Quantitative Finance and Economics

http://www.aimspress.com/journal/QFE

QFE, 8(2): 347–363.
DOI:10.3934/QFE.2024013
Received: 30 December 2023
Revised: 07 April 2024
Accepted: 15 May 2024
Published: 23 May 2024

Research article

A proposal of concentration measures for discount functions

Salvador Cruz Rambaud*, Piedad Ortiz Fernández, Javier Sánchez García and Paula Ortega Perals

Departamento de Economía y Empresa, Universidad de Almería, La Cañada de San Urbano, s/n, 04120, Almería, Spain

* **Correspondence:** Email: scruz@ual.es; Tel: +34950015184; Fax: +34950015178.

Abstract: The framework of this paper was intertemporal choice, that is to say, the process whereby people were required to choose between a smaller-sooner reward and a larger-later income. In this study, the selection of rewards was supported by a discount function instead of direct preferences between the involved rewards. The objective of this paper was to measure the discounting concentration of a discount function through a variant of the Gini index and the Lorenz curve usually used in statistics. Both measures allowed for the comparison of the discounting concentration corresponding to two discount functions. The methodology employed in this paper was based on the parallelism between a discount function and the distribution function of an absolutely continuous random variable. This similarity allowed us to export the measures of concentration from the field of statistics to finance. The main result of this work was the analysis of the discounting concentration depending on other characteristics of the shape of a discount function (regularity and super-additivity) and the total area under the discount function curve.

Keywords: intertemporal choice; discount function; regularity; super-additivity; area under the discounting curve

JEL Codes: C02, G40

1. Introduction

Some characterizations and properties of the discount functions involved in intertemporal choices are based on either their shape or the area under the curve of their corresponding graphs. In effect, Caliendo (2014) introduces the concept of *overall level of impatience of a discount function F from the perspective of age τ*, denoted by $I(\tau)$, as the area above the discount function and below the horizontal line at unity, i.e., $I(\tau) := \int_0^{\bar{T}-\tau} D(t)\mathrm{d}t$, where $D(t) := 1 - F(t)$ is the discount associated to $F(t)$ and $\bar{T}$ is the end of economic life span. This magnitude allows us to compare hyperbolic and exponential discount functions in dynamic economic models by ensuring that the overall level of impatience of the two discount functions is the same, thus isolating the pure effect of self-control on intertemporal choices. Additionally, several scholars have highlighted the parallelism between the expressions of a discount function and a probability distribution function (see Takeuchi (2011) or Cruz Rambaud et al. (2018)). There is a scarce literature on intertemporal choice focusing on the shape of the discount function F involved in the decision-making process. However, taking into account the parallelism between the discount associated to F ($D := 1 - F$) and the distribution function of a random variable, and also the parallelism between $f := D'$ and the corresponding density function, we can take advantage from the shape measures associated to a random variable and export them to the context of discount functions. This approach could be interesting because, to the extent of our knowledge, there is not a single figure which summarizes the behavior of a discount function. Specifically and following this methodology, we could consider a measure of asymmetry, bias, or concentration of the discount function. Focusing on the last measure, this paper demonstrates that the value of the so-called discounting concentration has implications on the type of nonadditivity (sub-additivity or super-additivity) of the discount function. The objective of this paper is to propose a measure of the discounting concentration of a given discount function and compare several discount functions according to their degrees of concentration. To do this, we are going to be based on the parallelism between the concepts of discount function and probability distribution function. Specifically, we are going to import the concept of Gini concentration index from the field of statistics to finance. It is well-known that this index is a measurement of inequality (Atkinson, 1970) which, in the case of a discount function, will represent a measure of the discounting concentration in certain parts of the domain of such discount function. Additionally, we will introduce the concept of the Lorenz curve associated to a discount function as a more complete tool to analyze the discounting concentration.

