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Article

Reimagining multiplication as diagrammatic and dynamic concepts via cutting, pasting and rescaling actions

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Abstract: Recently, Tisdell [48] developed some alternative pedagogical perspectives of multiplication strategies via cut-and-paste actions, underpinned via the principle of conservation of area. However, the ideas therein were limited to problems involving two factors that were close together, and so would not directly apply to a problem such as 17×93. The purpose of the present work is to establish what diagrammatic and dynamic perspectives could look like for these more complex classes of multiplication problems. My approach to explore this gap is through an analysis and discussion of case studies. I probe several multiplication problems in depth, and drill down to get at their complexity. Through this process, new techniques emerge that involve cut-and-paste and rescaling actions to enable a reimagination of the problem from diagrammatic and dynamic points of view. Furthermore, I provide some suggestions regarding how these ideas might be supplemented in the classroom through the employment of history that includes Leonardo Da Vinci's use of conservation principles in his famous notebooks. I thus establish a pedagogical framework that has the potential to support the learning and teaching of these extended problems from diagrammatic and dynamic perspectives. groups.

Keywords: mathematics education, multiplication, conservation of area, cutting, pasting, rescaling

1. Introduction

Opportunities to shine new light through the old windows of multiplication continue to fascinate the mathematics education community. The significance, the scale of challenge, and the educational needs and benefits of such illumination are well recognized. For example, West [52] advocates for multiplicative skill and understanding to be essential parts of preparation for life in the mathematical world of the 21st century. In addition, Larsson [31] and Larsson, Pettersson and Andrews [32] takes the position that multiplicative understanding is a core component for elementary arithmetic instruction and that such understanding supports higher-level mathematical topics, including fractions, ratio, proportionality and functions.

Multiplication forms one of the four basic mathematical operations and is learned as part of primary school education [38]. In countries such as the UK, US and Australia, students and teachers explore and develop more challenging aspects of multiplication in Year 4. For example, in England, students are expected to "multiply numbers up to 4 digits by a one- or two-digit number using a formal written method, including long multiplication for two-digit numbers" [14]. Furthermore, one of the standards in the US Common Core is for students to "Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models." [8]. Additionally, Australian students are called to "Develop efficient mental and written strategies and use appropriate digital technologies for multiplication…" and "using known facts and strategies, such as commutativity, doubling and halving for multiplication…" [1]

As we can see from the above examples, this deeper educational engagement with multiplication can involve problems that feature multi digit factors and mental techniques. There are a number of opportunities and challenges regarding this. For example, Izsk [25] identifies the research area of multi digit multiplication as an important but understudied domain, drawing on a range of studies, including that of Stigler, Lee, and Stevenson [42], who reported that only 54% of US fifth-grade students in "traditional" courses could solve 45×26 correctly. Beishuizen, Van Putten and Van Mulken [2] identifies mental arithmetic with two-digit numbers up to 100 as "a rather unexplored topic in research" and unmasks several gaps in our understanding therein. As such, advocates such as Beishuizen, Van Putten and Van Mulken [2] and McIntosh, Reys and Reys [35] call for "greater emphasis on mental computation with two-digit numbers up to 100, to stimulate the development of number sense and insightful flexible number operations" [2].

Sanne, Straatemeier, Jansen et al [38] also support the significance of mental multiplication as a process of "insightful procedures" rather than the application of rote-memorized steps. Furthermore, countries that emphasize the uses of mental calculation in mathematics education have performed well in international comparisons [16]. Verschaffel, Greer and De Corte [51] recognize that understanding a variety of strategies for arithmetic is a major goal of primary mathematics education across the world. The importance of this variety is expounded by mathematics education reformers [23] with the aim of developing creative and flexible approaches to arithmetical problems.

Recently, Tisdell [48] developed some alternative pedagogical perspectives of multiplication strategies via diagrammatic representations and dynamic rearrangements. Multiplication problems were modelled via rectangles and their area. By slicing and rearranging parts of these rectangles in strategic ways, Tisdell [48] afforded diagrammatic justifications of the underlying operations at work, and formed a dynamic process to carry out the multiplication itself. The strategies were underpinned via the principle of conservation of area.

However, the ideas in [48] were limited to multiplication problems where the two factors involved were close together, and so problems involving numbers that were further apart were out of scope. Thus, "a problem such as 17×93 would not appear to suit the particular algorithm" [48] and "there is an opportunity to explore what other approaches might be applicable to this kind of problem." [48]

Motivated by the above discussion, the following research questions drive the current article:

RQ1: How can diagrammatic and dynamic pedagogies be established for multiplication strategies when the numbers involved are not close together?

RQ2: What are the potential benefits and limitations of such pedagogies?

RQ3: How might these pedagogies be supplemented in a classroom setting?

To explore the above questions, I draw on a range of methodologies, including: case study research, exemplification, critique and problematization. For those who may be unfamiliar with some of these terms, let me provide some background and clarity.

