begin{aligned}
Z_n &= \frac{c}{a} + \frac{p}{a}\left(\frac{d}{b} + \frac{q}{b}Z_{n-2}\right) = \frac{\alpha}{a} + \frac{pd}{ab} + \frac{pq}{ab}Z_{n-2} \\
&= \frac{c}{a} + \frac{pd}{ab} + \frac{pqc}{a^2b} + \frac{p^2q}{a^2b}Z_{n-3} = \frac{c}{a} + \frac{pd}{ab} + \frac{pqc}{a^2b} + \frac{p^2qd}{a^2b^2} + \frac{p^2q^2}{a^2b^2}Z_{n-4} \\
&= \cdots = \frac{\frac{c}{a}\left[1 - \left(\frac{pq}{ab}\right)^k\right]}{1 - \frac{pq}{ab}} + \frac{\frac{pd}{ab}\left[1 - \left(\frac{pq}{ab}\right)^k\right]}{1 - \frac{pq}{ab}} + \left(\frac{pq}{ab}\right)^k Z_{n-2k} \\
&\le \frac{\frac{c}{a} + \frac{pd}{ab}}{1 - \frac{pq}{ab}} + Z_0 = \frac{bc + pd}{ab - pq} + Z_0 := \frac{1}{\rho}
\end{aligned} \tag{3.30}
$$

It follows from (3.27),(3.29) and (3.30) that

$$\xi \le y_n \le \frac{b}{d}, \quad \rho \le z_n \le \frac{a}{c}, \text{ for } n > 0. \tag{3.31}$$

This completes the proof of (i).

(ii) Let $(\bar{y}, \bar{z})$ be an equilibrium point of (3.24). It is clear to obtain that $(\bar{y}, \bar{z}) = (\frac{ab-pq}{qc+ad}, \frac{ab-pq}{pd+bc})$. The linearized equation of system (3.24) about the equilibrium point $(\bar{y}, \bar{z})$ is

$$\Psi_{n+1} = G\Psi_n. \tag{3.32}$$

where $\Psi_n = (y_n, z_n)^T$ and

$$G = \begin{pmatrix} 0 & \frac{qb}{(q+d\bar{z})^2} \\ \frac{pa}{(p+c\bar{y})^2} & 0 \end{pmatrix}$$

Thus the characteristic equation of (3.32) is

$$\lambda^2 - \frac{pqab}{(p+c\bar{y})^2(q+d\bar{z})^2} = 0.$$

Since (3.25) and (3.26) hold true, it is easy to obtain that the root of characteristic equation $|\lambda| < 1$. From [26, 27], thus the unique positive equilibrium point $(\bar{y}, \bar{z})$ is locally asymptotically stable.

Next, let (y_n, z_n) be an arbitrary positive solution of (3.24). From (3.29) and (3.30), we have $\{Y_n\}$, $\{Z_n\}$ are monotone increasing and have a upper bound. Namely, $\{y_n\}, \{z_n\}$ are monotone decreasing and have lower bound. So, set $\lim_{n\to\infty} y_n = y, \lim_{n\to\infty} z_n = z.$, From (3.24), it can follow that

$$\lim_{n\to\infty} y_n = y = \bar{y} = \frac{ab-pq}{qc+ad}, \quad \lim_{n\to\infty} z_n = z = \bar{z} = \frac{ab-pq}{pd+bc}.$$

Therefore it implies that the unique positive equilibrium point $(\bar{y}, \bar{z})$ is globally asymptotically stable.

Theorem 3.3. *Consider Beverton-Holt population model (1.1) with fuzzy uncertainty parameters and initial condition. If (3.16) and the following condition are satisfied*

$$\frac{B_{l,\alpha}}{B_{r,\alpha}} \le \frac{A_{l,\alpha} - \tilde{1}_{l,\alpha}}{A_{r,\alpha} - \tilde{1}_{r,\alpha}}, \ \forall \alpha \in (0,1]. \tag{3.33}$$

Then the following statements are true
(i) Every positive solution x_n of Beverton-Holt population model (1.1) is bounded and persistence.
(ii) Every positive solution x_n of Beverton-Holt population model (1.1) tends to the positive equilibrium point x as $n \to +\infty$.