This parallelism is not new. Let us present a well-known case related to some properties of discount functions related to nonadditivity. Given a discount function, $F(t)$, the discount associated to $F(t)$, say $D(t)$, satisfies the conditions required to be a probability distribution function. This parallelism justifies the relevance of the discount when trying to apply some concepts from a probability distribution function to discounting processes. For example, a discount function is said to be sub-additive if, for every s and $t \geq 0$,

$$F(s)F(t) < F(s+t). \tag{1}$$

Taking natural logarithms, the sub-additivity remains:

$$\ln F(s) + \ln F(t) < \ln F(s+t). \tag{2}$$

As the logarithmic function is increasing, inequalities (1) and (2) are equivalent. However, by considering the discount, one has:

$$D(s) + D(t) - D(s)D(t) > D(s+t), \tag{3}$$

which implies:

$$D(s) + D(t) > D(s+t). \tag{4}$$

Equation (4) describes the concept of sub-additivity of continuous random variables which shows the aforementioned parallelism (Shaked, 1994). Unfortunately, the sub-additivity of the discount is not a sufficient condition for the sub-additivity of the discount function. However, it can be shown that, for values of s and t small enough, inequalities (1) and (4) are equivalent since

$$\ln F(t) = \ln[1 - D(t)] \sim [-D(t)]$$

are equivalent infinitesimal functions.

This paper has been organized as follows. After this Introduction, Section 2 introduces a generalization of the concept of discount function without requiring the condition of continuity. Later, Section 3 presents the so-called discounting concentration based on the parallelism between the discount associated to a discount function and a probability distribution function. A specific consideration of the discounting concentration of sub-additive and super-additive discount functions has been made in Subsection 3.1. Section 4 describes the way of comparing the discounting concentration corresponding to two discount functions through their corresponding Lorenz curves. Finally, Section 5 summarizes and concludes.

2. A generalization of the concept of discount function

Following Lueddekens et al. (2022), discounting is a measure for changing background concentrations and, therefore, for changing impact characterization and normalization. Zhou et al. (2025) presented a consumer search model with prospective utility, in which the consumer's subjective evaluations for each company's product are assumed to be independent but not identically distributed uncertain variables, and the evaluation of a reserved product with probabilistic exhaustion is characterized consequently as an uncertain random variable.

Usually, the concept of discount function requires continuity. The following definition generalizes the concept of discount function without requiring such a condition.

Definition 1. *A discount function is a real-valued function*

$$F : \mathbb{R}^+ \cup \{0\} \longrightarrow]0, 1],$$

such that:

$$t \mapsto F(t)$$

satisfying the following three conditions:

1. $F(0) = 1$.
2. *F is strictly decreasing.*
3. *F is piecewise continuous and exhibits a finite number of discontinuities.*

Definition 2 describes real situations in which the interest rate is increasing with the term of the investment. In effect, this situation could occur for discounting bills of exchange, in which the interest is higher as the term increases. For example:

- Bills with a maturity up to 15 days: 3%.
- Bills with a maturity up to 1 month: 5%.
- Bills with a maturity up to 3 months: 6.5%.

- Bills with maturities greater than 3 months: 8%.

In effect, this is logical because the longer the due date, the risk of the operation increases, and so the discount interest is higher (see Figure 2).

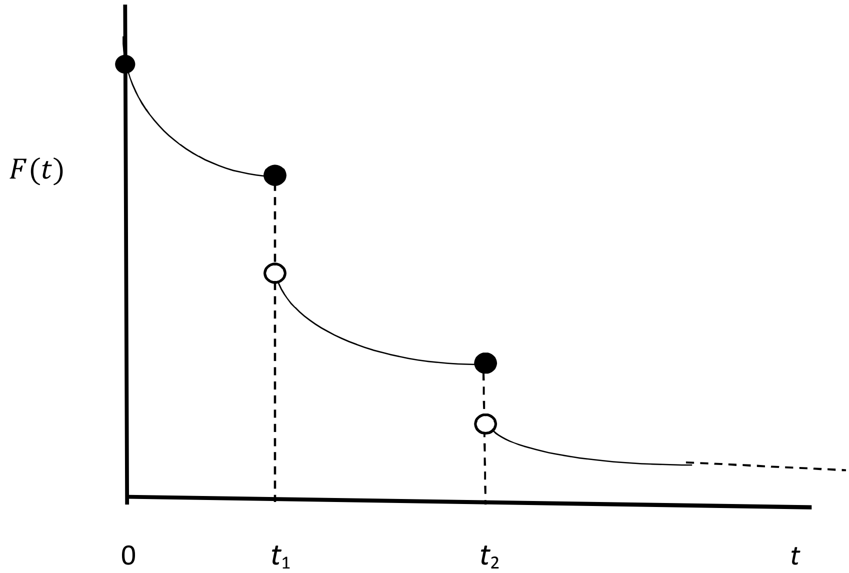

Figure 1. Plotting a discount function with discontinuities. Source: Own elaboration.