Case study research is a well-known methodology in the social sciences that can also be viewed as a strategy, a design framework, and a research genre [13, p114]. Since my research questions above are of an "explore and explain" nature involving particular phenomena that are not well understood, it strongly aligns with the purpose of case study research [13, p114]. Exemplification as a methodology can be traced (at least) back to Aristotle [4]. It features an approach of unique sample selection where the examples under consideration exemplify the construct of interest in a highly developed manner [4].

Problematization and critique are methods that challenge existing focalized viewpoints to create a dialogue that enables new perspectives, reflection and action to emerge [9, pp.155-156]. My style of critique within mathematics education positions itself as a counterpoint to what I regard as over-simplistic thinking [47].

Through exemplification, problematization and critique I probe several multiplication problems in depth herein, aiming to drill down to get at their complexity. These procedures, logic and research design align with the above research questions, to enable "a disciplined, balanced enquiry, conducted in a critical spirit" [45, p24].

This paper is organised in the following way. In Section 2 I identify the gaps from [48] through critique and problematization. In Section 3 I examine various examples of multiplication problems. Through case studies and exemplification, I reimagine these classes of problems from diagrammatic and dynamic perspectives to establish new and alternative strategies. Section 4 is dedicated to exploring the limitations of such perspectives. Section 5 develops some ideas for the classroom that can potentially supplement the mathematical ideas here by drawing on history. I make some conclusions within Section 6, and raise some open questions for further research.

2. Problematizing an Example

Tisdell [48] explored geometric perspectives of multiplication linked with the following algebraic identity

$$(a + b)(a + c) = a(a + b + c) + bc. \tag{1}$$

In (1), a is a natural number known as a comparison number (or a base number), and b and c are integers so that $a + b$ forms one factor and $a + c$ forms the second factor in the multiplication. Tisdell [48] discussed 13×12 where a was chosen to be 10, b was 3 and c was 2. The identity (1) would then become

$$(10 + 3)(10 + 2) = 10(13 + 2) + 32 = 150 + 6 = 156.$$

Tisdell [48] reimagined the multiplication from a diagrammatic and dynamic perspective, moving from the situation that can be represented by the area of a rectangle (i.e., the left side of the identity) to become the area of a polygon involving two other rectangles, each whose area was potentially easier to calculate (i.e., the right side of the identity).

However, the ideas in [48] appear to be limited to multiplication problems where the two numbers involved are close together, that is, when b and c are "small". This then ensures the calculation $b \times c$ is not too difficult. Indeed, the examples discussed by Tisdell (2021) [48] involved problems such as: 8×7; 13×12; and 9×14. Observe that each of these pairs lie close to a = 10 which is easier to multiply with, and more generally, a is known as a comparison number or base number that helps to simplify the calculations. Problems involving 103×109 (both close to $a = 100$) or 23×26 (both close to 20 or 25) would also fit into the above framework.

Allow me to scrutinize the example of 17×93 within the context of (1). The challenge here is to choose a value of a such that the calculation bc is manageable, however this presents a serious challenge as the following discussion illustrates. If we choose a value roughly in the middle, say $a = 50$, then $b = -34$ and $c = 43$ leads to $b \times c = -34 \times 43$. This is certainly not as simple as a multiplication involving single digits, but could be evaluated by running the process again to handle the calculation of 34 × 43 (say, with the choice $a = 40$). However, this lengthens the process by adding more steps.

If we choose a value closer to one of the factors, say 17 with $a = 10$, then $b = 7$ and $c = 83$. Here b is a single digit, but c is not. If we choose $a = 100$ then $b = -83$ and $c = 7$ and we run into the same problem regarding the difficulty of $b \times c$.

From a diagrammatic perspective, determining the area of the second rectangle with "sides" band c is not as simple as we might hope.

We also note that 93 can be expressed as 3×31, however in this case we require additional steps to first navigate 17×31 and then multiply this by 3.

Arithmetical strategies do exist that can help learners to navigate the above problem. For example, Santhamma [39, Ch 9ii], Handley [20, Ch 10; 21, Ch 7] and Doerfler [15, p13] have discussed the situation where the two factors in the multiplication are not close together. They draw on the algebraic identity

$$(ax + b)(a + c) = a(ax + b + cx) + bc \tag{2}$$

where x can be thought of as a natural number acting as a scaling factor. Thus $ax + b$ is not necessarily close to $a + c$.

In the aforementioned texts, algebraic perspectives dominate the discussion of (2). For example, Handley [20, pp235-236] justifies why his methods work by expanding the left hand side of the identity (2) via algebraic operations, and then recombining to the right hand side via regrouping and factorization methods. Santhamma [39, Ch 9ii] takes a similar approach. In addition, Handley [21] and Doerfler [15] make repeated use of, and reference to, the algebraic identity (2), however they remain silent on any kind of justification. These focalized algebraic viewpoints possess some mathematical advantages in that they are widely applicable, and they can facilitate justifications that are quite short and compact.

However, there are a number of potential disadvantages of the above approaches when viewed from educational perspectives. Remaining silent on justifications for why the methods work jeopardizes deeper learning by favouring skill over understanding.