Proof. (i) The proof is similar to those of Theorem 3.2. Let x_n be a positive solution of (1.1), from (3.16), (3.17) and Lemma 3.3, we get

$$L_{n,\alpha} \ge \frac{L_{0,\alpha}(A_{l,\alpha}A_{r,\alpha} - \tilde{1}_{l,\alpha}\tilde{1}_{r,\alpha})}{L_{0,\alpha}(A_{r,\alpha}B_{l,\alpha} + \tilde{1}_{l,\alpha}B_{r,\alpha}) + (A_{l,\alpha}A_{r,\alpha} - \tilde{1}_{l,\alpha}\tilde{1}_{r,\alpha})} \ge \frac{M_0(M_A^2 - Q^2)}{N_0(N_AN_B + QN_B) + (N_A^2 - P^2)} =: K \tag{3.34}$$

$$R_{n,\alpha} \le \frac{A_{l,\alpha}}{B_{l,\alpha}} \le \frac{N_A}{M_B} =: L \tag{3.35}$$

From (3.34) and (3.35), we get, for $n \ge 1$, $\bigcup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}] \subset [K, L]$. And so

$$\overline{\bigcup_{\alpha\in(0,1]} [L_{n,\alpha}, R_{n,\alpha}]} \subseteq [K, L].$$

Thus the positive solution x_n of (1.1) is bounded and persistent.

(ii) Suppose that there exists a fuzzy number x such that (3.19) is satisfied. Then from (3.19) and Case II, we have

$$L_\alpha = \frac{A_{r,\alpha}R_\alpha}{\tilde{1}_{r,\alpha} + B_{r,\alpha}R_\alpha}, \quad R_\alpha = \frac{A_{l,\alpha}L_\alpha}{\tilde{1}_{l,\alpha} + B_{l,\alpha}L_\alpha}. \tag{3.36}$$

It follows from (3.36) that

$$L_\alpha = \frac{A_{l,\alpha}A_{r,\alpha} - \tilde{1}_{l,\alpha}\tilde{1}_{r,\alpha}}{\tilde{1}_{r,\alpha}B_{l,\alpha} + A_{l,\alpha}B_{r,\alpha}}, \quad R_\alpha = \frac{A_{l,\alpha}A_{r,\alpha} - \tilde{1}_{l,\alpha}\tilde{1}_{r,\alpha}}{\tilde{1}_{l,\alpha}B_{r,\alpha} + A_{r,\alpha}B_{l,\alpha}}. \tag{3.37}$$

Let x_n be a positive solution of (1.1) such that Case II holds. Namely.

$$L_{n+1,\alpha} = \frac{A_{r,\alpha}R_{n,\alpha}}{\tilde{1}_{r,\alpha} + B_{r,\alpha}R_{n,\alpha}}, \quad R_{n+1,\alpha} = \frac{A_{l,\alpha}L_{n,\alpha}}{\tilde{1}_{l,\alpha} + B_{l,\alpha}L_{n,\alpha}}. \tag{3.38}$$

Since (3.38) is satisfied, we can apply Lemma 3.3 to system (3.38) and so we have

$$\lim_{n\to\infty} L_{n,\alpha} = L_\alpha, \quad \lim_{n\to\infty} R_{n,\alpha} = R_\alpha, \tag{3.39}$$

Therefore from (3.39) we have

$$\lim_{n\to\infty} D(x_n, x) = \lim_{n\to\infty} \sup_{\alpha\in(0,1]} \{\max\{|L_{n,\alpha} - L_\alpha|, |R_{n,\alpha} - R_\alpha|\}\} = 0.$$

This completes the proof of Theorem 3.3.

Remark 3.2. In population dynamical model, the parameters of model derived from statistic data with vagueness or uncertainty. It corresponds to reality to use fuzzy parameters in population dynamical model. In contrast with classic population model, the solution of fuzzy population model is within a range of value (approximate value), which are taken into account fuzzy uncertainties. Furthermore the global asymptotic behaviour of discrete Beverton-Holt population model are obtained in fuzzy context.

