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Classroom note

A practical strategy to improve performance of Newton's method in solving nonlinear equations

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Abstract: Newton's method is a popular numeric approach due to its simplicity and quadratic convergence to solve nonlinear equations that cannot be solved with exact solutions. However, the initial point chosen to activate the iteration of Newton's method may cause difficulties in slower convergence, stagnation, and divergence of the iterative process. The common advice to deal with these special cases was to choose another inner point to repeat the process again or use a graph of the nonlinear equation (function) to choose a new initial point. Based on the recent experiences in teaching preservice secondary-school mathematics teachers, this classroom note presents a simple and practical strategy to avoid many of the difficulties encountered in using Newton's method during solving nonlinear equations. Instead of plotting the graph of an equation to help choose the initial point, the practical strategy is to use the middle point of the range as the initial point to start the iterative process of Newton's method. By solving ten different nonlinear equations using Newton's method initiated from the endpoints and the middle point respectively, the results show that choosing the middle point of a defined range to initiate the iterative process for Newton's method seems a general and practical strategy to avoid the difficulties encountered in using the method to solve nonlinear equations with faster convergence in most cases.

Keywords: nonlinear equation, Newton's method, numeric iteration, convergence and divergence, initial point, middle point

1. Introduction

In real-world applications, a great number of nonlinear equations cannot be solved with exact solutions. The common strategy to solve such complicated problems is to obtain its approximate solution using an appropriate procedure known as '*numeric approach*' or '*numeric computation*'.

There are many different established numeric methods for solving various nonlinear equations. Newton's method is a popular approach among the numeric methods due to its simplicity and quadratic convergence. Hence, it has been widely presented in many mathematics textbooks [1–6] and books of numeric computation [7–9].

According to [10], Newton's method was originally presented as an immature form by Newton as a special case of the method in the 1670s, which was actually different from the current method. After a number of attempts by other mathematicians since then, Joseph Raphson found a simplified expression for this method in 1690, from which Thomas Simpson formulated the method as an iterative process to solve general nonlinear equations using the derivative of a function. This iterative process became the current formula of Newton's method, sometimes also called the Newton-Raphson method, but Simpson was missed in the brand of the modern method he formulated.

Despite the simplicity of Newton's method in iterative numeric computation through computer or calculator programs with a general quadratic convergence [9], a number of difficulties have been noticed over the years. For example, there is no guarantee that the solution (or root) for the nonlinear equation is unique in a given range $[a, b]$ even the process is certain to be convergent. A chosen initial point, usually a or b, does not ensure the process converging to the root for special cases where the derivative at the initial point is zero. In other instances, the process fails in progression where the derivative at the initial point does not exist.

Given the diverse causes behind the difficulties, there has been no concrete strategy proposed to effectively deal with the difficulties mentioned in most mathematics textbooks. For instance, the 'normal cases' were used to demonstrate the iterative process for Newton's method in [4,7] to avoid addressing any of the difficult situations. For the circumstances where the derivative at the initial point is zero or does not exist, a new initial point should be chosen as suggested in [1,2,8,9], but no concrete strategies were offered for how to choose the new initial point. Using graphs was suggested to choose a new initial point in [3,5] when encountering divergence in the iterative process or the derivative at the initial point is zero or does not exist. This works but also means a halt in automatic processing of iteration when the execution is through a computer program.

In this note, based on the classroom experience in teaching solving nonlinear equations using Newton's method to preservice secondary-school mathematics teachers at Central Queensland University (CQU) of Australia in recent years, a practical strategy for choosing the initial point to the iterative process for both the normal and special cases is shared with the mathematics teaching and learning communities around the world.

2. Fundamentals of Newton's method

Solving an equation $f(x) = 0$ within $[a, b]$ is equivalent to finding the point $x = x_r$ ($\in [a, b]$) where $f(x)$ intersects the x-axis (Figure 1). Such a point $x = x_r$ is called *the root* for $f(x) = 0$ if it exists. It is not easy to judge if any root exists for a general equation $f(x) = 0$, but in a situation where $f(a)$ and $f(b)$ have opposite signs, such point (or points) indeed exists. This is because for a *continuous and differentiable function* $f(x)$ defined within $[a, b]$, the curve of the function must cross the x-axis at least once so that $f(a)$ and $f(b)$ have opposite signs.

