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Case study

Beyond the compass: Exploring geometric constructions via a circle arc template and a straightedge

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Abstract: For thousands of years, the compass-and-straightedge tools have dominated the learning and teaching of geometry. As such, these inherited, long-standing instruments have gained a lustre of naturalized pedagogical value. However, mounting evidence suggests that many learners and teachers struggle to efficiently, effectively and safely use compasses when constructing geometric figures. Compasses are difficult for learners to use, can lead to inaccurate drawings, and can be dangerous. Thus, there is value in reconsidering the role of the compass in the learning and teaching of geometric constructions and to offer better tools as alternatives. The purpose of this work is to address the aforementioned need by proposing an alternative tool to the compass that is safer, more efficient and more effective. We will argue that a circle arc template forms such an alternative tool, and we will illustrate how learners and teachers can add value to their classrooms by using it, in conjunction with a straightedge, to establish the well-known constructions seen in geometry curricula around the world.

Keywords: beyond the compass, circle arc template, geometric constructions, mathematics education

1. Introduction

The area of geometric construction is usually taught and learnt in early high school. Its importance is recognized via its inclusion in mathematics curricula across the world. For example, learning goals and activities involving geometric construction appear in various national mathematics curricula, such as: the UK's Key Stage 3 (see p. 8 in [1]), the USA's Common Core (see CCSS.MATH.CONTENT.HSG.CO.D.12, CCSS.MATH.CONTENT.HSG.CO.D.13 in [2]), Australia's Year 7 (see Elaboration ACMMG164 in [3]), Singapore's Secondary Two and O-Levels (see p. 16 and p. 28 in [4]) and New Zealand's Level 5 [5].

In the aforementioned geometric construction curricula, students are asked to create some new geometric entity (a point, or a set of points) from a given starting set of data. For example, students are challenged: to construct the perpendicular bisector to a given line segment; to construct the bisector to a given angle; and to construct a line that passes through a given point and is perpendicular or parallel to a given line.

A method of solution to each of these kinds of problems is defined to be a construction. A construction is a theorem, requiring a proof and an algorithm (see Chapter 1 in [6]). Furthermore, constructions are constructive in the sense that they not only ensure that the point(s) asked for in the problem do actually exist, but in addition, constructions also explicitly illustrate how students can create and locate the point(s) of interest.

Students are expected to represent the particular theorem under consideration by creating an illustration that is carefully drawn with geometric tools. This drawing is also referred to as a construction. So, following Martin (Chapter 1 in [6]), we will consider the term *construction* as having these two technical meanings (i.e. proof and drawing).

Enabling students to produce accurate drawings is an important aspect in the learning, teaching and undertaking of geometric constructions. This position is, for instance, supported by Brown et al. [7]: "in order to develop geometric understanding and to demonstrate (not prove) theorems, accurate drawings are required. For example, to demonstrate that two of the angles are equal in an isosceles triangle, an accurate sketch is needed" (p. 5) [7].

Software is referred to in some of the aforementioned curricula as a potential digital tool for geometric constructions. However, access to digital means are not always readily available due to cost and complexity of the infrastructure. Many curricula specify hands-on learning through using physical materials in the form of particular instruments to undertake these constructions, namely a pair of compasses and a straightedge. The implied expectations here are that students and teachers will learn and teach how to physically undertake and understand these geometric constructions effectively and accurately with these two physical tools alone.

The restriction to only compass-and-straightedge instruments in geometric constructions appears to have been commonly attributed to Plato. According to Schaaf (p. iii) [8], Plato objected to other tools being employed for geometrical constructions because he considered these tools and their related solutions to be "mechanical and not geometrical". Thus, in Plato's view, in using tools other than compass-and-straightedge instruments for geometric construction "The good of geometry is set aside and destroyed, for we again reduce it to the world of sense, instead of elevating it and imbuing it with the eternal and incorporeal images of thought, even as it is employed by God, for which reason He always is God" (p. iii) [8].

If we combine this historical tradition with the inherited, long-standing commitment attributed to Plato that continues today in modern educational curricula, classrooms and textbooks regarding compass-and-straightedge instruments [9], then it seems that the exclusive use of these specific tools in physical geometric construction has gained a natural lustre of worth, forming what appears to be a state of pedagogical consensus.

However, this over-stabilisation has led to significant challenges that have been recognized within the mathematics education community. In particular, there is mounting evidence to question the efficiency, efficacy and safety of compasses when constructing geometric figures in the classroom. Let us unpack five of the problems with compasses: difficulty of use; inaccuracies in the drawings; the separation from the straightedge tool; the need to adjust and readjust the compass; and safety concerns.

Compasses can be difficult for learners to use. Indeed, the dynamic process of operating a compass is not simple from a kinesiological point of view. The learner must successfully roll the top of the compass between forefinger and thumb to enable the rotation, whilst maintaining pressure on the leg with the needle so it stays firmly in place, and simultaneously apply enough pressure on the other leg that holds the pencil so as to draw a clearly visible curve during the movement. But, the learner cannot apply too much pressure or there is a risk that the opening of the legs will change, or the legs will bend or flex during operation. If this wasn't enough to keep track of, the learner usually angles the top of the compass in the direction of rotation to assist with the movement. As we can see, a collection of significant fine motor skills are required for this sort of operation. In particular, it has meant that younger learners encounter significant challenges when trying to use compasses.

However, it is not just younger learners who face challenges with operating compasses. Hendroanto and Fitriyani [10] recently reported the findings of a survey that was administered in Yogyakarta, Indonesia indicating that a significant percentage of teachers struggle to efficiently, effectively and safely use traditional tools when constructing geometric figures. They reported over 40% of teacher respondents disagreeing with the statement that traditional instruments such as compasses were "efficient and practical to use", and more than 40% of teacher-respondents agreed with the statement that compasses tend to slip on a surface when in use [10].

Indeed, the tendency of the needle point or pencil to slip when in use is something that probably all learners and teachers could identify with, and is aptly captured in the following edited cartoon [11].

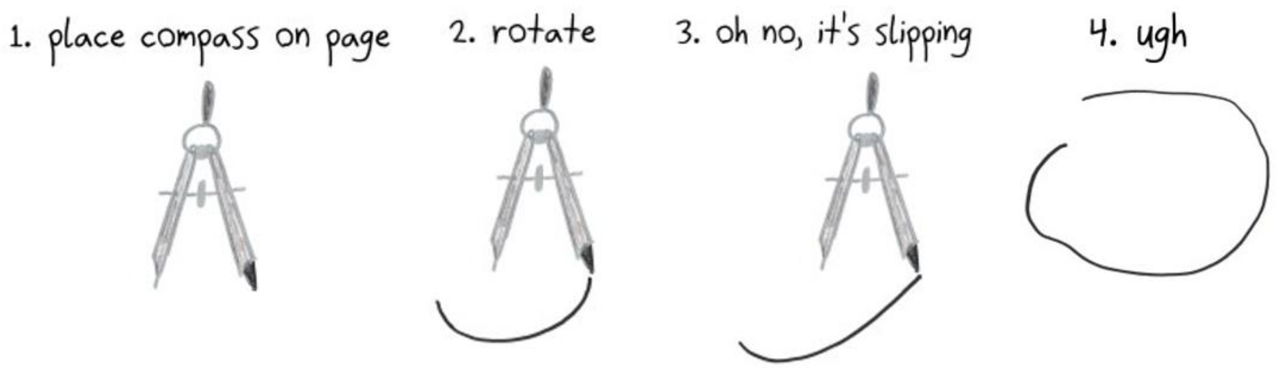

Figure 1. Compass constructions made easy [11]

Another problem is that the resulting arcs or circles produced when using a compass may be inaccurate. Even if learners possess the significant fine motor skills needed to operate a compass, then there are challenges remaining regarding the quality of the tool itself that is being used. For example, Richeson [12] takes the position that "If it is a cheap discount-store compass that stays in place using friction only, then after it has swept out 360 degrees, the jaws may have opened or closed slightly and the pencil doesn't close up the circle" (p. 178) [12]. Thus, even the most dexterous student may struggle to produce accurate geometric constructions if the compass they are using is of low quality.

In addition to the above, compasses present several inconveniencies for learners. Firstly, they are separate entities from the straightedge tool, so that learners need to keep track of two tools when learning geometric constructions. As such, there is a risk that a compass can become forgotten or lost.

Secondly, it can be tedious for students to keep adjusting and then re-adjusting the opening of a compass. As Richeson [12] says "Students must constantly adjust it to the desired opening. If it is an expensive compass that is held fast with a screw, then it is a tiresome and time-consuming process to adjust it" (p. 178) [12].

Another inconvenience is that compasses can damage the paper. The sharp needle point on one leg of a compass tends to poke holes in the paper on which students are using it. While this may be a somewhat minor irritation, we believe it would be preferable for learners to undertake geometry without inflicting these "pin prick scars" of construction.

Furthermore, it is not just paper that compasses can damage. There are genuine safety concerns regarding traditional compasses, most notably due to their ability to be used as a weapon by using the pointed leg. Recent media stories involving compasses being weaponized include: alleged bullying [13], reported stabbings [14] and even alleged murder [15].

Additionally, Albrecht, Jr. [9] offers a critique of the role of ruler and compass constructions in the teaching of high school geometry within the United States. He takes the position that "although ruler-and-compass constructions played a large role in the geometry of the ancients, and although the influence of the restriction has extended over more than twenty centuries, this does not constitute a reason for requiring modern boys and girls to confine themselves entirely" (p. 121) [9] and calls for a more inclusive approach to learning and teaching with practical instruments of geometry [9].

Indeed, Albrecht Jr.'s position echoes that of Welkowitz [16], to wit, that in many cases the use of alternative instruments help students to undertake constructions in a simpler and no less accurate way. Furthermore, "The constructions performed with non-Platonic instruments afford, in addition, practice in deduction by trying to explain why they 'work'. The great variety of problems that can be offered as a consequence of this extension... are more concrete, practical, and appealing to the average pupil" (p. 107) [16] and he advocates for us to "liberate ourselves from the classical restrictions to straight edge and compasses and thereby obviate the constant need for apologizing to our pupils for what they may not do" (p. 112) [16]. Learners also recognize the need to consider alternatives, for example, over 86% of students who participated in the survey of Hendroanto and Fitriyani [10] identified value in having alternative geometric tools at their disposal.

In synthesizing the above discussion, it is clear that there is a need to look beyond the compass in the learning and teaching of geometric constructions, and to offer a deeper understanding of alternatives. The purpose of this article is to respond to the aforementioned need, and to evaluate an alternative educational instrument that has the potential to form a safer, more efficient and more effective tool for geometric constructions.

A circle template is a clear plastic sheet with cut-out circular shapes. It serves as a tool for learners and teachers to draw circular arcs with a pen or pencil. We will introduce and analyse the design and potential use of a particular circle arc template as an instrument for the learning and teaching of geometric construction. We will explore if and how this type of tool can be used in conjunction with a straightedge to establish the classical constructions that students see in high school curricula. Thus, the research questions driving this article are:

RQ1: How can a circle arc template be designed and used to form a potential instrument for basic geometric constructions?

RQ2: What are the potential benefits and limitations of learning and teaching geometric constructions with such an instrument?

To explore the above questions, we will draw on a research design framework known as case study research. Case study research is a popular methodology in the social sciences that can also be viewed as a strategy and a research genre [17]. Indeed, there is strong alignment between the nature of our research questions and the purpose of case study research because the research questions driving this article are of an “explore and explain” nature and involve particular phenomena that are not well understood (p. 114) [17].

This paper is organised in the following way.

In section 2 we introduce the circle arc template as an instrument, probe some of its design properties, and discuss how students can use it. We also make some comparisons with the traditional compass tool. In particular, by drawing on the literature of the rusty compass and the broken compass, we show that it is theoretically possible for learners to use a straightedge and a circle arc template to undertake the usual constructions.

In section 3 we examine how learners can perform specific geometric constructions using a circle arc template and a straightedge. We probe several geometric construction problems in depth, aiming to drill down and to get at their complexity.

Section 4 is dedicated to exploring the limitations of geometric constructions with circle templates and a straightedge. We make some conclusions within section 5, and raise some open questions for further research.

2. Circle arc template

It is well known that it is difficult for humans to draw circles accurately via freehand due to the cognitive load involved [18, 19]. For instance, one part of our brain *recognises* the perfect circle, however another part is responsible for the *movements* involved with drawing circles. Tools known as drawing templates (a piece of clear plastic with cut-out shapes) have filled this gap by providing the means to accurately draw shapes with a pen or pencil. Architects, engineers, designers and drafters have long known about the power of templates to assist with technical drawing and illustrations. Younger learners may have had experiences with drawing templates when learning to write letters, numbers, and other forms. However, to our knowledge, circle arc templates have not yet appeared in the learning and teaching of geometric constructions.

The particular tool that we will focus in this work is captured in the following figure:

Figure 2. Circle arc template

As we can see, a large sector of the circle has been removed, and a small sector remains. On first inspection, the simple shape of the tool might remind some readers of a cheese wheel, a pizza, or Pac-Man, who was the main character from the 1980s computer game of the same name.

Particular design elements that we believe are of interest are:

- A translucent, polycarbonate material makes up the tool. This is designed so that the tool has strength when in use and can be positioned accurately.
- A positionable centre point. This is designed to enable students to accurately place the centre of the circle arc on a given point. Given that the template is translucent, it is easy to align the centre point of the tool with the point of interest.
- A fixed radius. The circle arc template in Figure 2 above has a fixed radius of 20mm however any convenient radius length will suffice. Once the tool is positioned in an appropriate place on the page, students can trace around the inside of the opening to produce an arc.
- A circular arc with an opening of 300 degrees ($5\pi/3$ radians). This is designed to enable students to draw extended arcs that are suitable for the basic constructions outlined earlier.
- A smooth, strong circular edge with an absence of marked graduations. The absence of graduations is designed to encourage student thinking and experimentation, rather than to facilitate a measurement of angles. In this way, it acknowledges some Platonic principles. The edge is manufactured to very precise standards which means it is smooth and consistent in its curvature. Its strength is designed to enable significant amounts of pressure to be applied against the edge of the arc with their pencil without deviations in the final drawing.

Students can operate the circle arc tool much like they would use any other drawing template, or indeed, a ruler. A student places the centre point of the circle arc template on a given point. One hand then holds the template in place and the other hand holds the pencil, and they can draw the arc by moving the pencil around the inside of the circle arc opening. Given our earlier discussion on the difficulties learners face in operating a compass, there are several comparisons that we now make with a compass.

A circle arc template is potentially easier to use than a compass. The tool stays fixed and does not demand the operator to rotate it in a way like a compass does. Rather, it is just a case of the learner's hand holding the pencil and moving their hand so that the pencil traces the circle arc. This is an action that is familiar to almost all learners and is similar to that of using a ruler or any other kind of drawing template. In this sense, there are fewer moving parts for the learner to control, which reduces the kinesiological and cognitive loads. As we can see, fine motor skills are not required to the same degree of operating a compass.

The resulting arcs or circles produced by learners using a circle arc template have a high degree of accuracy. Due to the tool being manufactured to precise standards from polycarbonate, it is strong, accurate and resistant to any kind of flexing. As such, there is a reduced risk of the inaccuracies that learners see when compass legs bend or flex during use. Learners have the option to keep the template in place with the whole of their non-drawing hand, which has the capability to naturally exert the pressure to keep the template from moving during use. This appears to be much easier for learners to do than maintaining the pressure with the tips of the thumb and forefinger when operating a compass. Thus, there is less tendency of the circle arc template to move during operation.

In contrast to the adjustable compass, there are no adjustments for learners to make when using the circle arc template. As such, this invariance fosters accuracy, and time is potentially saved. Due to the circle arc template having no moving parts, there is less chance of wear and failure of the tool.

Furthermore, it is possible to design a circle arc template as part of a larger tool that also has a straightedge. In this way, the two tools are conveniently together and thus less likely to become separated, forgotten or lost.

A circle arc template is less likely to damage the paper. Unlike the compass, there are no sharp points or edges in the circle arc template and so students will not damage the paper by producing holes when in use. Although no instrument is completely safe, the circle arc template does not have sharp, needle points like the compass does, and thus we believe that it has a reduced risk of being weaponized.

The circle arc template in Figure 2 is currently subject to a patent application by the first author and John Lawton of Objective Learning Materials.

3. Circle arc template in geometric constructions

Is it possible for students to perform all of the common constructions seen in high-school curricula with just a straightedge and a circle-arc template? The answer is yes. Francesco Severi [20] built on the famous Poncelet-Steiner theorem which states that if students have one circle—any circle—together with its centre point, then they can construct every compass-and-straightedge constructible point using only a straightedge. Severi [20] showed that it is not necessary to start with an entire circle: if students have just an arc of any circle (however short in length) and its centre point, then the conclusion of the Poncelet-Steiner Theorem still holds.

So, it is possible for students to place the circle arc template on a piece of paper, draw an arc and from that then use their straightedge to complete the geometric constructions in the curricula. This is significant from a theoretical point of view because we know that one arc and its centre point is enough, and that the circle arc template can certainly produce this.

However, from a practical perspective, having students draw a circle arc, and then discard the circle arc tool for a straightedge alone does not necessarily lead to an efficient solution method for the constructions involved. This is because solely relying on a straightedge after an initial arc is drawn can significantly increase the number of steps in the solution algorithm, even for "basic" constructions. As an example, consider a student constructing the perpendicular bisector or a midpoint to a given line segment with one arc and then only a straightedge. There are many moves to keep track of before these can be found, see Steiner [21]. As we will see, using both tools (a circle arc template and a straightedge) offers convenience since there are fewer steps to undertake if students work with both tools.

Although there are many who have worked with the rusty compass and straightedge, including Pappus, Abu al-Wafa, Duerer and Leonardo Da Vinci (see [12]), we are particularly inspired by the work of the Italian geometers: Tartaglia, Ferraro and Cardano [22]. We believe their work is practical and relevant for the younger learner within the school setting of geometric constructions because their ideas are direct and accessible.

We now examine three basic geometric constructions (plus a variation) in greater detail, utilising the circle arc template seen in Figure 2 in place of the traditional compass.

3.1. Perpendicular bisector of a given line segment

Suppose A and B are given points and students are asked to construct the perpendicular bisector to the line segment $\overline{AB}$. That is, students are challenged to construct a second line segment that is perpendicular to $\overline{AB}$ and intersects $\overline{AB}$ at its midpoint. The initial data is given in Figure 3.