4. Numerical examples

Example 4.1 Consider the following fuzzy discrete time Beverton-Holt population model

$$x_{n+1} = \frac{Ax_n}{\tilde{1} + Bx_n}, \quad n = 0, 1, \cdots, \tag{4.1}$$

we take $A, B, \tilde{1}$ and the initial values x_0 such that

$$A(x) = \begin{cases} \frac{5}{4}x - 1, & 0.8 \le x \le 1.6 \\ -\frac{5}{2}x + 5, & 1.6 \le x \le 2 \end{cases}, \quad \tilde{1}(x) = \begin{cases} 2x - 1, & 0.5 \le x \le 1 \\ -5x + 6, & 1 \le x \le 1.2 \end{cases} \tag{4.2}$$

$$B(x) = \begin{cases} 10x - 4, & 0.4 \le x \le 0.5 \\ -10x + 6, & 0.5 \le x \le 0.6 \end{cases}, \quad x_0(x) = \begin{cases} x - 3, & 3 \le x \le 4 \\ -x + 5, & 4 \le x \le 5 \end{cases} \tag{4.3}$$

From (4.2), we get

$$[A]^\alpha = \left[0.8 + \frac{4}{5}\alpha, 2 - \frac{2}{5}\alpha\right], \ [\tilde{1}]^\alpha = \left[0.5 + \frac{1}{2}\alpha, 1.2 - \frac{1}{5}\alpha\right], \ \alpha \in (0, 1]. \tag{4.4}$$

From (4.3) we get

$$[B]^\alpha = \left[0.4 + \frac{1}{10}\alpha, 0.6 - \frac{1}{10}\alpha\right], \ [x_0]^\alpha = [3 + \alpha, 5 - \alpha], \ \alpha \in (0, 1]. \tag{4.5}$$

Therefore, it follows that

$$\overline{\bigcup_{\alpha \in (0,1]} [A]^\alpha} = [0.8, 2], \ \overline{\bigcup_{\alpha \in (0,1]} [\tilde{1}]^\alpha} = [0.5, 1.2], \ \overline{\bigcup_{\alpha \in (0,1]} [B]^\alpha} = [0.4, 0.6], \ \overline{\bigcup_{\alpha \in (0,1]} [x_0]^\alpha} = [3, 5]. \tag{4.6}$$

From (4.1), it results in a coupled system of difference equations with parameter α,

$$L_{n+1,\alpha} = \frac{A_{l,\alpha} L_{n,\alpha}}{\tilde{1}_{l,\alpha} + B_{l,\alpha} L_{n,\alpha}}, \ R_{n+1,\alpha} = \frac{A_{r,\alpha} R_{n,\alpha}}{\tilde{1}_{r,\alpha} + B_{r,\alpha} R_{n,\alpha}}, \ \alpha \in (0, 1]. \tag{4.7}$$

Therefore, $A_{l,\alpha} \geq \tilde{1}_{l,\alpha} > 0, A_{r,\alpha} \geq \tilde{1}_{r,\alpha}, \forall \alpha \in (0, 1]$, and initial values x_0 are positive fuzzy numbers, so from Theorem 3.2, we have that every positive solution x_n of Eq.(4.1) is bounded and persistence.

