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Research article

Convergence and data dependence results of the nonlinear Volterra integral equation by the Picard's three step iteration

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Abstract: Picard's three step iteration algorithm was one of the iteration algorithms that was recently shown to be faster than some other iterative algorithms in the existing literature. The purpose of this paper was to study using this iteration algorithm for the solution of nonlinear Volterra integral equations. It was investigated that the sequences obtained from this iteration algorithm converged to the solution of nonlinear Volterra integral equations. Moreover, data dependence was obtained for nonlinear Volterra integral equations. An example was given that confirmed the applicability of the newly proven theorems.

Keywords: fixed point; nonlinear Volterra integral equations; Picard's three step iteration; convergence; data dependency
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1. Introduction

"Fixed Point Theory" studies, which are important in modern mathematics, date back to the early 19th century [1]. Fixed point theory has found application in many branches of mathematics as well as in other branches of natural sciences, namely, general topology, functional analysis, mathematical analysis, differential equations, physics, chemistry, biology, engineering and statistics. Apart from this, medicine, communication, economics, and many other fields have benefited from "fixed point theory". It has a wide application area in the past and today because it is used in solving many problems [2–4].

One of the areas where the fixed point approximation is most commonly used in mathematics is integral equations. An equation in which the unknown function is under the integral sign is called an integral equation. Integral equations, both linear and nonlinear, occur in many areas of science and engineering [5,6]. Indeed, many physical processes and mathematical models can be described by them, thus providing an important tool for modeling processes, particularly fluid mechanics, solid state physics, kinetic chemistry, biological models, etc. Integral equations arise in fields. In this study, we will focus on some special integral equations that are widely applied, such as the Volterra integral equations. Frequent applications of such integral equations are seen in mathematical physics, engineering, and mathematics. For example, there are many problems with renewable and accumulator power systems, load balancing problem, and energy stores based on discontinuous core Volterra integral equations. We can also find more applications in the basic characteristics of storages such as capacity, efficiency, number of cycles and discharge/charge ratio, load distribution between existing storages based on electrical load estimation, production from renewable energy sources, and conventional generation [7–10].

The main purpose of fixed point theory is to determine the appropriate conditions to be added on the set X or the mapping S in order for the fixed points of the mapping $S: X \to X$ to be different from the null. Once the existence of a fixed point of a mapping has been shown, algorithms called iterations have been defined to find this point.

However, the iteration method was first defined by the Italian mathematician Picard [11]. Until today, many iterations methods have been introduced and the properties of strong convergence, equivalence of convergence, data dependencies, and convergence rates of these iteration methods for certain classes of transformations have been examined in detail. These include Picard iteration, Mann iteration, Krasnosel'skii iteration, generalized Krasnosel'skii iteration, Kirk iteration, Ishikawa iteration, Noor iteration, multistep iteration, S iteration, two-step Mann iteration, SP iteration, CR iteration, S* iteration, Abbas and Nazir iteration, Thakur et al. iteration, three-step iteration, M iteration, M* iteration, and Picard's three-step iteration method. In addition, many hybrid iteration methods have been defined based on these iteration methods. These include Kirk-Ishikawa, Kirk-Mann, Kirk-Noor, Kirk-SP, Kirk-CR, Picard-Mann, Multistep-SP, Kirk-Multistep, Kirk-S, and Picard-S [12–42]. Of course, there are two basic questions that researchers focus on when defining these iteration methods:

(1) Question: Is it possible to define an iteration method that converges faster than the iteration methods introduced previously?

(2) Question: Is it possible to define a more practical and simpler iteration method than the iteration methods introduced previously?

Considering these questions, let's recall some studies that provide information about the equivalence and convergence fast between the iteration methods mentioned above. In the paper [25], a new three-step iterative method called CR iterative scheme was introduced for the class of quasi-contraction operators. It is also shown that the CR iterative method is faster than the Picard, Mann, Ishikawa, Noor, and SP iterative. Karahan and Özdemir in [27] introduced the S* iteration method for non-expansive mappings in Banach spaces and proved that this method is faster than Picard, Mann, and S iterations. A new iteration method called Picard-S iteration method was defined for contractions by Gürsoy and Karakaya in [32], and it was numerically shown that it converges faster than Picard, Mann, Ishikawa, Noor, SP, S, and some other iteration methods. Inspired by the studies mentioned above, Ali et al. described Picard's three-step iteration method for the approximation of fixed points of Zamfirescu operators in an arbitrary Banach space [40]. Moreover, they proved that Picard's three-step iteration process converges faster than all iteration methods such as Picard, Mann, Ishikawa, Noor, Picard-S, SP, S, CR, and M*. Apart from the studies mentioned here, there are studies on the convergence speed of many iteration methods in the literature. Interested researchers can look at the sources mentioned above and the references in these sources.

In the second part, the basic concepts that will be used in the study are stated. In the third part, the convergence and data dependence of nonlinear Volterra integral equations are investigated with the Picard's three-step iteration algorithm defined by Ali et al. Finally, examples are given to explain the result obtained.

