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Case study

Solving word problems involving triangles and implications on training pre-service mathematics teachers

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Abstract: Triangles and trigonometry are always difficult topics for both mathematics students and teachers. Hence, students' performance in solving mathematical word problems in these topics is not only a reflection of their learning outcomes but also an indication of teaching effectiveness. This case study drew from two examples of solving word problems involving triangles by pre-service mathematics teachers in a foundation mathematics course delivered by the author. The focus of this case study was on reasoning implications of students' performances on the effective training of pre-service mathematics teachers, from which a three-step interactive explicit teaching-learning approach, comprising teacher-led precise and inspiring teaching (or explicit teaching), student-driven engaged learning (or imitative learning), and student-led and teacher-guided problem-solving for real-world problems or projects (or active application), was summarized. Explicit teaching establishes a solid foundation for students to further their understanding of new mathematical concepts and to conceptualize the technical processes associated with these new concepts. Imitative learning helps students build technical abilities and enhance technical efficacy by engaging in learning activities. Once these first two steps have been completed, students should have a decent understanding of new mathematical concepts and technical efficacy to analyze, formulate, and finally solve real-world applications with assistance from teachers whenever required. Specially crafted professional development should also be considered for some in-service mathematics teachers to adopt this three-step interactive teaching-learning process.

Keywords: pre-service mathematics teacher, word problem, triangles, problem-solving, explicit teaching, imitated learning, active applications, professional development

1. Introduction

Triangles form an important part of geometry, being closely associated with trigonometry in mathematics. Solving problems involving triangles is frequently required in scientific, engineering, and technological applications in the real world. Hence, triangles and solving problems involving triangles are key themes in the mathematics curriculum for elementary and secondary schooling in most countries [1–7]. Correspondingly, triangles and their applications have been included in tertiary mathematics curricula for training pre-service mathematics teachers worldwide [8–13].

However, triangles and trigonometry are always challenging and tricky topics for all students who need to succeed in mathematics courses, particularly in STEM disciplines and mathematics education [6–13]. Triangular and trigonometric questions are often solvable by combining techniques from diverse mathematical topics, demanding that mathematics teachers possess advanced domain knowledge in multiple areas to handle problem-solving effectively, particularly with worded problems. Most existing studies adequately focus on students' performances in solving problems involving triangles and their implications for learning, but attention should also be paid to analyzing the links between students' performances in problem-solving and teachers' teaching efficiency to maximize the teaching-learning process for achieving optimal outcomes in mathematics education for students.

Based on the performances of 27 pre-service mathematics teachers in solving two-worded problems involving triangles in their assignments, this case study aims to associate students' performances with the teaching activities designed and conducted by the lecturer. The goal is to identify useful approaches for mathematics teachers to effectively facilitate mathematics learning, stimulate students' curiosity in applying various techniques for problem-solving, appreciate the beauty and power of mathematics, and boost students' confidence and participation in mathematics learning and applications. This study shows that explicit teaching leads to inspired and engaged learning, fostering a spiral progression in mathematical knowledge building for students, particularly pre-service mathematics teachers.

This case study draws from the author's experience in teaching a foundation mathematics course to undergraduate students specializing in secondary mathematics teaching within a Bachelor of Education program at a regional university in Australia. This specialty aims to train students to become mathematics teachers for secondary schools and consists of one statistics course and five mathematics courses across multiple academic levels over three years of full-time study. The five mathematics courses include one foundation mathematics course, two intermediate mathematics courses, and two advanced mathematics courses.

The foundation course serves as a prerequisite for the first intermediate mathematics course in the second year and aims to consolidate basic topics in algebra, geometry, and trigonometry that students had learned in secondary schools. This is achieved through systematic reviews combined with further conceptual reasoning, logical articulation, and real-world applications. The course helps students refresh their previously acquired mathematical knowledge and/or bridge gaps in their original mathematics learning. Students enrolled in the foundation mathematics course range from 17 to 50 years old, with classes typically comprising 20–40 students living in various regional, remote, and rural areas of Australia and, sometimes, abroad. Consequently, this course is delivered through weekly live online classes, with edited recordings uploaded to the course website soon after each session for all students to access.

Since this study directly examines students' attempts at the assigned questions, a comparative case study approach is adopted [14], supported by simple statistics. Sections 2 and 3 of this paper present two cases of word problems involving triangle solving, along with students' performances and interactions between the lecturer and students. Section 4 discusses students' results, focusing on training pre-service mathematics teachers, while Section 5 summarizes the study.

2. The first case study

2.1. The first word problem

The problem shown in Problem 1 was one of the 12 questions assigned to the 27 first-year pre-service mathematics teachers enrolled in the foundation mathematics course in a past teaching term. The 12 questions in this assignment covered a wide range of topics in foundation mathematics, in which only Problems 1 & 2 were worded problems and associated with solving triangles. This first word problem was intended to test students' understanding of the basic characteristics of isosceles triangles and their ability to solve the unknowns by streamlining two steps in a logical way. There should not be any problem in handling each of the two steps because all students had just completed the review of fundamentals of triangles and the Pythagorean theorem, which should also have been intensively studied in high schools. Furthermore, the solution to various problems involving right triangles had been demonstrated in the recent review classes to help students refresh and consolidate the related pre-learned knowledge. The real challenge of this problem was whether they could first translate the simple scenario described in words to a correct representation and then work out a logical plan to solve the problem step-by-step using pre-learned knowledge. Note that one of the general requirements for answering all questions in the assignment was to show sufficient working and/or key steps to support a clear presentation of problem-solving. This is particularly important in training pre-service mathematics teachers who would teach mathematics to school students using a step-by-step approach.

The area of an isosceles triangle is 30 cm^2 and its unequal side is 10 cm long. Find the lengths of two other sides for this isosceles triangle. [keep 2 decimal places in the final result]

Problem 1. The first word problem assigned to the pre-service mathematics teachers.