Newton's method uses a sequence of x-intercepts of the tangent lines from an initial point x_0 (usually either a or b) to approximate the root for equation $f(x) = 0$ within $[a, b]$. Suppose the initial point to be $x_0 = b$ (Figure 1). The tangent line on $f(x)$ at $(x_0, f(x_0))$ crosses the x-axis at x_1 and the tangent of this line is the derivative of $f(x)$ at $x_0 = b$, i.e.,

$$f'(x_0) = \frac{f(x_0)}{x_0 - x_1} \longrightarrow x_0 - x_1 = \frac{f(x_0)}{f'(x_0)} \longrightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

This means that x_1 can be determined by the values of x, y and y' at the initial point x_0. The new point x_1, closer to x_r compared to x_0, can be used to find a new x-intercept x_2 in a similar way as follows

$$f'(x_1) = \frac{f(x_1)}{x_1 - x_2} \longrightarrow x_1 - x_2 = \frac{f(x_1)}{f'(x_1)} \longrightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

Obviously x_2 is even closer to x_r compared to x_1. Repeating this process will take the consecutive x-intercepts approaching the root of the equation. This iterative procedure is captured by the following recurrent relation, called *Newton's method*:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}, \quad i = 0, 1, 2, 3, ... \tag{1}$$

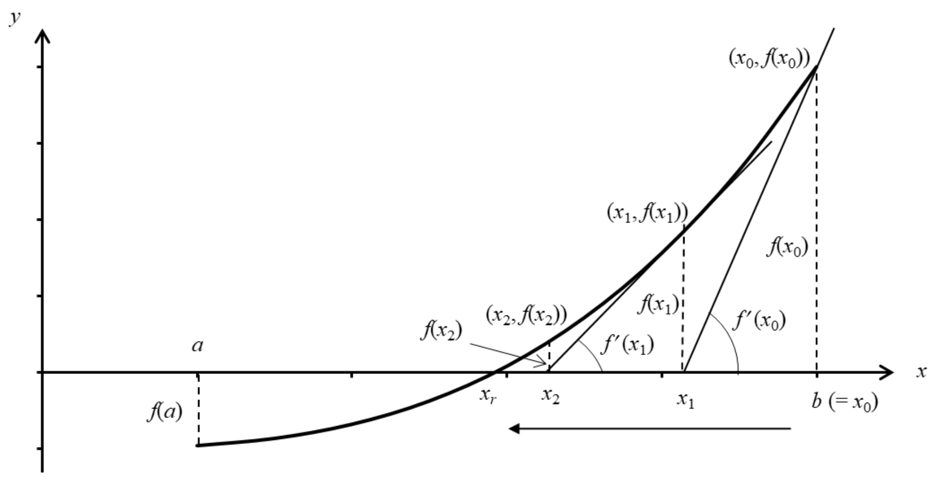

Figure 1. Process of approaching the root $x = x_r$ for equation $f(x) = 0$ by Newton's method

The iteration terminates when the difference of two consecutive x-intercepts is smaller than a pre-set error limit, i.e., $|x_{i+1} - x_i| < \varepsilon$, which indicates how close the approximate value of x_i is to the root x_r. If such condition is met after the ith iteration, an acceptable approximate solution is found for $f(x) = 0$, i.e., $x_r \approx x_{i+1}$. Note that i begins from 0.

3. Solving nonlinear equations by Newton's method: Examples

This section presents different examples of nonlinear equations solved using Newton's method to demonstrate some difficulties in using this method.

Example 1: Solve equation $f(x) = \frac{1}{x} - \ln x = 0$ in [1, 3] using Newton's method with $\varepsilon = 0.001$.

Since $f(a) = f(1) = 1$ and $f(b) = f(3) = 1/3 - \ln 3 \approx -0.7653$, Newton's method can be used to solve this nonlinear equation.

$$f(x)=\frac{1}{x}-\ln x \longrightarrow f(x_i)=\frac{1}{x_i}-\ln x_i=\frac{1-x_i\ln x_i}{x_i}$$

$$f'(x)=(\frac{1}{x}-\ln x)'=-\frac{1}{x^2}-\frac{1}{x}=-\frac{1+x}{x^2} \longrightarrow f'(x_i)=-\frac{1+x_i}{x_i^2}$$

Formula (1) becomes

$$x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)}=x_i-\frac{\frac{1-x_i\ln x_i}{x_i}}{-\frac{1+x_i}{x_i^2}}=x_i+\frac{x_i(1-x_i\ln x_i)}{1+x_i}.$$

Let $x_0 = a = 1$.

$$x_{0+1}=x_1=x_0+\frac{x_0(1-x_0\ln x_0)}{1+x_0}=1+\frac{1\times(1-1\times\ln 1)}{1+1}=1+\frac{1}{2}=1.5.$$

Since $|x_1-x_0|=|1.5-1|=0.5>\varepsilon=0.001,$ let $x_1 = 1.5$.

$$x_2=x_1+\frac{x_1(1-x_1\ln x_1)}{1+x_1}=1.5+\frac{1.5\times(1-1.5\times\ln 1.5)}{1+1.5}\approx 1.7351.$$

Since $|x_2-x_1|=|1.7351-1.5|=0.2351>\varepsilon=0.001,$ let $x_2 = 1.7351$.