2. Materials and methods

In this section, let's express some basic concepts that will be used in the next section.

Definition 2.1. Let (X,d) be a metric space and $S: X \to X$ a mapping. If for all $x, y \in X$ there exists a number $\delta > 0$ such that $d(Sx, Sy) \leq \delta d(x,y)$, then S is called a Lipschitzian mapping. If $\delta \in (0,1)$, then S is called a contraction mapping [2].

Also, if $S: X \to X$ is a Lipschitzian mapping defined on the normed space X, and for all $x, y \in X$ there exists real number $\delta \in (0,1)$ such that $\|Sx - Sy\| \leq \delta \|x - y\|$, then S is called a contraction mapping [38].

Now, let's state the Banach fixed point theorem, which is the most fundamental theorem of fixed point theory [1].

Theorem 2.2. Let $S : X \to X$ be a contraction mapping on a complete metric space (X,d). Then the following holds:

i) S mapping has only one fixed point $x \in X$.

ii) Iteration sequence $\{S^n x_0\}$ for any $x_0 \in X$ converges to the unique fixed point of S.

Definition 2.3. Let $S: X \to X$ be a mapping defined on the metric space (X,d). The fixed point iteration method is the most elementary form for $\forall n \in \mathbb{N}$ and $x_0 \in X$, where f is a function defined by the relation [2]

$$x_{n+1} = f(S, x_n). \tag{2.1}$$

The reason why we used Picard's three-step iteration algorithm in our study is that this algorithm was proven by Ali et al. in 2021 to be faster than many iteration algorithms such as Picard, Mann, Ishikawa, Noor, Picard-S, SP, S, CR, and M*. Now, let's express the definition of this algorithm.

Definition 2.4. Let X be a metric space or a Banach space, and let $S: X \to X$ be a defined mapping. The iteration method (x_n) given by the estimated starting point $x_0 \in X$ and for $n \in \mathbb{Z}^+$

$$\begin{cases} x_{n+1} = f(S, x_n) = Sy_n, \\ y_n = Sz_n, \\ z_n = Sx_n, \end{cases} \tag{2.2}$$

is called the Picard's three-step iteration [40].

Definition 2.5. The integral equations in the form of

$$\varphi(x) = f(x) + \alpha \int_a^x K\big(x, t, \varphi(t)\big) dt, \tag{2.3}$$

where $K(x, t, \varphi)$ is a known function defined over the region $D = \{(x, t, \varphi) \in \mathbb{R}^3 : a \leq x, t \leq b, -\infty < \varphi < \infty\}$ and $\varphi(t)$ is an unknown function whose solution is sought, and α is any numerical parameter, are called the second type of nonlinear Volterra integral equations. Here, K is called the kernel of the integral equation [5].

Thus, in [6] thet obtained the existence and uniqueness of the solution (2.3) by using the contraction principle under the following conditions.

Theorem 2.6. Let the nonlinear Volterra integral Eq (2.3) be given such that $\varphi(x)$ is an unknown function, the function $K(x,t,\varphi)$ is a continuous function on $D=\{(x,t,\varphi)\in\mathbb{R}^3 : 0\le x,t\le T, -\infty<\varphi<\infty\}$, and the function $f(x)$ is a continuous function on $[0,T]$. Suppose all of the following conditions are satisfied.

i) There exists a nonnegative and continuous function $\theta(x,t)$ in $[0,T]^2$ such that

$$|K(x,t,\varphi)-K(x,t,\psi)|\le\theta(x,t)|\varphi-\psi|,$$

for $\forall(x,t)\in[0,T]^2$ and $\forall\varphi,\psi\in(-\infty,\infty)$.

ii) When $\theta(x,t)=L>0$ is $L\le L_1$, ρ is defined as

$$\rho=\max\left\{e^{-L_1x}\int_0^x\theta(x,t)e^{L_1t}dt : x\in[0,T]\right\}.$$

Let $\rho<1$. Then, the integral equation given by (2.3) has a solution and it is unique.

Definition 2.7. Let's define the operator $S: C[0,T]\to C[0,T]$ for $\varphi(x)\in C[0,T]$ as

$$S(\varphi(x)) = f(x)+\alpha\int_0^x K(x,t,\varphi(t))dt, \tag{2.4}$$

and the norm of the function $\varphi(x)\in C[0,T]$ as [6]

$$\|\varphi\|_* = \max\{e^{-L_1x}|\varphi(x)| : x\in[0,T]\}.$$

Definition 2.8. Let $A,S : X\to X$ be operators. If $\|Ax-Sx\|\le\varepsilon$ for all $x\in X$ and constant $\varepsilon>0$, then S is called the approximation operator of A [21].

We end this section with two important lemmas that we will use to prove our main results.

Lemma 2.9. Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be two nonnegative real sequences. If $\mu_n\in(0,1)$ for each $n\ge n_0$, $\sum_{n=0}^{\infty}\mu_n=\infty$, and $\frac{\beta_n}{\mu_n}\to 0$ as $n\to\infty$, satisfying conditions such that $\alpha_{n+1}\le(1-\mu_n)\alpha_n+\beta_n$, then $\lim_{n\to\infty}\alpha_n=0$ [17].