2.2. The reference solution

Although not mandatory, it would be a better approach to draw a diagram to assist in solving this word problem with clarity, as shown in Figure 1.

Step 1: Assume the unequal side is the base b = 10 cm and the corresponding height is h for this isosceles triangle. Area (A) of this isosceles triangle should be

$$A=\frac{1}{2}bh=30\longrightarrow h=\frac{2A}{b}=\frac{2\times 30}{10}=6\ (cm).$$

Step 2: Since two other sizes are equal in length (a), the height and half of the base form a right triangle, which leads to determining the size a by the Pythagorean theorem.

$$a^2 = h^2 + (\frac{b}{2})^2 \longrightarrow a = \sqrt{h^2 + (\frac{b}{2})^2} = \sqrt{6^2 + 5^2} = \sqrt{61} = 7.81\ (cm)\ .$$

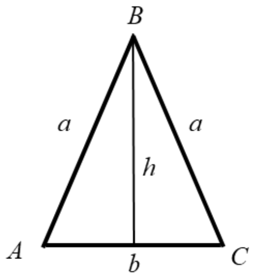


Figure 1. A sketch of isosceles triangle for the first word problem.

This would be the most efficient way to solve this problem. Of course, once Step 1 is correctly done, other methods can also be used to solve the problem with extra step(s).

2.3. The performance of pre-service teachers

The overall performance

Twenty-two out of the twenty-seven students solved this problem correctly, meaning that more than 80% of the students obtained the correct answer following the logical procedure. However, five others obtained incorrect answers to this problem. The overall performance of solving this problem by the student teachers is summarized in Table 1.

Table 1. Summary of the overall performance of the student teachers in solving the first problem.

Outcome	**Number**	**Percentage (%)**
Correct	22	81
Incorrect	5	19
No attempt	0	0
Total	27	100

The 22 students who solved this problem correctly all drew a diagram of isosceles triangle to assist them in solving the problem, and 21 of them followed the same two steps as outlined in the reference solution. One student adopted a three-step approach to solve this problem, which is illustrated as follows.

Step 1: Same as the reference solution.

Step 2: Work out angle A (or B) using a trigonometric ratio.

$$\tan A = \frac{h}{b/2} = \frac{6}{5} \longrightarrow A = \tan^{-1}\frac{6}{5} \approx 50.19°.$$

Step 3: Work out the length of the equal size *a* by either sine or cosine of angle A,

$$a = \frac{h}{\sin A} = \frac{6}{\sin 50.19°} = 7.81\ (cm)\ \text{ or }\ a = \frac{b/2}{\cos A} = \frac{5}{\cos 50.19°} = 7.81\ (cm).$$

The result was the same as the reference solution despite taking an extra step in the process.

Among the five students who obtained incorrect solutions, two students made careless errors in calculating the numbers in both/either Step 1 and/or Step 2 without checking their outcomes. Three others did not draw a diagram to assist in the attempts and hence made vital mistakes during the process. One student simply regarded the isosceles triangle as a right triangle, whereas another student regarded the isosceles triangle as an equilateral triangle. The other student combined the two steps together into one formula but forgot that only half of the base should be used in the Pythagorean theorem. This student did not realize the mistake he made even after reading the note the lecturer wrote on his assignment. The student argued that he had been a relief mathematics teacher in a remote high school for a few years and instructed his students in solving triangles in the way he did for this problem. He was silenced only after the lecturer redirected his attention to the sketch of the isosceles triangle showing that only half of the base should be used in the Pythagorean theorem.

Interactions between the lecturer and some students

After releasing the reference solutions and the marked assignments to the students, one of the 22 students who solved the problem correctly by the same way shown in the reference solution asked the lecturer by email *if this problem could be solved by obtaining one of the equal angles first and then the side by trigonometric ratio, because he had failed to reach the same solution by this way.*

To answer this question raised by the student, the lecturer prepared a new solution for the student by an alternative way to solve this problem, as follows.

Step 1: Find the height *h* by means of angle $\angle A$.

$$h = \frac{b}{2}\tan\angle A.$$

Step 2: The area of triangle ABC should be

$$A = \frac{1}{2}bh = \frac{1}{2}b\left(\frac{b}{2}\tan\angle A\right) = \frac{1}{4}b^2\tan\angle A \longrightarrow \tan\angle A = \frac{4A}{b^2} = \frac{4\times 30}{10^2} = \frac{120}{100} = 1.2$$

$$\therefore \angle A = \arctan 1.2 = 50.19°.$$

Step 3: Use trigonometric ratio to find the length of the two equal sizes.

$$a = \frac{b}{2\cos\angle A} = \frac{10}{5\cos 50.19°} = 7.81\ (cm).$$

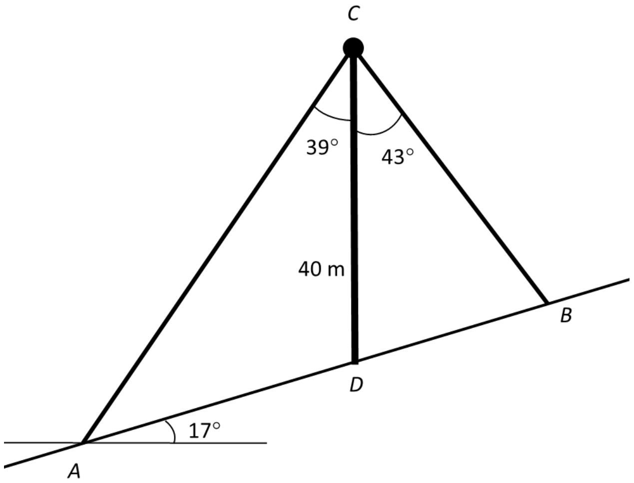
This was the same result as given in the reference solution. The student appreciated the lecturer for providing this alternative way, and he felt that this new way was not as simple as the one demonstrated in the reference solution.

