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Case study

The Laplace transform as an alternative general method for solving linear ordinary differential equations

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Abstract: The Laplace transform is a popular approach in solving ordinary differential equations (ODEs), particularly solving initial value problems (IVPs) of ODEs. Such stereotype may confuse students when they face a task of solving ODEs without explicit initial condition(s). In this paper, four case studies of solving ODEs by the Laplace transform are used to demonstrate that, firstly, how much influence of the stereotype of the Laplace transform was on student's perception of utilizing this method to solve ODEs under different initial conditions; secondly, how the generalization of the Laplace transform for solving linear ODEs with generic initial conditions can not only break down the stereotype but also broaden the applicability of the Laplace transform for solving constant-coefficient linear ODEs. These case studies also show that the Laplace transform is even more robust for obtaining the specific solutions directly from the general solution once the initial values are assigned later. This implies that the generic initial conditions in the general solution obtained by the Laplace transform could be used as a point of control for some dynamic systems.

Keywords: Laplace transform, linear ordinary differential equations, initial value problem, generic initial conditions, convolution, engineering mathematics

1. Introduction

The Laplace transform, a family of integral transforms, is a popular approach in solving ordinary differential equations (ODEs) and applications in science and engineering [1-7]. By transferring a differential equation (DE) in the time domain to the s-domain of complex frequency (or the state space) by the Laplace transform, linear ODEs and systems of linear ODEs, particularly the constant-coefficient linear ODEs, can be manipulated by algebraic operations in the s-domain for purposefully oriented processing (or filtering in engineering terms) before transferring the manipulated formula in the s-domain back to the time domain. Furthermore, the Laplace transform is particularly useful in system control and automation owing to its ability to deal with piece-wise and periodic control functions [1,8-10], with which the traditional methods would be difficult to deal.

In engineering, many dynamic systems are represented by differential equations to describe the real-world scenarios in the time domain. Engineers often modify or adjust some parameters of a system (or a DE) to obtain the desired outcome from the system. Such actions can be taken in the state space because of the relatively simpler algebraic operations or matric operations in the s-domain through the Laplace transform [10-12].

For example, given the linear ODEs $m\frac{d^2x}{dt^2}+kx=f(t)$ that describes the responses of a simple spring-mass system (on the left) to the external force (on the right), x can be regarded as the output from the system and $f(t)$ can be regarded as the input to the system. In the time domain, this system is complicated because the input and output are linked together through a second-order ODE. However, in the s-domain, this linear ODE is transferred into an algebraic equation as

$$X(s)=G(s)F(s)\longrightarrow G(s)=\frac{X(s)}{F(s)}, \tag{1}$$

where $X(s)=L[x(t)]$, $F(s)=L[f(t)]$ and $G(s)$ are the output, input and transfer function of this system in the s-domain. The transfer function $G(s)$ defined by the ratio of the Laplace transform of the output $X(s)$ to the Laplace transform of the input $F(s)$ varies depending on individual ODEs or systems. In other words, each transfer function represents a different dynamic system. By modifying the transfer function, engineers can obtain the desired output using the known input. This relationship can be shown using a single block diagram in Figure 1 [1,9,10,13]. The solution or output in the time domain can be obtained by the inverse Laplace transform

$$x(t)=L^{-1}\left[X(s)\right]=L^{-1}\left[G(s)F(s)\right]. \tag{2}$$

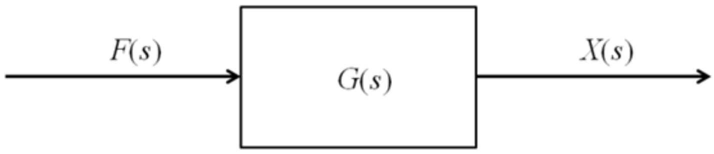

Figure 1. A system represented by a single block diagram in the state space

This seems like a simple process for the Laplace transform. However, in engineering education, the Laplace transform has been regarded as a highly difficult technique for the teachers to teach and for the students to learn [14,15]. Furthermore, in many advanced engineering mathematics textbooks, the Laplace transform is specified as a technique best suited for solving the initial value problem (IVP) of ODEs as it naturally embeds the initial values in the transfer process [1,10,13,16]. A very few textbooks [e.g., 17,18] introduced the Laplace transform through a generic example, but it was immediately followed by more IVP examples and applications. In a similar way, many academic publications involving the Laplace transform were also full of solving IVPs of ODEs [3,19-21]. Such a focused presentation is indeed highlighted the advantage of the Laplace transform but this stereotype also confuses students when they face a task to solve ODEs without explicit initial condition(s), such being demonstrated by the student's experience in the following sections in this article.

In this paper, after outlining the differences between using the conventional method and the Laplace transform to solve a resistor-inductor (*RL*) circuit in Section 2, Section 3 presents two cases of solving ODEs using the Laplace transform demonstrated by undergraduate engineering students at a regional university in Australia. This aims to demonstrate how much influence of the stereotype of the Laplace transform on student's perception of utilizing this method to solve ODEs under different conditions. In Section 4, the generalization of the Laplace transform for solving linear ODEs with generic initial values is presented using Case 1 and Case 2 appeared in Sections 2 and 3, and a new case of system of linear ODEs, aiming to not only break down the stereotype but also broaden the applicability of the Laplace transform for solving linear ODEs, practically the constant-coefficient linear ODEs. Brief discussion and conclusion are made in Section 5.

At many regional universities, students enrolled in STEM programs are diverse in ages, mathematical abilities, study modes, time availabilities, and levels of commitment to their learning due to various reasons. Hence, many students prefer a full and detailed presentation of solving a mathematical problem, even some steps seem too basic and unnecessary for students studying at prestigious or metropolitan universities. Since all the cases here are prepared to serve the needs of the diverse student cohorts at many regional universities as the bottom line, the essence and the step-by-step processes for solving a problem are all included to provide all detailed information to meet the needs of such students.

2. Modelling a *RL* circuit with an explicit initial value by two different methods

Let us begin with the case of modelling a *RL* circuit with an initial value by the conventional method and the Laplace transform, respectively.

Case 1: A series *RL* circuit with source voltage or input $f(t) = E_0 \sin \omega t$ and an initial value $i(0) = 0$ ampere is shown in Figure 2, in which E_0 and ω are the amplitude and angular frequency of the voltage supply, respectively. The current $i(t)$ in the circuit is described by the first-order linear ODE

$$\frac{di}{dt} + \frac{R}{L} i = \frac{E_0}{L} \sin \omega t , \tag{3}$$

where R and L are the resistance and inductance, respectively. Solve this ODE for $i(t)$ and model the electrical current in the circuit.