Lemma 2.10. Let $\{\alpha_n\}_{n=0}^{\infty}$ be a nonnegative real sequence. There exist for each $n\ge n_0, n_0\in\mathbb{N}$, $\mu_n\in(0,1)$, $\sum_{n=0}^{\infty}\mu_n=\infty$, and $\xi_n\ge 0$ satisfying conditions such that $\alpha_{n+1}\le(1-\mu_n)\alpha_n+\mu_n\xi_n$, then the following inequality holds: $0\le\limsup_{n\to\infty}\alpha_n\le\limsup_{n\to\infty}\xi_n$ [21].

3. Results

In this section, we show that the convergence result can be obtained for Eq (2.3) under suitable conditions, and we also examine the data dependence result through the iterative algorithm (2.2). For this, let the following iteration algorithm of convergence and data dependence of nonlinear Volterra integral equations be defined. Reconstruct iteration (2.2) using $S : C[0,T]\to C[0,T]$, $S(\varphi(x))=$

$f(x)+\alpha\int_0^x K\big(x,t,\varphi(t)\big)dt$ as follows:

$$\begin{cases}\varphi_{n+1}(x)=f(x)+\alpha\int_0^x K\big(x,t,\psi_n(t)\big)dt,\\ \psi_n(x)=f(x)+\alpha\int_0^x K\big(x,t,\tau_n(t)\big)dt,\\ \tau_n(x)=f(x)+\alpha\int_0^x K\big(x,t,\varphi_n(t)\big)dt.\end{cases} \tag{3.1}$$

Our first theorem proves that the sequence $\{\varphi_n\}_{n=0}^{\infty}$ obtained from the (2.2) iteration algorithm strongly converges to the fixed point of (2.4).

Theorem 3.1. Let $S:(C[0,T],\|\cdot\|_*)\to(C[0,T],\|\cdot\|_*)$ be an operator. So, the integral equation given by (2.4) has a unique solution $p\in C([0,T])$. Also, let ρ be defined as follows:

$$\rho=\max\left\{e^{-L_1x}\int_0^x \theta(x,t)\,e^{L_1t}dt\right\}.$$

If $|\alpha|\rho<1$, then the sequence $\{\varphi_n\}_{n=0}^{\infty}$ converges to p.

Proof: Let $\{\varphi_n\}_{n=0}^{\infty}$ be an iterative sequence generated by iteration algorithm (3.1) for the operator $S:(C[0,T],\|\cdot\|_*)\to(C[0,T],\|\cdot\|_*)$, $S\big(\varphi(x)\big)=f(x)+\alpha\int_0^x K\big(x,t,\varphi(t)\big)dt$. It will be shown that $\varphi_n\to p$ as $n\to\infty$. If the necessary calculations are made using algorithm (3.1) and the conditions of Theorem 2.6,

$$\begin{aligned}|\varphi_{n+1}(x)-p(x)|&=|S\psi_n(x)-Sp(x)|\\
&=\left|f(x)+\alpha\int_0^x K\big(x,t,\psi_n(t)\big)dt-\left(f(x)+\alpha\int_0^x K\big(x,t,p(t)\big)dt\right)\right|\\
&=|\alpha|\left|\int_0^x K\big(x,t,\psi_n(t)\big)dt-\int_0^x K\big(x,t,p(t)\big)dt\right|\\
&\le|\alpha|\int_0^x\left|K\big(x,t,\psi_n(t)\big)-K\big(x,t,p(t)\big)\right|dt\\
&\le|\alpha|\int_0^x \theta(x,t)|\psi_n(t)-p(t)|dt\\
&\le|\alpha|\|\psi_n-p\|_*\int_0^x\theta(x,t)e^{L_1t}dt,\end{aligned}$$

and $|\varphi_{n+1}(x)-p(x)|\le|\alpha|\|\psi_n-p\|_*\int_0^x\theta(x,t)e^{L_1t}dt$.

If both sides of this last inequality are multiplied by e^{-L_1x} and the maximums are taken,

$$e^{-L_1x}|\varphi_{n+1}(x)-p(x)|\le|\alpha|\|\psi_n-p\|_*e^{-L_1x}\int_0^x\theta(x,t)e^{L_1t}dt,$$

$$\max\{e^{-L_1x}|\varphi_{n+1}(x)-p(x)|\}\le|\alpha|\|\psi_n-p\|_*\max\left\{e^{-L_1x}\int_0^x\theta(x,t)e^{L_1t}dt\right\},$$

$$\|\varphi_{n+1}-p\|_*\le|\alpha|\|\psi_n-p\|_*\max\left\{e^{-L_1x}\int_0^x\theta(x,t)e^{L_1t}dt\right\}$$

is obtained. Hence, if $\theta(x,t)=L>0$ and $L\le L_1$, $\rho=\frac{L(1-e^{-L_1T})}{L_1}$ so that