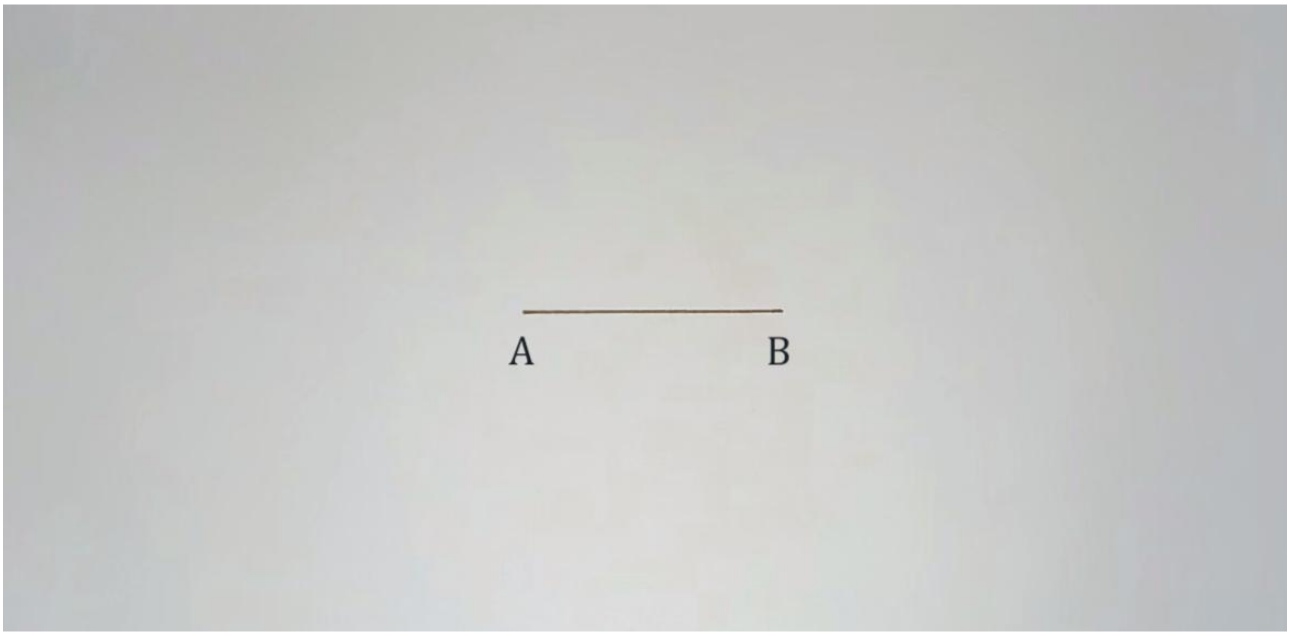

Figure 3. A given line segment $\overline{AB}$

Students can place the centre of the circle arc template at one of the end points as in Figure 4 below. Note that our circle arc template forms part of a larger template with various shapes. These additional shapes appear in our pictures, but are not relevant to the constructions herein.

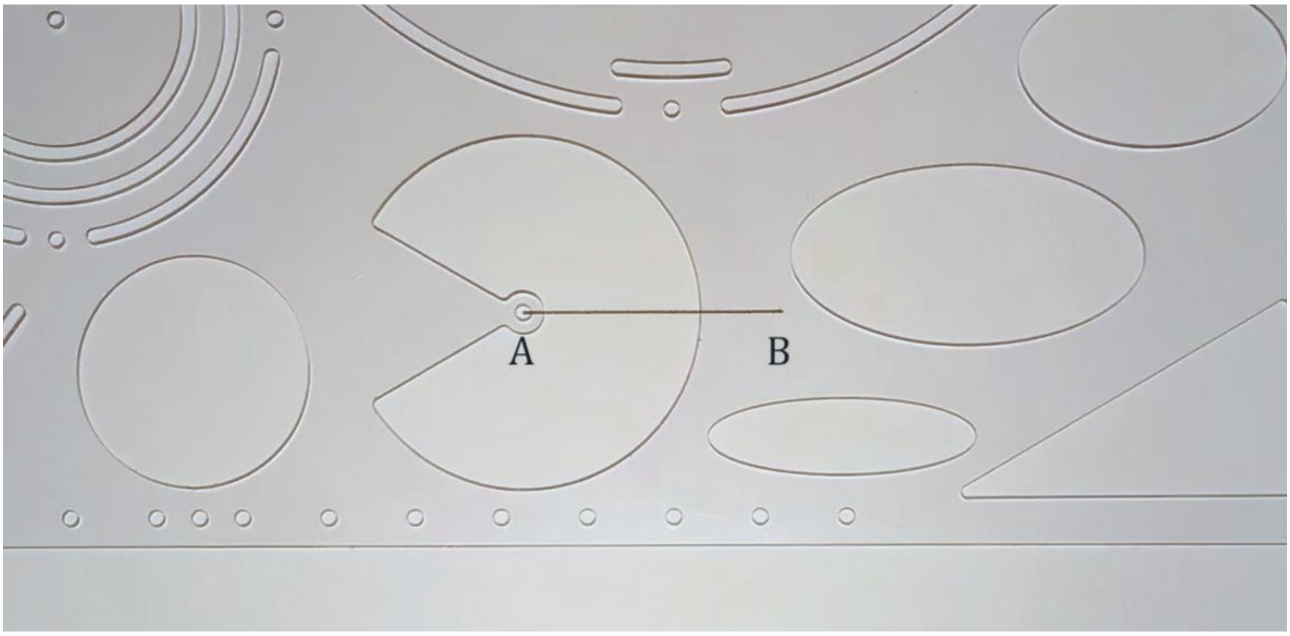

Figure 4. Circle arc template with centre at A

Students can draw the arc as per Figure 5 below.

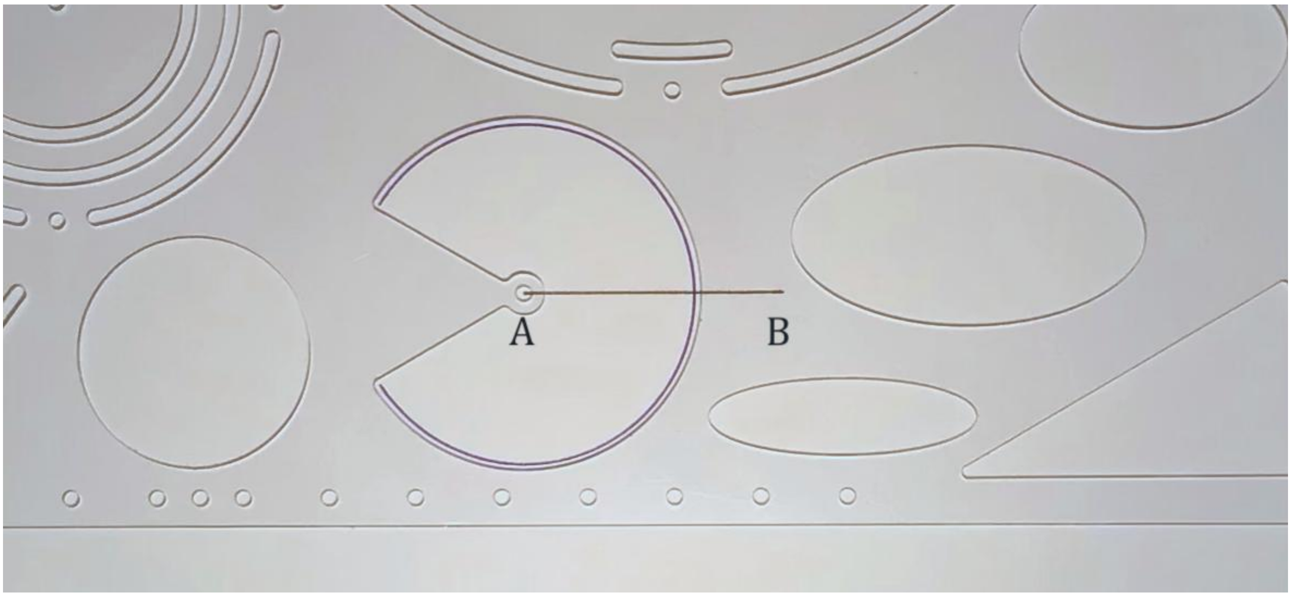

Figure 5. Circle arc template with centre at *A*

This first arc is given below in Figure 6.

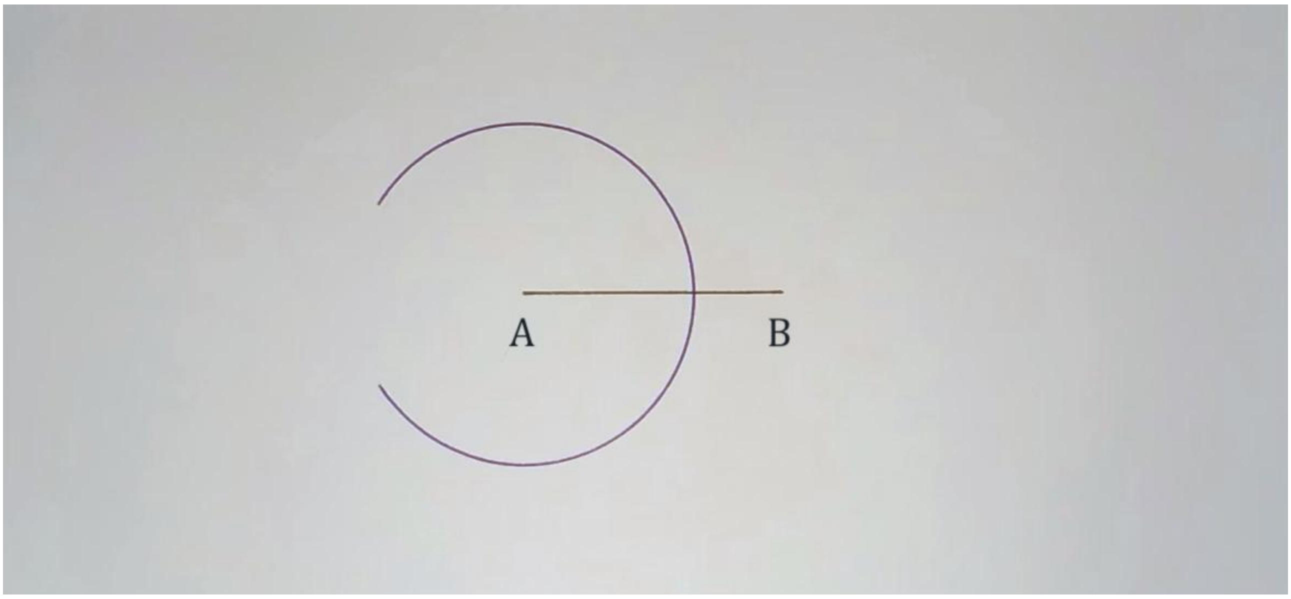

Figure 6. A first arc

Students can then repeat the above process with the same circle arc template at the remaining endpoint. A second arc can then be constructed as per Figure 7 and Figure 8.

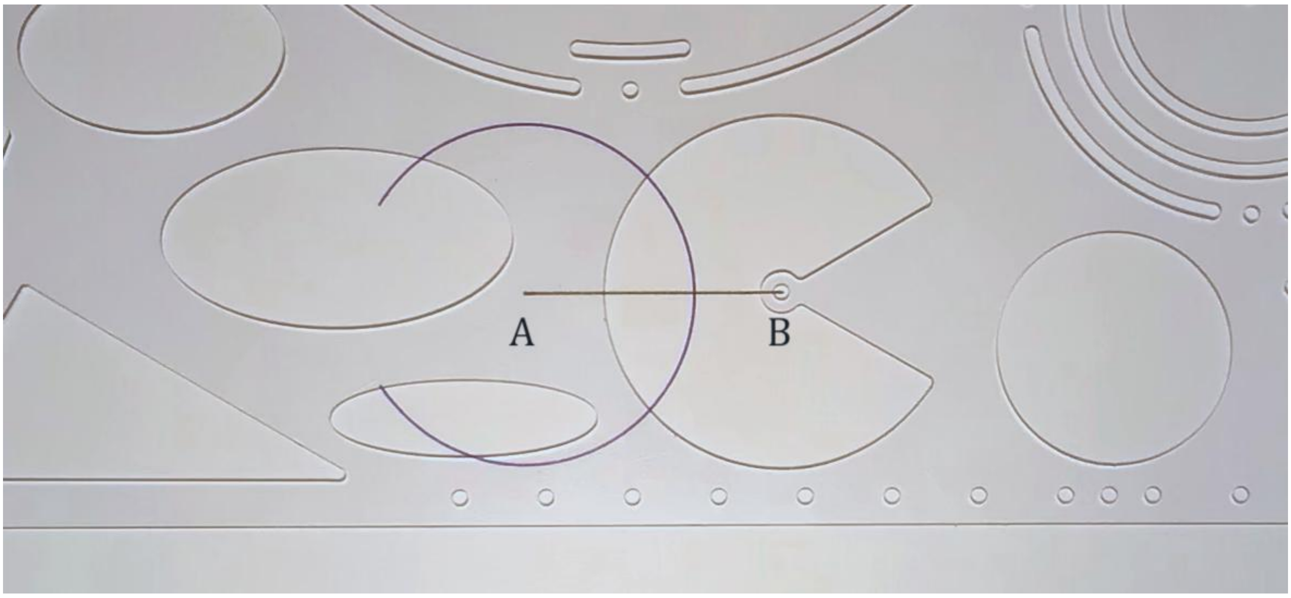

Figure 7. Circle arc template with centre at *B*

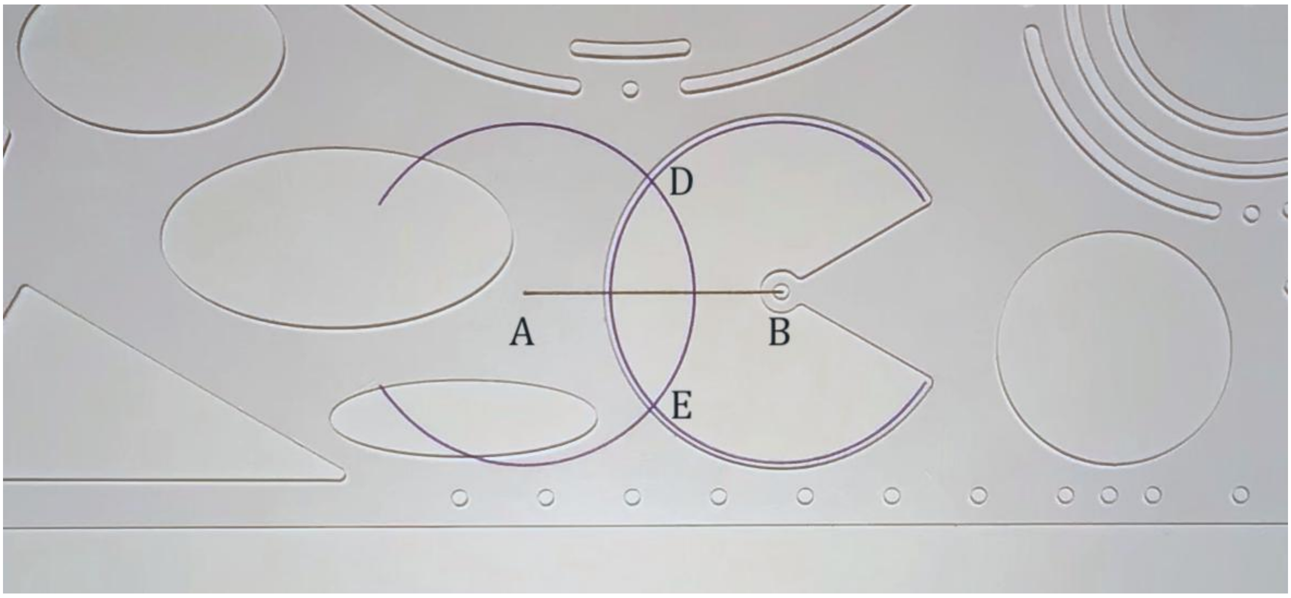

Figure 8. Second arc with intersection points labelled

Students can then see the two points that have been constructed as the intersections between the arc of their first circle and the arc of their second circle. Students can label these two points D and E (see Figure 9 and Figure 10).

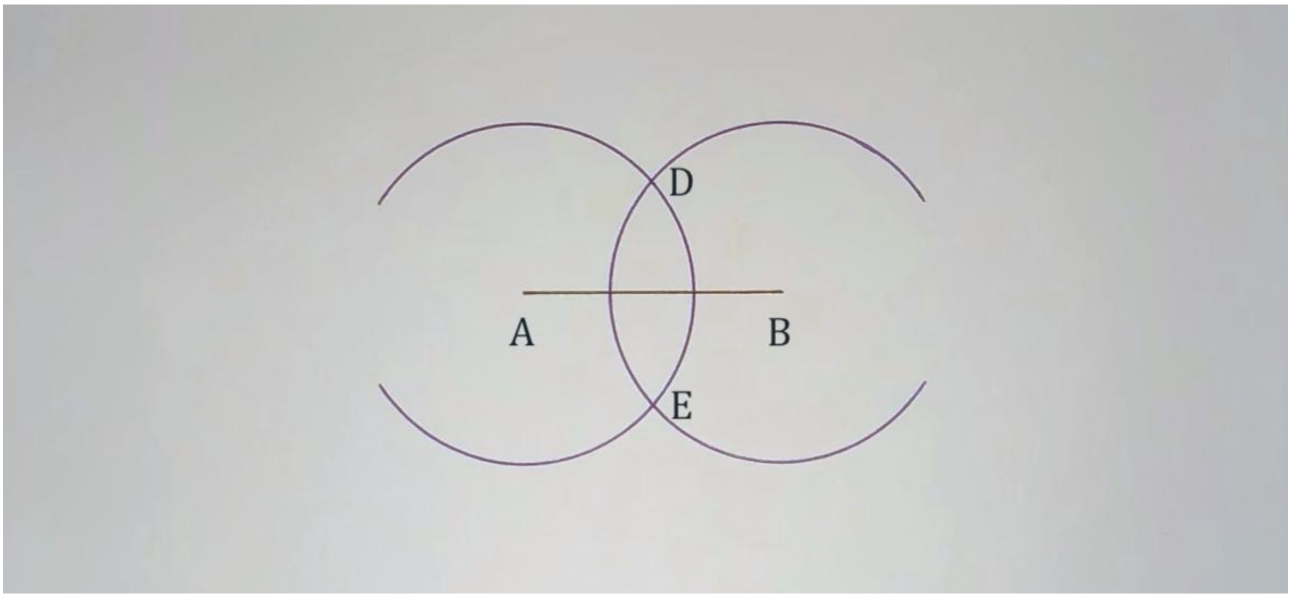

Figure 9. Intersection points, D and E

Students can then form the perpendicular bisector to $\overline{AB}$ by constructing the line segment $\overline{DE}$ as seen in Figure 10 and Figure 11.