3. The second case study

3.1. The second word problem

The second word problem shown in Problem 2 was another word problem among the 12 questions in the same assignment assigned to the same 27 student teachers. The problem presented a scenario involving two associated obtuse triangles with known parameters labeled in the figure embedded in the problem. This problem was primarily designed to test students' ability to apply the law of sines to solve the obtuse triangles reviewed in the past couple of weeks. Similar examples were demonstrated in both the lectures and tutorials before students' attempts on this problem.

An antenna mast of 40 m high is placed on sloping ground with an elevation angle of 17°. The two cables stabilizing the mast make angles with the mast of 39° and 43°, respectively (see details in the figure below). Find the length for the two cables, respectively. [keep 2 decimal places in the final result]



Problem 2. The second word problem assigned to the pre-service mathematics teachers.

3.2. The reference solution

The most efficient approach to solve this problem is to use the law of sines through the following two steps.

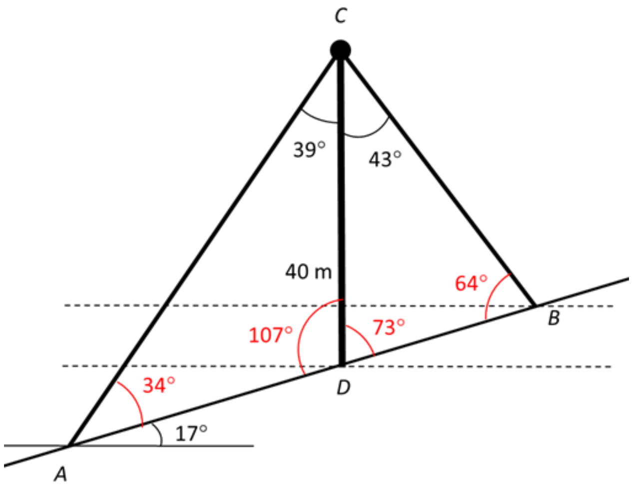


Figure 2. A reworked sketch for the second problem with derived angles (in red).

Step 1: Derive the interior angles for triangles ACD and BCD by geometric properties of triangles and/or parallel lines as follows, assisted by the reworked diagram shown in Figure 2.

$$\begin{aligned}
&\Delta ACD: \\
&\angle CAD + 17^\circ + 39^\circ = 90^\circ \longrightarrow \angle CAD = 90^\circ - 56^\circ = 34^\circ \\
&\angle ADC = 180^\circ - \angle CAD - 39^\circ = 180^\circ - 34^\circ - 39^\circ = 107^\circ \\
&\Delta BCD: \\
&\angle CBD + \angle CAD + 39^\circ + 43^\circ = 180^\circ \longrightarrow \angle CBD = 180^\circ - 82^\circ - 34^\circ = 64^\circ \\
&\angle BDC = 180^\circ - \angle CBD - 43^\circ = 180^\circ - 64^\circ - 43^\circ = 73^\circ
\end{aligned}$$

Alternatively,

$$\begin{aligned}
&\Delta ACD: \quad \angle ADC = 90^\circ + 17^\circ = 107^\circ \longrightarrow \angle CAD = 180^\circ - 39^\circ - 107^\circ = 34^\circ \\
&\Delta BCD: \quad \angle BDC = 90^\circ - 17^\circ = 73^\circ \longrightarrow \angle CBD = 180^\circ - 73^\circ - 43^\circ = 64^\circ
\end{aligned}$$

Step 2: With the known interior angles and the shared size CD = 40 m, directly use the law of sines to triangles ACD and BCD to obtain the length for AC and BC, respectively, as follows.

$$\Delta ACD: \quad \frac{AC}{\sin \angle ADC} = \frac{CD}{\sin \angle CAD} \longrightarrow AC = \frac{CD \sin \angle ADC}{\sin \angle CAD} = \frac{40 \times \sin 107^\circ}{\sin 34^\circ} = 68.41\ (m)$$

$$\Delta BCD: \quad \frac{BC}{\sin \angle BDC} = \frac{CD}{\sin \angle CBD} \longrightarrow BC = \frac{CD \sin \angle BDC}{\sin \angle CBD} = \frac{40 \times \sin 73^\circ}{\sin 64^\circ} = 42.56\ (m).$$

Of course, other methods can also be used to solve this problem with extra step(s).

3.3. The performance of pre-service teachers

The overall performance

Twenty out of the twenty-seven students solved this problem correctly, a correct rate of about 74% (Table 2). One student did not attempt this question, whereas six others made major mistakes in their attempts that led to wrong outcomes.

Table 2. Summary of the overall performance of the student teachers in solving the second problem.

Outcome	Number	Percentage (%)
Correct	20	74
Incorrect	6	22
No attempt	1	4
Total	27	100

The 20 students who solved this problem correctly all adopted the law of sines for solving the two associated obtuse triangles in the same way as the processes presented in the reference solution. One student applied this correct strategy and method but made errors in deriving some of the interior angles, which led to wrong outcomes. Another student applied both the laws of sines and cosines to the obtuse triangles but chose the wrong angles and sizes in the process, which demonstrated the student’s poor understanding of these methods. Three more students attempted this problem with different combinations of subdivided right triangles. Unfortunately, no one reached the correct solutions after tedious processes. The remaining student just wrote the Pythagorean theorem and the law of sines without any other workings. All these cases of applying different strategies and methods to solve this problem are summarized in Table 3. It is clear that the effective and correct use of the law of sines was the most efficient way to solve this word problem involving obtuse triangles.

Table 3. Summary of strategies and methods adopted by the students in solving the second problem.