$$\|\varphi_{n+1}-p\|_*\le|\alpha|\rho\|\psi_n-p\|_* \tag{3.2}$$

is found. Similarly,

$$\begin{aligned}|\psi_n(x)-p(x)|&=|S\tau_n(x)-Sp(x)|\\&=\left|f(x)+\alpha\int_0^x K(x,t,\tau_n(t))dt-Ap(x)-\left(f(x)+\alpha\int_0^x K(x,t,p(t))dt\right)\right|\\&\le|\alpha|\int_0^x\left|K(x,t,\tau_n(t))-K(x,t,p(t))\right|dt\\&\le|\alpha|\int_0^x\theta(x,t)|\tau_n(t)-p(t)|dt\\&\le|\alpha|\|\tau_n-p\|_*\int_0^x\theta(x,t)e^{L_1t}dt.\end{aligned}$$

If both sides of this last inequality are multiplied by $e^{-L_1x}$and the maximums are taken,

$$\|\psi_n-p\|_*\le|\alpha|\rho\|\tau_n-p\|_* \tag{3.3}$$

is found.

$$\begin{aligned}|\tau_n(x)-p(x)|&=|S\varphi_n(x)-Sp(x)|\\&=\left|f(x)+\alpha\int_0^x K(x,t,\varphi_n(t))dt-\left(f(x)+\alpha\int_0^x K(x,t,p(t))dt\right)\right|\\&=|\alpha|\left|\int_0^x K(x,t,\varphi_n(t))-K(x,t,p(t))dt\right|\\&\le|\alpha|\int_0^x\left|K(x,t,\varphi_n(t))-K(x,t,p(t))\right|dt\end{aligned}$$

$$\leq |\alpha| \int_0^x \theta(x,t)|\varphi_n(t) - p(t)|dt$$

$$\leq |\alpha| \|\varphi_n - p\|_* \int_0^x \theta(x,t) e^{L_1 t} dt$$

and $|\tau_n(x) - p(x)| \leq |\alpha| \|\varphi_n - p\|_* \int_0^x \theta(x,t) e^{L_1 t} dt$.

If both sides of this last inequality are multiplied by $e^{-L_1 x}$ and the maximums are taken,

$$\max\{e^{-L_1 x}|\tau_n(x) - p(x)|\} \leq |\alpha| \|\varphi_n - p\|_* \max\{e^{-L_1 x} \int_0^x \theta(x,t) e^{L_1 t} dt\},$$

$$\|\tau_n - p\|_* \leq |\alpha|\rho \|\varphi_n - p\|_* \tag{3.4}$$

is found. If inequality (3.4) is used in inequality (3.3),

$$\|\psi_n - p\|_* \leq (|\alpha|\rho)^2 \|\varphi_n - p\|_* \tag{3.5}$$

is found. If inequality (3.5) is used in inequality (3.2), $\|\varphi_{n+1} - p\|_* \leq (|\alpha|\rho)^3 \|\varphi_n - p\|_*$ is obtained. If induction is applied to the last inequality, inequality (3.6) is easily seen.

$$\begin{aligned} \|\varphi_{n+1} - p\|_* &\leq (|\alpha|\rho)^3 \|\varphi_n - p\|_* \\ &\leq (|\alpha|\rho)^6 \|\varphi_{n-1} - p\|_* \\ &\vdots \end{aligned}$$

$$\|\varphi_{n+1} - p\|_* \leq (|\alpha|\rho)^{3(n+1)} \|\varphi_0 - p\|_* \tag{3.6}$$

Thus, if the limit is taken when $n \to \infty$ in inequality (3.6), since $|\alpha|\rho < 1$, $\lim_{n\to\infty} \|\varphi_{n+1} - p\|_* = 0$ is obtained. This completes the proof.

This result leads to the next theorem. Now, we prove the result on data dependence for the Picard's three-step iteration algorithm. Thus, consider the integral equation

$$u(x) = g(x) + \alpha_1 \int_0^x H(x,t,u(t))dt, \tag{3.7}$$

where $g(x)$ is a continuous function in $[0,T]$, $u(x)$ is a function whose solution is desired in $[0,T]$, $H(x,t,u(t))$ is a function given continuously on the domain $D = \{(x,t,u) \in \mathbb{R}^3 : 0 \leq x,t \leq T, -\infty < u < \infty\}$, and α_1 is a parameter. (3.8) operators can be written from (3.7) integral equations. $A: (C[0,T], \|\cdot\|_*) \to (C[0,T], \|\cdot\|_*)$,

$$A(u(x)) = g(x) + \alpha_1 \int_0^x H(x,t,u(t))dt. \tag{3.8}$$

Thus, recreate the iteration algorithm given in (2.2) with the operator (3.8).

$$\begin{cases} u_{n+1}(x) = g(x) + \alpha_1 \int_0^x H(x,t,u_n(t))dt, \\ v_n(x) = g(x) + \alpha_1 \int_0^x H(x,t,w_n(t))dt, \\ w_n(x) = g(x) + \alpha_1 \int_0^x H(x,t,u_n(t))dt. \end{cases} \tag{3.9}$$

In the following theorem, data dependency is revealed by using (3.1) and (3.9) iteration algorithms for nonlinear Volterra integral equations. This result obtained immediately afterward is supported by an example.