2.1. The conventional method

In the ODE (3), let $a = R/L$ and $b = E_0/L$. This ODE can be rewritten as

$$\frac{di}{dt} + ai = b \sin \omega t . \tag{4}$$

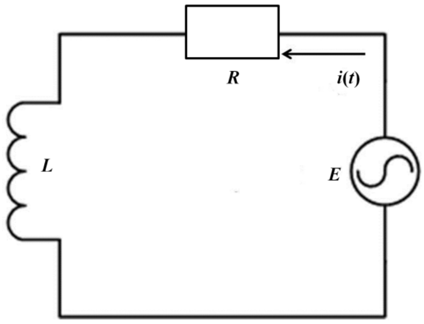

Figure 2. The series *RL* circuit with input $f(t) = E = E_0 \sin \omega t$

Since this is a first-order linear ODE in the standard form with $P(t) = a$ and $Q(t) = b\sin\omega t$, its general solution can be obtained using the explicit formula in [22] or [23]. Note the asterisk indicates that the details can be found in the Appendix of this article.

$$i(t) = ce^{\int -P(t)dt} + e^{\int -P(t)dt}\int Q(t)e^{\int P(t)dt}dt = ce^{\int -adt} + e^{\int -adt}\int b\sin(\omega t)e^{\int adt}dt = ce^{-at} + e^{-at}\int be^{at}\sin(\omega t)dt$$

$$= e^{-at}\left(c + \int be^{at}\sin(\omega t)dt\right) = e^{-at}\left[c + \frac{b}{a^2+\omega^2}e^{at}(a\sin\omega t - \omega\cos\omega t)\right]^*$$

$$= ce^{-at} + \frac{b}{a^2+\omega^2}(a\sin\omega t - \omega\cos\omega t) = ce^{-\frac{R}{L}t} + \frac{\frac{E_0}{L}}{\left(\frac{R}{L}\right)^2 + \omega^2}\left(\frac{R}{L}\sin\omega t - \omega\cos\omega t\right)$$

$$= ce^{-\frac{R}{L}t} + \frac{\frac{E_0}{L}}{\left(\frac{R}{L}\right)^2 + \frac{L^2\omega^2}{L^2}}\left(\frac{R}{L}\sin\omega t - \frac{L\omega}{L}\cos\omega t\right) = ce^{-\frac{R}{L}t} + \frac{\frac{E_0}{L^2}}{\frac{R^2+L^2\omega^2}{L^2}}(R\sin\omega t - L\omega\cos\omega t)$$

$$= ce^{-\frac{R}{L}t} + \frac{E_0}{R^2+\omega^2L^2}(R\sin\omega t - \omega L\cos\omega t).$$

or

$$i(t) = ce^{-\frac{R}{L}t} + \frac{E_0}{R^2+\omega^2L^2}(R\sin\omega t - \omega L\cos\omega t). \quad (5)$$

This is the general solution to the *RL* circuit. The first part is the response of the homogeneous ODE. This response i_h approaches zero as $t \to \infty$. The second part is the particular response i_p to the input that varies with time.

To find the specific solution satisfying the initial value $i(0) = 0$, substitute $t = 0$ and $i(0) = 0$ into the general solution (5).

$$0 = ce^{-\frac{R}{L}\times 0} + \frac{E_0}{R^2+\omega^2L^2}(R\sin 0 - \omega L\cos 0) \longrightarrow 0 = c + \frac{E_0}{R^2+\omega^2L^2}(0 - \omega L)$$

$$0 = c + \frac{-\omega LE_0}{R^2+\omega^2L^2} \longrightarrow c = \frac{\omega LE_0}{R^2+\omega^2L^2}.$$

Thus, the specific solution is

$$i_s = \frac{E_0}{R^2 + \omega^2 L^2}(\omega L e^{-\frac{R}{L}t} + R\sin\omega t - \omega L\cos\omega t)\,. \tag{6}$$

The separation between the general solution and the initial condition requires extra steps to get a specific solution for a given initial condition. However, such separation keeps the general solution as a universal module for the ODE regardless of what the initial condition(s) would be applied with respect to different circumstances.

Figure 3 shows the responses of the homogeneous solution i_h, the particular solution i_p and the combined solution i_s with time with $R = 10\ \Omega$, $L = 5$ H, $\omega = 2$, and $E_0 = 10$ V. It clearly shows that the current is dependent on the input after the initial period, in which the influence of the i_h component fades out quickly. Therefore, the response i_h correlating to the homogeneous ODE is called the ***transient*** current whereas the particular response correlating to the input in the inhomogeneous ODE (3) is called the ***steady-state*** current.

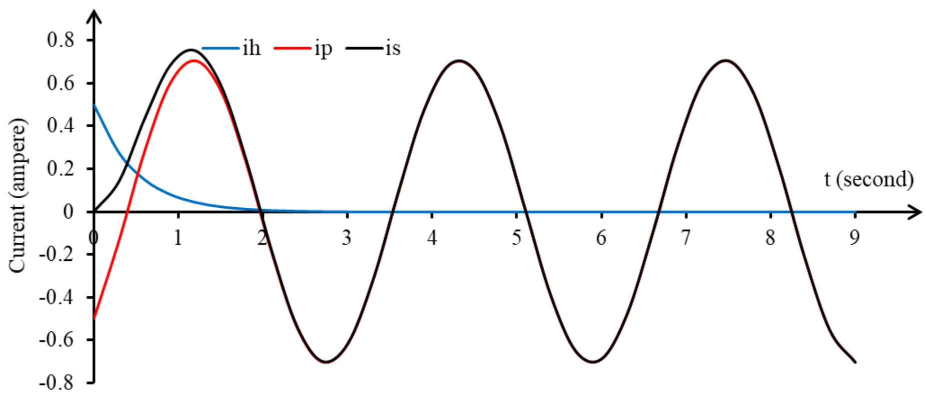

Figure 3. Electric currents in the *RL* circuit with $R = 10\ \Omega$, $L = 5$ H, $\omega = 2$, and $E_0 = 10$ V

2.2. The Laplace transform

Apply the Laplace transform to both sides of the ODE (4)

$$L\left[\frac{di}{dt}\right] + aL[i] = bL[\sin\omega t]\,. \tag{7}$$

Let $I(s) = L[i]$. This translates the ODE to its state space form

$$sI(s) - i(0) + aI(s) = bL[\sin\omega t] \tag{8}$$

$$(s+a)I(s) = \frac{b\omega}{s^2+\omega^2} \longrightarrow I(s) = \frac{1}{s+a}\frac{b\omega}{s^2+\omega^2} \longrightarrow I(s) = G(s)F(s)\,.$$

Here the input, transfer function, and the output in the state space are

$$F(s) = \frac{b\omega}{s^2+\omega^2},\quad G(s) = \frac{1}{s+a},\quad I(s) = \frac{1}{s+a}\frac{b\omega}{s^2+\omega^2}\,.$$

The corresponding functions in the time domain can be found by the inverse Laplace transform.

$$f(t) = L^{-1}\left[F(s)\right] = L^{-1}\left[\frac{b\omega}{s^2+\omega^2}\right] = b\sin\omega t, \quad g(t) = L^{-1}\left[G(s)\right] = L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$$

$$i(t) = L^{-1}\left[I(s)\right] = L^{-1}\left[G(s)F(s)\right] = g(t)*f(t).$$

Here $g(t)*f(t)$ is the convolution between the two functions. By convolution, the output in the time domain can be determined as follows.

$$\begin{aligned}
i(t) &= L^{-1}[G(s)F(s)] = g(t)*f(t) = e^{-at}*b\sin(\omega t) = b\int_0^t e^{-a(t-p)}\sin(\omega p)dp \\
&= be^{-at}\int_0^t e^{ap}\sin(\omega p)dp = be^{-at}\int_0^t \sin(\omega p)e^{ap}dp = \frac{be^{-at}}{a^2+\omega^2}\left[e^{ap}\left(a\sin(\omega p)-\omega\cos(\omega p)\right)\right]\Big|_0^{t*} \\
&= \frac{be^{-at}}{a^2+\omega^2}\left[e^{at}(a\sin\omega t-\omega\cos\omega t)-e^0(a\sin 0-\omega\cos 0)\right] \\
&= \frac{b}{a^2+\omega^2}\left(\omega e^{-at}+a\sin\omega t-\omega\cos\omega t\right) \longleftarrow \quad a=\frac{R}{L}, b=\frac{E_0}{L} \\
&= \frac{E_0}{R^2+\omega^2L^2}(\omega L e^{-\frac{R}{L}t}+R\sin\omega t-\omega L\cos\omega t).
\end{aligned}$$

This is the same outcome as obtained by the traditional method. Since the initial condition $i(0) = 0$ was embed in the equation (8) during the process, the result from the inverse Laplace transform is the specific solution satisfying this specific initial condition only. This process seems relatively simpler if students know the convolution well. Hence, many students may have a misunderstanding that the Laplace transform would be only applicable for solving ODEs with explicit initial condition(s), which is demonstrated by the student's attempts to the questions in the next section.