Strategy	Obtuse triangle		Right triangle	
Method	Law of sines	Mixture	Trigonometric ratio	Mixture
Correct	19	0	0	0
Incorrect	1	1	4	1

Interactions between the lecturer and some students

After releasing the reference solutions and the marked assignments to the students, one of the three students who incorrectly solved the problem using subdivided right triangles asked the lecturer by email *if this problem could be solved through subdivided right triangles, because the reference solution was only by solving obtuse triangles*. The student also mentioned that her mathematics teacher in high school said that *all triangles were able to be solved by means of right triangles, so laws of sine and cosine were somewhat redundant*. The student tried solving the obtuse triangles by subdividing them into right triangles but just did not know how to solve them correctly.

To address this question, the lecturer shared an alternative way of obtaining the same outcomes through the divided right triangles sketched in Figure 3 with the following processes.

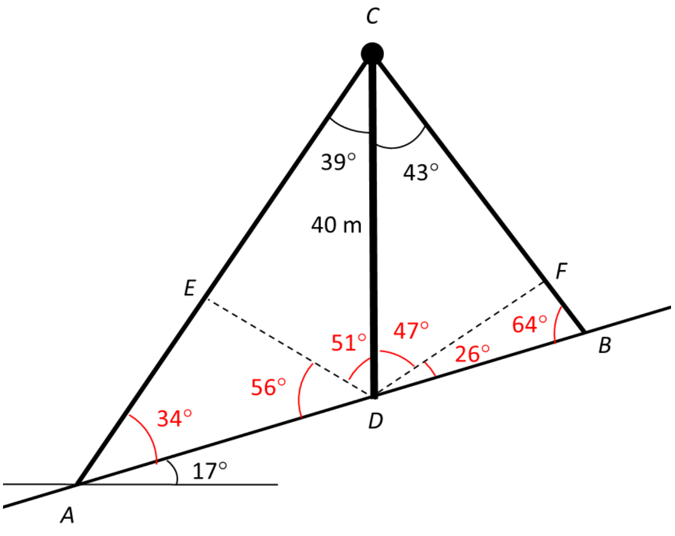


Figure 3. The first reworked sketch for solving the second problem through right triangles.

Step 1: For triangle ACD, draw a line from point D to point E on line AC so that line DE is perpendicular to line AC. This divides triangle ACD into two right triangles ADE and CDE. The interior angles for right triangles ADE and CDE can be derived from the geometric properties of triangles, as shown in Figure 3.

Step 2: With the known interior angles and the shared size CD = 40 m,

$$\Delta CDE: \quad DE = CD\sin 39° = 40\sin 39° \longrightarrow CE = CD\cos 39° = 40\cos 39°$$
$$\Delta ADE: \quad AE = DE\tan 56° = 40\sin 39°\tan 56°$$
$$\therefore AC = AE + CE = 40\sin 39°\tan 56° + 40\cos 39° = 40(\sin 39°\tan 56° + \cos 39°) = 68.41\ (m).$$

Step 3: For triangle BCD, draw a line from point D to point F on line BC so that line DF is perpendicular to line BC. This divides triangle BCD into two right triangles BDF and CDF. The interior angles for right triangles BDF and CDF can be derived from the geometric properties of triangles, also shown in Figure 3.

Step 4: With the known interior angles and the shared size CD = 40 m,

$$\Delta CDF: \quad DF = CD\sin 43° = 40\sin 43° \longrightarrow CF = CD\cos 43° = 40\cos 43°$$
$$\Delta BDF: \quad BF = DF\tan 26° = 40\sin 43°\tan 26°$$
$$\therefore BC = BF + CF = 40\sin 43°\tan 26° + 40\cos 43° = 40(\sin 43°\tan 26° + \cos 43°) = 42.56\ (m).$$

Hence, the same results were obtained through such subdivided right triangles. However, advanced knowledge and skills are required for properly dividing the obtuse triangles into suitable right triangles and carefully choosing trigonometric relationships to correctly handle the entire process for the right outcomes. This is even more demanding than simply using the law of sines in this case, which was indirectly verified by private email communications from three students who solved this problem correctly by the law of sines. One student said "*I never thought this problem could be solved by right triangles with such divisions. It is mind-blowing*." Another student reflected by saying "*you mentioned that mathematics is art too in the beginning of this term. I now understood what you meant. The way you solved this problem by right triangles looks elegant.*" Another student in his 50s wrote "*I believed in my ability in triangles and trigonometry and felt your assignment questions were not challenging enough to me until I saw how you solved this problem this way*".

After checking through the new reference solution based on right triangles, another student who obtained wrong results using right triangles privately asked the lecturer *if the same solutions could be reached by dividing the two obtuse triangles into a mixture of three right triangles and one obtuse triangle sketched in Figure 4a*.

To answer this question raised by the student, the lecturer provided the student with a new solution based on the divisions shown in Figure 4 as follows. Note that a couple of extensions are required for triangle ACD as shown in Figure 4b.

Step 1: Derive all the interior angles for the triangles shown in Figure 4b.