$$x_3=x_2+\frac{x_2(1-x_2\ln x_2)}{1+x_2}=1.7351+\frac{1.7351\times(1-1.7351\times\ln 1.7351)}{1+1.7351}\approx 1.7629.$$

Since $|x_3-x_2|=|1.7629-1.7351|=0.0278>\varepsilon=0.001,$ let $x_3 = 1.7629$.

$$x_4=x_3+\frac{x_3(1-x_3\ln x_3)}{1+x_3}=1.7629+\frac{1.7629\times(1-1.7629\times\ln 1.7629)}{1+1.7629}\approx 1.7631.$$

Since $|x_4-x_3|=|1.7631-1.7629|=0.0002<\varepsilon=0.001,$ the approximate solution is $x \approx x_4 = 1.7631$.

The iterative process above can also be realized using Excel as a table shown below.

i	x_i	x_{i+1}	$\lvert x_{i+1}-x_i \rvert$	$\lvert x_{i+1}-x_i \rvert < \varepsilon$
0	1	1.5	0.5	N
1	1.5	1.7351	0.2351	N
2	1.7351	1.7629	0.0278	N
3	1.7629	**1.7632**	**0.0003**	**Y**
		$x \approx x_4 = 1.7632$		

Of course, if choosing $x_0 = b = 3$, the similar iterative process should lead to the same solution for this case as shown in the table below. Note this choice takes one more iteration to reach the same root. This is a ‘normal case’ where choosing any of the two end points ensures convergence of iteration to the same root where the curve of $f(x)$ crosses the x-axis showing in Figure 2.

i	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$	$\lvert x_{i+1} - x_i \rvert < \varepsilon$
0	3.0000	1.2781	1.7219	N
1	1.2781	1.6632	0.3851	N
2	1.6632	1.7593	0.0961	N
3	1.7593	1.7632	0.0039	N
4	1.7632	**1.7632**	**0.0000**	**Y**
		$x \approx x_5 = 1.7632$		

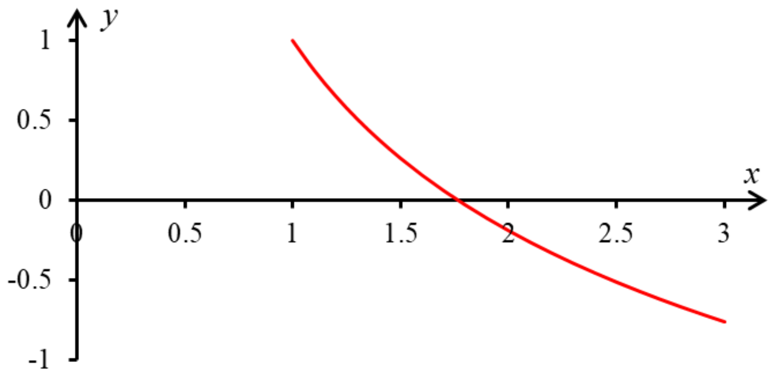

Figure 2. The root crossing the *x*-axis for equation *f*(*x*) = 0 in Example 1

Example 2: Solve equation $x^3 - 2x + 2 = \sqrt{x+2}$ in [0, 1] using Newton's method with $\varepsilon = 0.001$. We first rearrange $x^3 - 2x + 2 = \sqrt{x+2}$ to $f(x) = x^3 - 2x + 2 - \sqrt{x+2}$. Solving the equation becomes solving equation $f(x) = x^3 - 2x + 2 - \sqrt{x+2} = 0$. Since $f(a) = f(0) = 2 - \sqrt{2} \approx 0.5858$ and $f(b) = f(1) = 1 - 2 + 2 - \sqrt{3} \approx -0.7321$, Newton's method can be used to solve this nonlinear equation.

$$f(x) = x^3 - 2x + 2 - \sqrt{x+2}. \longrightarrow f(x_i) = x_i^3 - 2x_i + 2 - \sqrt{x_i + 2}.$$

$$f'(x) = (x^3 - 2x + 2 - \sqrt{x+2})' = 3x^2 - 2 - \frac{1}{2\sqrt{x+2}} \longrightarrow f'(x_i) = 3x_i^2 - 2 - \frac{1}{2\sqrt{x_i+2}}$$

Formula (1) becomes

$$x_{i+1} = x_i - \frac{x_i^3 - 2x_i + 2 - \sqrt{x_i+2}}{3x_i^2 - 2 - \dfrac{1}{2\sqrt{x_i+2}}} = x_i - \frac{(2\sqrt{x_i+2})(x_i^3 - 2x_i + 2 - \sqrt{x_i+2})}{(2\sqrt{x_i+2})(3x_i^2 - 2) - 1}.$$

Let $x_0 = a = 0$. The iterative process using the formula above is tabulated in the following table.