A next step in the definition of a discount function is requiring continuity in all its domain; and, finally, the most strict case is considering that the discount function is derivable in its domain. This distinction is important and is going to be justified in Section 3 when pointing our the parallelism between the discount associated to a discount function and the distribution function of a random variable.

3. Defining an index of discounting concentration

As stated, given a discount function, $F(t)$, the discount associated to $F(t)$, say $D(t)$, can be considered as the distribution function of the variable representing the random lifetime of 1$ in an investment throughout time. Observe that the first case of a discount function considers a mass of probability at a finite number of times. The second case corresponds to a continuous random variable which is differentiable except at a finite number of "corners". Finally, the third case allows us to have a valid density function at our disposal (Cramér, 1961; Kotz, 2006). This parallelism justifies the following definition of discounting concentration based on the Gini concentration index of a probability distribution function.

Definition 2. *Let $F(t)$ be a discount function. The discounting concentration of $F(t)$ is defined as (Calot, 1965):*

$$i := \frac{1}{m}\int_0^{+\infty} t\mathrm{d}D^2(t) - 1,$$

where:

$$m := \int_0^{+\infty} t f(t)\mathrm{d}t$$

is the mean and

$$f(t) := \frac{\mathrm{d}}{\mathrm{d}t} D(t)$$

is the density function corresponding to $D(t)$.

It can be shown that $0 \leq i \leq 1$. The following lemma gives three alternative expressions of the concentration. One of them is the most suitable when dealing with discount functions.

Lemma 1. *The following expressions of the discounting concentration of $F(t)$ are equivalent:*

(i) $i = \frac{1}{m} \int_0^{+\infty} t \mathrm{d}D^2(t) - 1.$
(ii) $i = \frac{1}{m} \int_0^{+\infty} F(t)D(t)\mathrm{d}t.$
(iii) $i = 1 - \frac{1}{m} \int_0^{+\infty} F^2(t)\mathrm{d}t.$

Moreover, the mean can be written as $m = \int_0^{+\infty} F(t)\mathrm{d}t$.

Proof. See Lubrano (2017).

The expression of the discounting concentration included in Lemma 1(ii) facilitates the interpretation of this concept (included in Lemma 1(i)) as the weighted average of the discount $D(t)$ at each instant t by using the values of the discount function as weights. On the other hand, we will use the formula contained in Lemma 1(iii) and that of the mean to calculate the discounting concentration due to its simplicity and because it only depends on the expression of the discount function $F(t)$.

The finiteness of the area under the curve of a discount function $F(t)$ is an important property to characterize the process of intertemporal choice described by $F(t)$ (Chambers, 2023). In the next propositions, we will see the relevance of this condition when determining the value of the discounting concentration. In order to organize the development of this section, we are going to provide Figure 2.