3. How students attempted to solve linear ODEs by the Laplace transform

The two selected cases presented below are from undergraduate student's assignments in the past several years at a regional university in Australia. Each case is from a different cohort and demonstrates student's strengths and weaknesses in using the Laplace transform to solve ODEs under different conditions.

3.1. Solving a second-order linear ODE with explicit initial conditions

This case was part of a group assignment for 35 teams totalling 91 students in an advanced engineering mathematics course. The assigned question is a second-order linear ODE with two initial values. The question is presented below, followed by the step-by-step working with the Laplace transform. Student's performances on solving this question are then analysed.

Case 2: Solve the following ODE with the initial values by the Laplace transform and the convolution theorem.

$$y'' + 9y = 2\cos 3t, \quad \text{given} \quad y(0) = y'(0) = 0 \tag{9}$$

Apply the Laplace transform to both sides of the ODE (9).

$$L[y''] + 9L[y] = 2L[\cos 3t]. \tag{10}$$

Let $Y(s) = L[y]$. This translates the ODE to its state space form

$$s^2Y(s) - sy(0) - y'(0) + 9Y(s) = \frac{2s}{s^2+3^2} \tag{11}$$

$$(s^2+9)Y(s) = \frac{2s}{s^2+9} \longrightarrow Y(s) = \frac{1}{s^2+9}\frac{2s}{s^2+9} \longrightarrow Y(s) = G(s)F(s).$$

These are equivalent to

$$F(s) = \frac{2s}{s^2+3^2}, \quad G(s) = \frac{1}{s^2+3^2}, \quad Y(s) = \frac{1}{s^2+3^2}\frac{2s}{s^2+3^2}.$$

The corresponding functions in the time domain can be found by the inverse Laplace transform.

$$f(t) = L^{-1}[F(s)] = L^{-1}\left[\frac{2s}{s^2+3^2}\right] = 2\cos 3t \longrightarrow g(t) = L^{-1}[G(s)] = L^{-1}\left[\frac{1}{s^2+3^2}\right] = \frac{1}{3}\sin 3t$$

$$y(t) = L^{-1}[Y(s)] = L^{-1}[G(s)F(s)] = g(t) * f(t). \tag{12}$$

By convolution, the output in the time domain can be determined as follows.

$$y(t) = L^{-1}[G(s)F(s)] = g(t) * f(t) = 2\cos 3t * \frac{1}{3}\sin 3t = \frac{2}{3}\sin 3t * \cos 3t$$

$$= \frac{2}{3}\int_0^t \sin(3t-3p)\cos(3p)dp = \frac{1}{3}\int_0^t [\sin(3t-3p+3p) + \sin(3t-3p-3p)]\,dp^{**}$$

$$= \frac{1}{3}\left[\int_0^t \sin(3t)dp - \int_0^t \sin(6p-3t)dp\right] = \frac{1}{3}\left[\sin 3t \cdot p + \frac{1}{6}\cos(6p-3t)\right]\Bigg|_0^t$$

$$= \frac{1}{3}\left[t\sin 3t + \frac{1}{6}[\cos 3t - \cos(-3t)]\right] = \frac{1}{3}\left[t\sin 3t + \frac{1}{6}[\cos 3t - \cos 3t]\right]$$

$$= \frac{1}{3}t\sin 3t. \tag{13}$$

** $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$

The plot of this specific solution is shown in Figure 4, which indicates a periodic sine pattern amplified by time.

Twenty-four out of the 35 teams solved this question correctly, i.e., correct in applying the Laplace transform, using the convolution theorem, and carrying out integrations during the entire process. One such example is shown in Figure 5. This team demonstrated not only a good understanding of the procedure of using the Laplace transform to solve the initial value ODE but also the efficacy of utilizing the mathematical skills obtained from previous mathematics courses.

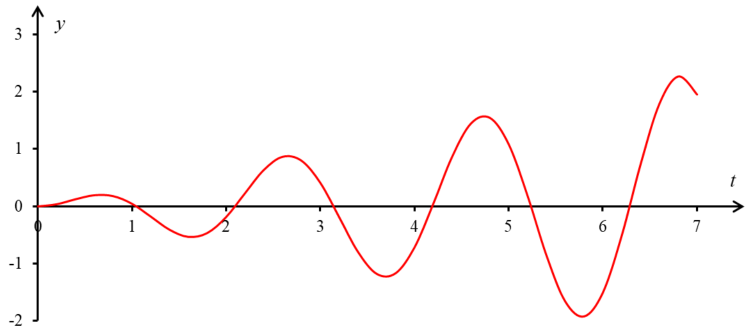

Figure 4. Plot of the output for Case 2 with $y(0) = 0$ and $y'(0) = 0$

$$y'' + 9y = 2\cos 3t, \qquad y(0) = y'(0) = 0$$

Insert all terms back into equation:

$$s^2Y(s) + 9Y(s) = 2\left(\frac{s}{s^2+3^2}\right)$$

$$(s^2+9)Y(s) = 2\left(\frac{s}{s^2+3^2}\right)$$

$$Y(s) = 2\left(\frac{s}{s^2+3^2}\right)\left(\frac{1}{s^2+3^2}\right)$$

$$y(t) = L^{-1}[Y(s)] = L^{-1}\left[\frac{2s}{s^2+3^2} \times \frac{1}{s^2+3^2}\right]$$

$$= L^{-1}\left[\frac{2s}{s^2+3^2}\right] * L^{-1}\left[\frac{1}{s^2+3^2}\right]$$

$$= 2(\cos 3t) * \frac{1}{3}\sin 3t$$

$$= \int_0^t 2\cos 3(t-p) \times \frac{1}{3}\sin 3p\, dp$$

$$= \int_0^t 2\cos 3p \times \frac{1}{3}\sin 3(t-p)\, dp$$

$$= \frac{2}{3}\int_0^t \cos 3p \times \sin 3(t-p)\, dp$$

$$= \frac{2}{3}\int_0^t \frac{1}{2}[\sin(3t-3p+3p) + \sin(3t-3p-3p)]dp$$

$$= \frac{1}{3}\int_0^t \sin(3t) + \frac{1}{3}\int_0^t \sin(3t-6p)\, dp$$

$$y(t) = \frac{1}{3}\sin 3t\,[t] + 0$$

$$y(t) = \frac{1}{3}t\sin 3t$$

Apply Laplace to all terms separately - (2.2):