i	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$	$\lvert x_{i+1} - x_i \rvert < \varepsilon$
0	0	0.2489	0.2489	N
1	0.2489	0.2573	0.0084	N
2	0.2573	**0.2573**	**0.0000**	**Y**
		$x \approx x_3 = 0.2573$		

If choosing $x_0 = b = 1$, however, the similar iterative process produces a different solution of $x = 1.3713$ outside of the range [0, 1] after 6 iterations showing in the table below. If we plot $f(x)$ beyond the range [0, 1], it shows that $x = 1.3713$ is another root of this equation (Figure 3). This is an 'abnormal case' or a special case for using Newton's method. This non-uniqueness is simply because the right end $x = b = 1$ is located closer to the root on its right and sharing the same ascending tendency in tangent to the right side of the curve. That is, the tangent of the left section is negative and approaches to root $x = 0.2573$ whereas the tangent of the right section is positive and approaches to the other root $x = 1.3713$.

i	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$	$\lvert x_{i+1} - x_i \rvert < \varepsilon$
0	1.00000	2.02914	1.02914	N
1	2.02914	1.60459	0.42454	N
2	1.60459	1.41715	0.18745	N
3	1.41715	1.37367	0.04347	N
4	1.37367	1.37132	0.00236	N
5	1.37132	**1.37131**	**0.00001**	**Y**
		$x \approx x_6 = 1.3713$		

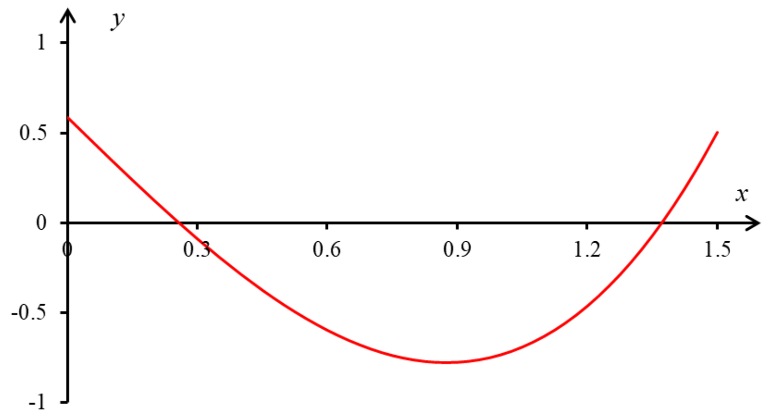

Figure 3. Two roots crossing the x-axis for equation $f(x) = 0$ in Example 2

Example 3: Solve equation $\cos x - \sqrt{x} = 0$ in [0, 1] with $\varepsilon = 0.001$ using Newton's method.
Since $f(a) = \cos 0 - \sqrt{0} = 1$ and $f(b) = \cos_r(1) - \sqrt{1} \approx 0.5403 - 1 = -0.4597$, Newton's method can be used to solve this nonlinear equation.

$$f(x) = \cos x - \sqrt{x} \longrightarrow f(x_i) = \cos x_i - \sqrt{x_i}$$

$$f'(x) = -\sin x - \frac{1}{2\sqrt{x}} = -(\sin x + \frac{1}{2\sqrt{x}}) \longrightarrow f'(x_i) = -(\sin x_i + \frac{1}{2\sqrt{x_i}}) = -\frac{2\sqrt{x_i}\sin x_i + 1}{2\sqrt{x_i}}$$

We can see the derivative at the left end $x = a = 0$ does not exist. Still using Formula (1),

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{\cos x_i - \sqrt{x_i}}{-\frac{2\sqrt{x_i}\sin x_i + 1}{2\sqrt{x_i}}} = x_i + \frac{2\sqrt{x_i}(\cos x_i - \sqrt{x_i})}{2\sqrt{x_i}\sin x_i + 1}.$$

Let $x_0 = a = 0$.

$$x_1 = x_0 + \frac{2\sqrt{x_0}(\cos x_0 - \sqrt{x_0})}{2\sqrt{x_0}\sin x_0 + 1} = x_0 + \frac{0}{1} = x_0.$$

This means that the iteration does not progress if choosing the left endpoint as the initial point where $f'(0)$ does not exist.

However, by choosing the right endpoint as the initial point, i.e., $x_0 = b = 1$, the iteration converges to $x \approx x_5 = 0.6419$ showing in the table below. Note that the input value for trigonometric functions should be in radians.

i	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$	$\lvert x_{i+1} - x_i \rvert < \varepsilon$
0	1.0000	0.6935	0.30647	N
1	0.6935	0.6537	0.03986	N
2	0.6537	0.6446	0.00905	N
3	0.6446	0.6424	0.00219	N
4	0.6424	**0.6419**	**0.00054**	**Y**
$x \approx x_5 = 0.6419$				

Example 4: Solve equation $f(x) = x^2 - \sin x = 0$ in [0.5, 1] using Newton's method with $\varepsilon = 0.001$.