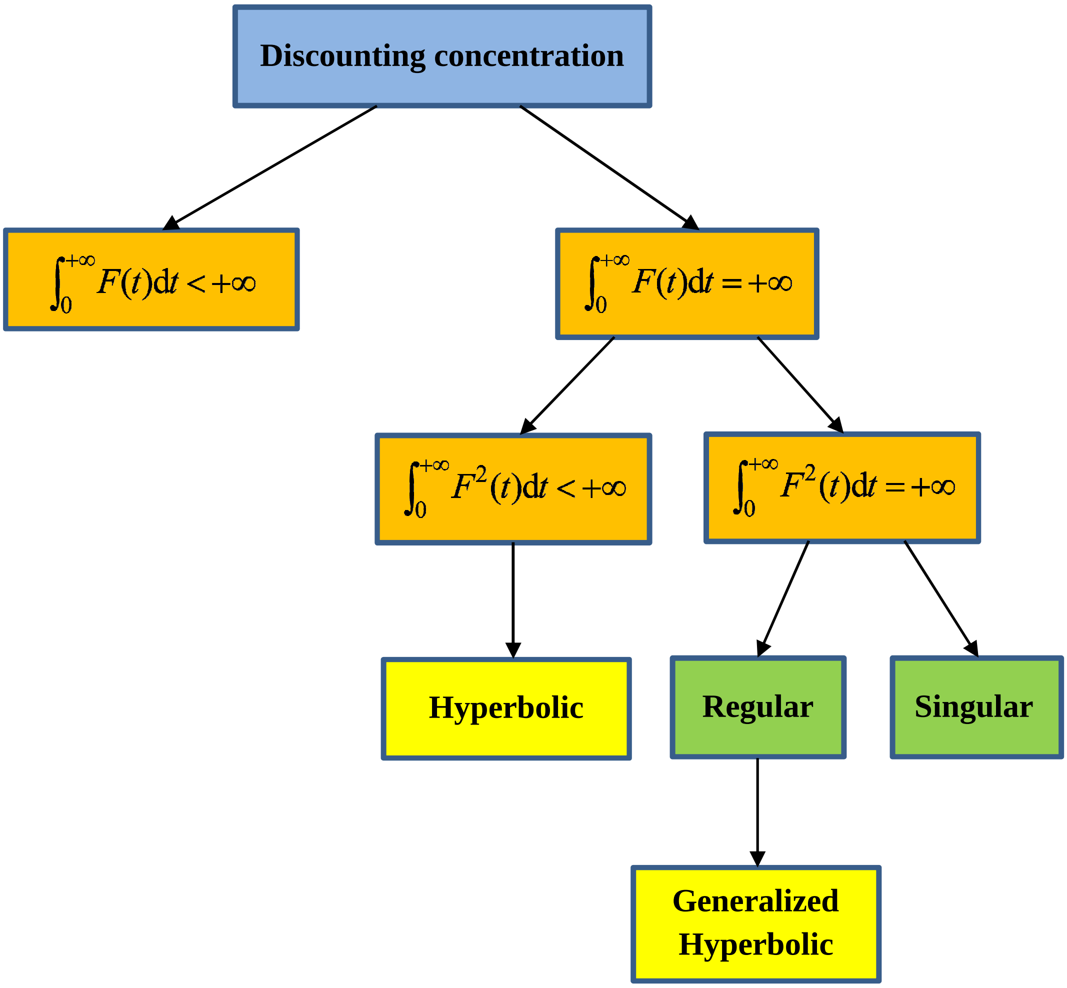

Figure 2. Scheme of the results included in this section. Source: Own elaboration.

Proposition 1. *Let $F(t)$ be a discount function such that $\int_0^{+\infty} F(t)\mathrm{d}t < +\infty$. In this case,* $0 < i < 1$.

Proof. In effect, as $0 < F(t) \leq 1$, then:

$$\int_0^{+\infty} F^2(t)\mathrm{d}t < \int_0^{+\infty} F(t)\mathrm{d}t < +\infty.$$

Therefore,

$$0 < \frac{1}{m}\int_0^{+\infty} F^2(t)\mathrm{d}t < 1$$

and, consequently,

$$0 < i < 1,$$

as required.

The following example calculates the discounting concentration when the discount function is additive.

Example 1. *Let us calculate the discounting concentration of the exponential discount function:*

$$F(t) := \exp\{-kt\},\ k > 0.$$

In this case, one has:

$$m := \int_0^{+\infty} \exp\{-kt\}\mathrm{d}t = -\frac{1}{k}\exp\{-kt\}]_0^{+\infty} = \frac{1}{k}.$$

On the other hand,

$$\int_0^{+\infty} F^2(t)\mathrm{d}t = \int_0^{+\infty} \exp\{-2kt\}\mathrm{d}t = \frac{1}{2k}\exp\{-2kt\}]_0^{+\infty} = \frac{1}{2k}.$$

Consequently,

$$i = 1 - k\frac{1}{2k} = \frac{1}{2}.$$

In the following propositions, we are going to assume that $\int_0^{+\infty} F(t)\mathrm{d}t = +\infty$.

Proposition 2. *Let $F(t)$ be a discount function such that $\int_0^{+\infty} F(t)\mathrm{d}t = +\infty$ and $\int_0^{+\infty} F^2(t)\mathrm{d}t < +\infty$. In this case, $i = 1$.*

Proof. It is obvious taking into account that, in this case,

$$\frac{1}{m}\int_0^{+\infty} F^2(t)\mathrm{d}t = 0$$

and, consequently,

$$i = 1.$$

The following example calculates the discounting concentration of another relevant discount functions, viz, the hyperbolic discount function.