Theorem 3.2. Let the sequence obtained from Eq (3.1) and the sequence obtained from Eq (3.9) be $\{\varphi_n\}_{n=0}^{\infty}$ and $\{u_n\}_{n=0}^{\infty}$, respectively. Under the conditions of Theorem 3.1, let the solution of (2.4) integral equations be p and the solution of (3.8) integral equations be q. Also, suppose the following conditions are met

- **i)** There exists a nonnegative and continuous function $\gamma(x,t)$ in $[0,T]^2$. Let there exist a positive number N_1 such that $\gamma(x,t) = N > 0$ and $N \leq N_1$. Thus, for $\forall(x,t) \in [0,T]$, $|K(x,t,\mu) - H(x,t,\omega)| \leq \gamma(x,t)|\mu - \omega|$.
- **ii)** $M = \max\{e^{-N_1 x}\int_0^x \gamma(x,t)\, e^{N_1 t}dt\}$ and $\varepsilon_1 = \max\{e^{-N_1 x}|f(x) - g(x)|\}$.
- **iii)** Let $\alpha_0 = \max\{|\alpha|, |\alpha_1|\}$ for $\forall n \in \mathbb{N}$ and a constant ε_2 such that $0 < \alpha_0 M < \varepsilon_2 < 1$.

In this case, if $\varphi_n \to p$ and $u_n \to q$ as $n \to \infty$, the $\|p - q\|_* \leq \frac{3\varepsilon_1}{1-\varepsilon_2}$ inequality holds.

Proof. If the necessary calculations are made by considering the hypotheses of Theorem 3.2, the following inequalities are obtained.

$$|\varphi_{n+1}(x) - u_{n+1}(x)| = \left| f(x) + \alpha \int_0^x K(x,\mathrm{t},\psi_n(t))dt - \left(g(x) + \alpha_1 \int_0^x H(x,t,v_n(t))dt \right) \right|$$

$$\leq |f(x) - g(x)| + \left| \alpha_0 \int_0^x \Big(K(x,t,\psi_n(t)) - H(x,t,v_n(t)) \Big)\, dt \right|$$

$$= |f(x) - g(x)| + |\alpha_0| \left| \int_0^x \Big(K(x,t,\psi_n(t)) - H(x,t,v_n(t)) \Big)\, dt \right|$$

$$\leq |f(x) - g(x)| + |\alpha_0| \int_0^x \big| K(x,t,\psi_n(t)) - H(x,t,v_n(t)) \big| dt$$

$$\leq |f(x) - g(x)| + |\alpha_0| \int_0^x \gamma(x,t) |\psi_n(t) - v_n(t)|\, dt$$

$$= |f(x) - g(x)| + |\alpha_0| \int_0^x \gamma(x,t) |\psi_n(t) - v_n(t)|\, e^{-N_1 t} e^{N_1 t}\, dt$$

$$|\varphi_{n+1}(x) - u_{n+1}(x)| \leq |f(x) - g(x)| + |\alpha_0| \|\psi_n - v_n\|_* \int_0^x \gamma(x,t) e^{N_1 t}\, dt.$$

If both sides of this last inequality are multiplied by $e^{-N_1 x}$, then the maximums are taken,

$$\max\{e^{-N_1 x}|\varphi_{n+1}(x) - u_{n+1}(x)|\} \leq \max\{e^{-N_1 x}|f(x) - g(x)|\}$$

$$+\max\left\{|\alpha_0| \|\psi_n - v_n\|_* e^{-N_1 x} \int_0^x \gamma(x,t) e^{N_1 t}\, dt\right\},$$

$$\|\varphi_{n+1} - u_{n+1}\|_* \leq \varepsilon_1 + |\alpha_0| M \|\psi_n - v_n\|_* \tag{3.10}$$

is obtained. Similarly,

$$|\psi_n(x) - v_n(x)| \leq \left| f(x) + \alpha \int_0^x K(x,t,\tau_n(t)) dt - \left(g(x) + \alpha_1 \int_0^x H(x,t,w_n(t)) dt \right) \right|$$

$$\leq |f(x) - g(x)| + |\alpha_0| \left| \int_0^x \left(K(x,t,\tau_n(t)) - H(x,t,w_n(t)) \right) dt \right|$$

$$\leq |f(x) - g(x)| + |\alpha_0| \|\tau_n - w_n\|_* \int_0^x \gamma(x,t) e^{N_1 t}\, dt.$$