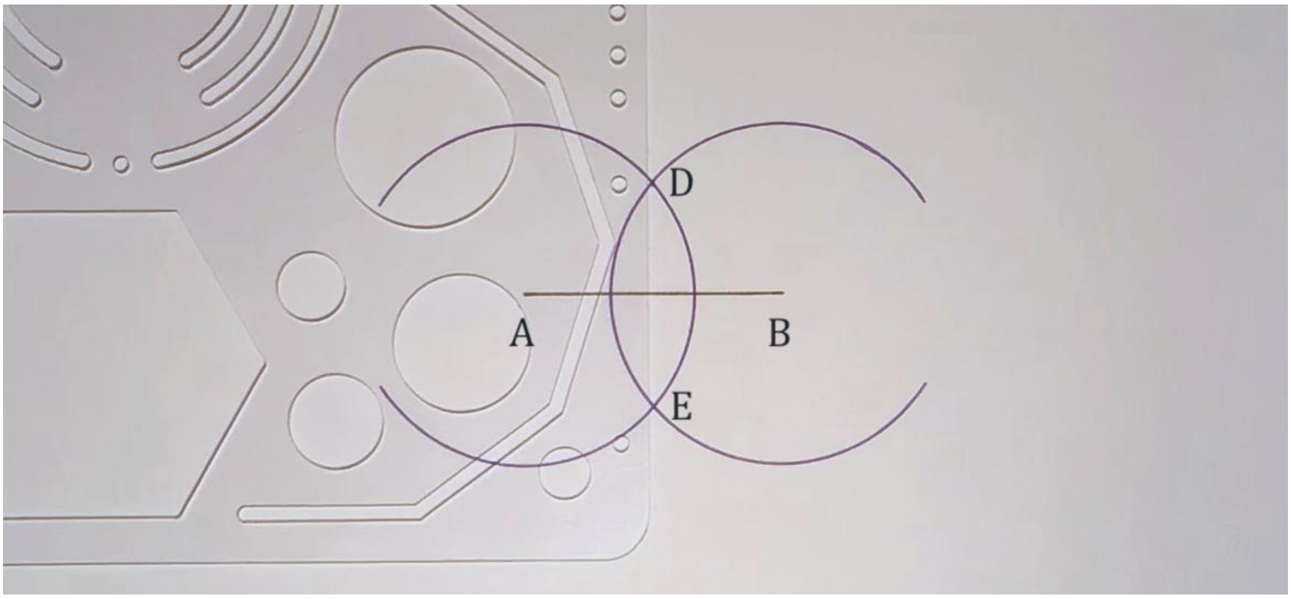

Figure 10. Straightedge aligned with points D and E

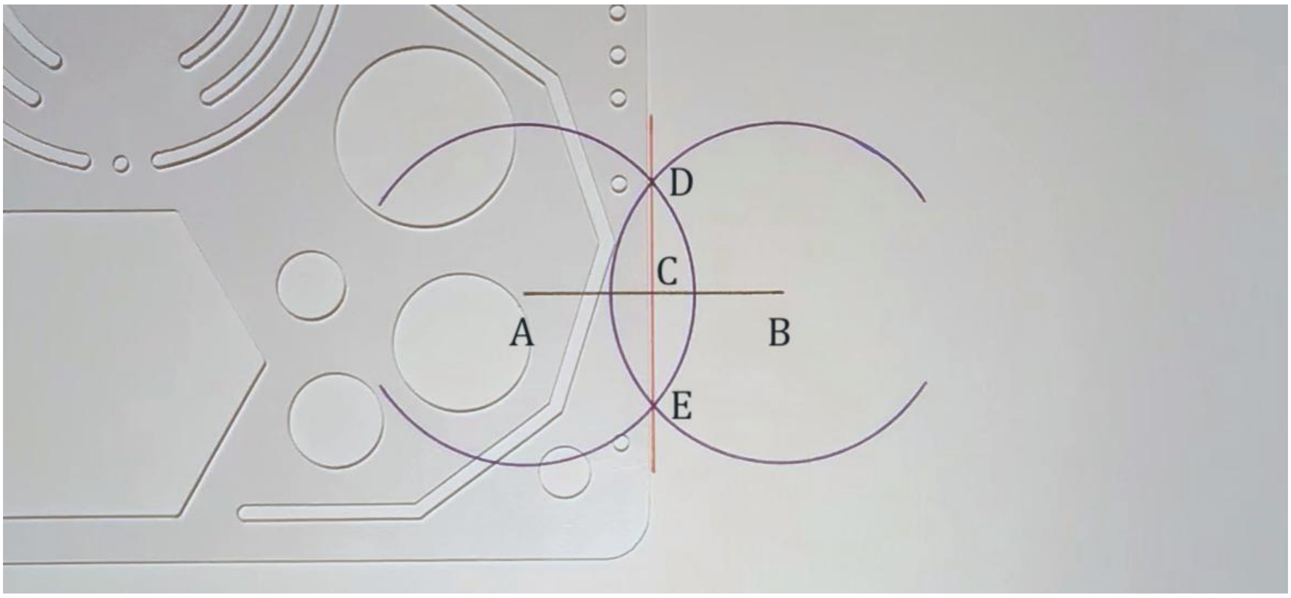

Figure 11. Line segment $\overline{DE}$ constructed

The constructed point of intersection, labelled as C, between the two line segments in Figure 11 is the midpoint of $\overline{AB}$ (that is $|\overline{AC}| = |\overline{BC}|$) and $\angle DCB$ is right angled. Figure 12 shows the finished construction, with $\overline{DE}$ being the desired perpendicular bisector of $\overline{AB}$.

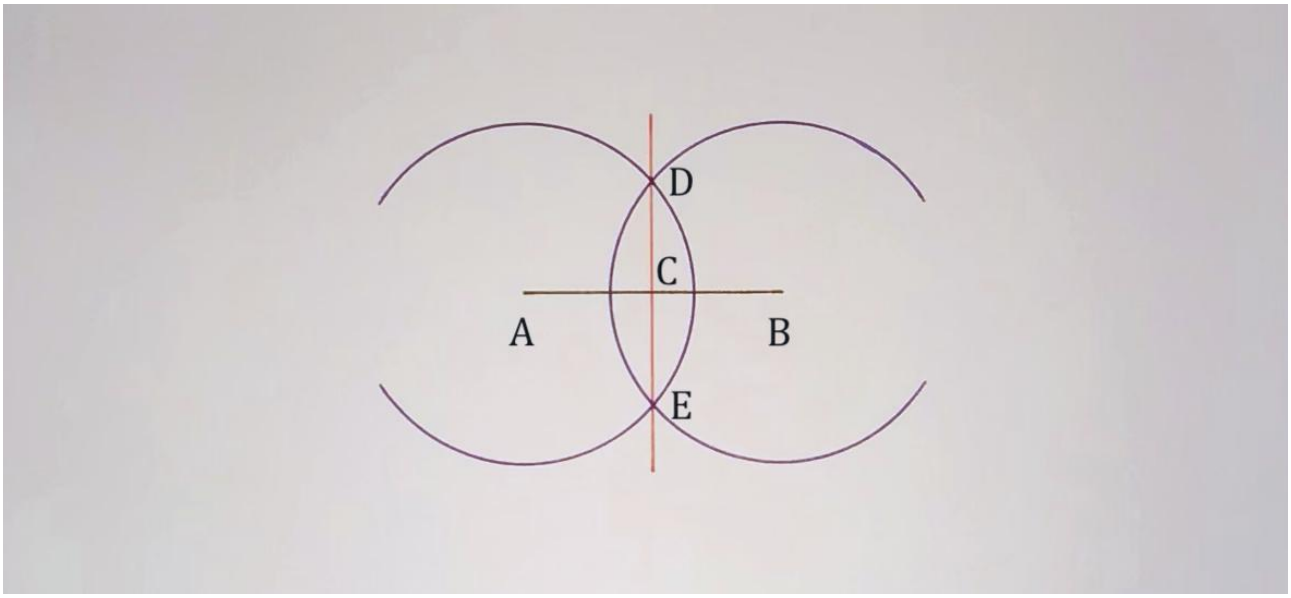

Figure 12. Finished construction. $\overline{DE}$ is the perpendicular bisector of $\overline{AB}$

Justification. The nature of any justification for the previous construction and the remaining constructions is to be linked with the interests and abilities of the learners at hand. For this particular construction this could involve formally drawing on the theory of congruent triangles, or a more informal approach of simply folding the paper.

3.1.1. Variation: Perpendicular bisector of a 'long' line segment

If the given line segment is longer than twice the radius of the circle arc template being used then students face an additional challenge, since the method detailed above does not work given that the two circle arcs will not intersect. However, the construction can be modified to accommodate this case. A student may instead use the following process to find the perpendicular bisector of such a line segment.

Suppose A and B are given points and students are asked to construct the perpendicular bisector to the line segment $\overline{AB}$. Suppose too that $\overline{AB}$ is longer than twice the radius of the circle arc template being used, such that the arcs drawn with the endpoints A and B as centres do not intersect. The given line segment $\overline{AB}$ is given below in Figure 13.

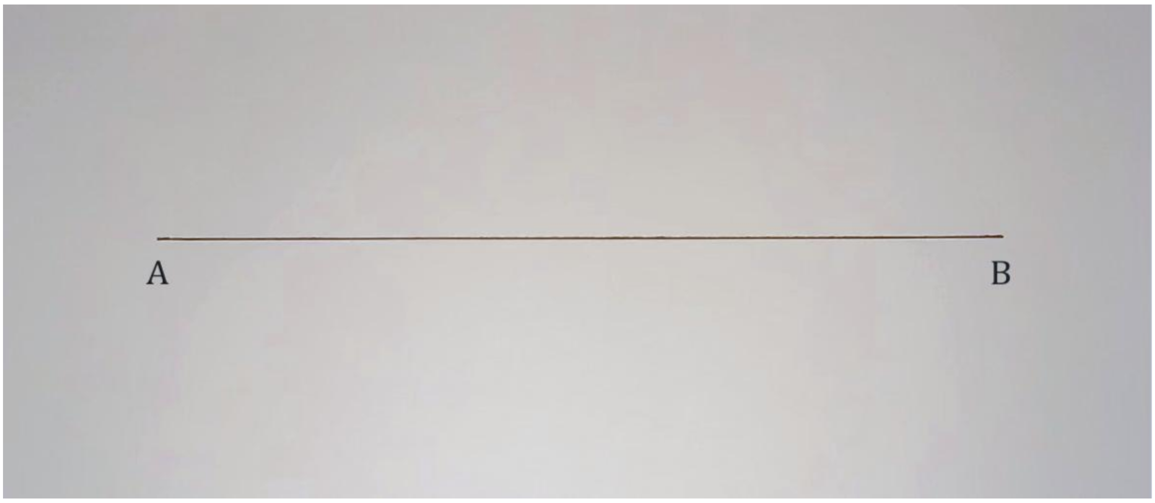

Figure 13. Line segment $\overline{AB}$

Students can then position the circle arc template with its centre at one of the endpoints as shown in Figure 14 below.

Figure 14. Circle arc template with centre at A

Students can now mark the point where the arc intersects $\overline{AB}$. Students can label this point, P. See Figure 15 and Figure 16 below.

Figure 15. Marking point of intersection made with $\overline{AB}$

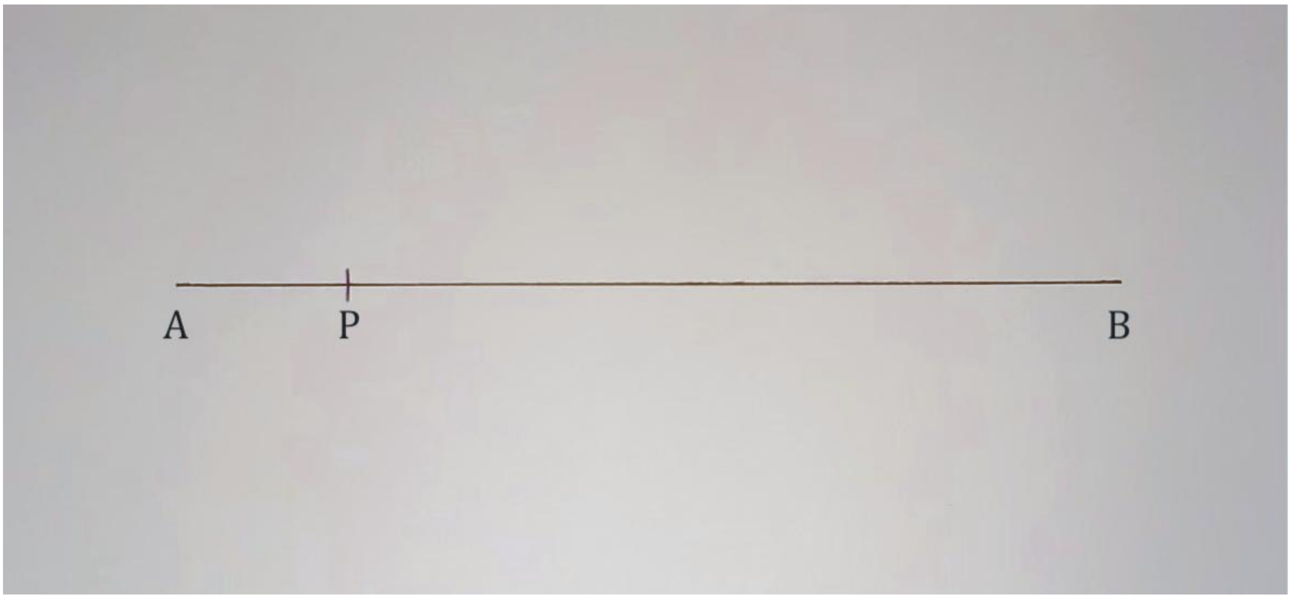

Figure 16. Point of intersection labelled P

Students can repeat the same procedure for the remaining endpoint of $\overline{AB}$. In Figure 17 below, the centre of the circle arc template is repositioned with its centre now at B.

Figure 17. Circle arc template with centre at B

As before, students can mark the point where the arc intersects $\overline{AB}$. This is labelled S. See Figure 18 and Figure 19 below.

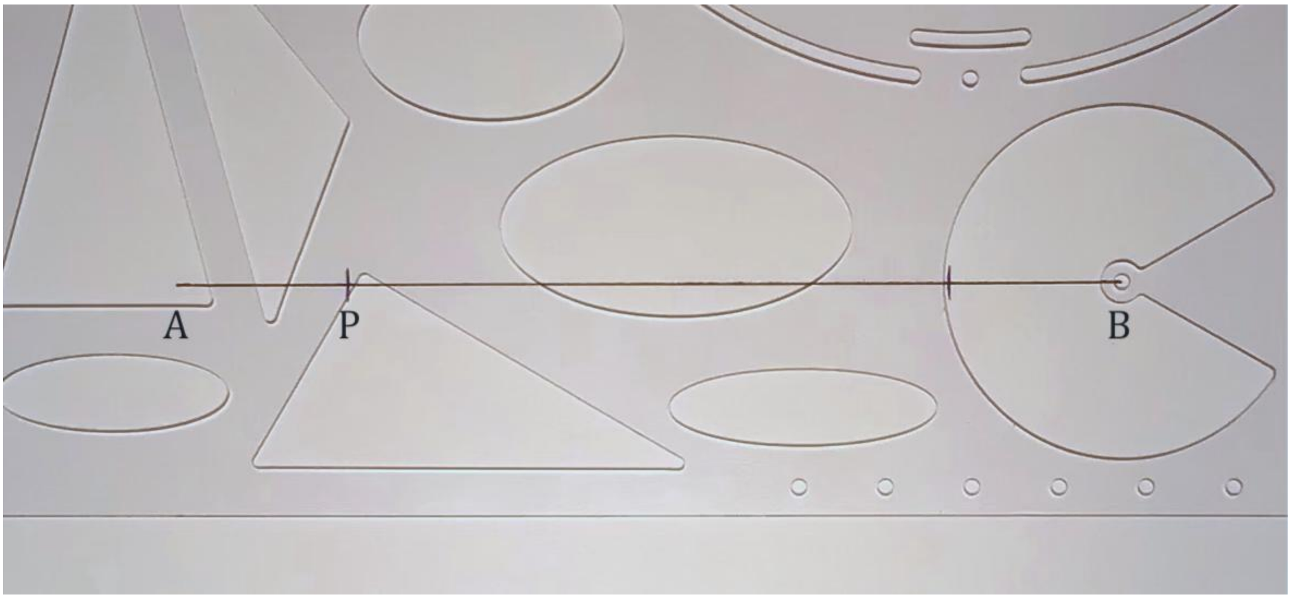

Figure 18. Marking point of intersection made with $\overline{AB}$

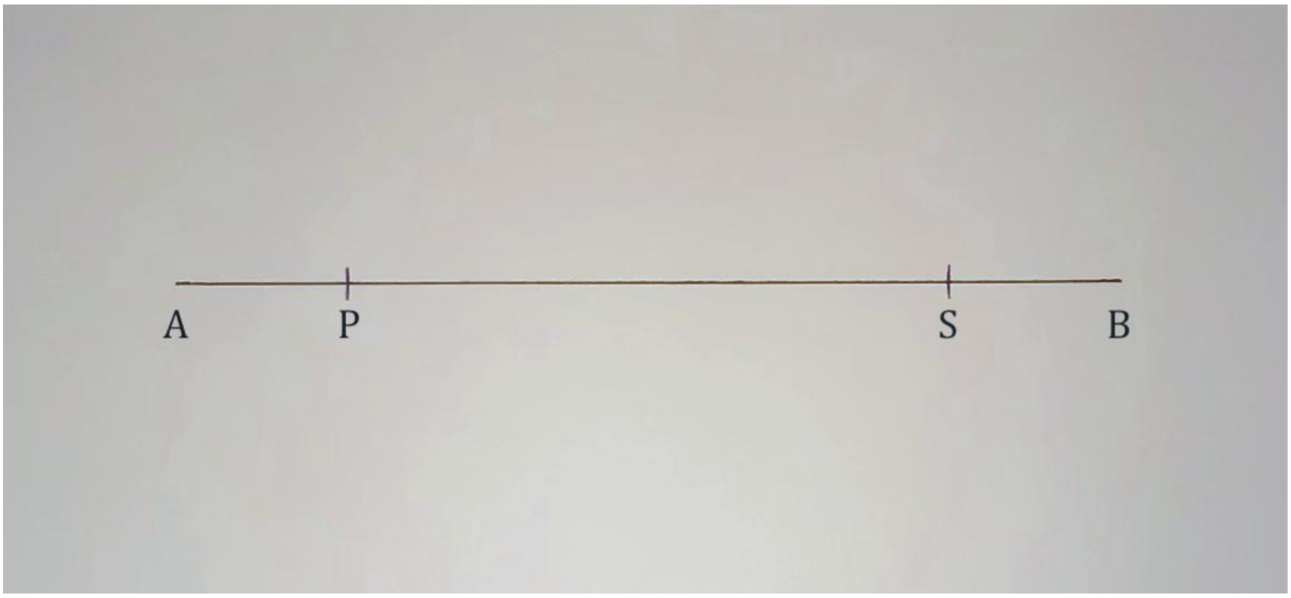

Figure 19. Point of intersection labelled S

Students may now place the circle arc template with its centre at point P.