Example 2. *Let us calculate the discounting concentration of the hyperbolic discount function:*

$$F(t) := \frac{1}{1+kt},\ k > 0.$$

In this case, one has:

$$m := \int_0^{+\infty} \frac{1}{1+kt}\mathrm{d}t = \frac{1}{k}\ln(1+kt)]_0^{+\infty} = +\infty.$$

On the other hand,

$$\int_0^{+\infty} F^2(t)\mathrm{d}t = \int_0^{+\infty} \frac{1}{(1+kt)^2}\mathrm{d}t = \frac{1}{k}\left.\frac{1}{1+kt}\right]_0^{+\infty} = \frac{1}{k}.$$

Consequently,

$$i = 1 - \frac{1}{+\infty}\frac{1}{k} = 1.$$

In order to enunciate the next proposition, we need to introduce the following concepts.

Definition 3. *A discount function $F(t)$ is said to be regular if* $\lim_{t\to+\infty} = 0$.

Definition 4. *A discount function $F(t)$ is said to be singular if* $\lim_{t\to+\infty} \neq 0$.

Remark 1. *A singular discount function is a peculiar discount function which has a horizontal asymptote at $y = l$, where l can be interpreted as the mass of probability at infinity of the corresponding distribution function (Cruz Rambaud, 2014).*

Proposition 3. *Let $F(t)$ be a regular discount function such that $\int_0^{+\infty} F^2(t)\mathrm{d}t = +\infty$. In this case,* $i = 1$.

Proof. Observe that $\int_0^{+\infty} F(t)\mathrm{d}t = \int_0^{+\infty} F^2(t)\mathrm{d}t = +\infty$. In this case, the value of i is an indeterminate limit of the form $1 - \frac{\infty}{\infty}$, which can be solved by using L'Hôpital's rule and the fundamental theorem of calculus:

$$i = 1 - \frac{\lim_{t\to+\infty} \int_0^t F^2(z)\mathrm{d}z}{\lim_{t\to+\infty} \int_0^t F(z)\mathrm{d}z} = 1 - \frac{\lim_{t\to+\infty} F^2(t)}{\lim_{t\to+\infty} F(t)} = 1 - \lim_{t\to+\infty} \frac{F^2(t)}{F(t)} = 1 - 0 = 1.$$

Consequently, $i = 1$.

Example 3. *Let us calculate the discounting concentration of the following generalized hyperbolic discount function:*

$$F(t) := \frac{1}{(1+kt)^{1/2}},\ k > 0.$$

In this case, one has:

$$m := \int_0^{+\infty} \frac{1}{(1+kt)^{1/2}}\mathrm{d}t = \frac{1}{k}\ (1+kt)^{1/2}\Big]_0^{+\infty} = +\infty.$$

On the other hand,

$$\int_0^{+\infty} F^2(t)\mathrm{d}t = \int_0^{+\infty} \frac{1}{1+kt}\mathrm{d}t = \frac{1}{k}\ \ln(1+kt)]_0^{+\infty} = +\infty.$$

By Proposition 3,

$$i = 1.$$

Proposition 4. *Let $F(t)$ be a singular discount function. In this case,* $0 < i < 1$.

Proof. Take into account that, in this case, $\lim_{t\to+\infty} F(t) = l$, where $0 < l < 1$. Consequently, $\lim_{t\to+\infty} F^2(t) = l^2 > 0$, and then $\int_0^{+\infty} F(t)\mathrm{d}t = \int_0^{+\infty} F^2(t)\mathrm{d}t = +\infty$. Therefore, by using the same reasoning as Proposition 3, one has:

$$i = 1 - l \neq 0$$

and, consequently, $0 < i < 1$.

Example 4. *Let us calculate the discounting concentration of the following singular discount function:*

$$F(t) := \frac{1 + at}{1 + bt},\ 0 < a < b.$$

By Proposition 4, one has:

$$i = 1 - \frac{a}{b}.$$

Observe that this family offers a variety of discount functions whose concentration indices can go from 0 to 1. In effect, keeping constant the value of a:

- If $b \to a$, then $i \to 0$, and
- If $b \to +\infty$, then $i \to 1$.