First Term: $L[y''] = s^2 L[y] - sy(0) - y'(0)$

$$L[y''] = s^2 L[y]$$

Second Term: $L[9y] = 9L[y]$

Third Term: $L[2\cos 3t] = 2\left(\frac{s}{s^2+3^2}\right)$

Find the L^{-1} *of each term:*

$2\left(\frac{s}{s^2+3^2}\right) = 2(\cos 3t)$ *Given*

$$\left(\frac{1}{s^2+3^2}\right) = \left(\frac{3}{s^2+3^2}\right)\left(\frac{1}{3}\right) = \left(\frac{1}{3}\right)L[\sin 3t]$$

Trig Identity:

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$$

where $\alpha = (3t-3p)$, $\beta = 3p$

First Term $\frac{1}{3}\int_0^t \sin(3t)\, dp$

$$= \frac{1}{3}\sin 3t \int_0^t 1\, dp$$

$$= \frac{1}{3}\sin 3t\,[t]$$

Second Term $\frac{1}{3}\int_0^t \sin(3t-6p)\, dp$

$$= \frac{1}{3}\left[\frac{-\cos(3t-6p)}{6}\right]_0^t$$

$$= \frac{1}{18}\{-\cos(3t-6t) - (-)\cos(3t)\}$$

$$= \frac{1}{18}\{-\cos(-3t) + \cos(3t)\}$$

$$= \frac{1}{18} \times 0 = 0$$

Figure 5. An example of student's work on solving the ODE in Case 2 by the Laplace transform

For the eleven teams who presented incorrect or partly correct solutions to the question, two teams were completely wrong with applying the Laplace transform; six teams encountered problems in using the convolution theorem; three teams made mistakes in integration during convolution. The student's overall performances in solving this question are summarized in Table 1. With an overall correct rate of 69% (24/35), plus a few teams with partly correct solutions, student's ability to solve ODEs with explicit initial values by the Laplace transform was reasonably satisfactory.

Table 1. Summary of the performances in solving the ODE in Case 2 by students

Mistake in	Laplace transform	Convolution	Integration
Incorrect	2	6	3
Correct	24	24	24

3.2. Solving a second-order linear ODE with generic initial conditions

The second case was part of an individual assignment for 124 students in the same advanced engineering mathematics course in another year. The assigned question is also a second-order linear ODE with only one explicit initial condition. Students were asked to solve this question using the Laplace transform but convolution was not compulsory. Without providing the full explicit initial conditions, this case became a partly open question. Students would need their reasoning to start applying the Laplace transform.

This question is presented below, followed by the step-by-step working with the Laplace transform for the preferred outcome. Student's performances on solving this question are then analysed.

Case 3: Use the Laplace transform to solve the following ODE

$$y'' - y = e^{2t} + 1, \text{ given } y(0) = 1. \tag{14}$$

Apply the Laplace transform to both sides of the ODE (14)

$$L[y''] - L[y] = L[e^{2t} + 1]. \tag{15}$$

Let $Y(s) = L[y]$ and assume $y'(0) = v_1$. This translates the ODE to its state space form

$$s^2 Y(s) - sy(0) - y'(0) - Y(s) = \frac{1}{s-2} + \frac{1}{s} \tag{16}$$

$$(s^2 - 1)Y(s) - s - v_1 = \frac{1}{s-2} + \frac{1}{s} \longrightarrow (s^2 - 1)Y(s) = \frac{1}{s-2} + \frac{1}{s} + s + v_1$$

$$Y(s) = \frac{1}{(s^2-1)(s-2)} + \frac{1}{(s^2-1)s} + \frac{s}{s^2-1} + \frac{v_1}{s^2-1} = G(s)[F_1(s) + F_2(s)] + G_0(s) + v_1 G(s)$$

where

$$G(s) = \frac{1}{s^2-1},\ F_1(s) = \frac{1}{s-2},\ F_2(s) = \frac{1}{s},\ G_0(s) = \frac{s}{s^2-1}.$$

The corresponding functions in the time domain can be found by the inverse Laplace transform.

$$g(t) = L^{-1}\left[G(s)\right] = L^{-1}\left[\frac{1}{s^2-1}\right] = \sinh t,\ f_1(t) = L^{-1}\left[F_1(s)\right] = L^{-1}\left[\frac{1}{s-2}\right] = e^{2t}$$

$$f_2(t) = L^{-1}\left[F_2(s)\right] = L^{-1}\left[\frac{1}{s}\right] = 1,\ g_0(t) = L^{-1}\left[G_0(s)\right] = L^{-1}\left[\frac{s}{s^2-1}\right] = \cosh t$$

$$y(t) = L^{-1}\left[Y(s)\right] = L^{-1}\left[G(s)F_1(s) + G(s)F_2(s) + G_0(s) + v_1G(s)\right].$$

By convolution, the output in the time domain can be determined as follows.

$$\begin{aligned}
y(t) &= L^{-1}\left[Y(s)\right] = g(t) * f_1(t) + g(t) * f_2(t) + g_0(t) + v_1 g(t) \\
&= \int_0^t e^{2(t-p)} \sinh p dp + \int_0^t 1 \cdot \sinh p dp + \cosh t + v_1 \sinh t \\
&= e^{2t} \int_0^t e^{-2p} \left(\frac{e^p - e^{-p}}{2} \right) dp + \cosh p\Big|_0^t + \cosh t + v_1 \sinh t \\
&= \frac{e^{2t}}{2} \int_0^t (e^{-p} - e^{-3p}) dp + \cosh t - 1 + \cosh t + v_1 \sinh t \\
&= \frac{e^{2t}}{2} \left[-e^{-p} + \frac{1}{3} e^{-3p} \right]\Bigg|_0^t + 2\cosh t + v_1 \sinh t - 1 \\
&= \frac{e^{2t}}{2} \left[-e^{-t} + \frac{1}{3} e^{-3t} + 1 - \frac{1}{3} \right] + 2\cosh t + v_1 \sinh t - 1 \\
&= \frac{1}{3} e^{2t} - \frac{1}{2} e^{t} + \frac{1}{6} e^{-t} + 2 \cdot \frac{e^t + e^{-t}}{2} + v_1 \cdot \frac{e^t - e^{-t}}{2} - 1 \\
&= \frac{1}{3} e^{2t} - \frac{1}{2} e^{t} + \frac{1}{6} e^{-t} + e^t + e^{-t} + \frac{v_1}{2} e^t - \frac{v_1}{2} e^{-t} - 1 \\
&= \frac{1}{3} e^{2t} + \frac{(1+v_1)}{2} e^t + \frac{(7-3v_1)}{6} e^{-t} - 1 .
\end{aligned} \tag{17}$$

The consideration here is that the 'system' represented by the ODE would exhibit a certain pattern if all two initial values are given. By fixing one initial value and allowing the other to vary, the 'system' would become a dynamic system to some extent. For example, with $y(0) = 1$ already embed in the solution, by using –2, 0, 2 for $y'(0) = v_1$ respectively, three different outputs can be obtained with the solution (17) and each displays a unique pattern as shown in Figure 6. Of course, more variations can be determined similarly by only varying the value for $y'(0) = v_1$ in this solution, which cannot be achieved should both initial values be given in the beginning of solving this problem.