In addition, an over-reliance on algebraic perspectives risks excluding those learners who are unfamiliar with algebra, algebraic symbols, and the algebraic rules for manipulating these symbols. Such algebraic perspectives are problematic for younger learners of arithmetic who are yet to encounter algebra. In addition, those older learners who may be less algebraically-inclined also face challenges in accessing these algebraic justifications [48].

Furthermore, concentrating on algebra alone threatens learners missing or losing the connections between arithmetic, algebra and diagrams. Although the use of algebra may form a protective, comforting wall, it also encircles and potentially entraps those who focalise their ways of thinking. Even though algebra was divorced from the constraints of geometry by Al-Karaji and Diophantus [29], have we forgotten the connections, and have we come too far? When we speak of "the square of something", does the symbol x^2 immediately come to mind instead of a square itself, or both?

Fostering the ability to think from more diagrammatical perspectives is important and has significant advantages. For example, Brown [5] argues that pictures in mathematics are crucial, and "Trying to get along without them would be like trying to do theoretical physics without the benefit of experiments to test conjectures." [5, p29]. Giaquinto [18] argues that visual images and diagrams can serve as a resource for discovery, justification and proof. The famous mathematician Littlewood identifies his position and ways of working with pictures in helping him to see through the fog: "Some pictures are of course not rigorous, but I should say most are (and I use them whenever possible myself)....This is rigorous (and printable), in the sense that in translating into symbols no step occurs that is not both unequivocal and trivial. For myself I think like this wherever the subject matter permits." [33, pp35-36]. In summary, pictures do have important roles to play in the learning and teaching of mathematics due to their ability to: illuminate, justify, defend, explain, explore and make us wiser.

3. Case Studies

Let me probe several case studies. Some new dynamic and diagrammatic processes will consequently emerge from this analysis.

Example 1: Consider 12×34.

Drawing on (2) we could choose $a = 10$ so that $c = 2$, $x = 3$, and $b = 4$. Thus,

$$12 \times 34 = 10(34 + 2 \times 3) + 4 \times 2 = 10(40) + 8 = 408.$$

Let me reimagine this from diagrammatic perspectives. Students can represent the value 12×34 through an area model involving an appropriate rectangle. This can be drawn by hand as shown in Figure 1a, or realized through manipulatives such as blocks, or via computer software.

Let us aim to somehow transform the rectangle into a shape whose area will be the sum or difference of the area of two new rectangles, where the area of each of these new rectangles is easy to calculate.

The base of the rectangle in Figure 1 is partitioned to form two rectangular sub-parts, one of which has a base of length 10, that is, we essentially "slice" off $c = 2$, and our particular choice of base will correspond to our comparison number $a = 10$ for this example. Our diagram in Figure 1b represents the partitioning and distributive property

$$12 \times 34 = (10 + 2)\, 34 = 10 \times 34 + 2 \times 34.$$

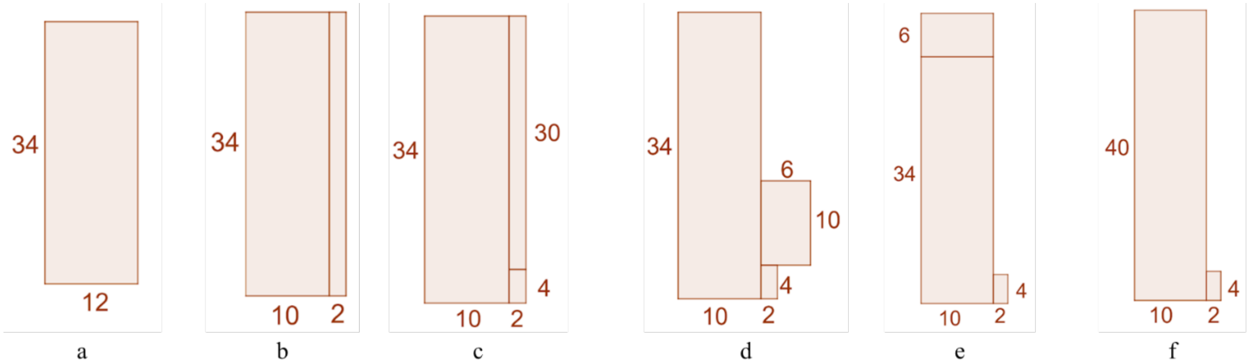

Figure 1. Diagram for Example 1

The vertical sides of the smaller rectangle in Figure 1b can be partitioned into tens and units, forming two smaller rectangles. The dimensions of the rectangle involving the units will involve the values for b and c. Our diagram in Figure 1c represents

$$10 \times 34 + 2 \times 34 = 10 \times 34 + 2(30 + 4) = 10 \times 34 + 2 \times 30 + 2 \times 4.$$