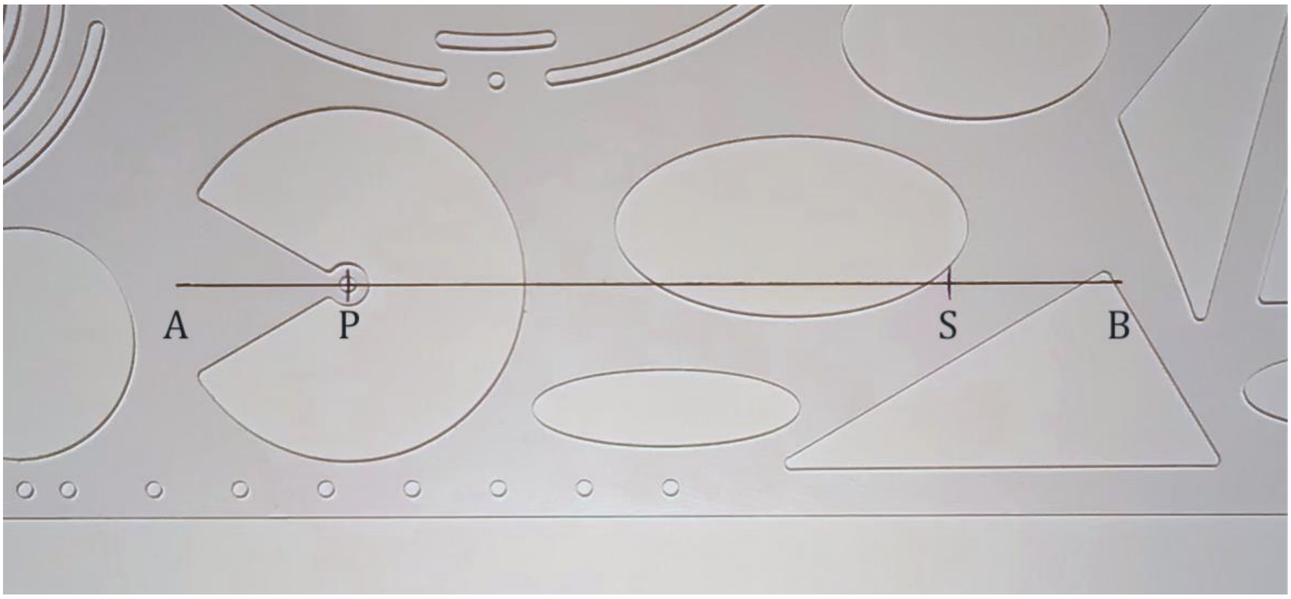

Figure 20. Circle arc template with centre at P

The point at which the arc intercepts $\overline{AB}$ is marked and labelled Q (see Figure 21 and Figure 22).

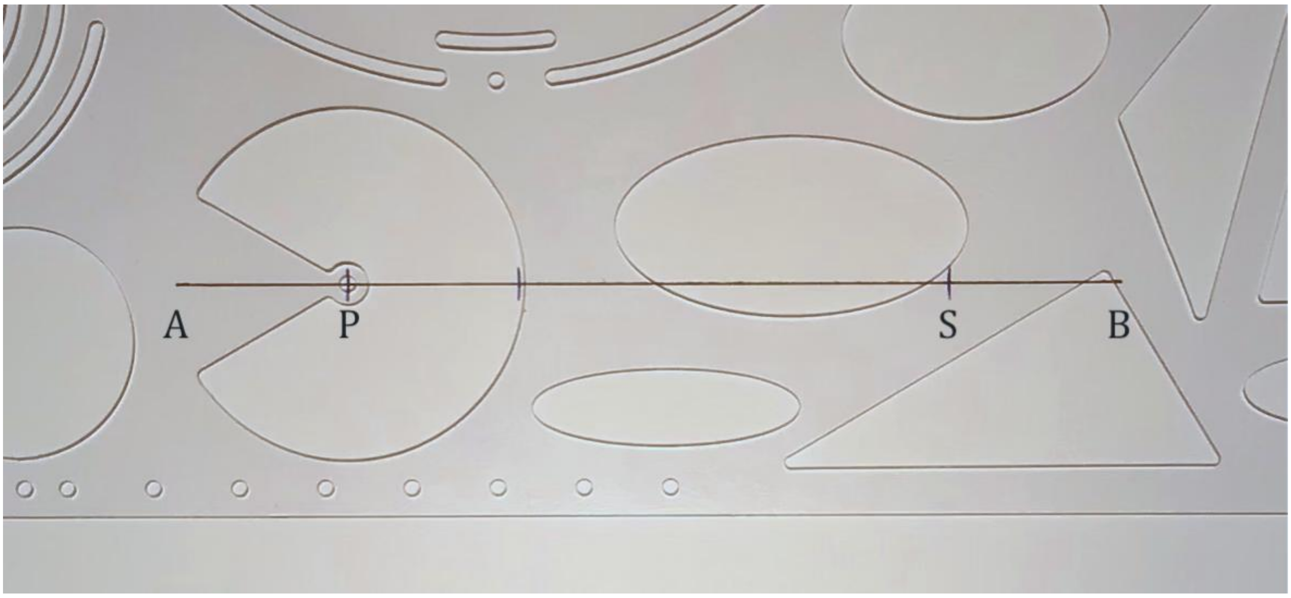

Figure 21. Marking point of intersection made with $\overline{AB}$

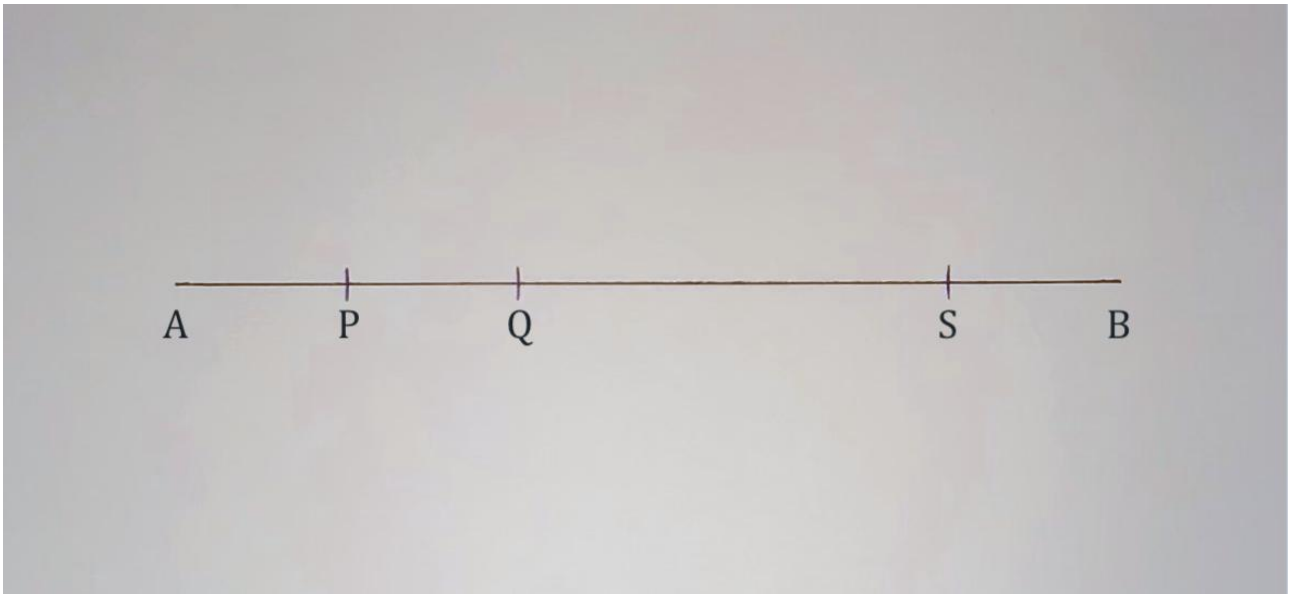

Figure 22. Point of intersection labelled Q

Students can use the same procedure with S as the centre of the circle arc template (Figure 23).

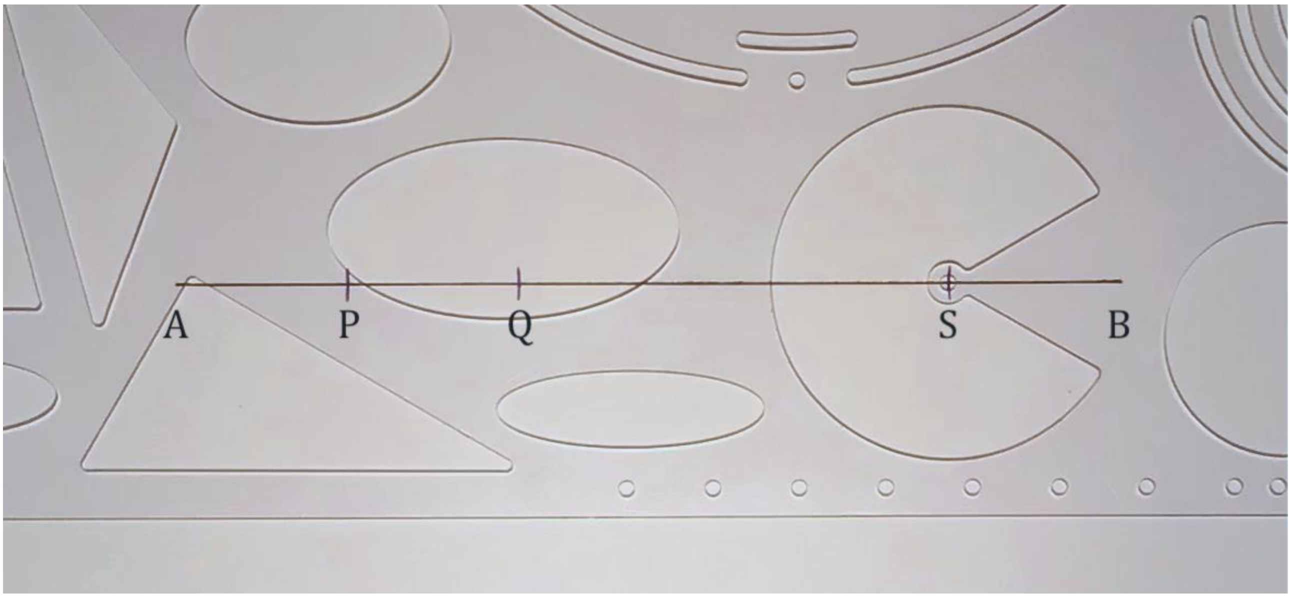

Figure 23. Circle arc template with centre at S

Students may mark the intersection with $\overline{AB}$ as before and label this point, R. See Figure 24 and Figure 25.

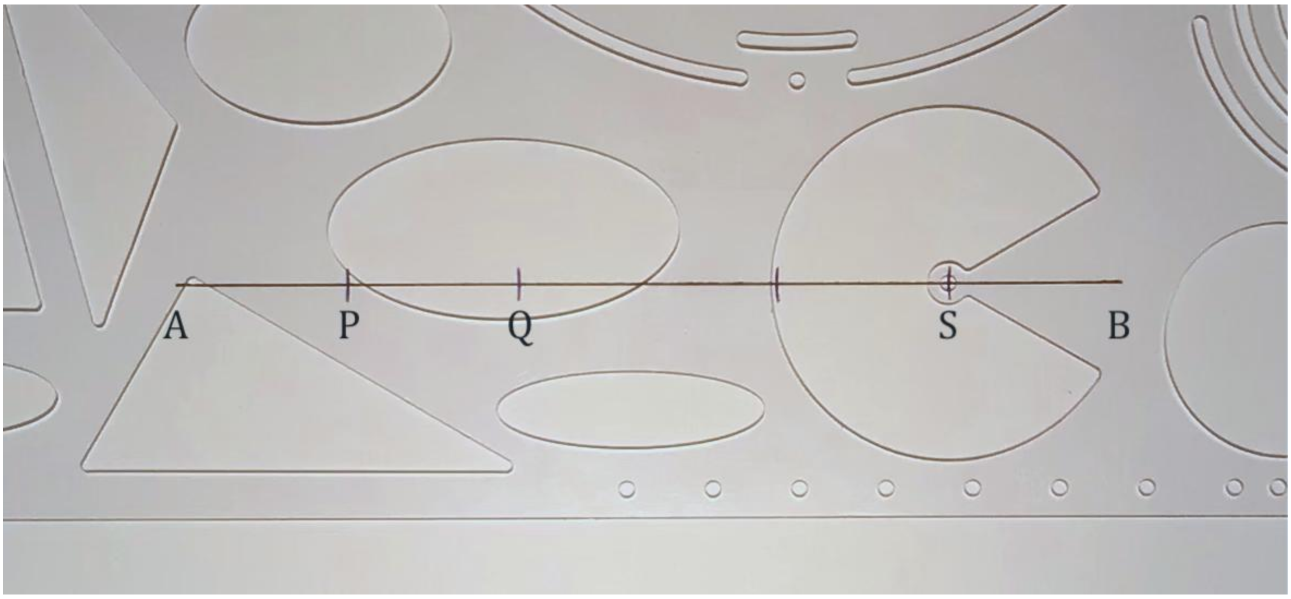

Figure 24. Marking point of intersection made with $\overline{AB}$

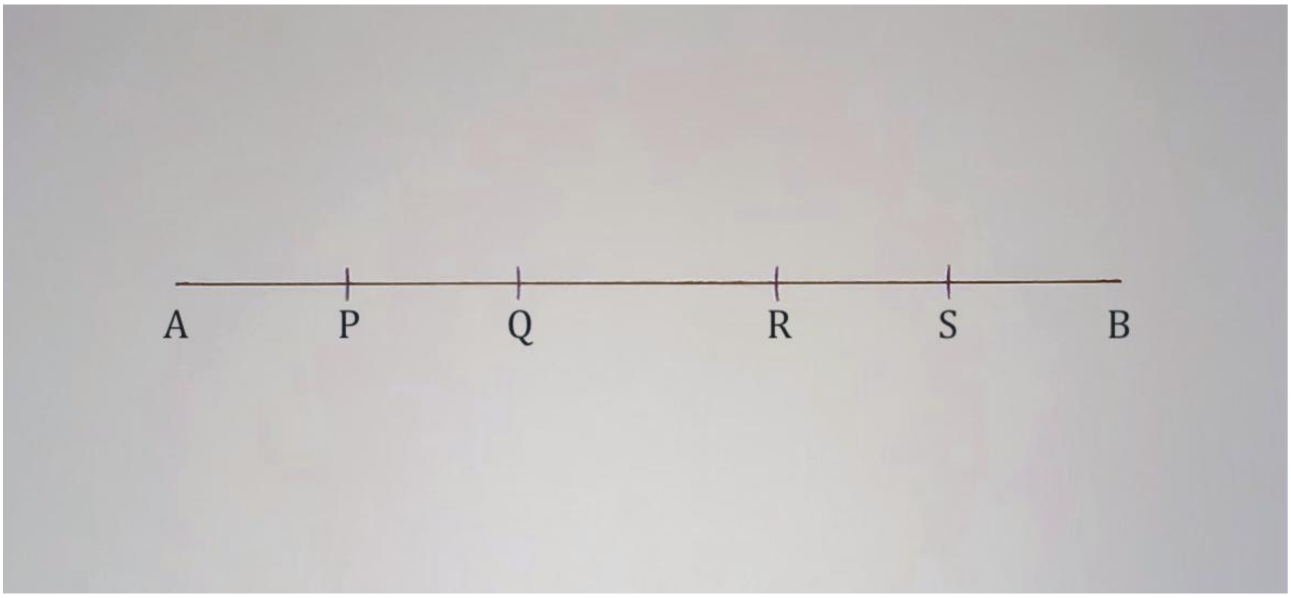

Figure 25. New point of intersection labelled R

Aside. Students may notice at this stage the repetitive nature of the method. Students may also notice that the line segment $\overline{QR}$ is less than twice the radius of the arc template. Thus, as might be predicted, the remainder of this construction follows identically to that outlined previously (Section 3.1) for a 'short' line segment. For completeness the remainder of the construction is included below.

Students can now place the arc template with its centre at Q.

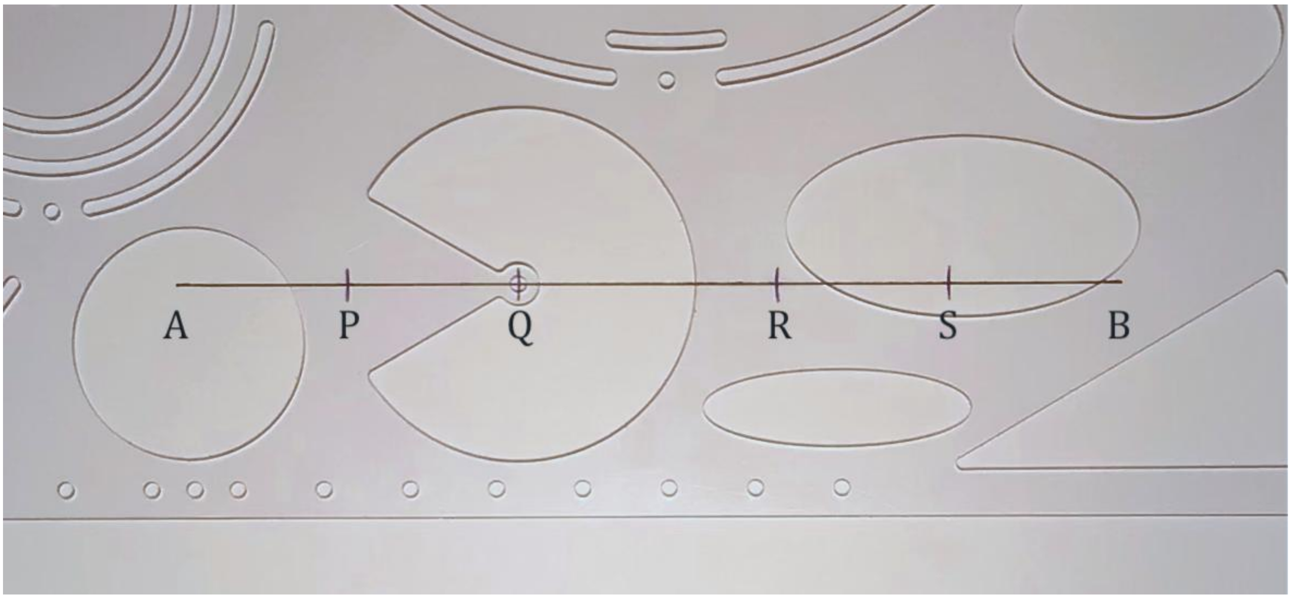

Figure 26. Circle arc template with centre at Q

Students can then construct the arc with centre at Q as shown in Figure 27.