If both sides of this last inequality are multiplied by $e^{-N_1 x}$, then the maximums are taken,

$$\max\{ e^{-N_1 x}|\psi_n(x) - v_n(x)|\} \leq \max\{e^{-N_1 x}|f(x) - g(x)|\}$$

$$+|\alpha_0| \|\tau_n - w_n\|_* \max\left\{e^{-N_1 x} \int_0^x \gamma(x,t)\, e^{N_1 t} dt\right\},$$

$$\|\psi_n - v_n\|_* \leq \varepsilon_1 + |\alpha_0| M \|\tau_n - w_n\|_* \tag{3.11}$$

is obtained.

$$|\tau_n(x) - w_n(x)| = \left| f(x) + \alpha \int_0^x K(x,t,\varphi_n(t)) dt - \left(g(x) + \alpha_1 \int_0^x H(x,t,u_n(t)) dt \right) \right|$$

$$\leq |f(x) - g(x)| + |\alpha_0| \left| \int_0^x K(x,t,\varphi_n(t)) - H(x,t,u_n(t)) dt \right|$$

$$\leq |f(x) - g(x)| + |\alpha_0| \int_0^x \left| K(x,t,\varphi_n(t)) - H(x,t,u_n(t)) \right| dt$$

$$\leq |f(x) - g(x)| + |\alpha_0| \int_0^x \gamma(x,t) |\varphi_n(t) - u_n(t)| dt$$

$$\leq |f(x) - g(x)| + |\alpha_0| \|\varphi_n - u_n\|_* \int_0^x \gamma(x,t) e^{N_1 t} dt.$$

If both sides of this last inequality are multiplied by $e^{-N_1 x}$, then the maximums are taken,

$$e^{-N_1 x}\max\{|\tau_n(x) - w_n(x)|\} \leq \max\{e^{-N_1 x}|f(x) - g(x)|\}$$

$$+|\alpha_0|\|\varphi_n - u_n\|_* \max\{e^{-N_1 x}\int_0^x \gamma(x,t)e^{N_1 t}dt\},$$

$$\|\tau_n - w_n\|_* \leq \varepsilon_1 + |\alpha_0|M\|\varphi_n - u_n\|_* \tag{3.12}$$

is obtained. Using the following hypotheses and combining (3.11) and (3.12), we obtain

- $\alpha_0 = \max\{|\alpha|, |\alpha_1|\}$.
- $0 < \alpha_0 M < \varepsilon_2 < 1$.

$$\begin{aligned}\|\psi_n - v_n\|_* &\leq \varepsilon_1 + |\alpha_0|M(\varepsilon_1 + |\alpha_0|M\|\varphi_n - u_n\|_*)\\ &= \varepsilon_1 + \varepsilon_1|\alpha_0|M + |\alpha_0|M|\alpha_0|M\|\varphi_n - u_n\|_*\\ &\leq 2\varepsilon_1 + |\alpha_0|M\|\varphi_n - u_n\|_* .\end{aligned} \tag{3.13}$$

If inequality (3.13) is written in inequality (3.10),

$$\begin{aligned}\|\varphi_{n+1} - u_{n+1}\|_* &\leq \varepsilon_1 + |\alpha_0|M(2\varepsilon_1 + |\alpha_0|M\|\varphi_n - u_n\|_*)\\ &= \varepsilon_1 + 2\varepsilon_1|\alpha_0|M + |\alpha_0|M|\alpha_0|M\|\varphi_n - u_n\|_*\\ &\leq 3\varepsilon_1 + |\alpha_0|M\|\varphi_n - u_n\|_*\\ &= \big(1 - (1 - |\alpha_0|M)\big)\|\varphi_n - u_n\|_* + 3\varepsilon_1\\ &= \big(1 - (1 - |\alpha_0|M)\big)\|\varphi_n - u_n\|_* + (1 - |\alpha_0|M)\frac{3\varepsilon_1}{1 - |\alpha_0|M}\\ &\leq \big(1 - (1 - |\alpha_0|M)\big)\|\varphi_n - u_n\|_* + (1 - |\alpha_0|M)\frac{3\varepsilon_1}{1-\varepsilon_2}\end{aligned} \tag{3.14}$$

is found. Considering the inequality (3.14), we get $\alpha_n = \|\varphi_n - u_n\|_*$, $\mu_n = (1 - |\alpha_0|M) \in (0,1)$, and $\xi_n = \frac{3\varepsilon_1}{1-\varepsilon_2} \geq 0$. Since the inequality (3.14) satisfies the conditions of Lemma 2.10, we found

$$\begin{aligned}0 &\leq \limsup\|\varphi_n - u_n\|_\infty\\ &\leq \limsup\left\{\frac{3\varepsilon_1}{1 - \varepsilon_2}\right\}\\ &= \frac{3\varepsilon_1}{1 - \varepsilon_2}\end{aligned}$$

and $0 \leq \limsup_{n\to\infty}\|\varphi_n - u_n\|_* \leq \limsup_{n\to\infty}\xi_n = \limsup_{n\to\infty}\frac{3\varepsilon_1}{1-\varepsilon_2}$. Since $\varphi_n \to p$ and $u_n \to q$ when $n \to \infty$,

$$\|p - q\|_* \leq \frac{3\varepsilon_1}{1 - \varepsilon_2}$$

is found.