Since $f(a) = f(0.5) = 0.25 - \sin_r 0.5 \approx 0.25 - 0.4794 = -0.2294$ and $f(b) = f(1) = 1 - \sin_r 1 \approx 1 - 0.8415 = 0.1585$, Newton's method can be used to solve this nonlinear equation.

$$f(x) = x^2 - \sin x \longrightarrow f(x_i) = x_i^2 - \sin x_i$$

$$f'(x) = (x^2 - \sin x)' = 2x - \cos x \longrightarrow f'(x_i) = 2x_i - \cos x_i$$

Formula (1) becomes

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{x_i^2 - \sin x_i}{2x_i - \cos x_i}.$$

Letting $x_0 = b = 1$, it takes 3 iterations to find an acceptable root $x \approx x_3 = 0.8767$ shown in the table below.

i	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$	$\lvert x_{i+1} - x_i \rvert < \varepsilon$
0	1	0.89140	0.10860	N
1	0.89140	0.87698	0.01442	N
2	0.87698	**0.87673**	**0.00025**	**Y**
$x \approx x_3 = 0.8767$				

The same solution can be reached by choosing $x_0 = a = 0.5$. However, the convergence of the process would be much slower by taking 6 iterations to find the first acceptable root $x \approx x_6 = 0.8767$ shown in the table below. It can be clearly seen in the plot of this function (Figure 4) that the left endpoint $x_0 = a = 0.5$ is near a local minimum on the curve where the tangent line is almost horizontal (green line) and its x-intercept is far beyond the right endpoint $b = 1$. In contrast, the tangent line from the right endpoint (blue line) $x_0 = b = 1$ has a larger value for the gradient (or the derivative of the function), leading to an x-intercept closer to the root. Hence, it reaches the root faster than beginning from the left endpoint.

i	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$	$\lvert x_{i+1} - x_i \rvert < \varepsilon$
0	0.50000	2.37412	1.87412	N
1	2.37412	1.47028	0.90385	N
2	1.47028	1.05948	0.41080	N
3	1.05948	0.90583	0.15365	N
4	0.90583	0.87771	0.02811	N
5	0.87771	**0.87673**	**0.00098**	**Y**
		$x \approx x_6 = 0.8767$		

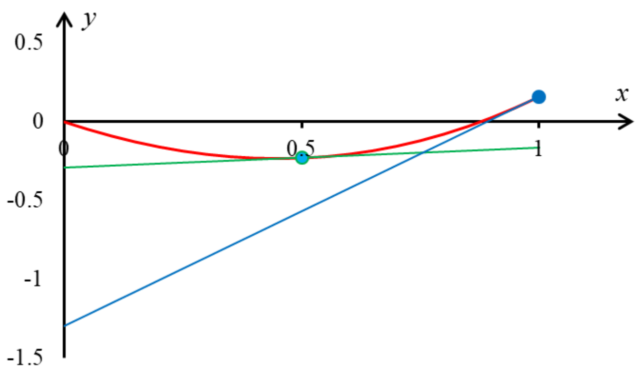

Figure 4. The graph of the function in range [0.5, 1] in Example 4 (red) and the tangent line crossing the right endpoint (1, 0.1585) in blue and that crossing point (0.5, –0.2294) in green

This example is still a normal case for using Newton's method. However, it demonstrates that the speed of convergence in iteration heavily depends on the choice of the initial point.

Example 5: Solve equation $\sin x - \frac{1}{x+1} = 0$ in [0, 1.5] with $\varepsilon = 0.001$ using Newton's method.

Since $f(a) = f(0) = \sin 0 - \frac{1}{0+1} = -1$ and $f(b) = f(1.5) = \sin_r 1.5 - \frac{1}{1.5+1} \approx 0.5975$, Newton's method can be used to solve this nonlinear equation.

$$f(x) = \sin x - \frac{1}{x+1} \longrightarrow f(x_i) = \sin x_i - \frac{1}{x_i+1}$$

$$f'(x) = \cos x + \frac{1}{(x+1)^2} \longrightarrow f'(x_i) = \cos x_i + \frac{1}{(x_i+1)^2}$$

By formula (1) and choosing $x_0 = a = 0$, it takes 4 iterations to find the first acceptable root $x \approx x_4 = 0.6508$ shown in the table below.

i	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$	$\lvert x_{i+1} - x_i \rvert < \varepsilon$
0	0.0000	0.5000	0.50000	N
1	0.5000	0.6416	0.14163	N
2	0.6416	0.6507	0.00908	N
3	0.6507	**0.6508**	**0.00004**	**Y**
		$x \approx x_4 = 0.6508$		

Choosing $x_0 = b = 1.5$, however, the process does not converge. Each iteration produced x-intercept outside the confined range [0, 1.5] shown in the table below.