In addition, from Theorem 3.2, Eq.(4.1) has a unique positive equilibrium $\overline{x} = (0.75, 1.2, 1.333)$. Moreover every positive solution x_n of Eq.(4.1) converges the unique equilibrium $\overline{x}$ with respect to D as $n \to \infty$. (see Figure 1–Figure 3)
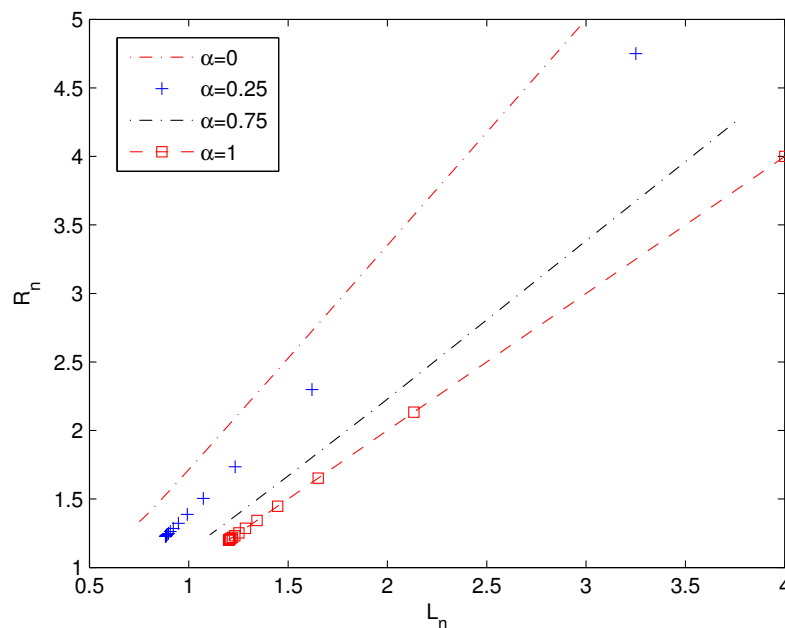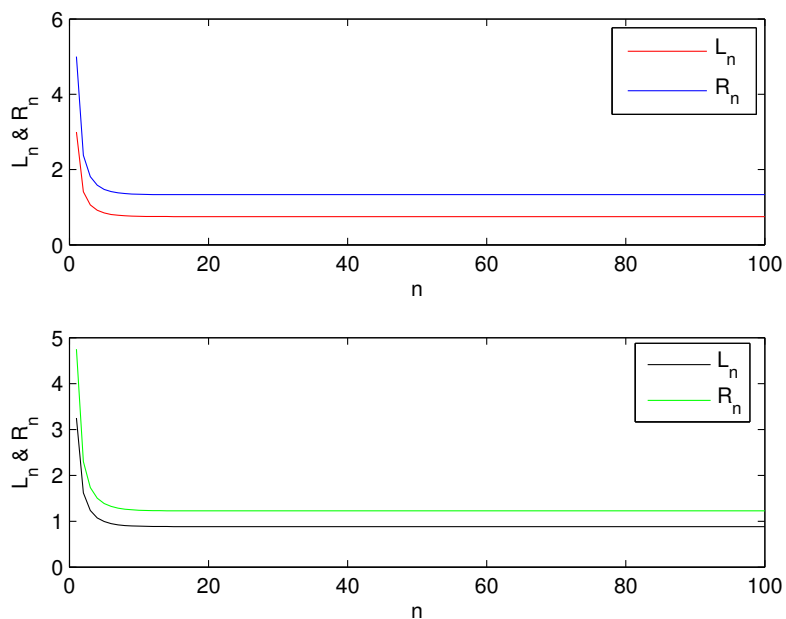

Figure 1. The dynamics of system (4.7).

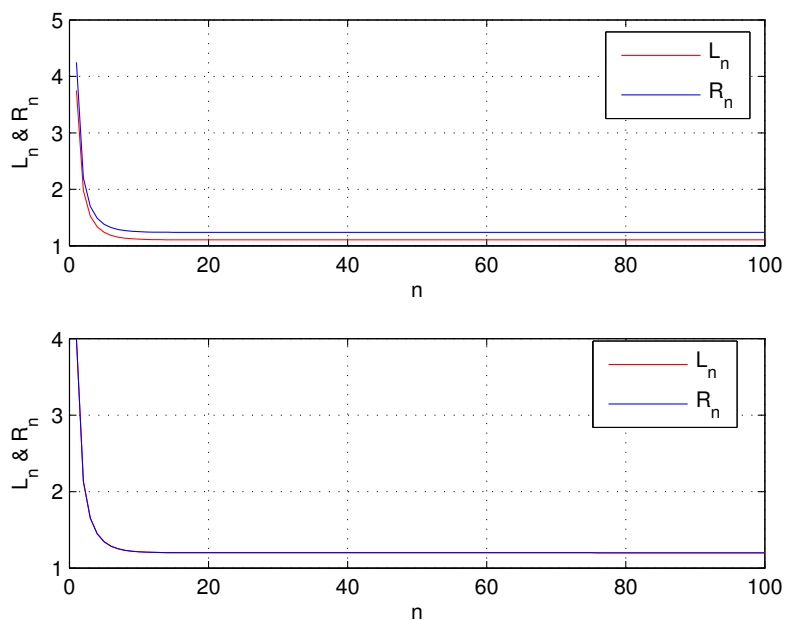**Figure 2.** The solution of system (4.7) at $\alpha = 0$ and $\alpha = 0.25$.