The results displayed in Propositions 1–4 can be summarized in the following scheme:

$$\text{Values of } i \begin{cases} \displaystyle\int_0^{+\infty} F(t)\mathrm{d}t < +\infty \ (0 < i < 1\text{: Proposition 1}) \\ \displaystyle\int_0^{+\infty} F(t)\mathrm{d}t = +\infty \begin{cases} \displaystyle\int_0^{+\infty} F^2(t)\mathrm{d}t < +\infty \ (i = 1 \text{ Proposition 2}) \\ \displaystyle\int_0^{+\infty} F^2(t)\mathrm{d}t = +\infty \begin{cases} \lim_{t\to+\infty} F(t) = 0 \ (i = 1 \text{ Proposition 3}) \\ \lim_{t\to+\infty} F(t) \neq 0 \ (0 < i < 1 \text{ Proposition 4}) \end{cases} \end{cases} \end{cases}$$

3.1. Sub-additivity, super-additivity and discounting concentration

Definition 5. *A discount function $F(t)$ is said to be super-additive if, for every s and t ($s \geq 0$ and $t \geq 0$), one has:*

$$F(s)F(t) > F(s + t). \tag{5}$$

In particular, taking $s = t = z$, inequality (5) results in:

$$F^2(z) = F(z)F(z) > F(2z). \tag{6}$$

Theorem 1. *The discounting concentration of a super-additive discount function is less than or equal to 0.5.*

Proof. In effect, let us consider the following ratio taking part of the definition of the discounting concentration:

$$l := \frac{\int_0^{+\infty} F^2(t)\mathrm{d}t}{\int_0^{+\infty} F(t)\mathrm{d}t}.$$

This quotient of integrals can be successively written as follows:

$$l = \frac{\lim_{t\to+\infty} \int_0^t F^2(z)\mathrm{d}z}{\lim_{t\to+\infty} \int_0^t F(z)\mathrm{d}z}$$

$$= \frac{\lim_{t\to+\infty}\int_0^t F^2(z)\mathrm{d}z}{2\lim_{t\to+\infty}\int_0^{t/2} F(2z)\mathrm{d}z}$$

$$= \frac{\lim_{t\to+\infty}\int_0^{t/2} F^2(z)\mathrm{d}z + \lim_{t\to+\infty}\int_{t/2}^t F^2(z)\mathrm{d}z}{2\lim_{t\to+\infty}\int_0^{t/2} F(2z)\mathrm{d}z} \text{ (by inequality (6))}$$

$$\geq \frac{1}{2} + \frac{\lim_{t\to+\infty}\int_{t/2}^t F^2(z)\mathrm{d}z}{2\lim_{t\to+\infty}\int_0^{t/2} F(2z)\mathrm{d}z}.$$

Therefore $l \geq \frac{1}{2}$ and then $i := 1 - l \leq \frac{1}{2}$.

Example 5. *Let us calculate the discounting concentration of the linear discount function:*

$$F(t) := 1 - kt, \ k > 0.$$

First, observe that the domain of F is the interval $[0, 1/d[$. *Simple calculations lead to:*

- $\int_0^{1/k}(1-kt)^2\mathrm{d}t = \frac{1}{3k}$, *and*
- $\int_0^{1/k}(1-kt)\mathrm{d}t = \frac{1}{2k}$.

Therefore, $l = \frac{2}{3}$, *and then* $i = \frac{1}{3} < 0.5$. *This result confirms Theorem 1 because* $F(t)$ *is super-additive.*

Example 6. *Let us calculate the discounting concentration of the following generalized exponential discount function:*

$$F(t) := \exp\{-kt^2\}, \ k > 0.$$

Taking into account that $F(t) := \exp\{-t^2\}$ *is the half of the so-called Euler-Poisson integral, simple calculations lead to:*

- $\int_0^{+\infty} \exp\{-2kt^2\}\mathrm{d}t = \frac{\sqrt{\pi}}{2\sqrt{2k}}$, *and*
- $\int_0^{+\infty} \exp\{-kt^2\}\mathrm{d}t = \frac{\sqrt{\pi}}{2\sqrt{k}}$.