Among the 124 students, only nineteen students (or 15% of all students) attempted this question, partly due to the assessment setting that allowed students to choose five out of the six questions in the assignment. All these nineteen students assumed $y'(0) = 0$, and eleven of them obtained correct solutions under this assumption (Table 2). Among the eight students who used convolution, six achieved the correct solution whereas five students out of the eleven students who used the conventional method of partial fractions also obtained the correct solution. Usually students who are confident in their mathematics abilities tend to choose the convolution method for the inverse Laplace transform; hence it is not surprising to see more of them have achieved the correct outcome.

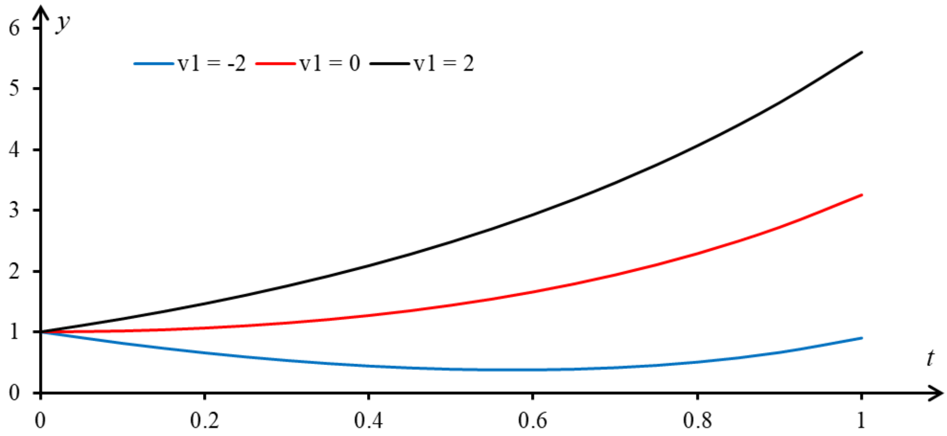

Figure 6. Plots of the solution to Case 3 with fixed $y(0) = 1$ and different values for $y'(0) = v_1$

The more profound observation from this case is that no student attempted to use a generic value for $y'(0) = v_1$. Instead, they all believed that the omission of a known value for $y'(0)$ was an unnoticed mistake made by the teacher in setting up this assignment because from their knowledge the Laplace transform should be used *only* for initial value problems. Upon the explanation of the preferred solution and exhibition of different results and implications with different values for $y'(0) = v_1$ shown in Figure 6, the students began to understand that the solution they obtained was only a special case for $y'(0) = 0$ that was also covered in the solution (17) for $y'(0) = v_1 = 0$. This case also made students rethink of the use of the Laplace transform as a general method for solving linear ODEs, both the general ODEs and the initial value ODEs with fully or partially known initial conditions.

Table 2. Summary of attempts to solve Case 3 by 124 students

	$y(0)=1$ & $y'(0)=0$	$y(0)=1$ & $y'(0)=v_1$
No attempt	105	105
Convolution	8 (*6*)	0
Partial fraction	11 (*5*)	0
Correct solution	*11*	*0*

Italic numbers indicate the correct solutions obtained by students

4. Generalization of the Laplace transform for solving constant-coefficient linear ODEs

To help students further their understanding of the capacity of the Laplace transform in solving ODEs with generic initial value(s), Case 1 and Case 2 presented in previous sections and a system of linear ODEs as Case 4 are solved by the Laplace transform for generalization in this section.

4.1. Solving the RL circuit in Case 1 with a generic initial value

Assuming $i(0) = v_0$, the equation (8) in the state space becomes

$$sI(s) - v_0 + aI(s) = bL[\sin \omega t] \tag{18}$$

$$(s+a)I(s) = \frac{b\omega}{s^2+\omega^2} + v_0 \longrightarrow I(s) = \frac{1}{s+a}\frac{b\omega}{s^2+\omega^2} + \frac{v_0}{s+a}$$

$$I(s) = G(s)F(s) + v_0 G(s),$$

where

$$F(s) = \frac{b\omega}{s^2+\omega^2}, \quad G(s) = \frac{1}{s+a}.$$

There is a new term $v_0 G(s)$ whose inverse Laplace transform should be added to the existing solution,

$$i(t) = L^{-1}[I(s)] = L^{-1}[G(s)F(s)] + L^{-1}[v_0 G(s)] = g(t) * f(t) + v_0 e^{-at}.$$

Therefore, the general output in the time domain is

$$i(t) = g(t) * f(t) + v_0 e^{-at} = \frac{b}{a^2+\omega^2}\left(\omega e^{-at} + a \sin \omega t - \omega \cos \omega t\right) + v_0 e^{-at}$$

$$= \frac{E_0}{R^2+\omega^2 L^2}(\omega L e^{-\frac{R}{L}t} + R \sin \omega t - \omega L \cos \omega t) + v_0 e^{-\frac{R}{L}t}. \tag{19}$$

By assigning $i(0) = 0$ ampere, its specific solution is decided as

$$i(t) = \frac{E_0}{R^2+\omega^2 L^2}(\omega L e^{-\frac{R}{L}t} + R \sin \omega t - \omega L \cos \omega t).$$

This is the same as the solution (6).

By assigning $i(0) = -10$ amperes, the corresponding specific solution is decided as

$$i(t) = \frac{E_0}{R^2+\omega^2 L^2}(\omega L e^{-\frac{R}{L}t} + R \sin \omega t - \omega L \cos \omega t) - 10 e^{-\frac{R}{L}t}. \tag{20}$$

By assigning $i(0) = 10$ amperes, the corresponding specific solution is decided as

$$i(t) = \frac{E_0}{R^2+\omega^2 L^2}(\omega L e^{-\frac{R}{L}t} + R \sin \omega t - \omega L \cos \omega t) + 10 e^{-\frac{R}{L}t}. \tag{21}$$

These three corresponding outputs are shown in Figure 7. The red curve represents the specific case with $i(0) = 0$ ampere, starting from the original till reaching the steady-state phase after the first period, the same as demonstrated in Figure 2. In the meantime, the solution (19) is more flexible in demonstrating the patterns resulted from other initial values. For example, with $i(0) = -10$ amperes, the blue curve begins from the initial position at 10 units below the original till reaching the steady-state phase after the first period. With $i(0) = 10$ amperes, the black curve begins from the initial position at 10 units above the original till reaching the steady-state phase after the first period.