Figure 3. The solution of system (4.7) at $\alpha = 0.75$ and $\alpha = 1$.

Example 4.2 Consider the following fuzzy discrete time Beverton-Holt population model (4.1).

where A, $\tilde{1}$, B and the initial values x_0 are satisfied

$$A(x) = \begin{cases} \frac{10}{3}x - 3, & 0.9 \le x \le 1.2 \\ \\ -\frac{10}{3}x + 5, & 1.2 \le x \le 1.5 \end{cases}, \quad \tilde{1}(x) = \begin{cases} \frac{10}{3}x - \frac{7}{3}, & 0.7 \le x \le 1 \\ \\ -\frac{10}{3}x + \frac{13}{3}, & 1 \le x \le 1.3 \end{cases} \quad (4.8)$$

$$B(x) = \begin{cases} 5x - 2, & 0.4 \le x \le 0.6 \\ \\ -5x + 4, & 0.6 \le x \le 0.8 \end{cases}, \quad x_0(x) = \begin{cases} \frac{5}{2}x - 1.5, & 0.6 \le x \le 1 \\ \\ -\frac{5}{2}x + 3.5, & 1 \le x \le 1.4 \end{cases} \quad (4.9)$$

From (4.8), we get

$$[A]^\alpha = \left[0.9 + \frac{3}{10}\alpha, 1.5 - \frac{3}{10}\alpha\right], \quad [\tilde{1}]^\alpha = \left[0.7 + \frac{3}{10}\alpha, 1.3 - \frac{3}{10}\alpha\right], \quad \alpha \in (0, 1]. \quad (4.10)$$

From (4.9), we get

$$[B]^\alpha = \left[0.4 + \frac{1}{5}\alpha, 0.8 - \frac{1}{5}\alpha\right], \quad [x_0]^\alpha = \left[0.6 + \frac{2}{5}\alpha, 1.4 - \frac{2}{5}\alpha\right], \quad \alpha \in (0, 1]. \quad (4.11)$$

Therefore, it follows that

$$\overline{\bigcup_{\alpha \in (0,1]} [A]^\alpha} = [0.9, 1.5], \quad \overline{\bigcup_{\alpha \in (0,1]} [\tilde{1}]^\alpha} = [0.7, 1.3], \quad \overline{\bigcup_{\alpha \in (0,1]} [B]^\alpha} = [0.4, 0.8], \quad \overline{\bigcup_{\alpha \in (0,1]} [x_0]^\alpha} = [0.6, 1.4]. \quad (4.12)$$

From (4.1), it results in a coupled system of difference equation with parameter α,

$$L_{n+1,\alpha} = \frac{A_{r,\alpha} R_{n,\alpha}}{\tilde{1}_{r,\alpha} + B_{r,\alpha} R_{n,\alpha}}, \quad R_{n+1,\alpha} = \frac{A_{l,\alpha} L_{n,\alpha}}{\tilde{1}_{l,\alpha} + B_{l,\alpha} L_{n,\alpha}}, \quad \alpha \in (0, 1]. \quad (4.13)$$