Step 2: For the triangles on the left side in Figure 4b, with the known interior angles and the shared size CD = 40 m,

$$CG = AC\cos 39° \longrightarrow AG = AC\sin 39° \longrightarrow DG = AG\tan 17° = AC\sin 39°\tan 17°$$
$$\therefore CD = CG - DG = AC\cos 39° - AC\sin 39°\tan 17° \longleftarrow CD = 40$$
$$40 = AC(\cos 39° - \sin 39°\tan 17°) \longrightarrow AC = \frac{40}{\cos 39° - \sin 39°\tan 17°} = 68.41\ (m).$$

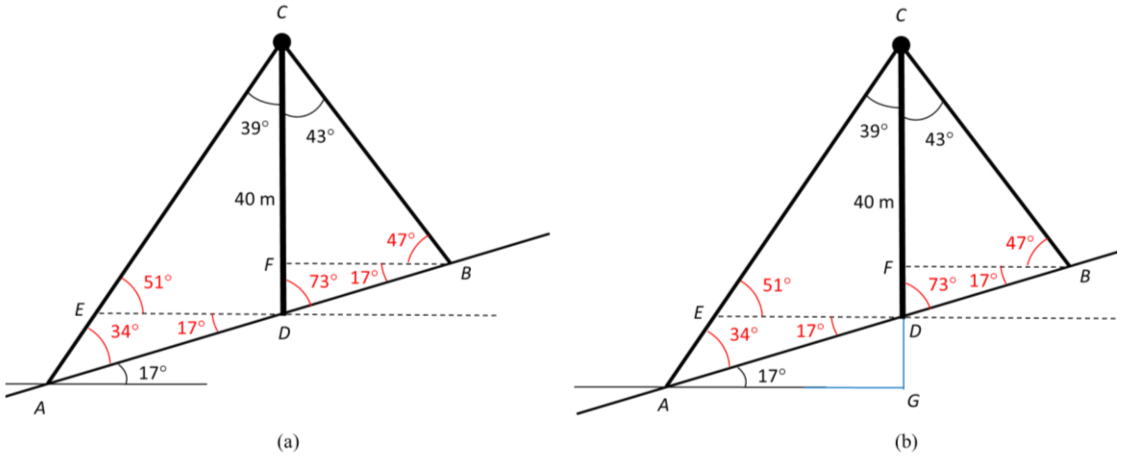


Figure 4. The second reworked sketch for solving the second problem through right triangles.

Step 3: For the triangles on the right side in Figure 4b, with the known interior angles and the shared size CD = 40 m,

$$CF = BC\cos 43^\circ \longrightarrow BF = BC\sin 43^\circ \longrightarrow DF = BF\tan 17^\circ = BC\sin 43^\circ \tan 17^\circ$$

$$\therefore CD = CF + DF = BC\cos 43^\circ + BC\sin 43^\circ \tan 17^\circ \longleftarrow CD = 40$$

$$40 = BC(\cos 43^\circ + \sin 43^\circ \tan 17^\circ) \longrightarrow DC = \frac{40}{\cos 43^\circ + \sin 43^\circ \tan 17^\circ} = 42.56\ (m).$$

Hence, the same results were obtained through the divisions in Figure 4. This approach is even more challenging than that sketched in Figure 3 because this new approach requires setting up a trigonometric equation first and then solving the equation to get the solution, which requires students to integrate knowledge and skills from different areas together.

In the meantime, one more student who also obtained wrong results using right triangles privately asked the lecturer *why she could not reach the correct solutions using similarity of right triangles divided as shown in Figure 5*.

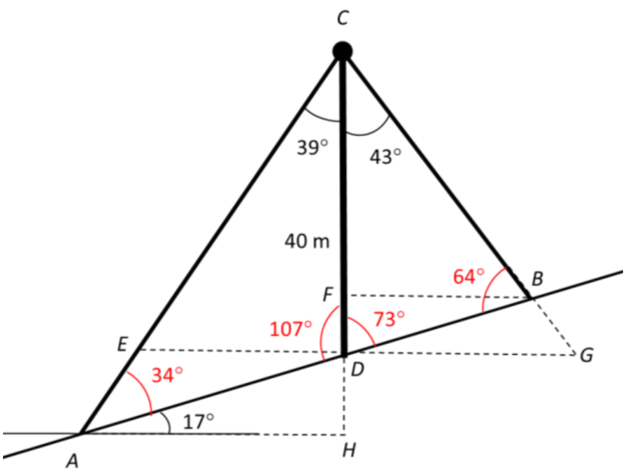


Figure 5. The third reworked sketch for solving the second problem through right triangles.

To answer this question, the lecturer provided the student with another new solution based on the divisions shown in Figure 5, as follows.

Step 1: Derive all the interior angles for the triangles shown in Figure 5.

Step 2: For the triangles on the left side in Figure 5, with the known interior angles and the shared size CD = 40 m,

$$AH = AC\sin 39° \longrightarrow DH = AH\tan 17° = AC\sin 39°\tan 17°$$
$$DE = CD\tan 39° \longrightarrow CH = CD + DH = 40 + AC\sin 39°\tan 17°$$
$$\because \Delta CDE \sim \Delta CHA \longrightarrow \therefore \frac{CD}{CH} = \frac{DE}{AH} \longrightarrow \frac{CD}{40 + AC\sin 39°\tan 17°} = \frac{CD\tan 39°}{AC\sin 39°}$$
$$\frac{1}{40 + AC\sin 39°\tan 17°} = \frac{1}{AC\cos 39°} \longleftarrow \frac{\tan 39°}{\sin 39°} = \frac{\tan 39°}{\cos 39°\tan 39°} = \frac{1}{\cos 39°}$$
$$40 + AC\sin 39°\tan 17° = AC\cos 39° \longrightarrow AC\cos 39° - AC\sin 39°\tan 17° = 40$$
$$AC(\cos 39° - \sin 39°\tan 17°) = 40 \longrightarrow AC = \frac{40}{\cos 39° - \sin 39°\tan 17°} = 68.41\ (m).$$

Step 3: For the triangles on the right side in Figure 5, with the known interior angles and the shared size CD = 40 m,

$$DG = CD\tan 43° \longrightarrow BF = BC\sin 43° \longrightarrow DF = BF\tan 17° = BC\sin 43°\tan 17°$$
$$CF = CD - DF = CD - BF\tan 17° = 40 - BC\sin 43°\tan 17°$$
$$\because \Delta CBF \sim \Delta CGD \longrightarrow \therefore \frac{CF}{CD} = \frac{BF}{DG} \longrightarrow \frac{40 - BC\sin 43°\tan 17°}{CD} = \frac{BC\sin 43°}{CD\tan 43°}$$

$$40 - BC\sin 43°\tan 17° = BC\cos 43° \longleftarrow \frac{\sin 43°}{\tan 43°} = \frac{\cos 43°\tan 43°}{\tan 43°} = \cos 43°$$
$$40 = BC(\cos 43° + \sin 43°\tan 17°) \longrightarrow BC = \frac{40}{\cos 43° + \sin 43°\tan 17°} = 42.56\ (m).$$

Hence, the same results were obtained through the divisions in Figure 5. In fact, the two methods based on the divisions in Figure 4 and Figure 5 proceeded to solve the same trigonometric equations. The similarity of triangles used in the last approach was only an alternative way to set up the same trigonometric equations.