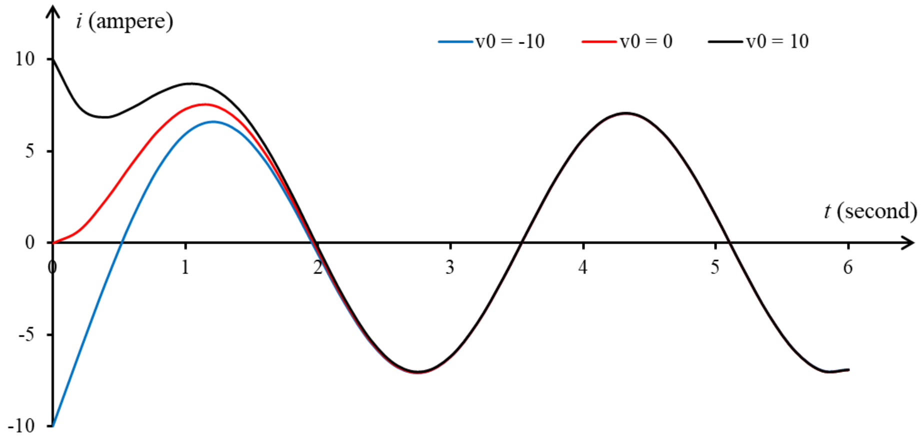

Figure 7. Electric currents of the *RL* circuit in Case 1 with $R = 10\ \Omega$, $L = 5$ H, $\omega = 2$, and $E_0 = 10$ V [$v_0 = i(0)$ for −10, 0, and 10 amperes, respectively]

In this case, simply replacing v_0 by the given initial value will produce the specific solution, without substituting the initial value into the general solution (5) obtained by the conventional method to define the unknown constant for the new cases.

4.2. Solving the second-order ODE in Case 2 with generic initial conditions

Assuming $y(0) = v_0$ and $y'(0) = v_1$, the second-order ODE in Case 2 in the state space becomes

$$\begin{aligned} &s^2Y(s) - sy(0) - y'(0) + 9Y(s) = \frac{2s}{s^2+3^2} \longrightarrow (s^2+9)Y(s) = \frac{2s}{s^2+3^2} + sv_0 + v_1 \\ &Y(s) = \frac{1}{s^2+9}\frac{2s}{s^2+9} + \frac{sv_0}{s^2+9} + \frac{v_1}{s^2+9} \longrightarrow Y(s) = G(s)F(s) + G_0(s) + v_1G(s) \end{aligned} \tag{22}$$

where

$$F(s) = \frac{2s}{s^2+3^2}, \quad G(s) = \frac{1}{s^2+3^2}, \quad G_0(s) = \frac{sv_0}{s^2+3^2}.$$

The corresponding functions in the time domain can be found by the inverse Laplace transform.

$$\begin{aligned} &f(t) = L^{-1}\left[F(s)\right] = L^{-1}\left[\frac{2s}{s^2+3^2}\right] = 2\cos 3t, \quad g(t) = L^{-1}\left[G(s)\right] = L^{-1}\left[\frac{1}{s^2+3^2}\right] = \frac{1}{3}\sin 3t \\ &g_0(t) = L^{-1}\left[G_0(s)\right] = L^{-1}\left[\frac{sv_0}{s^2+3^2}\right] = v_0\cos 3t \end{aligned}$$

Considering the convolution outcome (13) in Section 3, the output in the time domain can be determined as follows.

$$y(t) = L^{-1}[Y(s)] = g(t) * f(t) + g_0(t) + v_1 g(t) = \frac{1}{3} t \sin 3t + v_0 \cos 3t + \frac{v_1}{3} \sin 3t$$
$$= v_0 \cos 3t + \frac{1}{3}(v_1 + t) \sin 3t. \quad (23)$$

By assigning $y(0) = 0$ and $y'(0) = 0$ to the solution (23), the corresponding specific solution is defined as

$$y(t) = \frac{1}{3} t \sin 3t\ .$$

This is the same as the solution (13) obtained in Section 3.

By assigning $y(0) = -3$ and $y'(0) = -3$ to the solution (23), the corresponding specific solution is defined as

$$y(t) = -3\cos 3t + \frac{1}{3}(t-3)\sin 3t\ . \quad (24)$$

By assigning $y(0) = 5$ and $y'(0) = 5$ to solution (23), the corresponding specific solution is defined as

$$y(t) = 5\cos 3t + \frac{1}{3}(t+5)\sin 3t\ . \quad (25)$$

These three specific outputs are shown in Figure 8. The red curve represents the specific case with $y(0) = y'(0) = 0$, starting from the original with a periodic sine pattern amplified by time, the same as that demonstrated in Figure 4.

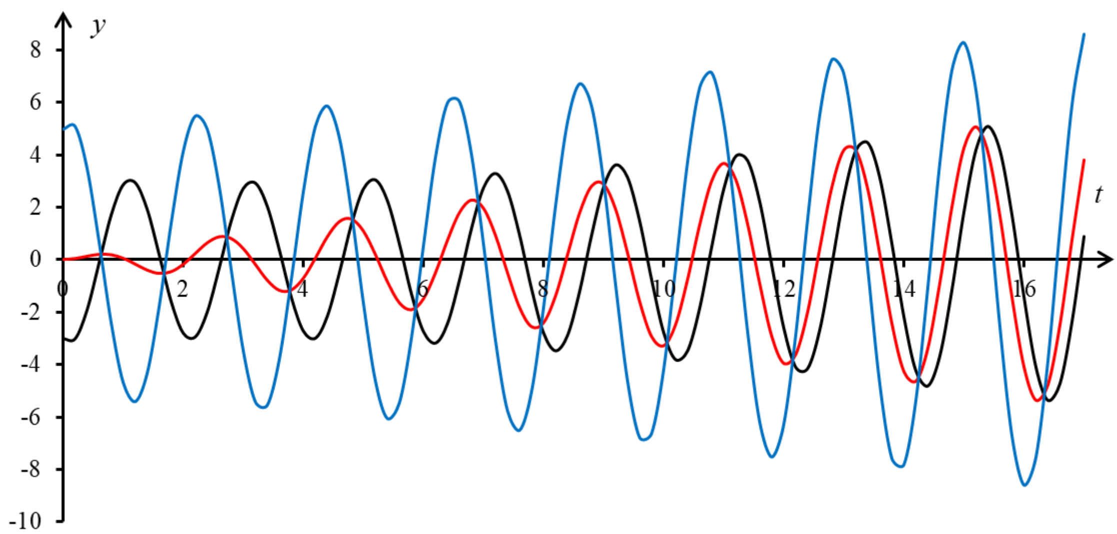

Figure 8. Plots of the output of the ODE in Case 2 by Laplace transform with different initial values [Black: $v_0 = -3$, $v_1 = -3$; Red: $v_0 = 0$, $v_1 = 0$; Blue: $v_0 = 5$, $v_1 = 5$]

The black curve represents the pattern with $y(0) = -3$ and $y'(0) = -3$ which begins from the initial position at 3 units below the original with a cosine pattern. This is because in the solution (24) the sine component is much smaller than the cosine component in the early stage when t is small. With the increase in time, the sine component gradually surpasses the cosine component and eventually reaches the similar sine pattern with a slightly larger amplitude compared with the red curve.

The blue curve represents the pattern with $y(0) = 5$ and $y'(0) = 5$ which begins from the initial position at 5 units above the original with a cosine pattern. This is because in the solution (25) the sine component is smaller than the cosine component in the early stage when t is small. With the increase in time, the sine component strengthens quickly and soon dominates the pattern with a much larger amplitude compared with the red curve.

In this case, simply replacing v_0 and/or v_1 for different initial conditions in the solution (23) will produce the anticipated specific solution without a need to substitute the initial values into the general solution determined by the conventional method to define the unknown constants for the new case. Therefore, the Laplace transform is a general method suitable to solve constant-coefficient linear ODEs not only with explicit initial values for the corresponding specific solution but also with generic initial values for the general solution.