It is clear that (3.33) is satisfied and initial values x_0 are positive fuzzy numbers, so from Theorem 3.3, Eq.(4.1) has a unique positive equilibrium $\overline{x} = (0.333, 0.3548, 0.3793)$. Moreover every positive solution x_n of Eq.(4.1) converges the unique equilibrium $\overline{x}$ with respect to D as $n \to \infty$. (see Figre 4–Figure 6)
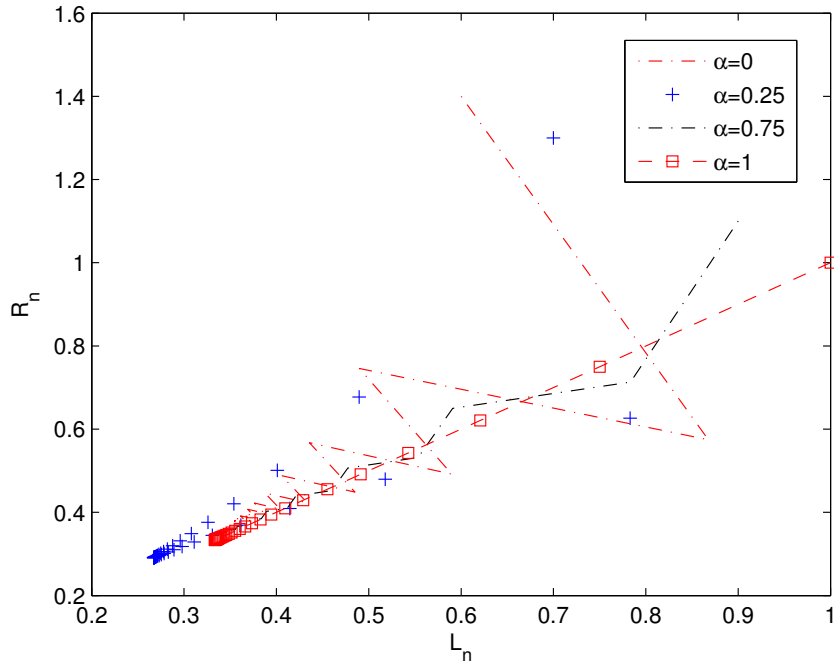
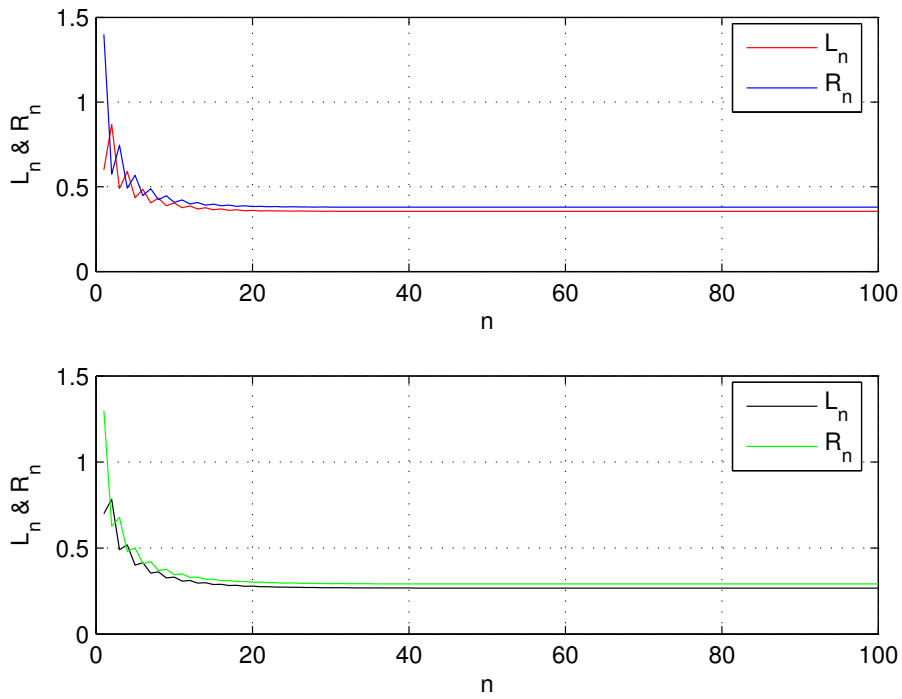**Figure 4.** The dynamics of system (4.13).