i	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$	$\lvert x_{i+1} - x_i \rvert < \varepsilon$
0	1.5000	-1.0895	2.58950	N
1	-1.0895	-1.1716	0.08210	N
2	-1.1716	-1.3144	0.14284	N
3	-1.3144	-1.5279	0.21345	N

Observing the curve of this function (red line) and the tangent line crossing the right end (1.5, 0.5975) shown as the blue line in Figure 5, its initial x-intercept projects to the negative half outside of the range, further away from the solution around $x = 0.65$. The divergence of the iterative process is caused by the small gradient of the tangent line at the right endpoint. If choosing another inner point, for instance $x_0 = 1$, the gradient of the line crossing this point is larger (green line) and the process will converge to the same root after 4 iterations shown in the table below.

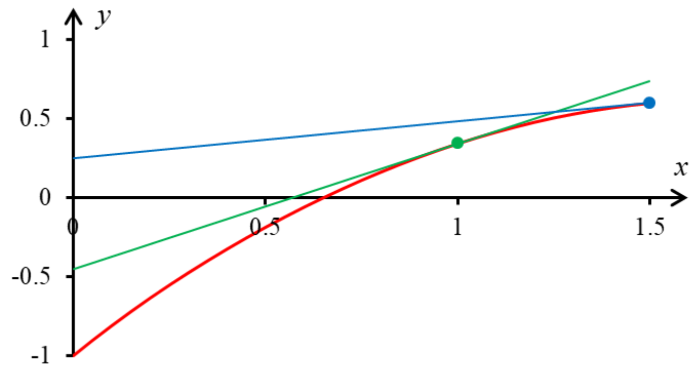

Figure 5. The graph of the function in range [0, 1.5] in Example 5 (red) and the tangent line crossing the right endpoint (1.5 0.5975) in blue and that crossing point (1, 0.3415) in green

i	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$	$\lvert x_{i+1} - x_i \rvert < \varepsilon$
0	1.0000	0.5679	0.43208	N
1	0.5679	0.6479	0.07994	N
2	0.6479	0.6507	0.00289	N
3	0.6507	**0.6508**	**0.00000**	**Y**
		$x \approx x_4 = 0.6508$		

4. A practical strategy to improve performance of Newton's method

The examples presented in Section 3 were all from the worked samples and exercises in the tailored textbook, *Essentials and Examples of Applied Mathematics* (2nd ed) [6], for the preservice secondary-school mathematics teachers at CQU. Many of the problems with Newton's method were gradually exposed by either the teacher (also the author of the textbook) or students since 2020. The advice provided to students for dealing with those special issues was initially the same as the general suggestions made in other books, such as '*choosing the other endpoint if one endpoint did not work*' or '*choosing a new point close to the endpoint if the end point did not work*', or '*using a graph of the function to help choose the initial point*'.

However, a conversation on the matters of using Newton's method with one talented student in 2022 triggered further thinking about a general but practical strategy for dealing with such matters in using Newton's method. The student simply augured that *any other method could be equally effective to find the acceptable root if we choose the initial point close enough to the 'true root' on the graph of the nonlinear equation or function within the range*. *If so, why should we still stick with Newton's method with the assistance of a graph*?

There are still difficulties in proving if a general rule exists as a specific guide to the users when encountering some of the difficulties in using Newton's method. Instead, a practical strategy is shared below to guide students and other users to choose the initial point to avoid encountering the difficulties exposed in Section 3 with Newton's method without plotting the nonlinear equation or function. This strategy is stated as follows:

> *By meeting the condition that the continuous and differentiable nonlinear function $f(x)$ within the range $[a, b]$ has values at the two endpoints with opposite signs, it is recommended to choose the middle point of the range as the initial point to start the iterative process of Newton's method, i.e.,* $x_0 = (a + b)/2$.

The check of the values at the two endpoints with opposite signs ensures existence of at least one root within the given range $[a, b]$ for a continuous and differentiable function. Of course, the smaller the range, the better the chance to find a unique root within the range. For example, in Example 2, only one root $x = 0.2573$ would be found if the range is defined as [0, 0.5].

Strictly proving the correctness of this simple strategy is beyond the author's capability. However, ten problems, including the five questions presented in Section 3, have been solved by Newton's method using this strategy with a smaller error limit $\varepsilon = 0.0001$, along with the results from starting the process at the two endpoints as comparisons in the performances of Newton's method for solving these questions. The performance data in term of the number of iterations for all questions are summarized in Table 1. All the questions, the iterative processes and outcomes are detailed in Appendix A.

Table 1. Number of iterations to reach the first acceptable root using different initial points

x_0	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10
a	5	3	*	7	4	5	5	5	*	4
b	5	#	7	4	+	4	4	4	4	5
$(a + b)/2$	4	4	7	4	3	2	3	4	3	3

#: Reached a different root shown in Example 2; *: Not progressing as the derivative was not defined at the point; +: Divergent as shown in Example 5.