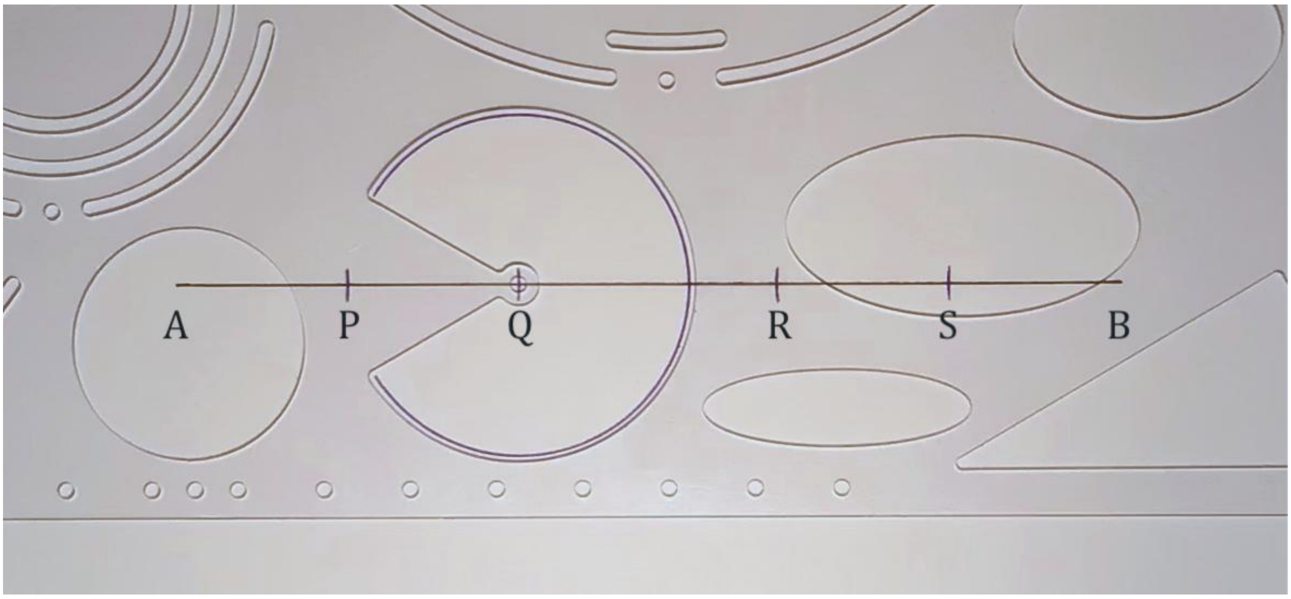

Figure 27. Arc with centre at point Q

Students can now construct a second arc using the circle arc template, centred at R.

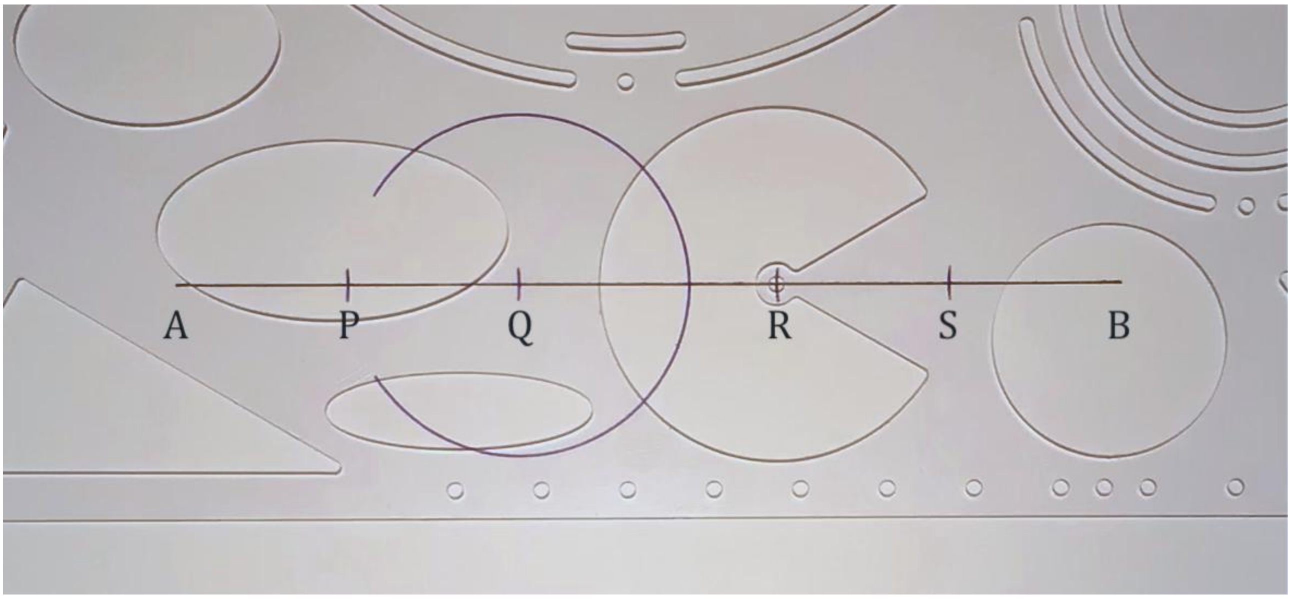

Figure 28. Circle arc template with centre at R

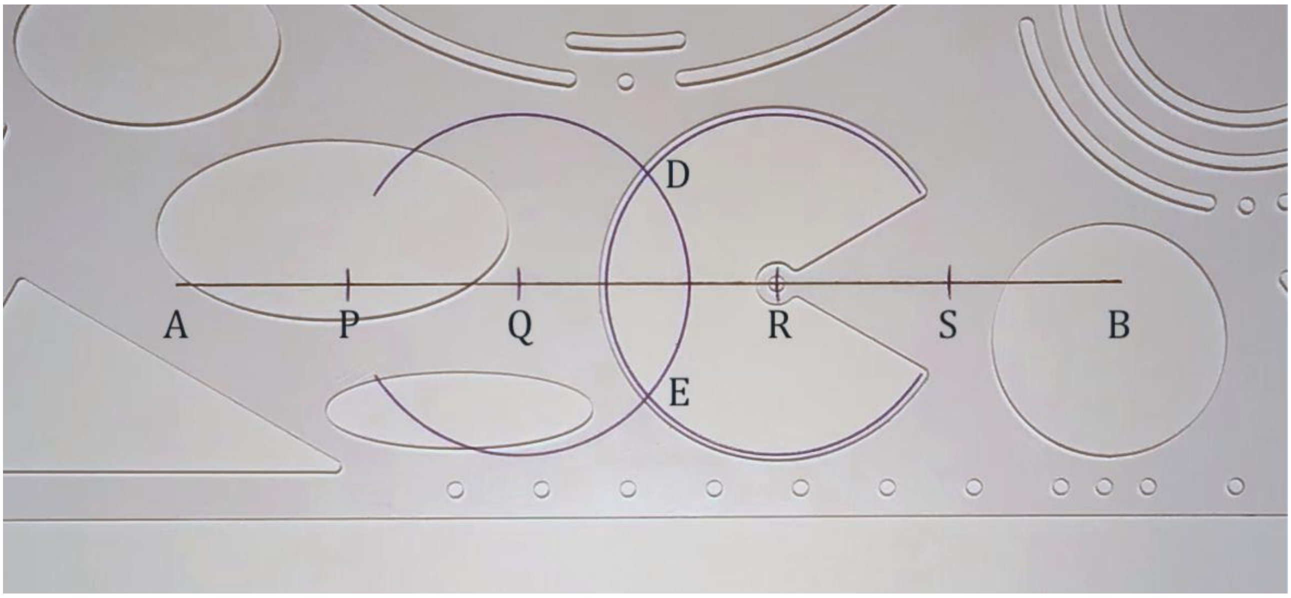

Figure 29. Arc with centre at point R and intersection points labelled

Students can label the two new points of intersection made between the first and second arc, and E (Figure 29).

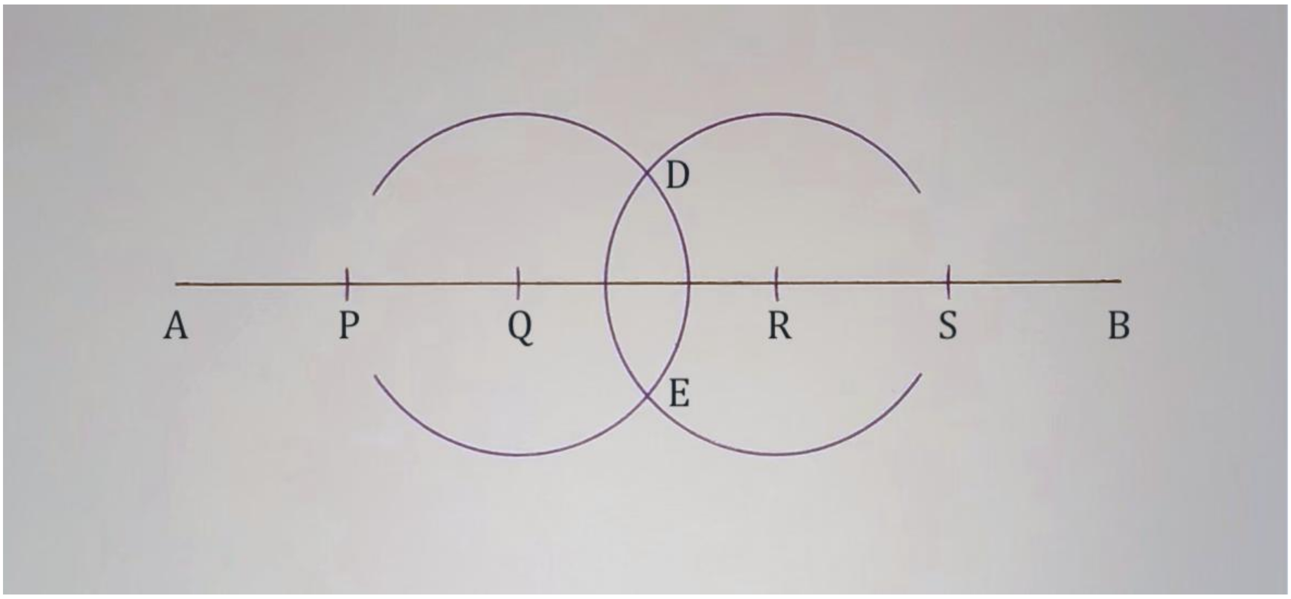

Figure 30. Intersection points D and E

Students can then form the perpendicular bisector to $\overline{AB}$ by constructing the line segment $\overline{DE}$ as seen in Figure 31 and Figure 32.

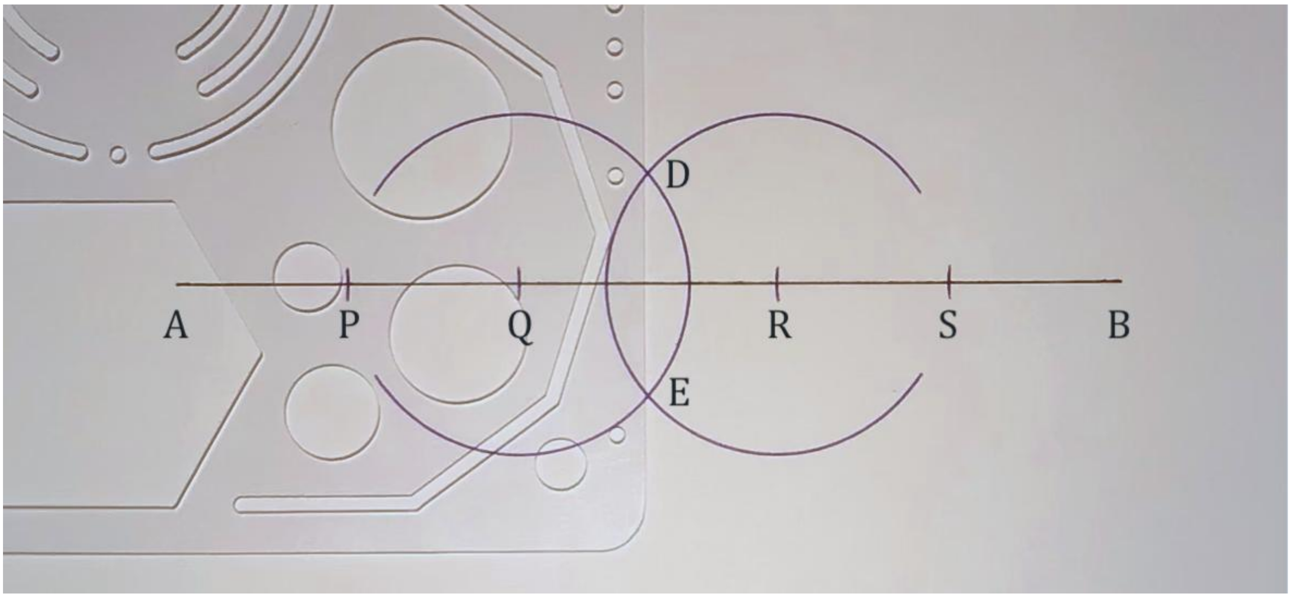

Figure 31. Straightedge aligned with points D and E

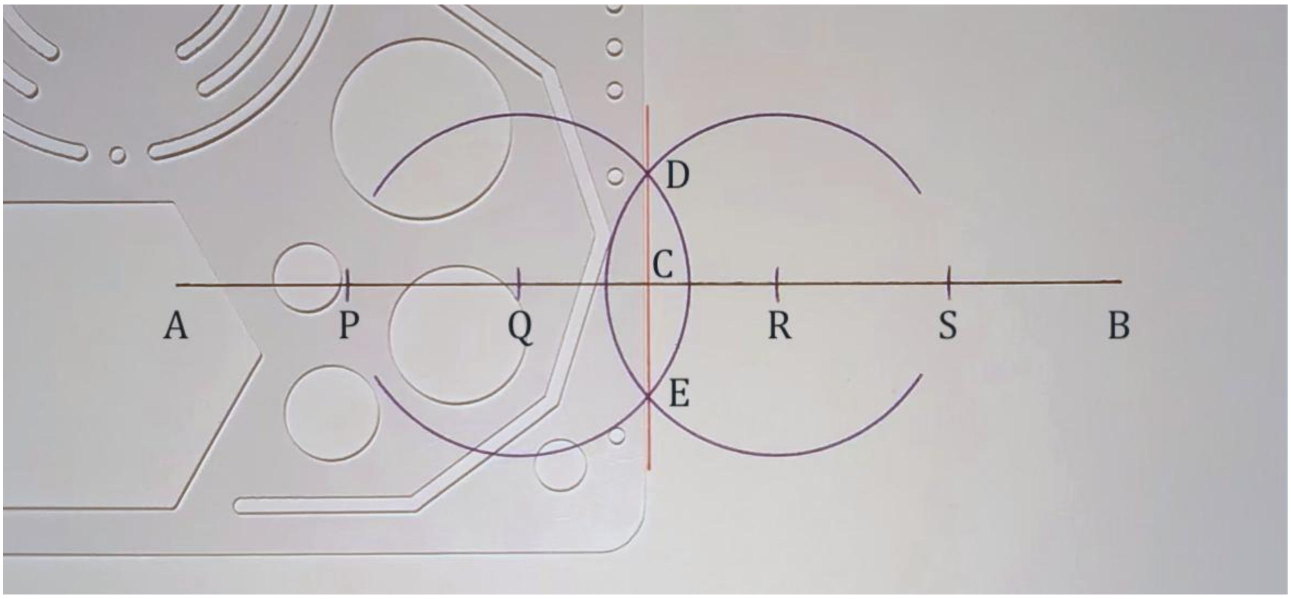

Figure 32. Straightedge aligned with points D and E

Figure 33 shows the finished construction, with $\overline{DE}$ being the desired perpendicular bisector of $\overline{AB}$.

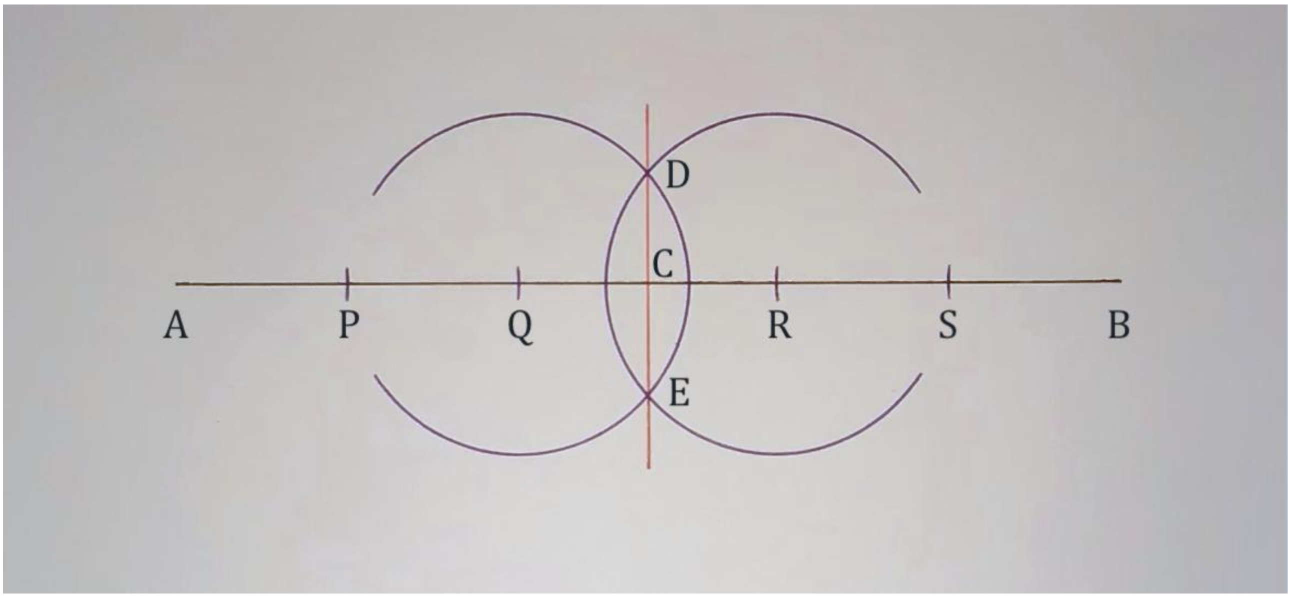

Figure 33. Finished construction. $\overline{DE}$ is the perpendicular bisector of $\overline{AB}$

3.2. Bisecting a given angle

Suppose A, O and B are given points and students are asked to construct the bisector to the angle $\angle AOB$. That is, students are challenged to construct a line segment so as to divide the angle $\angle AOB$ into two equal angles. The initial data is given in Figure 34.