Figure 5. The solution of system (4.13) at $\alpha = 0$ and $\alpha = 0.25$.
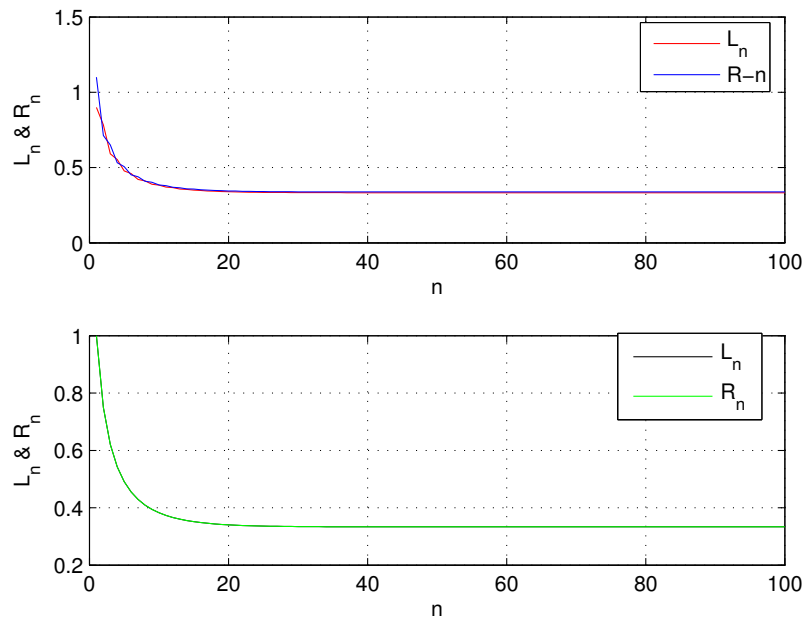
Figure 6. The solution of system (4.13) at $\alpha = 0.75$ and $\alpha = 1$.

5. Conclusion

In this work, according to a generalization of division (g-division) of fuzzy number, we study the fuzzy discrete time Beverton-Holt population model $x_{n+1} = \frac{Ax_n}{1+Bx_n}$. The existence of positive solution and qualitative behavior to (1.1) are investigated. The main results are as follows

(1) Under Case I, the positive solution is bounded and persists if $A_{l,\alpha} > \tilde{1}_{l,\alpha}, A_{r,\alpha} > \tilde{1}_{r,\alpha}, \alpha \in (0, 1]$. Every positive solution x_n tends to the unique equilibrium x as $n \to \infty$.

(2) Under Case II, the positive solution is bounded and persists if $A_{l,\alpha} > \tilde{1}_{l,\alpha}, A_{r,\alpha} > \tilde{1}_{r,\alpha}$, and $\frac{B_{l,\alpha}}{B_{r,\alpha}} \leq \frac{A_{l,\alpha}-\tilde{1}_{l,\alpha}}{A_{r,\alpha}-\tilde{1}_{r,\alpha}}, \alpha \in (0, 1]$. Every positive solution x_n tends to the unique equilibrium x as $n \to \infty$.

Acknowledgments

Conflict of interest

The authors declare that they have no competing interests in this paper.

References

1. M. Kot, Elements of Mathematical Ecology, Cambridge University Press, New York, 2001.

2. R. Beverton and S. Holt, On the dynamics of exploited fish populations, *Fish. Invest. Ser 2*, **19** (1957), 1–533.

3. M. D. L. Sen, The generalized Beverton-Holt equation and the control of populations, *Appl. Math. Model.*, **32** (2008), 2312–2328.

4. M. D. L. Sen and S. Alonso-Quesada, Control issues for the Beverton-Holt equation in ecology by locally monitoring the environment carrying capacity: Nonadaptive and adaptive cases, *Appl. Math. Comput.*, **215** (2009), 2616–2633.