Now, let's give one for both linear and nonlinear Volterra integral equations.

Example 3.3. Let's take the following integral defined over region $D = \{(x,t,\varphi) \in \mathbb{R}^3 :\ 0 \le x,t \le 1,\ -\infty < \varphi < \infty\}$,

$$\varphi(x) = \frac{(1+x)e^{2x}}{20} - \int_0^x (2-t)\varphi(t)dt,$$

where $K(x,t,\varphi(t)) = (2-t)\varphi(t)$ is a continuous function over the region D. Thus,

$$\begin{aligned} |K(x,t,\varphi) - K(x,t,\psi)| &\le \theta(x,t)|\varphi(t) - \psi(t)| \\ &\le |(2-t)\varphi(t) - (2-t)\psi(t)| \\ &\le (2-t)|\varphi(t) - \psi(t)|, \end{aligned}$$

there exists a nonnegative and continuous function $\theta(x,t) = 2-t$ in $[0,1]^2$. Defined on the space $C([0,1])$, if we take $\theta(x,t) = (2-t) = L$ and $L \le L_1$ with $L_1 = 2 > 0$, $\rho = \max\left\{\frac{\theta(x,t)(1-e^{-L_1 x})}{L_1}\right\} = \frac{2(1-e^{-2})}{2} < 1$, the integral equation has only one solution on $[0,1]$.

Also, consider the following integral equation defined over region $D = \{(x,t,u) \in \mathbb{R}^3 :\ 0 \le x,t \le 1,\ -\infty < u < \infty\}$,

$$u(x) = \frac{xe^{2x}}{20} - \int_0^x (1-t)u(t)dt\,,$$

where $H(x,t,u(t)) = (1-t)u(t)$ is a continuous function over the region D. Thus,

$$\begin{aligned} |H(x,t,u) - H(x,t,v)| &\le \theta(x,t)|u(t) - v(t)| \\ &\le |(1-t)u(t) - (1-t)v(t)| \\ &\le (1-t)|u(t) - v(t)|, \end{aligned}$$

there exists a nonnegative and continuous function $\theta(x,t) = 1-t$ in $[0,1]^2$. Defined on the space $C([0,1])$, if $\theta(x,t) = (1-t) = L$ and $L \le L_1$ with $L_1 = 1 > 0$, $\rho = \max\left\{\frac{\theta(x,t)(1-e^{-L_1 x})}{L_1}\right\} = \frac{(1-0)(1-e^{-1})}{1} < 1$, the given integral equation has one solution and is unique on $[0,1]$.

Thus, $\varphi(x) = \frac{(1+x)e^{2x}}{20} - \int_0^x (2-t)\varphi(t)dt$ and $u(x) = \frac{xe^{2x}}{20} - \int_0^x (1-t)u(t)dt$

$$|K(x,t,\varphi) - H(x,t,u)| \le (2-t)|\varphi(t) - u(t)|,$$

let there exist a nonnegative and continuous function $\gamma(x,t) = 2-t$ in $[0,1]^2$. So, there exists $N_1 = 2$ such that $\gamma(x,t) = N > 0$ and $N \le N_1$,

$$M = \max\left\{e^{-2x}\int_0^x (2-t)\, e^{2t}dt\right\} = \max\left\{e^{-2x}\left[\frac{5}{4}-\frac{x}{2}-\frac{5}{4}e^{-2x}\right]\right\} = 0{,}58$$

$$\varepsilon_1 = \max\left\{e^{-2x}\left|\frac{(1+x)e^{2x}}{20}-\frac{xe^{2x}}{20}\right|\right\} = \frac{1}{20}.$$

Also, $\alpha_0 = \max\{|-1|,|-1|\} = 1$ for $\forall n \in \mathbb{N}$

$$0 < \alpha_0 M < \varepsilon_2 < 1,$$

such that there is a constant ε_2. From the above processes, all the conditions of Theorem 3.2 are held. In this case, if $\varphi_n \to p$ and $u_n \to q$ as $n \to \infty$,

$$\|p-q\|_* \leq \frac{3\varepsilon_1}{(1-\varepsilon_2)} = \frac{3.\frac{1}{20}}{0{,}42} \leq 0{,}357$$

is provided.

Example 3.4. Let's take the following integral defined over region $D = \{(x,t,\varphi) \in \mathbb{R}^3 : 0 \leq x,t \leq 1, -\infty < \varphi < \infty\}$,

$$\varphi(x) = \frac{(\pi+x)e^{3x}}{10} + \int_0^x t\sin(\varphi(t))\, dt,$$

where $K(x,t,\varphi(t)) = t\sin(\varphi(t))$ is a continuous function over the region D. Thus,

$$|K(x,t,\varphi) - K(x,t,\psi)| \leq \theta(x,t)|\varphi(t)-\psi(t)|$$

$$\leq \left|t\sin(\varphi(t)) - t\sin(\psi(t))\right|$$

$$\leq t|\varphi(t)-\psi(t)|\left|2\cos\left(\frac{\varphi(t)+\psi(t)}{2}\right)\sin\left(\frac{\varphi(t)-\psi(t)}{2}\right)\right|$$