If we examine the top right rectangle in Figure 1c, then we see the vertical sides have length $30 = 3 \times 10$. Thus, we may interpret the 3 as a scaling factor that acts on 10. We can reform this rectangle by transferring this scale factor from the vertical sides to the horizontal sides, that is, this rectangle moves from a form of $2 \times (3 \times 10)$ to $(2 \times 3) \times 10$. Our diagram in Figure 4 represents the culmination of

$$10 \times 34 + 2 \times 30 + 2 \times 4 = 10 \times 34 + 2\ (10 \times 3) + 2 \times 4$$
$$= 10 \times 34 + 10\ (2 \times 3) + 2 \times 4 = 10 \times 34 + 10 \times 6 + 2 \times 4.$$

The height of this transformed rectangle in Figure 1d is now the same as the base of the biggest rectangle therein. Thus, we can rotate, move and reconnect this rectangle as per Figure 1e. Thus, our diagram in Figure 1e represents

$$10 \times 34 + 10 \times 6 + 2 \times 4 = 10(34 + 6) + 2 \times 4.$$

Finally, we can recombine two of the side-by-side rectangles so that the area of our original rectangle in Figure A1 will now be the sum of the area of two new rectangles, where the area of each of these new rectangles is easy to calculate. In Figure 1f we express

$$10 \times 40 + 2 \times 4 = 400 + 8 = 408.$$

If we bring everything together then we can see the following is captured by our sequence of diagrams:

$$12 \times 34 = (10 + 2)34 = 10 \times 34 + 2 \times 34$$
$$= 10 \times 34 + 2(30 + 4) = 10 \times 34 + 2 \times 30 + 2 \times 4$$
$$= 10 \times 34 + 2\ (10 \times 3) + 2 \times 4$$
$$= 10 \times 34 + 10 \times 6 + 2 \times 4 = 10(34 + 6) + 2 \times 4$$
$$= 10 \times 40 + 2 \times 4 = 400 + 8 = 408.$$

Note that in my translation of pictures into the above symbols, no step occurs that is not both unequivocal and trivial, and so this aligns with the position of Littlewood [33].

Example 2: Consider 8×29.

In (2) we could choose $a = 10$ so that $c = -2$, $x = 3$ and $b = -1$. Thus,

$$8 \times 29 = 10(29 - 2 \times 3) + 1 \times 2 = 10(23) + 2 = 232.$$

Once again, let me reimagine this from diagrammatic perspectives, with students starting with the initial area model represented in Figure 2a.

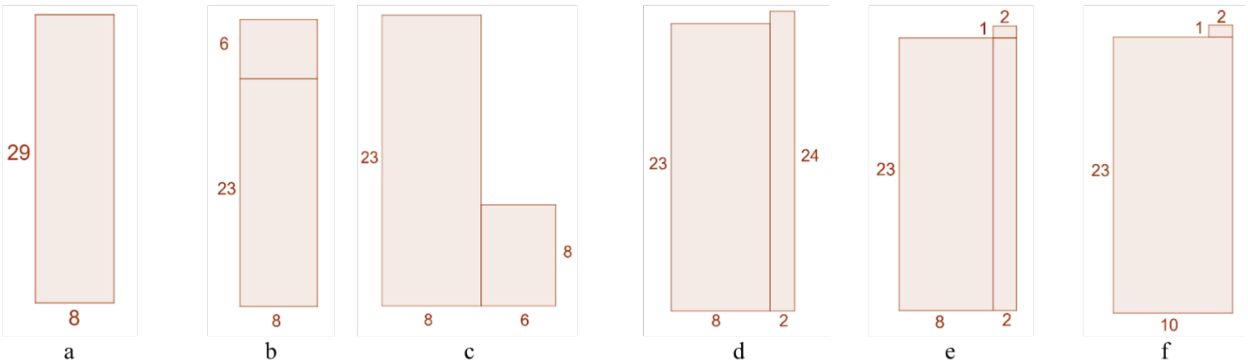

Figure 2. Diagram for Example 2

Once more, the idea is to transform the rectangle into a shape whose area will be the sum or difference of the area of two new rectangles, where the area of each of these new rectangles is easy to calculate. However, I will do this in a different way to the previous example.

Let us aim to construct a figure where the larger of the two rectangles has a base of 10. This choice of base will correspond to our comparison number $a = 10$ for this example. Since the length of the base is $8 < 10$ it is unclear how we can slide the horizontal sides as per the previous example. However, we can slice the vertical sides of the rectangle, partitioning into two rectangular sub-parts, one with height $6 = -cx = 2 \times 3$ where x is our scaling factor.

Our diagram in Figure 2b represents

$$8 \times 29 = 8 \ (23 + (2 \times 3)).$$

The upper rectangle in Figure 8 can be rotated and translated with its new position illustrated in Figure 2c.