5. M. Bohner and S. Streipert, Optimal harvesting policy for the Beverton-Holt model, *Math. Biosci. Eng.*, **13** (2016), 673–695.

6. L. A. Zadeh, Fuzzy set, *Inf. Control*, **8** (1965), 338–353.

7. E. Y. Deeba, A. De Korvin and E. L. Koh, A fuzzy difference equation with an application, *J. Differ. Equ. Appl.*, **2** (1996), 365–374.

8. E. Y. Deeba and A. De Korvin, Analysis by fuzzy difference equations of a model of CO_2 level in the blood, *Appl. Math. Lett.*, **12** (1999), 33–40.

9. G. Papaschinopoulos and B. K. Papadopoulos, On the fuzzy difference equation $x_{n+1} = A + B/x_n$, *Soft Comput.*, **6** (2002), 456–461.

10. G. Papaschinopoulos and B. K. Papadopoulos, On the fuzzy difference equation $x_{n+1} = A + x_n/x_{n-m}$, *Fuzzy Set. Syst.*, **129** (2002), 73–81.

11. G. Stefanidou, G. Papaschinopoulos and C. J. Schinas, On an exponential-type fuzzy difference equation, *Adv. Differ. Equ.*, **2010** (2010), 1–19.

12. Q. Din, Asymptotic behavior of a second order fuzzy rational difference equations, *J. Discrete Math.*, **2015** (2015), 1–7.

13. R. Memarbashi and A. Ghasemabadi, Fuzzy difference equations of volterra type, *Int. J. Nonlinear Anal. Appl.*, **4** (2013), 74–78.

14. K. A. Chrysafis, B. K. Papadopoulos and G. papaschinopoulos, On the fuzzy difference equations of finance, *Fuzzy Set. Syst.*, **159** (2008), 3259–3270.

15. Q. Zhang, L. Yang and D. Liao, Behaviour of solutions of to a fuzzy nonlinear difference equation, *Iranian J. Fuzzy Syst.*, **9** (2012), 1–12.

16. Q. Zhang, L. Yang and D. Liao, On first order fuzzy riccati difference equation, *Inform. Sciences*, **270** (2014), 226–236.

17. Q. Zhang, J. Liu and Z. Luo, Dynamical behaviour of a third-order rational fuzzy difference equation, *Adv. Differ. Equ.*, **2015** (2015).

18. S. P. Mondal, D. K. Vishwakarma and A. K. Saha, Solution of second order linear fuzzy difference equation by Lagranges multiplier method, *J. Soft Comput. Appl.*, **1** (2016), 11–27.

19. Z. Alijani and F. Tchier, On the fuzzy difference equation of higher order, *J. Comput. Complex. Appl.*, **3** (2017), 44–49.

20. A. Khastan, Fuzzy Logistic difference equation, *Iranian J. Fuzzy Syst.*, (2017), In Press.

21. C. Wang, X. Su, P. Liu, et al., On the dynamics of a five-order fuzzy difference equation, *J. Nonlinear Sci. Appl.*, **10** (2017), 3303–3319.

22. A. Khastan and Z. Alijani, On the new solutions to the fuzzy difference equation $x_{n+1} = A + B/x_n$, *Fuzzy Set. Syst.*, (2018), In Press.

23. A. Khastan, New solutions for first order linear fuzzy difference equations, *J. Comput. Appl. Math.*, **312** (2017), 156–166.

24. D. Dubois and H. Prade, Possibility theory: an approach to computerized processing of uncertainty, Plenum Publishing Corporation, New York, 1998.

25. L. Stefanini, A generalization of Hukuhara difference and division for interval and fuzzy arithmetic, *Fuzzy Set. Sys.*, **161** (2010), 1564–1584.

26. V. L. Kocic and G. Ladas, Global behavior of nonlinear difference equations of higher order with application, Kluwer Academic Publishers, Dordrecht, 1993.

27. M. R. S. Kulenonvic and G. Ladas, Dynamics of second order rational difference equations with open problems and conjectures, Chapaman & Hall/CRC, Boca Raton, 2002.