4.3. Solving a system of linear ODEs with generic initial conditions

Case 4: Tank 1 (T1) and Tank 2 (T2) initially contain V litres of saline. In the beginning, the saline in T1 contains x_0 kg of salt dissolved uniformly in T1 and the saline in T2 contains y_0 kg of salt dissolved uniformly in T2. The liquid is circulated at a constant rate of q litre/min and stirred to keep the mixture uniform. Find the amount of salt $x(t)$ in T1 and $y(t)$ in T2 with time, respectively.

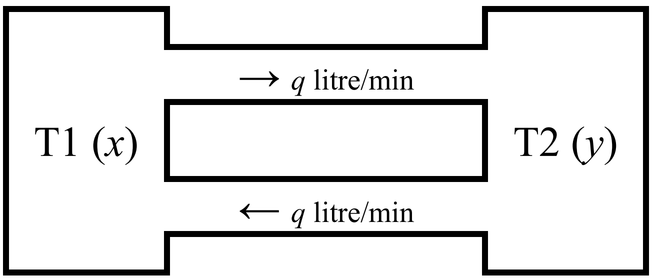

Figure 9. The mixing problem for Case 4

For T1, the rate of change in salt equals the amount of salt in the inflow from T2 by taking away the amount of salt in the outflow to T2, and so does for T2 (Figure 9), i.e.,

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{q}{V}y - \dfrac{q}{V}x, \ x(0) = x_0 \\ \dfrac{dy}{dt} = \dfrac{q}{V}x - \dfrac{q}{V}y, \ y(0) = y_0 \end{cases} \longrightarrow \begin{cases} x' = -\dfrac{q}{V}x + \dfrac{q}{V}y, \ x(0) = x_0 \\ y' = \dfrac{q}{V}x - \dfrac{q}{V}y, \ y(0) = y_0 \end{cases}. \tag{26}$$

Let $k = q / V$. Apply the Laplace transform to both sides of the system (26)

$$\begin{cases} L[x'] = -kL[x] + kL[y] \\ L[y'] = kL[x] - kL[y] \end{cases}. \tag{27}$$

Let $X(s) = L[x]$ and $Y(s) = L[y]$. This translates the ODEs to its state space form

$$\begin{cases} sX(s) - x_0 = -kX(s) + kY(s) \\ sY(s) - y_0 = kX(s) - kY(s) \end{cases}. \tag{28}$$

The vertical addition of the two equations in the system (28) becomes

$$s\left[X(s) + Y(s)\right] = x_0 + y_0 \longrightarrow X(s) + Y(s) = \frac{x_0 + y_0}{s} \longrightarrow Y(s) = \frac{x_0 + y_0}{s} - X(s). \tag{29}$$

Substitute $Y(s)$ into the first ODE in the system (28).

$$sX(s) - x_0 = -kX(s) + k\left[\frac{x_0 + y_0}{s} - X(s)\right] \longrightarrow (s + 2k)X(s) = x_0 + \frac{k(x_0 + y_0)}{s}, \tag{30}$$

or

$$X(s) = \frac{x_0}{(s + 2k)} + \frac{k(x_0 + y_0)}{s(s + 2k)} = x_0 G(s) + k(x_0 + y_0)F(s)G(s), \tag{31}$$

here

$$F(s) = \frac{1}{s}, \quad G(s) = \frac{1}{s + 2k}.$$

$$Y(s) = \frac{x_0 + y_0}{s} - X(s) = (x_0 + y_0)F(s) - x_0 G(s) - k(x_0 + y_0)F(s)G(s). \tag{32}$$

The corresponding functions in the time domain can be found by the inverse Laplace transform.

$$f(t) = L^{-1}\left[F(s)\right] = L^{-1}\left[\frac{1}{s}\right] = 1, \quad g(t) = L^{-1}\left[G(s)\right] = L^{-1}\left[\frac{1}{s + 2k}\right] = e^{-2kt}$$

The outputs in the time domain can be determined as follows.

$$x(t) = L^{-1}[X(s)] = x_0 g(t) + k(x_0 + y_0) f(t) * g(t) f(t) = x_0 e^{-2kt} + k(x_0 + y_0)\int_0^t 1 \cdot e^{-2kp} dp$$

$$= x_0 e^{-2kt} - \frac{x_0 + y_0}{2} e^{-2kp}\Big|_0^t = x_0 e^{-2kt} - \frac{x_0 + y_0}{2} e^{-2kt} + \frac{x_0 + y_0}{2} = \frac{x_0 + y_0}{2} + \frac{x_0 - y_0}{2} e^{-2kt}.$$

$$y(t) = L^{-1}[Y(s)] = L^{-1}[Y(s)] = (x_0 + y_0) f(t) - x(t) = (x_0 + y_0) - \frac{x_0 + y_0}{2} - \frac{x_0 - y_0}{2} e^{-2kt}$$

$$= \frac{x_0 + y_0}{2} - \frac{x_0 - y_0}{2} e^{-2kt}.$$

Substitute $k = q / V$ back.

$$\begin{cases} x(t) = \frac{1}{2}(x_0 + y_0) + \frac{1}{2}(x_0 - y_0)e^{-\frac{2q}{V}t} \\ y(t) = \frac{1}{2}(x_0 + y_0) - \frac{1}{2}(x_0 - y_0)e^{-\frac{2q}{V}t} \end{cases}. \tag{33}$$

This is the general solution to this mixing problem without specifying the initial values.

Assume the volume for both tanks is 4000 liters and in the beginning 600 kg of salt are dissolved in T1 and only pure water is in T2, i.e., $x_0 = x(0) = 600$ kg and $y_0 = y(0) = 0$ kg. If the two circulating rates, $q_1 = 40$ l/m and $q_2 = 100$ l/m, are used in the mixing process, two sets of specific solutions can then be determined by the general solution (33) as follows.

$q_1 = 40$ l/m:

$$\begin{cases} x(t) = 300 + 300e^{-0.02t} \\ y(t) = 300 - 300e^{-0.02t} \end{cases}. \tag{34}$$

$q_2 = 100$ l/m:

$$\begin{cases} x(t) = 300 + 300e^{-0.05t} \\ y(t) = 300 - 300e^{-0.05t} \end{cases}. \tag{35}$$

Figure 10 displays the progression of mixing for these two settings. Suppose the mixing is to achieve at least 99% (297–303 kg) of uniformity of salt in both tanks. The mixing process using the higher circulating rate of 100 l/m would reach this threshold in about 95 minutes, indicated by the red dot on the red curves. The lower circulating rate of 40 l/m would take more than 180 minutes to have a chance to achieve this standard as indicated by the blue curves in Figure 10.