$$\leq |\varphi(t)-\psi(t)|\left|2\cos\left(\frac{\varphi(t)+\psi(t)}{2}\right)\right|\left|\sin\left(\frac{\varphi(t)-\psi(t)}{2}\right)\right|$$

$$\leq 2|\varphi(t)-\psi(t)|,$$

and there exists a nonnegative and continuous function $\theta(x,t) = 2$ in $[0,1]^2$. Defined on the space $C([0,1])$, if we take $\theta(x,t) = 2 = L$ and $L_1 = 2$ with $L \leq L_1$, $\rho = \max\left\{\frac{\theta(x,t)(1-e^{-L_1 x})}{L_1}\right\} = \frac{2(1-e^{-2})}{2} < 1$, the integral equation has only one solution in $[0,1]$.

Also, consider the following integral equation defined over region $D = \{(x,t,u) \in \mathbb{R}^3 : 0 \leq x,t \leq 1, -\infty < u < \infty\}$,

$$u(x) = \frac{xe^{3x}}{10} + \int_0^x (1-t)cos\big(u(t)\big)\,dt\,,$$

where $H\big(x,t,u(t)\big) = (1-t)u(t)cos\big(u(t)\big)$ is a continuous function over the region D. Thus,

$$\begin{aligned}
|H(x,t,u) - H(x,t,v)| &\le \theta(x,t)|u(t)-v(t)| \\
&\le \big|(1-t)cos\big(u(t)\big) - (1-t)cos\big(v(t)\big)\big| \\
&\le |1-t||u(t)-v(t)|\left|-2sin\left(\frac{\varphi(t)+\psi(t)}{2}\right)sin\left(\frac{\varphi(t)-\psi(t)}{2}\right)\right| \\
&\le |-2||u(t)-v(t)|\left|sin\left(\frac{\varphi(t)+\psi(t)}{2}\right)\right|\left|sin\left(\frac{\varphi(t)-\psi(t)}{2}\right)\right| \\
&\le 2|u(t)-v(t)|,
\end{aligned}$$

and there exists a nonnegative and continuous function $\theta(x,t) = 2$ in $[0,1]^2$. Defined on the space $C([0,1])$, if $\theta(x,t) = 2 = L$ and $L \le L_1$ with $L_1 = 2$, $\rho = \max\left\{\frac{\theta(x,t)(1-e^{-L_1 x})}{L_1}\right\} = \frac{2(1-e^{-2})}{2} < 1$, the given integral equation has one solution and is unique in $[0,1]$.

Thus, $\varphi(x) = \frac{(\pi+x)e^{3x}}{10} + \int_0^x tsin\big(\varphi(t)\big)\,dt$ and $u(x) = \frac{xe^{3x}}{10} + \int_0^x (1-t)cos\big(u(t)\big)\,dt$

$$|K(x,t,\varphi) - H(x,t,u)| \le 3|\varphi(t) - u(t)|,$$

let there exist a nonnegative and continuous function $\gamma(x,t) = 3$ in $[0,1]^2$. So, there exists $N_1 = 3$ such that $\gamma(x,t) = N$ and $N \le N_1$

$$M = \max\left\{e^{-3x}\int_0^x \gamma(x,t)\,e^{3t}dt\right\} = \max\left\{e^{-3x}\int_0^x 3\,e^{3t}\,dt\right\} = 0{,}51,$$

$$\varepsilon_1 = \max\left\{e^{-3x}\left|\frac{(\pi+x)e^{3x}}{10} - \frac{xe^{3x}}{10}\right|\right\} = \frac{\pi}{10} = 0{,}31.$$

Also, $\alpha_0 = \max\{|1|,|1|\} = 1$ for $\forall n \in \mathbb{N}$, $0 < \alpha_0 M < \varepsilon_2 < 1$ such that there is a constant ε_2. From the above processes, all the conditions of Theorem 3.2 are held. In this case, if $\varphi_n \to p$ and $u_n \to q$ as $n \to \infty$,

$$\|p-q\|_* \le \frac{3\varepsilon_1}{(1-\varepsilon_2)} = \frac{3.\,(0{,}31)}{1-0{,}51} \le 1{,}89$$

is provided.

4. Conclusions

In this study, first, the solution of nonlinear Volterra integral equations was examined by using the sequence obtained from the iteration algorithm of Eq (2.2). In fact, Theorem 3.1 shows that the solution of nonlinear Volterra integral equations is converged using the Picard's three-step iteration algorithm. Additionally, in Theorem 3.2, data dependence for nonlinear Volterra integral equations is revealed, and this result is explained with an example. The findings presented here will contribute to many existing studies in the literature. Researchers can reconsider this iteration with different integral equations or apply a different iteration method for nonlinear Volterra integral equations. Thus, strong convergence and data dependence of integral equations can be achieved with different iteration methods.

Use of AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

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