Inspired by the alternative way to solve this problem demonstrated by the lecturer, a week later three students who solved this problem correctly by the law of sines dropped by the lecturer's office and showed their collective work on solving this problem by the law of cosines. They were unsure of the correctness of their approach, as the solutions had small errors compared to the reference solution. The lecturer confirmed that their approach using the law of cosines was correct and the small errors in their solutions were likely caused by the truncation errors of trigonometric values used in the intermediate calculations where they only kept 2–3 decimal places. Both the lecturer and the students were pleased with and proud of their extra effort in exploring alternative ways to solve the same problem, even though they admitted that the law of cosines was far less effective than the law of sines for solving this problem. For interested readers, this new alternative to solve the second word problem by the law of cosines is detailed as follows.

Step 1: Derive all the interior angles for the triangles shown in Figure 5.

Step 2: For the triangles on the left side in Figure 5, with the known interior angles and the shared size CD = 40 m,

$$AH = AC\sin 39° = 0.6293AC \longrightarrow AD = \frac{AH}{\cos 17°} = \frac{AC\sin 39°}{\cos 17°} = 0.6581AC$$
$$AC^2 = AD^2 + CD^2 - 2AD\cdot CD\cos 107° = (0.6581AC)^2 + 40^2 - 2\times 40\times 0.6581AC\cos 107°$$
$$AC^2 = 0.4331AC^2 + 1600 + 15.3928AC \longrightarrow 0.5669AC^2 - 15.3928AC - 1600 = 0$$
$$AC = \frac{15.3928 + \sqrt{(-15.3928)^2 + 4\times 0.5669\times 1600}}{2\times 0.5669} = 68.41\ (m).$$

Step 3: For the triangles on the right side in Figure 5, with the known interior angles and the shared size CD = 40 m,

$$BF = BC\sin 43° = 0.6820BC \longrightarrow BD = \frac{BF}{\sin 73°} = \frac{BC\sin 43°}{\sin 73°} = 0.7132BC$$
$$BC^2 = BD^2 + CD^2 - 2BD\cdot CD\cos 73° = (0.7132BC)^2 + 40^2 - 2\times 40\times 0.7132BC\cos 73°$$
$$BC^2 = 0.5087BC^2 + 1600 - 16.6816BC \longrightarrow 0.4913BC^2 + 16.6816BC - 1600 = 0$$
$$BC = \frac{-16.6816 + \sqrt{(16.6816)^2 + 4\times 0.4913\times 1600}}{2\times 0.4913} = 42.56\ (m).$$

4. Discussion

4.1. Explicit teaching (or instruction) for worded problem solving

The logical processes for effective learning of mathematics consist of understanding the concept(s) involved, following the sequential mathematical procedure to solve explicitly expressed problems, and finally being able to apply them to solve real-world problems. Accordingly, the explicit teaching of mathematics consists of clearly explaining the new concept(s) with links to real-world circumstances, demonstrating how to solve different types of well-expressed and related problems using a step-by-step procedure with pre-learned and newly learned mathematical techniques, and finally exhibiting the usefulness, power, and beauty of mathematics through capturing, formulating, and obtaining appropriate solutions for various real-world applications to further stimulate students' passion and enthusiasm for learning mathematics.

This foundation mathematics course has been delivered using such an explicit teaching approach since it was redeveloped in 2018. Many students initially felt strange about the teacher-led discourse as it contrasted with their learning experiences in high school through teacher-facilitated and student-led inquiry-based learning (IBL) or project-based learning (PBL), except for a few students in their 50s who experienced similar teacher-led explicit teaching approaches in their high school 30–40 years ago. Most students became more comfortable with the teacher-led explicit teaching after 3–4 weeks and adopted the corresponding learning approach recommended by the lecturer [15] for spiral progression in mathematics knowledge building.

During reviewing triangles in lectures and tutorials in this course, different triangles and their properties were systematically explained again and associated with various mathematical techniques, such as the Pythagorean theorem, trigonometric relationships, and the law of sines and the law of cosines, for the purposeful selection of methods to efficiently solve given problems involving triangles. These properties and techniques were then used to solve numerous explicitly expressed questions involving triangles. This allowed students to observe first-hand how to use these properties and mathematical techniques to carry out step-by-step procedures and reach correct solutions to different types of problems. More importantly, it inspired them to think about why and where a mathematical technique was chosen to address a specific task in problem-solving with clarity and efficiency. The next phase was to demonstrate how to use technical efficacy to solve word questions describing various real-world scenarios and complex problems that combine two or more different topics or techniques from different areas. Of course, students were required to complete the designated weekly exercise questions to consolidate their learning and build knowledge. By the time students began attempting the assignment questions that were set corresponding to the explicit teaching approach, they should have been well-prepared to handle mathematical challenges.