Our diagram in Figure 2c can be viewed as representing

$$8 \ \big(23 + (2 \times 3)\big) = 8 \times 23 + \ 8 \ (2 \times 3).$$

The sub-rectangle on the right in Figure 2c can be transformed so that its base has a length of 2. To do this, students can transfer the scale factor of $x = 3$ from the two horizontal sides to the two vertical sides. That is, this rectangle moves from a form of $8 \times (2 \times 3)$ to $2 \times (8 \times 3)$. Our diagram in Figure 2d represents the culmination of

$$8 \times 23 + 8\,(2 \times 3) = 8 \times 23 + 2\,(8 \times 3) = 8 \times 23 + 2 \times 24.$$

We can now partition the taller rectangle to "align" with the rectangle on its left. In Figure 2e we have illustrated

$$8 \times 23 + 2 \times 24 = 8 \times 23 + 2(23 + 1) = 8 \times 23 + 2 \times 23 + 2 \times 1.$$

Thus, recombining into two rectangles we have the situation in Figure 2f that can be captured by

$$8 \times 23 + 2 \times 23 + 2 \times 1 = (8 + 2)\ 23 + 2 \times 1 = 10\ 23 + 2 \times 1 = 232.$$

If we bring everything together then we can see the following sequence that is captured by our diagrams:

$$\begin{aligned} 8 \times 29 &= 8 \times (23 + 2 \times 3) = 8 \times 23 + 8 \times (2 \times 3) \\ &= 8 \times 23 + 2(8 \times 3) = 8 \times 23 + 2 \times 24 \\ &= 8 \times 23 + 2(23 + 1) \\ &= 8 \times 23 + 2 \times 23 + 2 \times 1 \\ &= (8 + 2) \times 23 + 2 \times 1 = 10 \times 23 + 2 \times 1 \\ &= 230 + 2 = 232. \end{aligned}$$

Example 3: Consider 17×41.

In (2) we could choose $a = 20$ so that $c = -3$, $x = 2$ and $b = 1$. Thus,

$$17 \times 41 = 20(41 - 2 \times 3) - 1 \times 3 = 20(35) - 6 = 700 - 6 = 694.$$

Once again, let me reimagine this from diagrammatic perspectives, with students starting with the initial area model represented in Figure 3a.

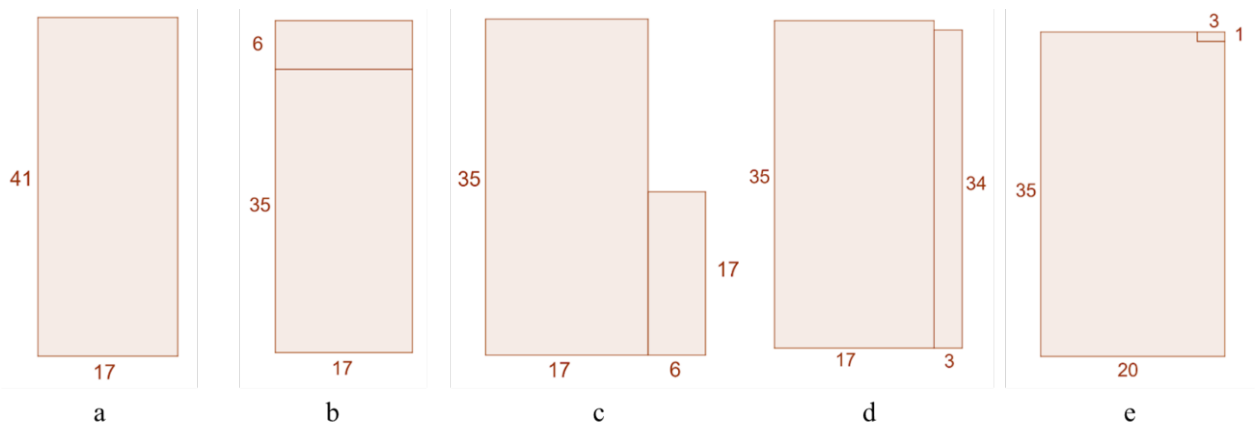

Figure 3. Diagram for Example 3

Once again, the idea is to transform the rectangle into a shape whose area will be the sum or difference of the area of two new rectangles, where the area of each of these new rectangles is easy to calculate. In this particular case it will be a difference of two areas.

Let us aim to construct a figure where the larger of the two rectangles has a base of 20. This choice of base will correspond to our comparison number $a = 20$ for this example. Since the length of the base is $17 < 20$ we can slice the vertical sides of the rectangle, partitioning them into two rectangular sub-parts, one with height $6 = -cx = 2 \times 3$ where x is our scaling factor. Our diagram in Figure 3b represents $17 \times 41 = 17 \times (35 + 6)$.

The upper rectangle in Figure 3b can be rotated and translated with its new position illustrated in Figure 15. Our diagram in Figure 3c represents

$$17\,(35 + 6) = 17 \times 35 + (2 \times 3)\ 17.$$

The rectangle on the right in Figure 3c can be transformed so that its base has a length 3. To do this, students can transfer the scale factor of $x = 3$ from the two horizontal sides to the two vertical sides. That is, this rectangle moves from a form of $(2 \times 3) \times 17$ to $3 \times (2 \times 17)$.