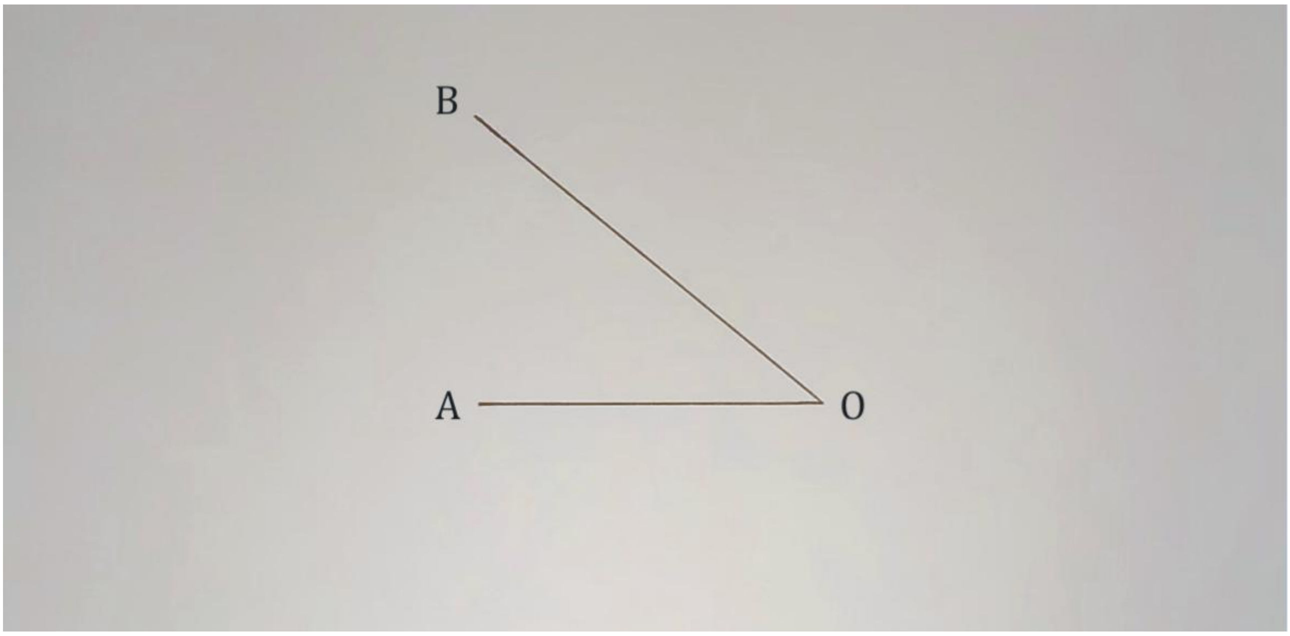

Figure 34. $\angle AOB$

Students can begin this construction by first placing the centre of the circle arc template at O.

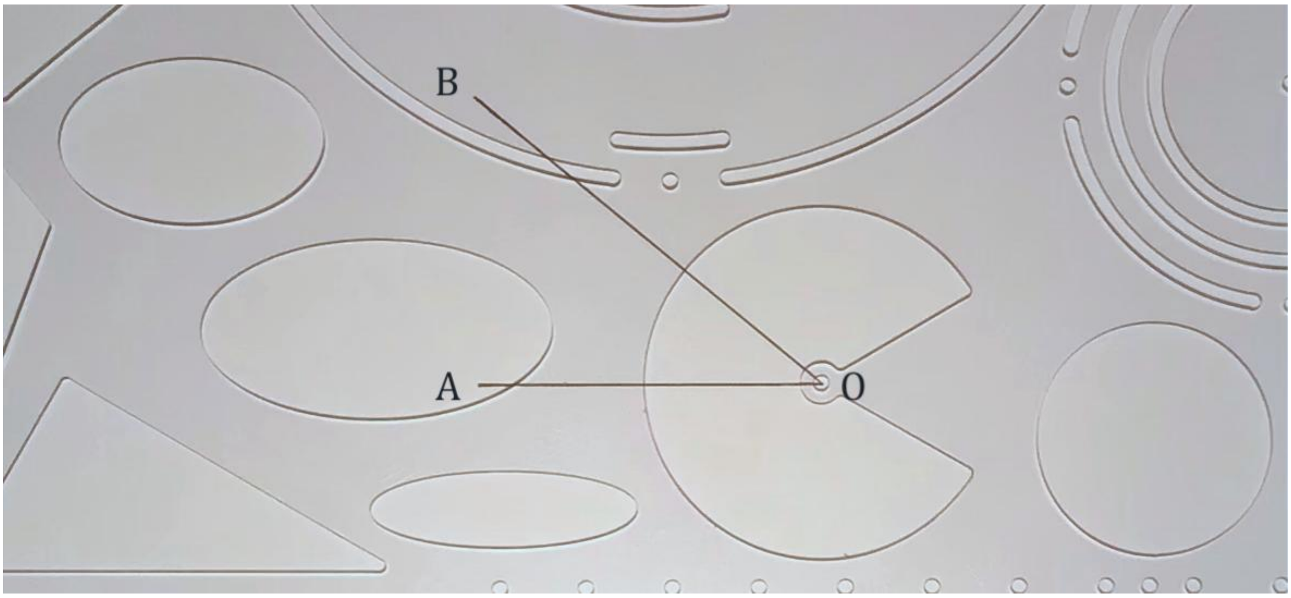

Figure 35. Circle arc template with centre at O

Students can then mark the point at which the arc intersects with the line segment $\overline{OB}$, labelling it P (Figure 36).

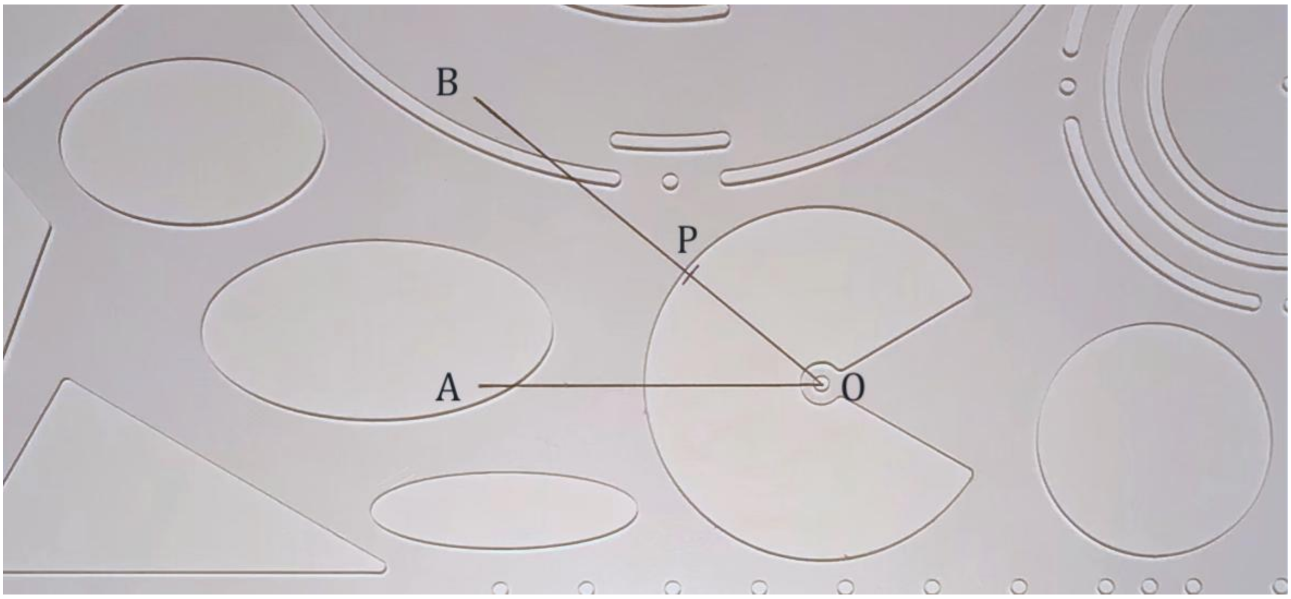

Figure 36. Marking point of intersection, P

Students can repeat the same procedure for point of intersection between the arc and line segment $\overline{OA}$. Students can label this point Q (Figure 37).

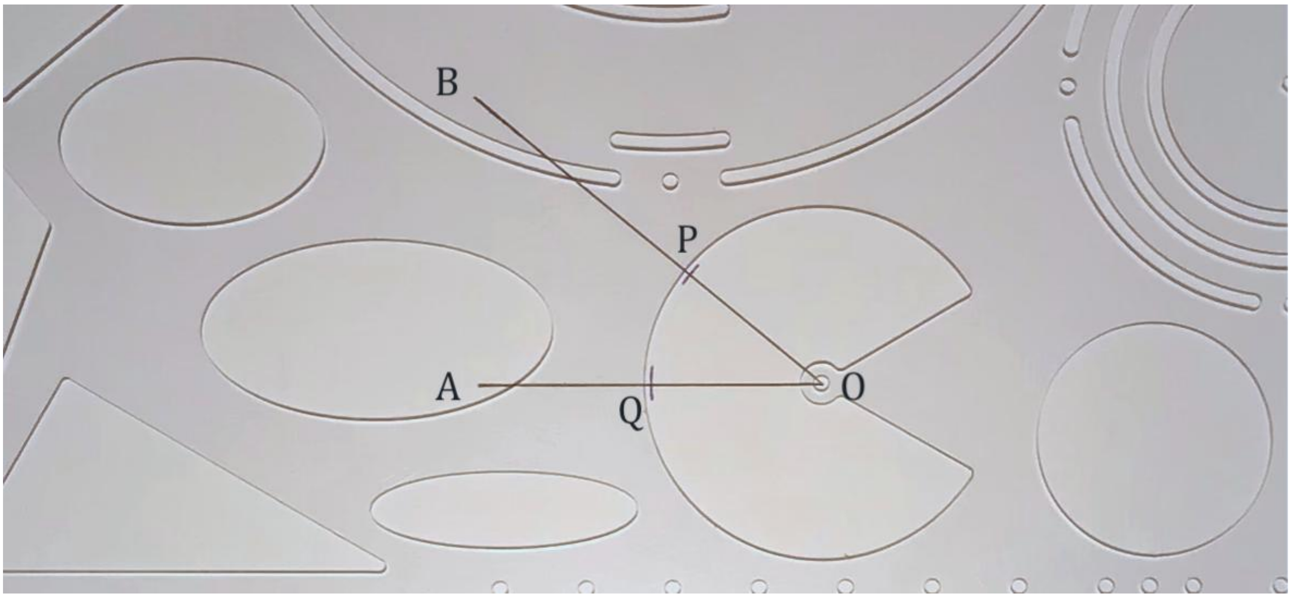

Figure 37. Marking point of intersection, Q

In following this procedure students should be left with the following figure.

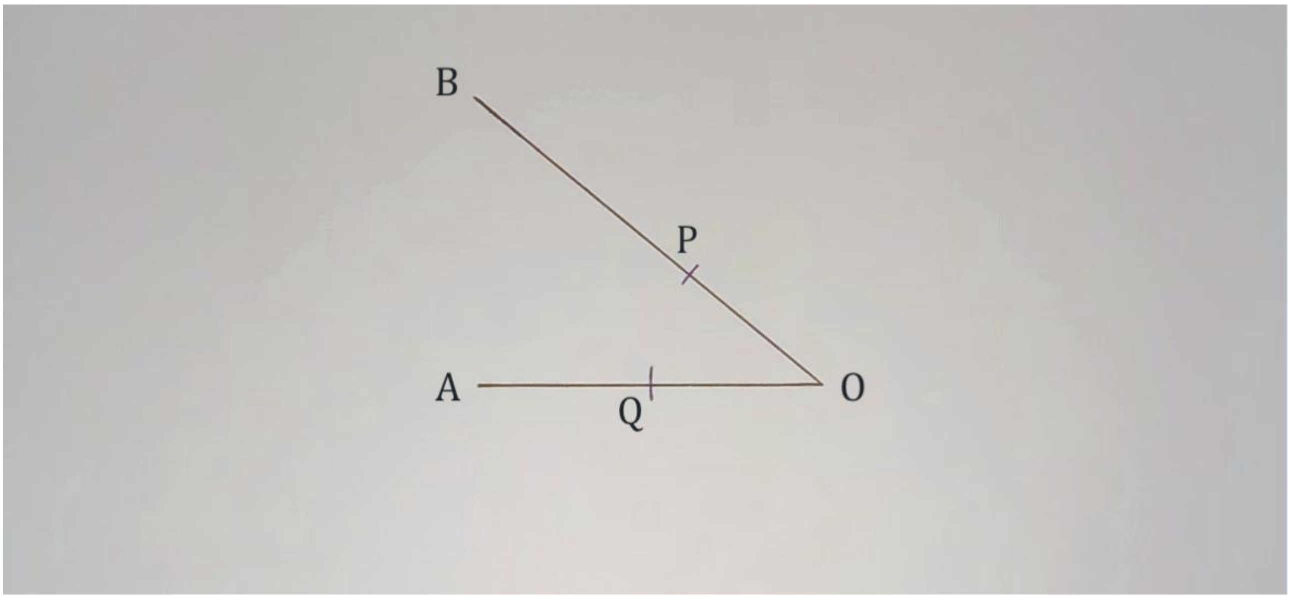

Figure 38. $\angle AOB$ with points P and Q

Students can now place the centre of the arc template at point P.

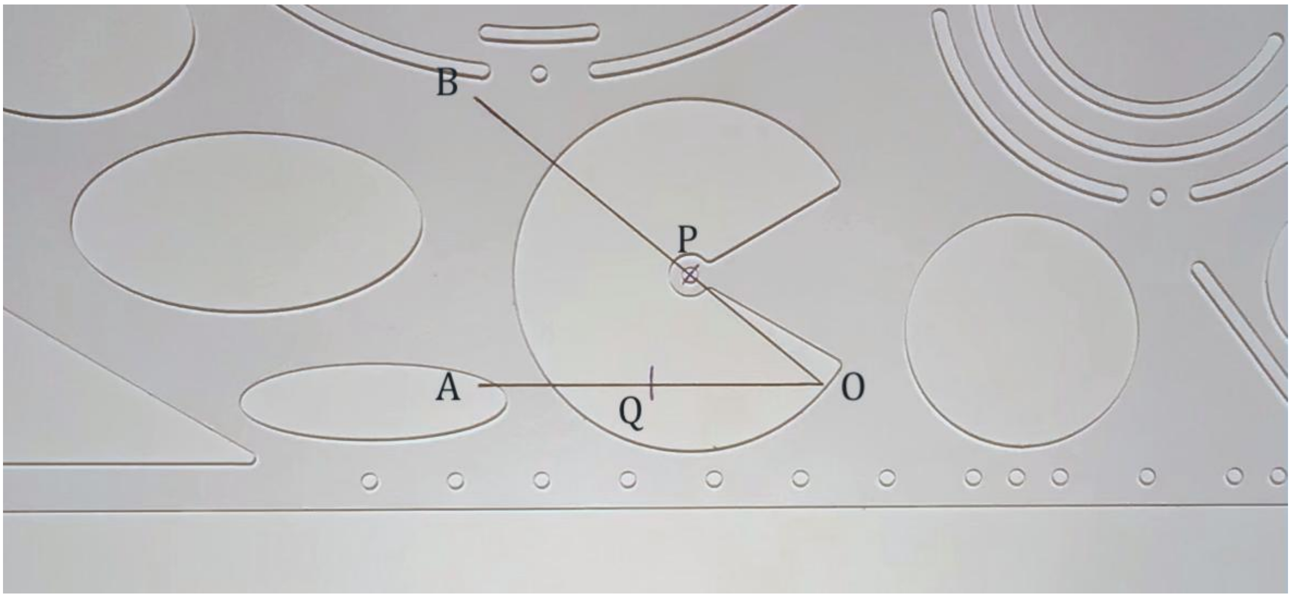

Figure 39. Circle arc template with centre at *P*

Students can now trace a small section of the arc in the proximity of where the same arc with centre at Q is expected to be (Figure 40).

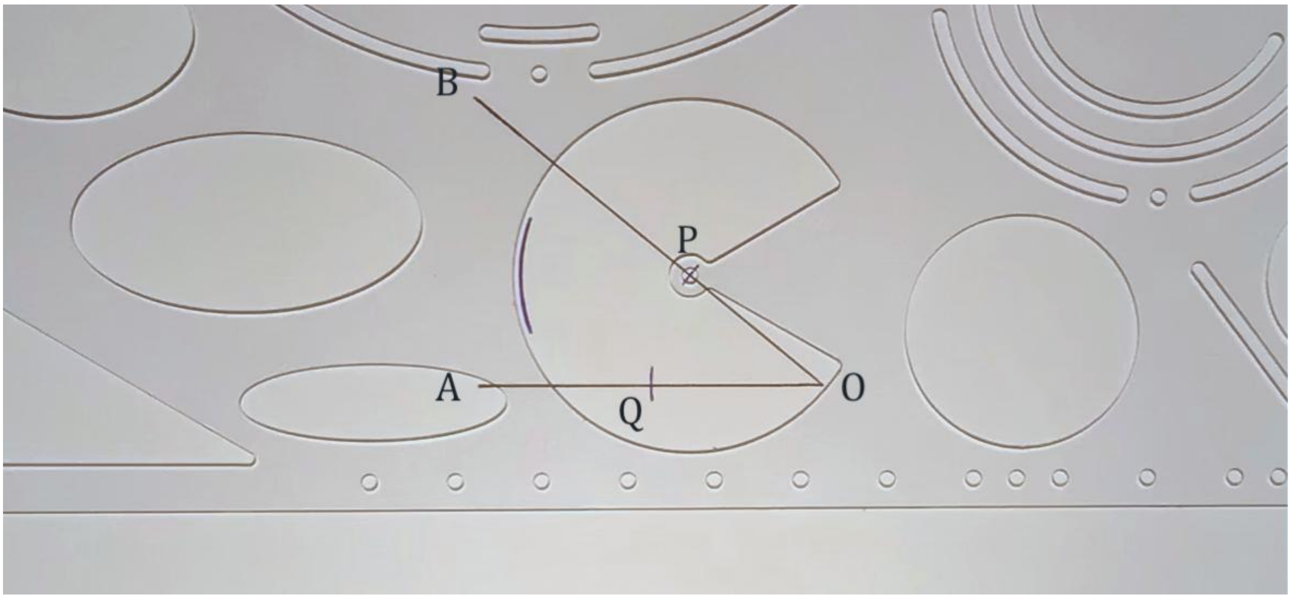

Figure 40. 'Small' section of arc traced, with centre at *P*

Students may now place the centre of the circle arc template at point Q (Figure 41).