In this study, the first case was a word question for a simple scenario that required using both the area of a triangle and then the Pythagorean theorem to solve the problem. This problem was naturally less challenging for those who knew the basics of right triangles. Among the 22 students who correctly answered this question, the course logs showed that 21 of the 22 students had constantly interacted with the teaching and learning materials hosted on the course website since the beginning of the term. Therefore, it was no surprise to see these students succeed in solving this problem in the same way as demonstrated in the explicit teaching. The five students who obtained wrong outcomes for this question had not been actively engaged in the teaching and learning of this course since the start of the term. Hence, they did not follow the way demonstrated in the explicit teaching.

The second problem was more challenging than the first problem, but it would have become an easier task once students correctly worked out the interior angles for the two obtuse triangles, which naturally guided students to choose the law of sines to solve the problem unless one had not been engaged in teaching and learning of solving obtuse triangles in this course. The five students who had not been engaged in teaching and learning in this course continued to struggle with handling this second problem, and all chose the right triangles to solve this problem with wrong outcomes. Of the other 22 students who had correctly solved the first problem, 20 of them solved this problem in the exact same manner as the reference solution shown; one student followed the correct process but made mistakes with numbers. Unfortunately, another student did not submit his work and withdrew from the course later.

The performances of students in solving these two worded problems involving triangles showed that those who were actively engaged in the explicit teaching and completed the corresponding weekly learning activities enhanced their ability to properly solve both regular and challenging word problems involving triangles, whereas those who stuck to their old learning habits and were disengaged in the explicit teaching and learning practices were not able to make any positive progress. Regarding the two worded problems involving triangles as assigned projects, those who were disengaged in the explicit teaching and learning practices in this course were likely to approach the problems through either IBL or PBL, which shifts the student's learning of a new topic from the teacher's explicit explanation and demonstration to guiding or facilitating students to find the information and use it [16,17]. As this was a foundation course to review what students learned in high school, these students relied on their existing knowledge and skills to solve the problems. As a result, their old learning habits fostered by IBL or PBL might mislead them into thinking that they already knew the "things" about triangles, thus believing that there was no need for them to make any further "inquiry" or "searching" for the new learning materials available in the course website. In fact, engaging in explicit teaching and learning practices first, followed by IBL or PBL, would be more beneficial to both the students and the teachers involved, as advocated by many mathematics teachers for decades [18–24].

4.2. Focusing on fulfilling the pedagogical goals whilst offering alternative approaches to deal with the same word problem

The pedagogical goal for the first word problem was to examine students' understanding of the basic properties of triangles (especially right triangles and isosceles triangles) and their ability to combine different techniques from two subareas of triangles: the area for an isosceles triangle and the Pythagorean theorem for right triangles, to solve the problem by a logically connected procedure. The correct rate of above 80% for this problem indicated the fulfillment of the pedagogical goal for the first word problem. However, since this problem posed little challenge to students, as right triangles form the basics of all geometric triangles and trigonometry, the invisible glue connecting the individual steps was a correct sketch with clear labeling of the sizes referred to during problem-solving. Not all students who sketched reached the correct outcome for this problem due to calculation errors by a couple of students, but all students who obtained incorrect results failed to draw a sketch to assist their attempts.

The pedagogical goal for the second word problem was to test students' ability to apply the law of sines to solve the two obtuse triangles presented in the problem based on a good understanding of the basic properties of triangles and parallel lines. Having been examined in the first problem, right triangles were not part of the primary pedagogical goal of the second problem. However, should a student solve this problem correctly using right triangles, the student would get the maximum marks assigned to this problem.

The second problem appeared more challenging than the first problem, but the technical process of solving it was relatively straightforward once the interior angles of the two obtuse triangles were derived, as shown in the reference solution. The effectiveness of adopting this straightforward approach was evident, with 20 out of 21 students choosing the law of sines to solve these obtuse triangles, while the other student made mistakes in deriving the interior angles.

In contrast, the five students who chose to solve this problem using right triangles failed to produce the correct outcomes. More worrying than obtaining wrong outcomes was that they remained confident in their results until the release of the solutions and marked assignments. Even then, none could figure out where they had made mistakes in their attempts using right triangles. This sentiment was evident in the tone of their requests for assistance, claiming that their high school mathematics teachers had taught them that obtuse triangles could be solved by subdivided right triangles.

As the main pedagogical goal for the second word problem was to apply the law of sines to solve obtuse triangles, the lecturer did not need to prepare multiple alternative solutions using subdivided right triangles. However, to demonstrate the power and beauty of mathematics in solving the same problem using different approaches, despite the alternatives being not as effective as the approach using the law of sines, the lecturer took extra time to provide all students with an alternative approach based on subdivided right triangles, as sketched in Figure 3. This alternative approach looks like the best way to solve the second problem using subdivided right triangles, but many may agree that the idea of dividing obtuse triangles and the logic of setting up the trigonometric equation would be more demanding than resolving the established equation. The other alternative approaches sketched in Figures 4 and 5 primarily aimed to help students find where they had made mistakes, although technically these approaches were still correct. However, these alternatives should not alter the primary pedagogical goal for the second word problem, that is, to use the law of sines to solve the obtuse triangles.

Providing multiple alternative solutions using different approaches sounds great in supporting students in effective learning of mathematics and opening students' minds in problem-solving. However, it must be understood that teachers' time and energy are also limited. Hence, focusing on fulfilling the main pedagogical goal at each stage would keep mathematics teaching and learning progressing within the planned timeframe for both teachers and students.