Our diagram in Figure 3d represents

$$17 \times 35 + (2 \times 3) \times 17 = 17 \times 35 + 3\ (2 \times 17) = 17 \times 35 + 3 \times 34.$$

We can now see that the desired area of the two rectangles in Fig 16 is the difference in areas of two other rectangles, with this captured via Figure 3e. In particular, Figure 3e captures the culmination of

$$17 \times 35 + 3 \times 34 = 17 \times 35 + 3\,(35 - 1) = (17 + 3)\ 35 - 3 \times 1 = 20 \times 35 - 3 \times 1.$$

If we bring everything together then we can see the following sequence that is captured by our diagrams:

$$17 \times 41 = 17 \times (35 + 3 \times 2) = 17 \times 35 + 17 \times (3 \times 2)$$
$$= 17 \times 35 + 3\,(17 \times 2) = 17 \times 35 + 3 \times 34$$
$$= 17 \times 35 + 3\,(35 - 1)$$
$$= 17 \times 35 + 3 \times 35 - 3 \times 1$$
$$= (17 + 3) \times 35 - 3 \times 1 = 20 \times 35 - 3 \times 1 = 700 - 3 = 697.$$

Alternatively, for this example we could choose a=10 so that

$$17 \times 41 = 10\,(41 + 4 \times 7) + 7 \times 1$$
$$= 10\,(41 + 28) + 7 = 10 \times 69 + 7 = 690 + 7 = 697.$$

4. Discussion

What are the advantages and limitations of the ideas in the previous section?

Looking back at our previous examples, we can see that conservation of area forms a key, underlying principle. Kospentanris, Spyrou and Lappas [30] draws on the idea that the actions of cutting -and-pasting to rearrange parts of a figure to produce another one with equal area may help the students to develop an understanding of the principle of conservation of area, forming a preliminary step in their process of mastering area measurement. Thus, an advantage of the ideas herein is their potential to support this type of development by fostering an awareness of conservation of area and its links with basic area models derived from the original multiplication problem.

However, the advantages of my ideas have the potential to reach further and wider. Note that the actions herein are not limited to cut-and-paste, but also involve a rescaling of rectangles; and the original problems are from arithmetic, not area measurement. In these senses the ideas herein also extend to building connections, awareness and understanding involving cutting-and-pasting actions, rescaling actions, conservation of area and multiplication problems from arithmetic (rather than a limitation to the measurement of area).

Conversely, the notions of conservation of area and rescaling of rectangles are non-trivial concepts for younger learners to grasp and apply, and so a reliance on these elements also form a potential limitation of the ideas and ways of working herein.

If we compare the strategies in Example 1 with Example 2, then we see there are differences in the processes therein. This illustrates that there is no universal strategy and that the rearrangements can be performed in multiple ways. One of the benefits of this observation is that students have the freedom to consider their options and then choose which method suits them best. The importance of this flexibility in problem-solving and understanding at an individual learner level is supported by Tisdell [46]. On the flip-side, having multiple options may risk overwhelming students with too many choices and there may be insufficient direction regarding when to choose which strategy.

If we reflect on the use of diagrams in my examples, then we see that the figures themselves can tell a story. On one hand, the cutting, scaling and rearrangements of the rectangles herein can be communicated, carried out and understood independently of the algebraic / arithmetical textual representations, and so in this sense it aligns with the suggestions of Casselman [7] and Tufte [49] regarding principles for making good illustrations in mathematics. On the other hand, the pictures and arithmetical representations herein need not be independent of each other in the sense that they can be used together, when the situation allows it (for example, when learners may be more comfortable with algebraic aspects). In particular, the operations involving my pictures are analogous to algebraic operations and so our use of pictures are rigorous, drawable and printable, "in the sense that in translating into symbols no step occurs that is not both unequivocal and trivial" and so align with the position of Littlewood [33] in having the potential to help learners to see through the fog.

Drawing on the position of Casselman [7] that "it is rare for there to be too many illustrations in a mathematics paper", I believe that it is rare for there to be too many diagrams in the learning and teaching of mathematics. Students and teachers can often understand more rapidly and conceptually what a certain part of the curriculum is about when illustrations are plentiful and meaningful. Students and teachers can skim the sequence of diagrams herein rather than reading the associated text in isolation. Furthermore, students have the potential to draw plenty of pictures regarding the ideas and processes herein that involve the conservation of area and multiplication problems. Their pictures can be modified, redrawn and explored. My establishment of what diagrammatic and dynamic perspectives could look like for more complex classes of multiplication problems forms a framework for practice, skill and understanding.

Naturally, I wish to avoid visual clutter and distraction through pictures, and the question of how many pictures are necessary for these kinds of strategies is dependent on the individual's relationship between the problem and the methods under consideration. For one learner, many pictures may be helpful in shining a light; whereas for others, one or two may suffice [46].

As mentioned earlier, students and teachers can draw the associated rectangles and rearranged polygons herein via various methods, such as pencil-and-paper or digital means. The question of variety in visualization methods in multiplication and conservation of area aligns with the position of Hanna and Sidoli [22] in the sense that "there is room for more effort aimed at better ways to use visualisation" in such contexts.