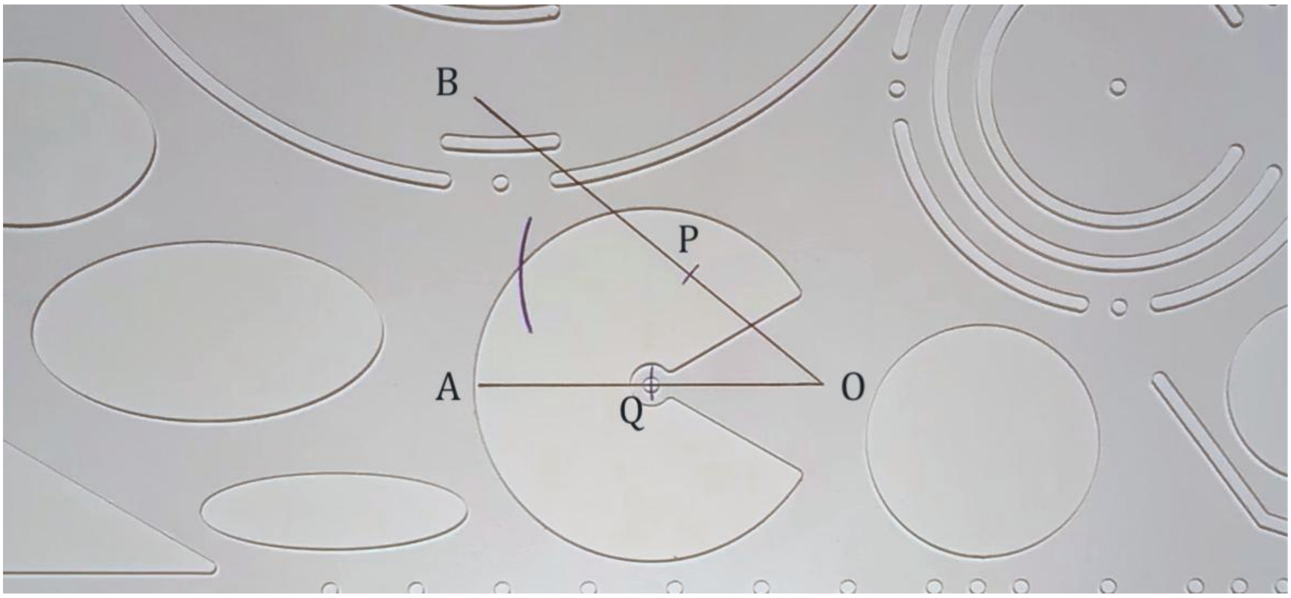

Figure 41. Circle arc template with centre at Q

Students can again trace a small section of the arc, thus constructing a point of intersection with the previous arc. Students can label this point C (Figure 42).

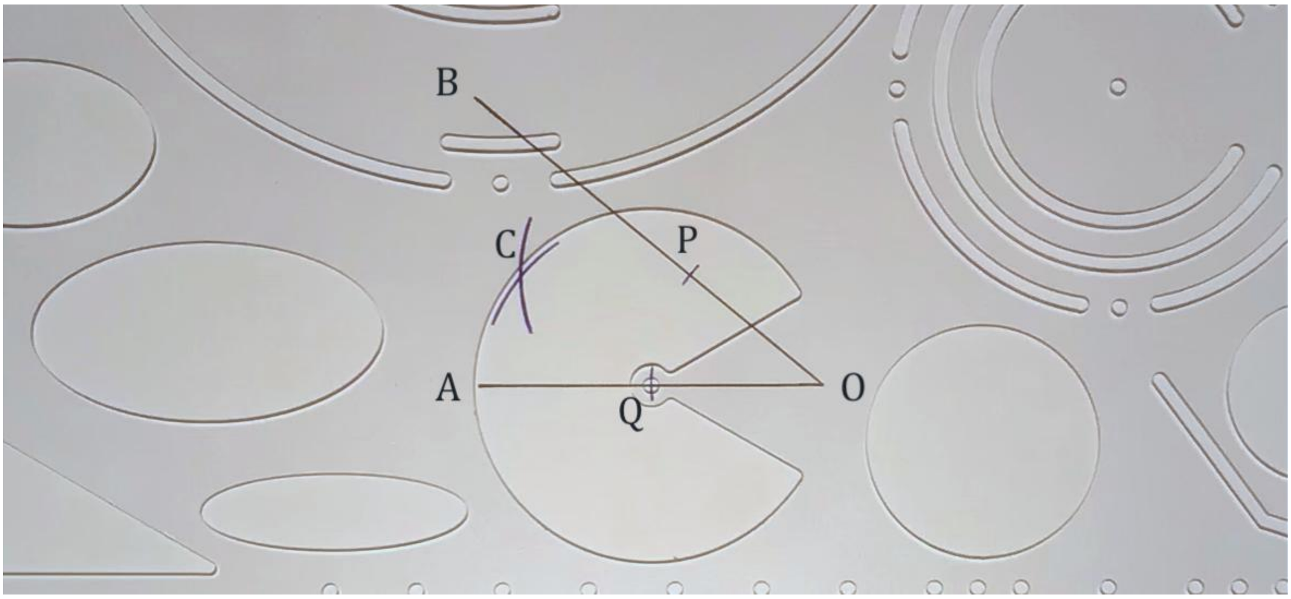

Figure 42. Arc with centre at Q intersecting previous arc to form point C

Students can now align the straightedge with O and C and then construct the line segment $\overline{OC}$ (Figure 43).

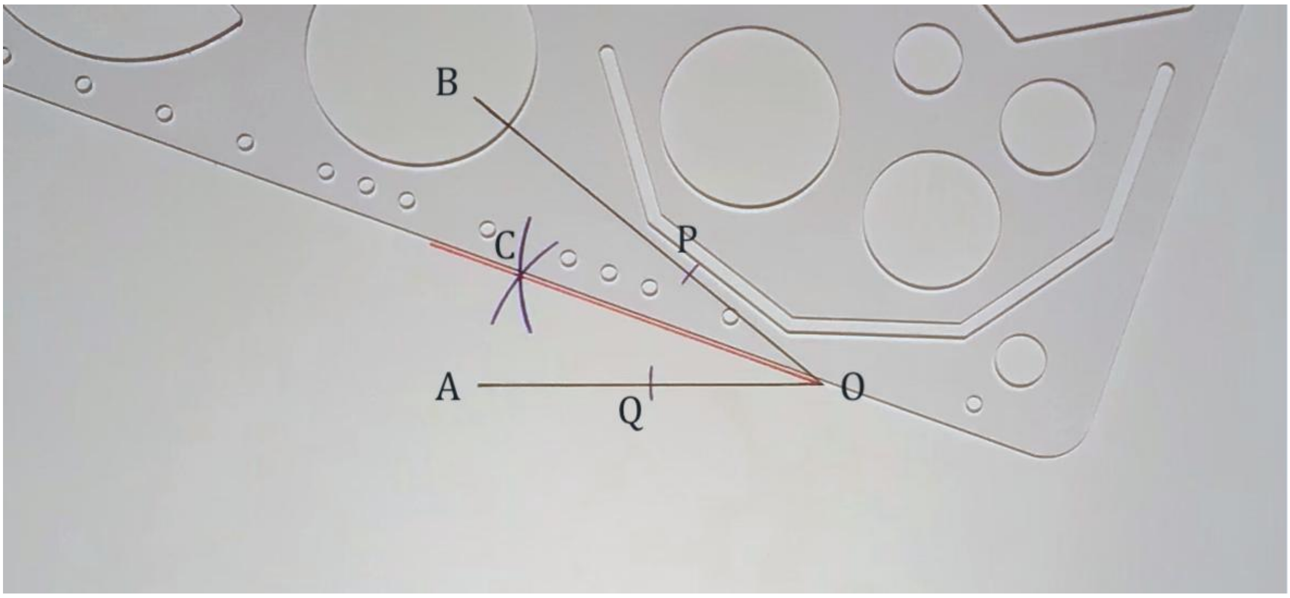

Figure 43. Line segment $\overline{OC}$ is constructed using straightedge

The line segment $\overline{OC}$ is the bisector of the angle $\angle AOB$. Equivalently, it is the case that $\angle AOC = \angle BOC$. This completes the construction.

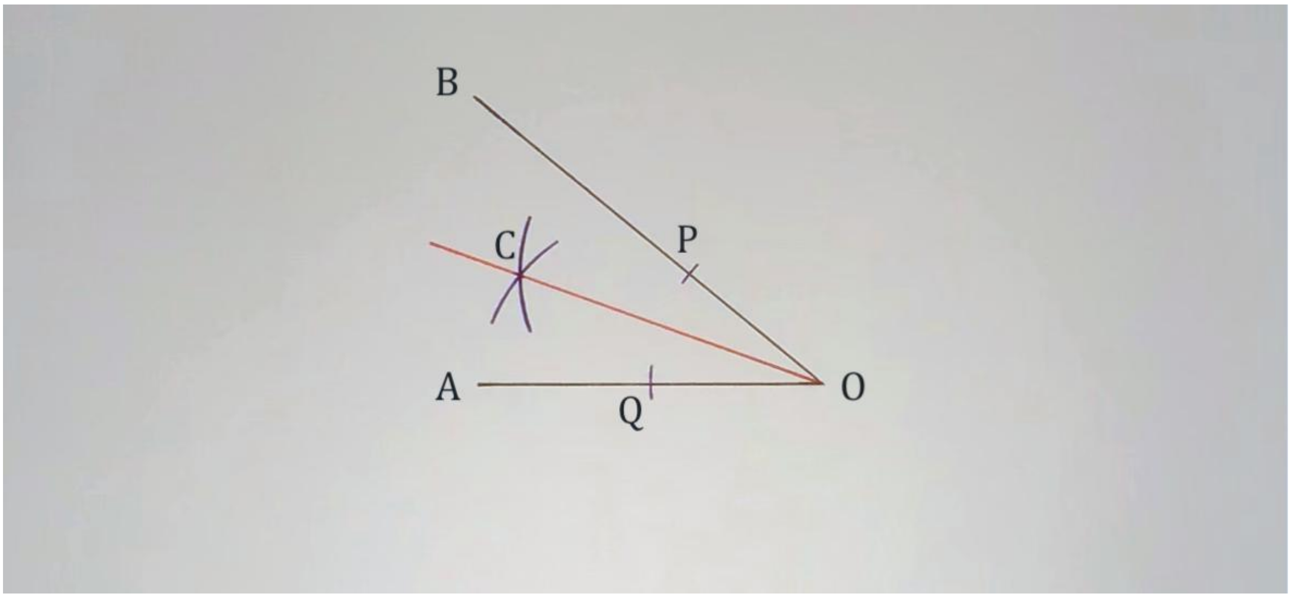

Figure 44. Finished construction. $\overline{OC}$ is the bisector of $\angle AOB$

3.3. Dropping a perpendicular line through a given point

Suppose A, B and P are given points and students are asked to construct a line through P which is perpendicular to $\overline{AB}$. The initial data is given in Figure 45.

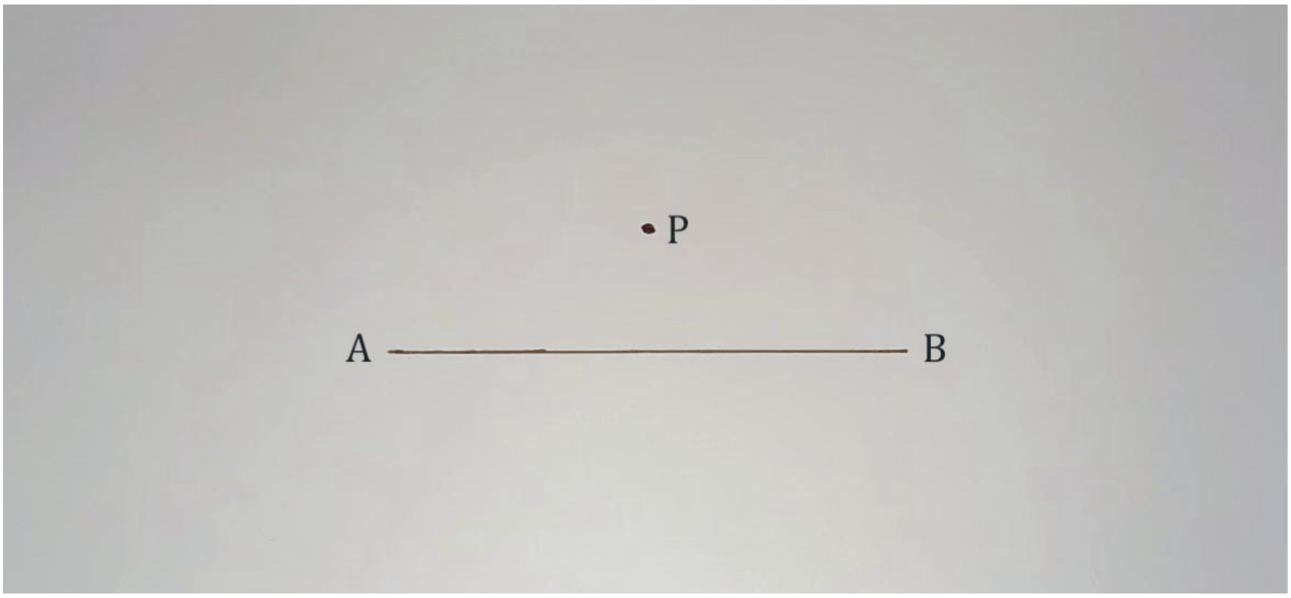

Figure 45. Line Segment $\overline{AB}$ and point P

Students can commence by placing the centre of the arc template at point P. See Figure 46.

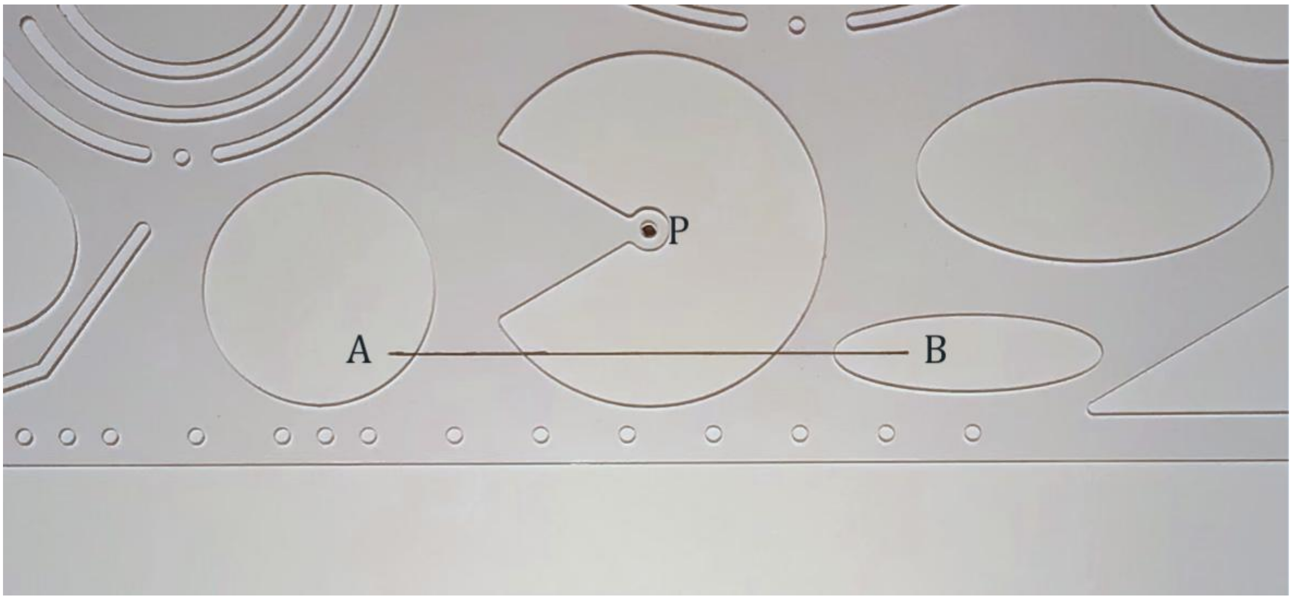

Figure 46. Circle arc template with centre at P

Students can then mark the points at which the arc intersect with the line segment $\overline{AB}$. These points can be labelled Q and R (Figure 47).

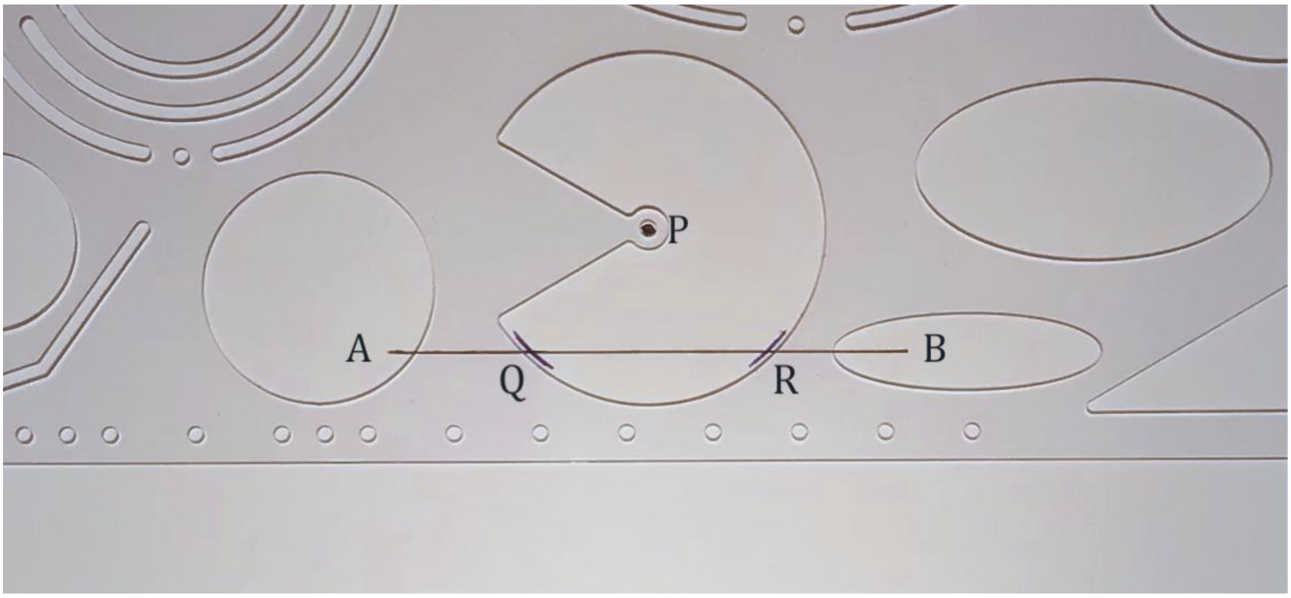

Figure 47. Points Q and R constructed

Students can then position the centre of the arc template at point Q and trace out the arc. Teachers should emphasise here that point P will lie along the arc.