5. Discussion and conclusion

A few observations can be made from the data in Table 1. First, by choosing the middle point as the initial point, the iterative process of solving the ten nonlinear equations by Newton's method converged to the desired root within the range without the assistance of the graph of a function. Using the middle point to initiate the process, there is no difficult to find the desired root within the range for the four special cases where an endpoint did not work as expected for Newton's method.

Second, in most cases, beginning with the middle point reaches the desired root with fewer iterations, or no slower than beginning from any of the two endpoints. This is because the middle point should be closer to the root than the endpoints in most cases, which in turn leads to reaching the root with fewer iterations. However, in cases where the solution is located near any of the two endpoints, like Example 2 where the root $x = 0.2573$ is near the left endpoint $x = 0$ and far away from the right endpoint $x = 1.5$, using the middle point as the initial point may be slightly slower than using the near endpoint in approaching the root.

Hence, choosing the middle point of a defined range to initiate the iterative process for Newton's method seems a general and practical strategy to avoid the difficulties encountered in using the method to solve nonlinear equations. However, more work needs to be done to prove how far this strategy could be extended in dealing with various special cases in using Newton's method in the future.

References

1. Stewart, J., *Calculus: Concepts and Contexts*, 4th ed. 2019. Boston, USA: Cengage.
2. Trim, D., *Calculus for Engineers*. 4th ed. 2008, Toronto, Canada: Pearson.
3. Gillett, P., *Calculus and Analytic Geometry*. 1981, Toronto, Canada: D.C. Heath and Company.
4. Croft, A., Davison, R., Hargreaves, M. and Flint J., *Engineering Mathematics*, 5th ed. 2017, Harlow, UK: Pearson.
5. Haeussler, E.F., Paul, R.S. and Wood, R.J., *Introductory Mathematical Analysis for Business, Economics and Life and Social Sciences*, 13^{th} ed. 2011. Boston, USA: Pearson.
6. Guo, W.W., *Essentials and Examples of Applied Mathematics*, 2^{nd} ed. 2020, Melbourne, Australia: Pearson.
7. Wheatley, G., *Applied Numerical Analysis*, 7^{th} ed. 2004, Boston, USA: Pearson.
8. Chapra, S.C., *Applied Numerical Methods with MATLAB for Engineers and Scientists*. 2005, Boston, USA: McGraw-Hill Higher Education.
9. Sauer, T., *Numerical Analysis*, 2^{nd} ed. 2014, Harlow, UK: Pearson.
10. Ypma, T.J., Historical development of the Newton–Raphson method. *SIAM Review*, 1995, 37(4): 531–551. https://doi.org/10.1137/1037125

Appendix A

The ten nonlinear equations to which Newton's method was applied. The error limit is $\varepsilon = 0.0001$ for all questions.

1. Solve equation $f(x) = \frac{1}{x} - \ln x = 0$ in [1, 3].

i	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$
0	1.00000	1.50000	0.50000	3.00000	1.27812	1.72188	**2.00000**	1.74247	0.25753
1	1.50000	1.73508	0.23508	1.27812	1.66320	0.38508	1.74247	1.76306	0.02059
2	1.73508	1.76292	0.02783	1.66320	1.75929	0.09609	1.76306	1.76322	0.00017
3	1.76292	1.76322	0.00031	1.75929	1.76322	0.00393	1.76322	**1.76322**	0.00000
4	1.76322	**1.76322**	0.00000	1.76322	**1.76322**	0.00001			

2. Solve equation $x^3 - 2x + 2 = \sqrt{x+2}$ in [0, 1].

i	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$
0	0.00000	0.24889	0.24889	1.00000	2.02914	1.02914	**0.50000**	0.20877	0.29123
1	0.24889	0.25728	0.00838	2.02914	1.60459	0.42454	0.20877	0.25654	0.04777
2	0.25728	**0.25730**	0.00003	1.60459	1.41715	0.18745	0.25654	0.25730	0.00076
3				1.41715	1.37367	0.04347	0.25730	**0.25730**	0.00000
4				1.37367	1.37132	0.00236			
5				1.37132	***1.37131***	0.00001			

3. Solve equation $\cos x - \sqrt{x} = 0$ in [0, 1].

i	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$
0	0.00000	0.00000	0.00000	1.00000	0.69353	0.30647	**0.50000**	0.59986	0.09986
1				0.69353	0.65367	0.03986	0.59986	0.63080	0.03093
2				0.65367	0.64462	0.00905	0.63080	0.63898	0.00818
3				0.64462	0.64243	0.00219	0.63898	0.64104	0.00206
4				0.64243	0.64189	0.00054	0.64104	0.64155	0.00051
5				0.64189	0.64176	0.00013	0.64155	0.64167	0.00013
6				0.64176	**0.64173**	0.00003	0.64167	**0.64170**	0.00003