The ideas discussed in the previous section herein can accommodate the original problem of 93×17 (choose, say, $a = 20$ so that $x = 5$, $b = -7$ and $c = -3$; or choose $a = 10$ so that $x = 9$, $b = 3$ and $c = 7$). However, it is not immediately apparent how the ideas could be applied to a problem such as 67×97 (where we now have both prime factors). One of the challenges here is to choose a and $x \geq 1$ in such a way to keep the calculation $b \times c$ under control (say, to single digit factors), which on the surface, does not seem possible. Thus, we see that although the strategies herein have the ability to solve problems that could not be successfully navigated before, they do not appear to be universally applicable.

Multiplication problems that involve three factors can be linked with the volume of a rectangular prism. These problems might be able to be reimagined as the sum or difference of the volumes of two new prisms, where the volume of each of these new prisms is easy to calculate. I acknowledge that these solids are harder to draw and lie outside the scope of the present article.

5. Ideas for the Classroom

In this section I offer some ideas for the classroom (8-12 year-olds) that teachers can draw on to supplement the previous multiplication concepts. In particular, I propose some small, specific uses of history in the mathematics classroom, guided by an "illumination approach" [27] that incorporates "historical snippets" [50, pp. 208, 214]. In this style, these smaller supplements align with Jankvist's analogy of educators adding "spices" to the mathematics education casserole [27].

Janquist [27] synthesises some of the classic arguments found in the literature that support the use of history as a tool for teaching mathematics, and includes its potential to act as an enabler of: motivation, excitement, interest and confidence-building in learners. Furthermore, Russ et al [37, p7] argue that if teachers employ historical approaches, then this potentially humanizes mathematics and can make it less frightening to learners. In particular, my ideas are guided by the position of Marshall and Rich [34], and Furner and Brewer [17] who identified the need for specificity and examples regarding what history in mathematics education might look like.

Spice #1: Conservation principles via Leonardo Da Vinci's Notebooks

Connolly (in [28]) describes Leonardo Da Vinci as the original Renaissance man. Over the past 500 years, Leonardo's profound and innovative contributions as an artist, sculptor, scientist, mathematician, architect and inventor have incited significant interest and admiration. Leonardo's legacy is so famous that some younger learners may recognize his name and some of his works, such as the Mona Lisa.

Leonardo's use of conservation principles in mathematics is in a similar spirit to the earlier ideas seen in this work and can be found in his famous notebooks. For example, in Codex Foster I, his intention was to dedicate the ideas therein to "transformation of a body into another one without decrease or increase of the matter" [24, p316]. The two types of transformations that we have seen earlier in this paper are skilfully explored by Leonardo in his notebooks: the cut-and-paste action; and the continuous deformation process.

For example, learners can explore Leonardo's mode of thought regarding cut-and-paste actions through his work on conserving the area between a triangle and a scythe (falcate). Leonardo captures this kind of conservation in Codex Madrid II, folio 111v [11] via Figure 4.

Leonardo's accompanying text in Codex Madrid II is translated by Capra [6, p268] "I shall take away portion b from triangle ab, and I will return it at c. If I give back to a surface what I have taken away from it, the surface returns to its former state." Another similar line of thought from Leonardo can be found in [12] that has been included in Figure 5.