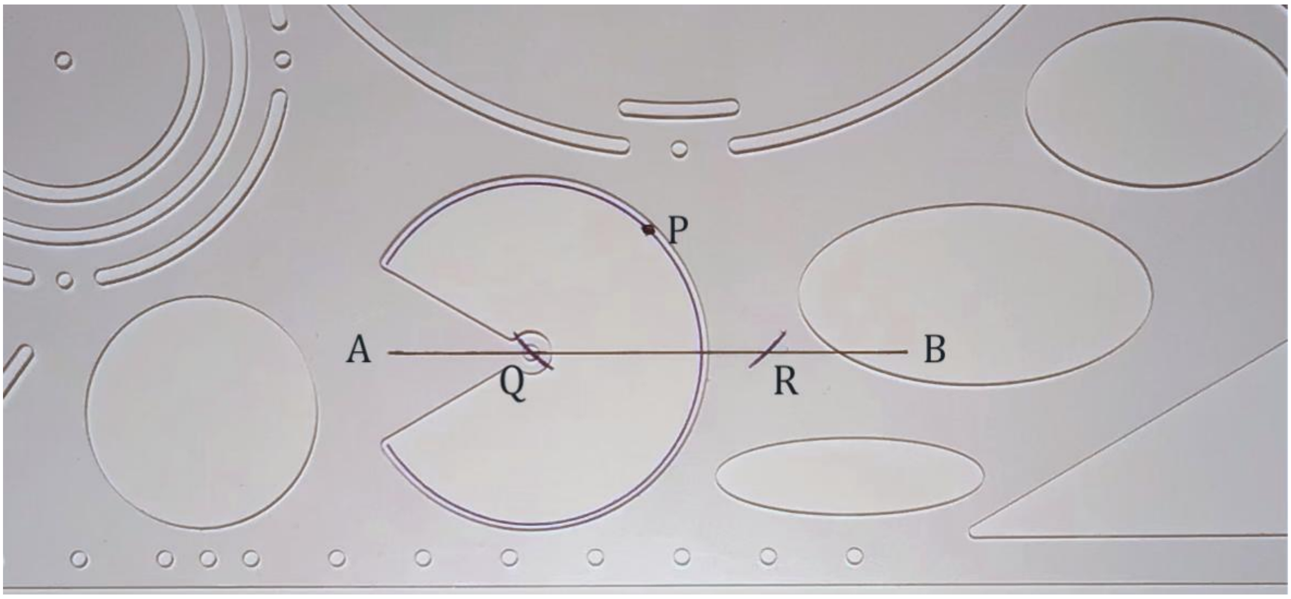

Figure 48. Drawing arc with centre at point Q

Students can now repeat the same process with the centre of the circle arc template at R instead. Again, students should note that the arc will pass through the point P. See Figure 49.

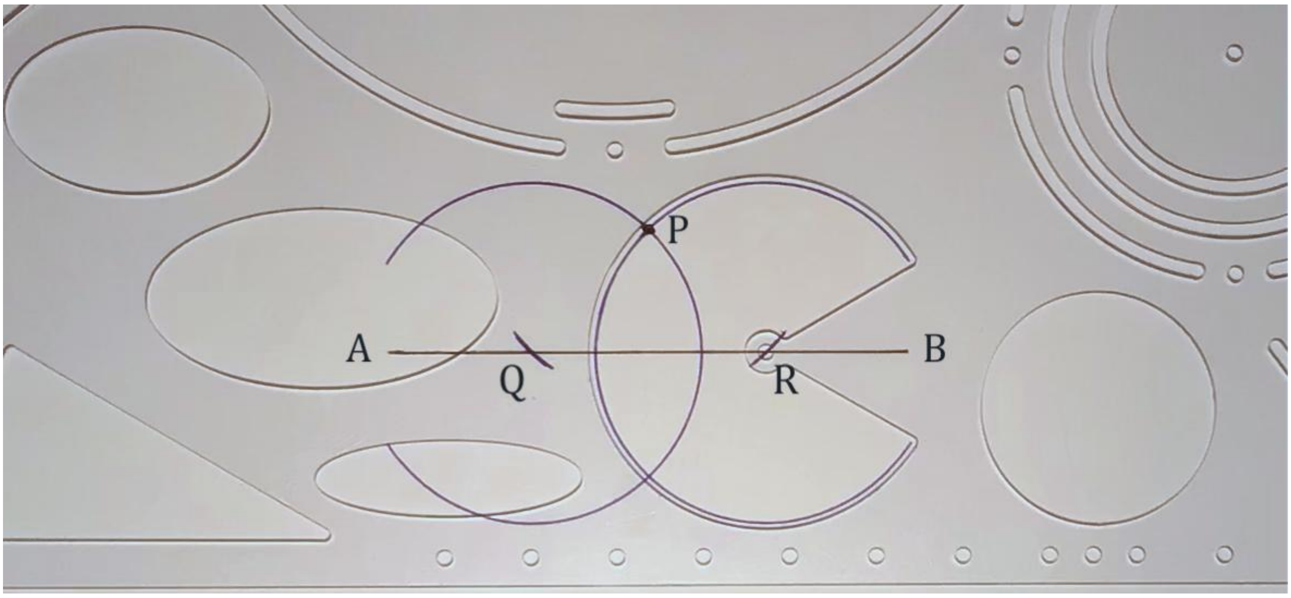

Figure 49. Drawing arc with centre at point *R*

Students can now label the new point of intersection between the first and second arc as *S* (Figure 50).

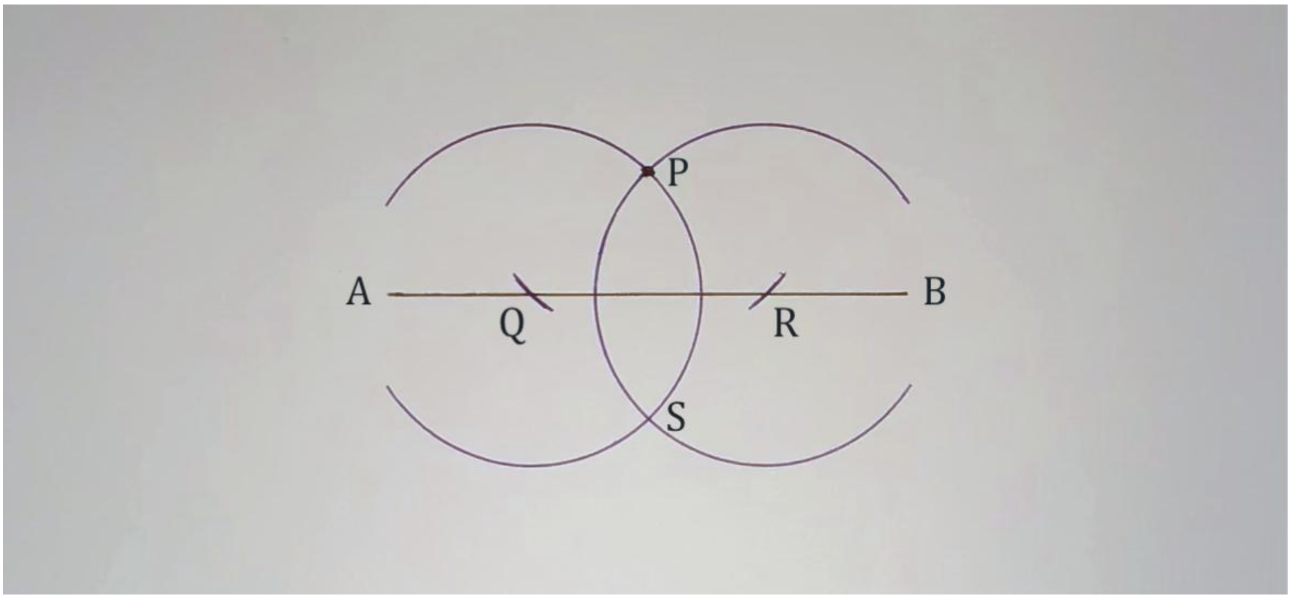

Figure 50. New point of intersection labelled *S*

Students can then construct the line segment $\overline{PS}$ using a straightedge. See Figure 51.

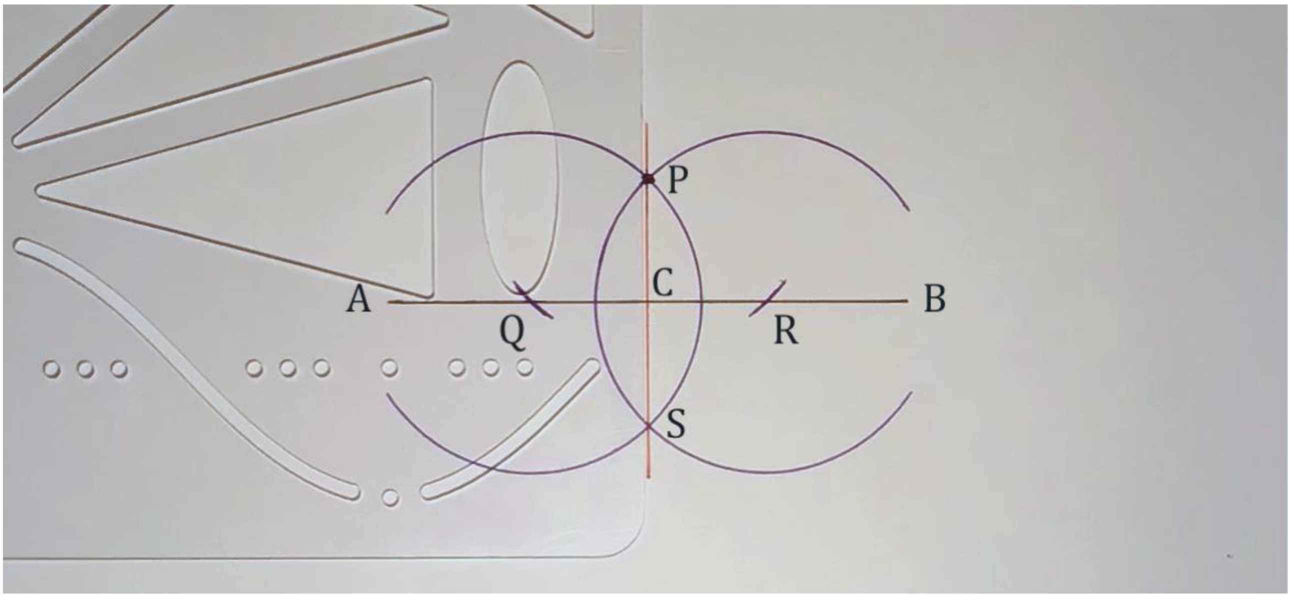

Figure 51. Line segment $\overline{PS}$ is constructed using straightedge

Students can now observe that the line segment $\overline{PS}$ is perpendicular to $\overline{AB}$ and passes through P. This completes the construction.

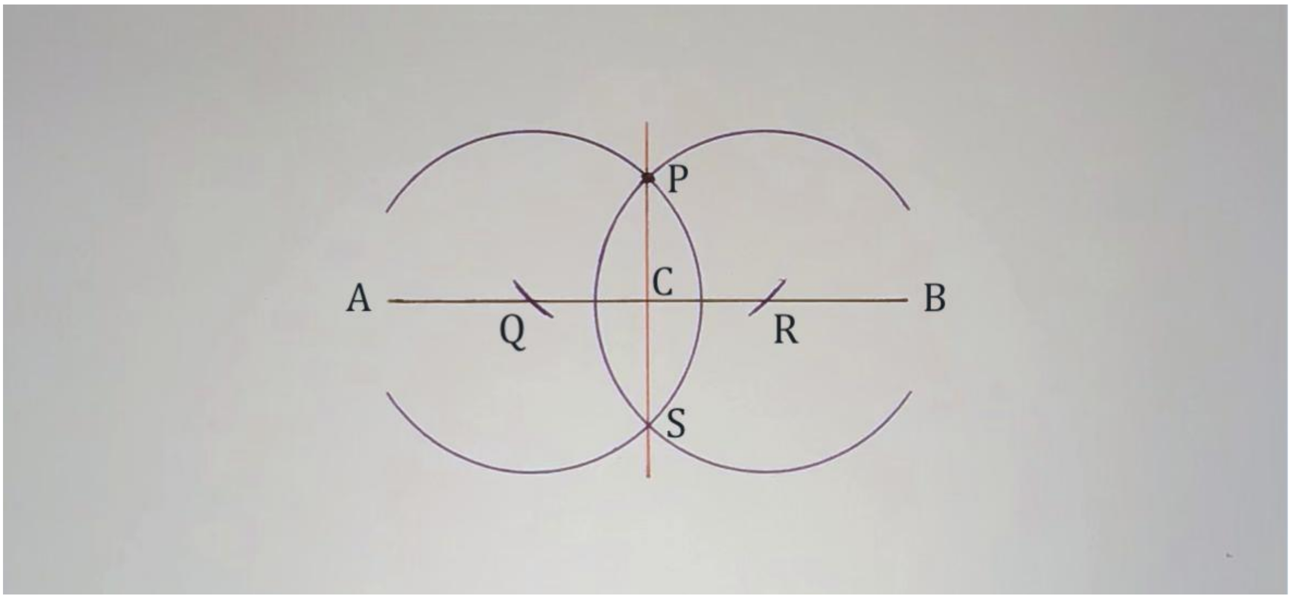

Figure 52. Finished construction. $\overline{PS}$ is perpendicular to $\overline{AB}$

Note: If P is too far from $\overline{AB}$ for the initial arc to intersect, then it is possible for students to construct a suitable line that is parallel to $\overline{AB}$ and close enough to P so that the original construction steps will work, for example, see Figure 4 in Mackay's 1886 paper [22].

This is the last of the geometric constructions that we consider.

4. Discussion and limitations

We are now in a position to address our original research questions.

The first of these questions (RQ1) regarded how a template could be designed and utilised to perform basic geometric constructions. In the previous section we proposed the use of a 'Pac-Man' style circle arc template seen in Figure 2. We saw that considerations of strength, smoothness, a positionable centre point, a fixed radius and fixed arc, and a translucent material were of importance. In addition, our three constructions illustrate how students can utilize the circle arc template in conjunction with a straightedge to perform the geometric constructions seen in high-school curricula in accurate, efficient and safe ways. Teachers and students can now see the potential of the circle arc template to replace the compass for geometric constructions. Let us unpack this a bit more below.

The potential benefits of using this template are mostly concerned with the instrument's ease of use and its ability to make accurate geometric constructions. Concerning this first point, the template was found to confirm all the predicted advantages stated in section 2 of this paper. No instrument-derived errors were found to occur during the generation of the constructions for Section 3. In particular it was found that no excessive amount of pressure was needed to be applied on the template to prevent accidental slippage during the constructions. Another benefit is in the ability to make geometric constructions using only a single writing instrument. Compasses require their own lead tips or use a pencil/pen grip thus requiring a second means of marking the page to complete the geometric constructions. The circle arc template requires no such additional writing instruments.

Concerning the ability to make accurate geometric constructions, the template's fixed radius design ensures that no variation in arc size occurs when repositioning the template. This in effect makes the geometric constructions which require such use of equally sized arcs more accurate. It should be noted that some compasses have the ability to *lock* the opening of the compass arms. This feature tends to be present only in more expensive compass sets, and does not eliminate the risk of the legs flexing while in use.

The design of a new instrument intended to replace the compass should be able to perform the same actions as the compass itself. Whilst the ability to choose the centre of an arc has been shown to be possible using the circle arc template, this new instrument is not completely able to replicate all the abilities of the compass. We treat these limitations individually now, in the context of how such limitations affect the students' capacity to perform the same geometric constructions as they would be able to using a compass.

A limitation is that the circle arc template draws arcs of a single radius, whilst a compass is able to draw arcs of varying radii. Notwithstanding, in the context of the geometric constructions there is more to be said. Although the circle arc template can only draw arcs of a certain radius one must consider that a compass cannot draw arbitrarily large (or small) circles. The compass is limited in how far it can be opened. Thus, for sufficiently large geometric constructions the compass would fare no better than a fixed radius circle arc template. It is for this reason that the variation of the first case involving constructing a perpendicular bisector of a *long* line segment (section 3.1.1 of this work) is not a demonstration of the limitation of the circle arc template. Rather it shows how given *only* a fixed radius circle arc template one is still able to perform the construction, albeit with more steps.

A further limitation is the inability to draw the full circumference of a circle using the template. The circle arc template allows the student to draw arcs of up to 300 degrees. This is in contrast to a compass which allows a full revolution to be drawn. One could rotate the template to complete the circle however this could introduce the opportunity for the template to be repositioned accidentally leading to inaccuracy. It is nonetheless worth mentioning that for the constructions demonstrated in this paper there is no need to be able to draw a full circle; arcs suffice. This is not always the case, for example if one wishes to find the circumcentre of a triangle, the final circumcircle cannot be drawn. The template suffices, however, to construct the centre and the radius of the final circle. Regarding this last point it may be argued that knowing the centre and radius of the final circle is sufficient to judge the construction as complete. If this is not to be conceded, then perhaps a template with many different sized circle arc templates may improve this limitation somewhat. This is a minor improvement given that one cannot have the necessary infinite number of differently sized templates required to fully rectify this problem.

Note that in students bisecting all three angles of a given triangle via our earlier construction and extending these to intersect to form the incentre, students can appreciate how our simple constructions can be synthesized to form more challenging constructions.

This last discussion of the potential benefits and limitations of circle arc templates in their application to geometric constructions directly answers our second research question (RQ2).

5. Conclusion and opportunities

An important outcome of this work is the establishment of new knowledge, and the generation of practical activities that teachers can incorporate into their geometry lessons that empower students to learn geometric constructions in new, effective and safe ways. Through this, teachers and students can become more aware of alternative and illuminating geometrical approaches that can move their understanding beyond the limitations of the traditional compass to circle arc templates. In this sense our work acknowledges and extends that of Gibb [23] and Hlavaty [24] by introducing a new tool into the learning and teaching of geometric constructions.

This work partially addresses the need to offer better alternatives to the compass in the learning and teaching of geometry via circle templates. It lays the necessary foundation for more specific work involving the roles of geometry templates in the learning and teaching of mathematics. Although only a small number of geometric constructions have been demonstrated, the techniques shown in these few examples apply to constructions beyond them. Further work is to be done to extend the catalogue of geometric construction demonstrations using these templates and set of instructional YouTube style videos are planned regarding this.

A further opportunity exists in the form of designing alternative and improved templates so as to combat the limitations found with the circle arc template of this work. One of the main challenges is to overcome the tension between enabling a positionable centre point, a full rotation, and a variable radius.

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