4. Solve equation $f(x) = x^2 - \sin x = 0$ in [0.5, 1].

i	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$
0	0.50000	2.37412	1.87412	1.00000	0.89140	0.10860	0.75000	0.90507	0.15507
1	2.37412	1.47028	0.90385	0.89140	0.87698	0.01441	0.90507	0.87766	0.02740
2	1.47028	1.05948	0.41080	0.87698	0.87673	0.00026	0.87766	0.87673	0.00094
3	1.05948	0.90583	0.15365	0.87673	**0.87673**	0.00000	0.87673	**0.87673**	0.00000
4	0.90583	0.87771	0.02811						
5	0.87771	0.87673	0.00098						
6	0.87673	**0.87673**	0.00000						

5. Solve equation $\sin x - \frac{1}{x+1} = 0$ in [0, 1.5].

i	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$
0	0.00000	0.50000	0.50000	1.50000	-1.08950	2.58950	**0.75000**	0.64585	0.10415
1	0.50000	0.64163	0.14163	-1.08950	-1.17160	0.08210	0.64585	0.65074	0.00489
2	0.64163	0.65071	0.00908	-1.17160	-1.31444	0.14284	0.65074	**0.65075**	0.00001
3	0.65071	**0.65075**	0.00004	-1.31444	-1.52789	0.21345			

6. Solve equation $e^x - x - 3 = 0$ in [1, 2].

i	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$
0	1.00000	1.74593	0.74593	2.00000	1.62607	0.37393	**1.50000**	1.50526	0.00526
1	1.74593	1.53768	0.20825	1.62607	1.51397	0.11210	1.50526	**1.50524**	0.00002
2	1.53768	1.50590	0.03177	1.51397	1.50529	0.00868			
3	1.50590	1.50524	0.00066	1.50529	**1.50524**	0.00005			
4	1.50524	**1.50524**	0.00000						

7. Solve equation $x^3 - 4x^2 + x - 7 = 0$ in [3.5, 4.5].

i	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$
0	3.50000	4.48718	0.98718	4.50000	4.20388	0.29612	**4.00000**	4.17647	0.17647
1	4.48718	4.20112	0.28606	4.20388	4.16430	0.03959	4.17647	4.16369	0.01278
2	4.20112	4.16421	0.03691	4.16430	4.16362	0.00068	4.16369	**4.16362**	0.00007
3	4.16421	4.16362	0.00059	4.16362	**4.16362**	0.00000			
4	4.16362	**4.16362**	0.00000						

8. Solve equation $xe^x = x^2 + 1$ in [0, 1].

i	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$
0	0.00000	1.00000	1.00000	1.00000	0.79099	0.20901	**0.50000**	0.78894	0.28894
1	1.00000	0.79099	0.20901	0.79099	0.74077	0.05022	0.78894	0.74060	0.04835
2	0.79099	0.74077	0.05022	0.74077	0.73844	0.00233	0.74060	0.73844	0.00216
3	0.74077	0.73844	0.00233	0.73844	**0.73843**	0.00000	0.73844	**0.73843**	0.00000
4	0.73844	**0.73843**	0.00000						

9. Solve equation $\sqrt[3]{x}-\frac{1}{\sqrt{x+1}}=0$ in [0, 1].

i	x_i	x_{i+1}	$\|x_{i+1}-x_i\|$	x_i	x_{i+1}	$\|x_{i+1}-x_i\|$	x_i	x_{i+1}	$\|x_{i+1}-x_i\|$
0	0.00000	0.00000	0.00000	1.00000	0.42582	0.57418	**0.50000**	0.52845	0.02845
1				0.42582	0.52228	0.09646	0.52845	0.52895	0.00050
2				0.52228	0.52892	0.00664	0.52895	**0.52895**	0.00000
3				0.52892	**0.52895**	0.00003			

10. Solve equation $2\sin x+1=2\cos x$ in [0, π/2].

i	x_i	x_{i+1}	$\|x_{i+1}-x_i\|$	x_i	x_{i+1}	$\|x_{i+1}-x_i\|$	x_i	x_{i+1}	$\|x_{i+1}-x_i\|$
0	0.00000	0.50000	0.50000	1.57080	0.07080	1.50000	**0.78540**	0.43184	0.35355
1	0.50000	0.42495	0.07505	0.07080	0.47030	0.39950	0.43184	0.42404	0.00780
2	0.42495	0.42403	0.00092	0.47030	0.42440	0.04590	0.42404	**0.42403**	0.00001
3	0.42403	**0.42403**	0.00000	0.42440	0.42403	0.00037			
4				0.42403	**0.42403**	0.00000			

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