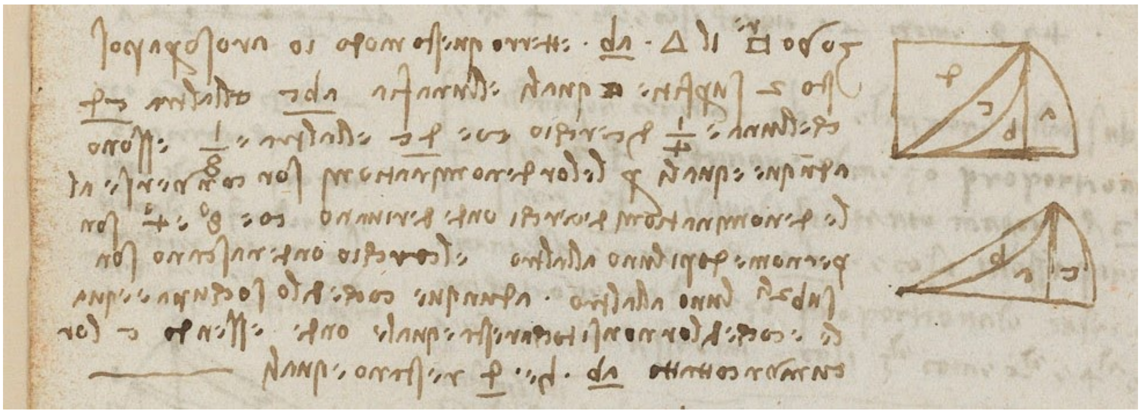

Figure 4. One example of conservation in Codex Madrid II captured by Leonardo

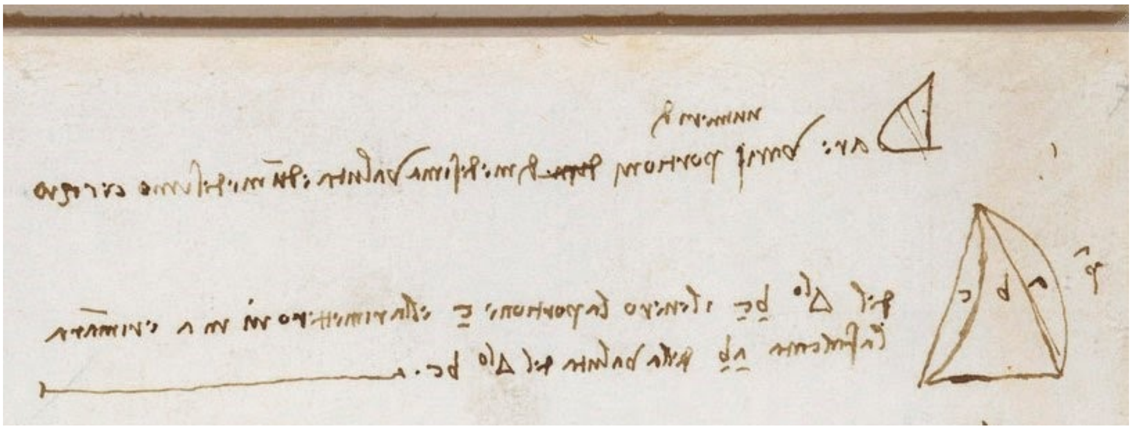

Figure 5. Another example of conservation in Codex Madrid II captured by Leonardo

In addition, Leonardo's notebooks enable learners to discover his conservation of volume method where he melts wax to transform from one solid to another. For example, on Codex Atlanticus folio 820v, he writes [10]: "Take some amount of wax and with that make a square, on it should be made a 4 side pyramid. What should be the height of the pyramid ? Transform the wax into a cube and take 2 parts of wax, and make a cylinder with the same base and height, then take 2/3 of that and you will get the wax for the proposed Pyramid." [24, p316]

In recent years, many notebooks of Leonardo have been scanned and the images have become publicly available online via various libraries. Using these scans as primary sources supporting history in mathematics education aligns with the position of Jahnke [26] by presenting ambitious and rewarding opportunities in the classroom. In line with the previous 'spices' analogy, the material from Leonardo's notebooks needs to be chosen with great care to ensure they are appropriate for the classroom situation. A few carefully chosen diagrams (such as those herein) blended with some anecdotes and stories could potentially play this role [40].

Spice #2: The lives and times of calculating prodigies

Another strategy that can form a supplement for the classroom setting is a discussion of the lives, times and achievements of lightning-fast calculators. Since the multiplication strategies herein can be built into mental strategies for multiplication, the connection with lightning-fast calculators is well-aligned.

Educators could refer to names, dates, famous works and events, time charts, biographies and famous problems, methods and so on (e.g., [43, 44]) of such lightning calculators. For example, there are several interesting characters who have developed skill and accuracy in multiplying numbers mentally. This includes: Yaashwin Sarawanan, who was the runner-up in the TV show of Asia's Got Talent in 2019; Scott Flansburg, who is dubbed "The Human Calculator" and is listed in the Guinness Book of World Records for speed of mental calculation; and Arthur Benjamin, who is a mathematics professor and well-known for his mental mathematics abilities with a strategy he calls "mathemagics" [3]. Furthermore, there are dozens of interesting stories of calculating prodigies in Smith's book [41] (1983) that can be shared.

Classroom discussions on The Mental Calculator World Cup [36] also provides an avenue for the "spices" to be added to the mathematics education casserole.

6. Conclusion

I am now in a position to respond to the original research questions posed in the Introduction.

RQ1: How can diagrammatic and dynamic pedagogies be established for multiplication strategies when the numbers involved are not close together?

Through an analysis of case studies, and drawing on the principle of conservation of area, new methods emerged herein that involved transforming a rectangle via cutting, rescaling and pasting into a shape whose area is the sum or difference of the area of two new rectangles, where the area of each of these new rectangles is easy to calculate. These actions can involve problems where the numbers are not close together and the rearrangement strategies are of a plural nature.

RQ2: What are the potential benefits and limitations of such pedagogies?

Part of the benefit in my ideas herein is in their potential to build connections, awareness and understanding of and multiplication problems from arithmetic via cutting-and-pasting actions, rescaling actions, conservation of area; and not being limited just to the measurement of area. Furthermore, my framework offers alternatives to the use of arithmetical text, so that learners and educators can utilize the sequence of diagrams herein rather than solely reading the associated text in isolation. In addition, our results can be applied to navigate multiplication problems that were not previously tractable.

However, the ideas herein do not appear to immediately apply to a problem such as 67×97 due to the factors being "not far enough" apart, and so this remains an avenue that warrants further exploration.

RQ3: How might these pedagogies be supplemented in a classroom setting?

I offered some illumination strategies with which the ideas herein might be supplemented in the classroom. A use of history and iconic people has the potential to enable students to appreciate cultural aspects associated with the concepts, and to humanize the ideas.

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