4.3. Becoming competent mathematics teachers to better facilitate mathematics learning

To explicitly teach mathematics and provide students with alternative ways to solve the same mathematical problem, mathematics teachers should not only possess soft skills from education sciences for classroom management, communication, and so forth but also have technical proficiency in a wide range of mathematical topics. Unlike disciplines in other social sciences that are primarily based on elaboration, common sense, life experiences, social norms, policies or regulations, or code of conduct, having little formulation and algorithmic processes, mathematics is primarily about logical reasoning and technical procedures involving one or multiple topics. The ultimate goal of mathematics education is to enable students to use mathematical reasoning and techniques to solve both practical exercises and real-world problems with confidence. Hence, mathematics teachers should not only explain mathematical concepts well but also demonstrate to students how a related mathematical problem can be solved by a step-by-step procedure or multiple procedures so that students can initially imitate the procedure(s) and then apply them to solve word problems involving real-world scenarios. It is not good enough for teachers to just mention that something else may also be useful without demonstrating how to use it to solve problems. For example, just telling students that obtuse triangles can be solved by right triangles without demonstrating how actually deviated the focus of some students from learning and applying the law of sines directly and most efficiently to solve the associated obtuse triangles in the second word problem. Those students attempted to solve this problem of obtuse triangles by means of right triangles, but their existing knowledge and skills were not sufficient to navigate through the even more demanding processes.

The other key fact of mathematics competence for school mathematics teachers is to be able to show young learners how to make a seemingly complicated mathematics problem into few relatively simple steps so that each of these steps can be dealt with clearly and confidently by students, rather than amalgamate the simple steps into one abstract formula that could only be understood by the teacher. In other words, making a complicated mathematical problem simple, rather than making simple processes even more difficult for students. For instance, in the first word problem, for the best interests of student learning, the two-step procedure assisted with a sketch, compared with the amalgamated one-step approach without a sketch, would be more beneficial to the students and help to identify where a mistake might have been made.

4.4. A three-step interactive explicit teaching-learning approach for effective mathematics education

This case study showed that the teacher-led explicit teaching helped most students (more than 75% of the 27 students), with almost all of the engaged students (more than 95%) achieving the correct solutions to the two worded problems involving triangles. This is a superb outcome compared with the performances of pre-service mathematics teachers in solving triangular problems reported in other studies [8–10]. For example, among 50 final-year pre-service secondary mathematics teachers, Walsh et al. [9] found that more than 80% of the participants had difficulties associated with solving oblique triangles, and more than 90% were not able to apply laws of sines and cosines to solve a scientific problem described in words.

In fact, the teacher-led explicit teaching conducted by lecturers in the past 10 years for hundreds of engineering and education students has consistently guided most engaged students to achieve satisfactory learning outcomes in mathematics courses, from foundation mathematics to advanced mathematics and applications [7,12,25,26]. This successful experience in mathematics education can be summarized by a three-step interactive explicit teaching-learning approach, comprising teacher-led precise and inspiring teaching (or explicit teaching), student-driven engaged learning (or imitative learning), and student-led and teacher-guided problem solving for real-world problems or projects (or active application). Explicit teaching sets a solid foundation for students to further their understanding of new mathematical concepts and conceptualize the technical processes associated with the new concepts. Imitative learning leads students to build technical ability and enhance technical efficacy by succeeding in learning activities. Once the first two steps have been completed, students should have a decent understanding of new mathematical concepts and technical efficacy to analyze, formulate, and finally solve real-world applications under the guidance of teachers.

Explicit teaching should not exclude or be against student-centric IBL or PBL, but the sequence of teaching-learning interactions must be logically rationalized. IBL or PBL would be most effective in the final step of problem-solving for real-world applications once the first two steps have been completed by students. This allows students to focus on achieving the goals of an assigned project or problem (rather than searching for the basic knowledge and techniques required by themselves), further enhancing technical efficacy, improving time efficiency for both students and teachers, appreciating the power and beauty of mathematics, and boosting their confidence in learning mathematics and applications. A good example of active application was exhibited by the three students' voluntary effort in solving the second word problem using the law of cosines, inspired by the alternative method demonstrated by the lecturer. The success of the students' active application and the lecturer's commendation for the students' effort in trying alternatives not only made the students feel satisfied with their effort and proud of the outcome but also helped them appreciate the power and efficiency of different mathematical methods in effectively solving real-world problems.

5. Conclusions

Teaching and learning are interactive and seamless interconnections among the planned teaching practices and learning activities, which facilitates steady and progressive mathematics knowledge building for students according to the curriculum timeframe. In the summarized three-step interactive explicit teaching-learning approach, explicit teaching or instruction should be the first step that leads students toward engaged and effective learning as the following step. Naturally, precise and inspiring teaching is the most important part of these three steps. Implementing this interactive approach should start in high schools so that students become familiar with such interactive processes from their early teenage years. Accordingly, this would require the in-service mathematics teachers to adopt an explicit teaching approach to foster a more effective learning habit for students. To keep students interested in learning mathematics, teachers should be able to demonstrate to students multiple ways to solve some selected challenging problems and particularly the application-oriented problems to stimulate student's curiosity and for them to experience the power of mathematics. Since inquiry-based mathematics education has been promoted in many countries for more than two decades [27], many in-service mathematics teachers might need to embed explicit teaching into their familiar inquiry-based teaching practices. Therefore, in addition to training the pre-service mathematics teachers who are currently enrolled in tertiary education programs with this explicit interactive teaching-learning approach, special professional development schemes should be crafted for some in-service mathematics teachers to assist them in making necessary adjustments in mathematics teaching.

Of course, the second step is also a key and integral part of this interactive explicit teaching-learning process. Engaged learning should be primarily driven by the students after going through the phase of teacher-led explicit teaching. Learning effectiveness is improved only for those students who are willing to embrace opportunities and challenges after following through the explicit teaching practices demonstrated by the teachers. For those students who stick to their old learning habits by disregarding the explicit teaching practices and completion of the necessary learning activities, they would have no chance to taste the success and appreciate the effectiveness of this interactive explicit teaching-learning process in their mathematics knowledge building.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflict of interest in any part of this article.

Ethics declaration

The author declared that the ethics committee approval was waived for the study.

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