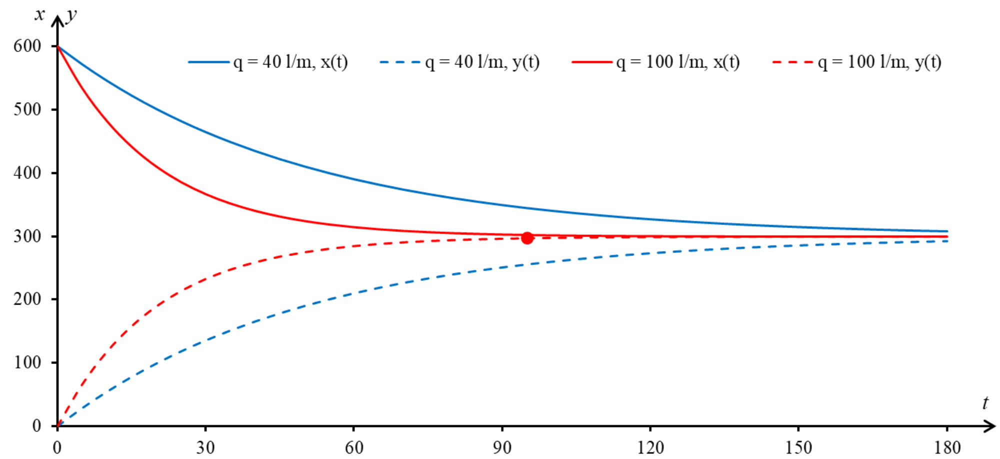

Figure 10. Plots of the mixing processes in the solutions (34) and (35)
$V = 4000$ liters, $x_0 = 600$ kg, $y_0 = 0$ kg; blue curves: $q_1 = 40$ l/m; red curves: $q_2 = 100$ l/m

If in the beginning 500 kg of salt are dissolved in T1 and 100 kg of salt are dissolved in T2, i.e., $x_0 = x(0) = 500$ kg and $y_0 = y(0) = 100$ kg. With the two circulating rates, $q_1 = 40$ l/m and $q_2 = 100$ l/m, two sets of new specific solutions can be determined by the general solution (33) as follows.

$q_1 = 40$ l/m:

$$\begin{cases} x(t) = 300 + 200e^{-0.02t} \\ y(t) = 300 - 200e^{-0.02t} \end{cases}. \tag{36}$$

$q_2 = 100$ l/m:

$$\begin{cases} x(t) = 300 + 200e^{-0.05t} \\ y(t) = 300 - 200e^{-0.05t} \end{cases}. \tag{37}$$

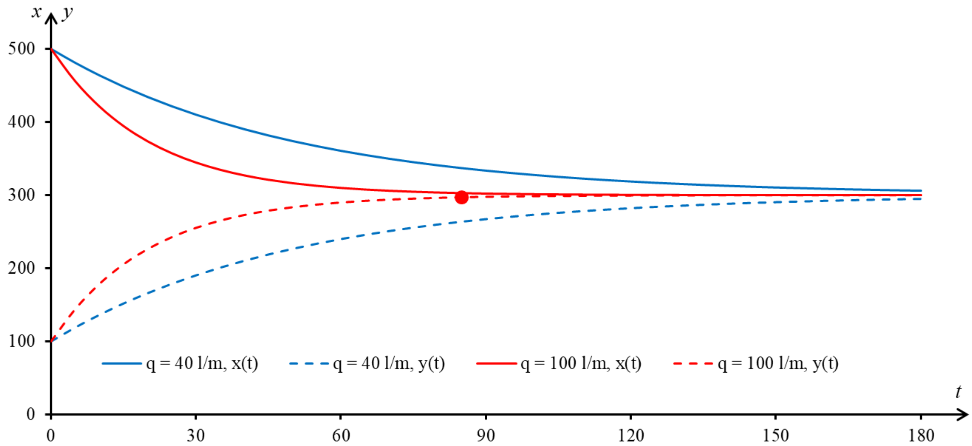

Figure 11. Plots of the mixing processes in the solutions (36) and (37)
$V = 4000$ liters, $x_0 = 500$ kg, $y_0 = 100$ kg; blue curves: $q_1 = 40$ l/m; red curves: $q_2 = 100$ l/m

Figure 11 displays the progression of mixing for these two settings. Suppose the mixing is to achieve at least 99% of uniformity of salt in both tanks. The mixing process using the higher circulating rate of 100 l/m would reach this threshold in about 85 minutes, indicated by the red dot on the red curves. This is faster than the case where all 600 kg of salt are dissolved in Tank 1 in the beginning. The lower circulating rate of 40 l/m would still take more than 180 minutes to have a chance to achieve this standard as indicated by the blue curves in Figure 11.

This case also demonstrates that the Laplace transform is well capable of solving systems of linear ODEs with generic initial values. The general solutions from the Laplace transform are even simpler to produce specific solutions by simply replacing the generic initial values with the given numbers, without a need to substitute the initial values into the general solution to define the unknown constants for the new case through the conventional method.

5. Discussion and conclusion

All the cases have demonstrated that the Laplace transform is not only useful in solving linear ODEs with explicit initial values, but also powerful in solving constant-coefficient linear ODEs with generic initial conditions. The general solution obtained by the Laplace transform is even more robust for obtaining the specific solutions directly once the initial values are assigned. This is because the generic initial values, even being symbolic in the general solutions, are already embed into the process of problem solving by the Laplace transform, no need to go through the substitution process with the initial values to determine the unknown constants contained in the general solution resulted from the conventional method. This implies that the generic initial conditions in the general solution obtained from the Laplace transform could be used as a point of control for some dynamic systems.

However, a good understanding of the power of the Laplace transform for solving ODEs is different from the effective use of the powerful tool to solve the real problems. This is because the nature of the Laplace transform already leans to a more difficult path than the conventional way, particularly through convolution, experienced by both the teachers and students [14,15]. The challenging route is often compounded by the inefficient retention of the skills and knowledge students gained in previous mathematics courses, such as integration by parts, trigonometric relationships etc. [23-25]. Hence, a high level of efficacy of mathematical skills and techniques is still the necessity for a smooth progression in advanced mathematics studies, including the Laplace transform.

Although four representative cases are closely examined in this study, more research should be done to explore the extra power and/or the limit that the Laplace transform may bring to solving other types of ODEs. For example, to what extent, the Laplace transform can deal with solving nonlinear ODEs with generic initial conditions? How will the piecewise functions as the input to linear ODEs with generic initial conditions affect the effectiveness of applying the Laplace transform as a general tool? How best the generic initial conditions in the general solution obtained through the Laplace transform can be integrated into the control of dynamic system? More questions are raised than solved, which should become the research questions for future studies.

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Appendix

$$I = \int b\sin\omega t e^{at}dt = b\sin\omega t\frac{e^{at}}{a} - \int\frac{\omega b}{a}\cos\omega t e^{at}dt = \frac{b}{a}\sin\omega t e^{at} - \frac{\omega b}{a}[\cos\omega t\frac{e^{at}}{a} + \int\frac{\omega}{a}\sin\omega t e^{at}dt]$$

$$= \frac{b}{a}\sin\omega t e^{at} - \frac{\omega}{a}[\frac{b}{a}\cos\omega t e^{at} + \frac{\omega}{a}\int b\sin\omega t e^{at}dt] = \frac{b}{a}\sin\omega t e^{at} - \frac{\omega b}{a^2}\cos\omega t e^{at} - \frac{\omega^2}{a^2}I$$

Hence

$$I + \frac{\omega^2}{a^2}I = \frac{b}{a}e^{at}(\sin\omega t - \frac{\omega}{a}\cos\omega t)$$

$$\frac{a^2+\omega^2}{a^2}I = \frac{b}{a^2}e^{at}(a\sin\omega t - \omega\cos\omega t)$$

$$(a^2+\omega^2)I = be^{at}(a\sin\omega t - \omega\cos\omega t)$$

Finally

$$I = \int b\sin\omega t e^{at}dt = \frac{b}{a^2+\omega^2}e^{at}(a\sin\omega t - \omega\cos\omega t)$$