Therefore, $l = \frac{1}{\sqrt{2}} < 0.5$, *and then* $i = \frac{\sqrt{2}-1}{\sqrt{2}}$. *This result also confirms Theorem 1 because* $F(t)$ *is super-additive.*

Definition 6. *A discount function* $F(t)$ *is said to be sub-additive if, for every s and t (*$s \geq 0$ *and* $t \geq 0$*), one has:*

$$F(s)F(t) < F(s+t). \tag{7}$$

In this case, contrarily to the super-additive case, the discounting concentration can take any value between 0 and 1. In effect, the discount function of Example 4 is sub-additive and, depending on the values of a and b, i ranges from 0 to 1. Therefore, all cases regarding the additivity and nonadditivity of the discount function can be summarized in Table 1.

Table 1. Values of i according to the additivity of F. Source: Own elaboration.

Additive	Nonadditive and non-super-additive	Super-additive
$i = 0.5$ (Example 1)	$i \geq 0.5$	$i < 0.5$ (Theorem 1)

4. Comparing discounting concentrations

The discounting concentration has the inconvenience that it is a figure which represents the discounting "behavior" of the discount function in its entire temporal domain. Therefore, it is necessary to introduce another more complete and detailed measure of discounting concentration.

The following definition has been inspired in Bonetti et al. (2009).

Definition 7. *The discount function F_2 with instantaneous discount rate δ_2 is said to dominate the discount function F_1 with instantaneous discount rate δ_1 (denoted by $F_1 \preceq_{dr} F_2$) if $\delta_1(t) \geq \delta_2(t)$, for every $t \geq 0$, and $\delta_1 \neq \delta_2$.*

Obviously, this definition is equivalent to require that $F_1 \leq F_2$ and $F_1 \neq F_2$.

Theorem 2. *Let F_1 and F_2 be two discount functions such that $F_1 \preceq_{dr} F_2$ and*

$$\int_0^{+\infty} F_1^2(t)\mathrm{d}t = \int_0^{+\infty} F_2^2(t)\mathrm{d}t = +\infty.$$

In this case, $i_1 \geq i_2$,

Proof. If $F_1 \preceq_{dr} F_2$, by definition, $\delta_1(t) \geq \delta_2(t)$, for every $t \geq 0$. Consequently, one has the following chain of inequalities:

$$-\frac{F_1'(t)}{F_1(t)} \geq -\frac{F_2'(t)}{F_2(t)},$$

$$-F_1'(t)F_2(t) \geq -F_2'(t)F_1(t),$$

$$-F_1'(t)F_2(t) + F_2'(t)F_1(t) \geq 0,$$

$$-\left[\frac{F_1(t)}{F_2(t)}\right]' \geq 0,$$

$$\left[\frac{F_1(t)}{F_2(t)}\right]' \leq 0.$$

Therefore, $\frac{F_1}{F_2}$ is decreasing starting from its initial value

$$\frac{F_1(0)}{F_2(0)} = 1,$$

which implies that there exists the following limit:

$$\lim_{t\to+\infty} \frac{F_1(t)}{F_2(t)} \leq 1.$$

On the other hand,

$$\int_0^{+\infty} F_1^2(t)\mathrm{d}t = \int_0^{+\infty} F_2^2(t)\mathrm{d}t = +\infty$$

implies

$$\int_0^{+\infty} F_1(t)\mathrm{d}t = \int_0^{+\infty} F_2(t)\mathrm{d}t = +\infty.$$

Thus,

$$\frac{1-i_1}{1-i_2} = \frac{\lim_{t\to+\infty} F_1(t)}{\lim_{t\to+\infty} F_2(t)} = \lim_{t\to+\infty} \frac{F_1(t)}{F_2(t)} \leq 1.$$

Consequently,

$$1-i_1 \leq 1-i_2,$$

from where

$$i_1 \geq i_2,$$

as stated.

Remark 2. *Observe that the former result is also valid if $F_1 \preceq_{dr} F_2$ in a neighborhood of infinity.*

Example 7. *Let us consider the generalized Weibull discount function of parameters k, α and β, all greater than zero, defined by:*

$$F(t) = \frac{1}{(1+kt^{\alpha})^{\beta}}.$$

In this case, the instantaneous discount rate is given by the following expression:

$$\delta(t) = \frac{k\alpha\beta t^{\alpha-1}}{1+kt^{\alpha}} = \frac{k\alpha\beta}{t^{1-\alpha}+kt} = \frac{1}{t}\frac{k\alpha\beta}{t^{-\alpha}+k}. \tag{8}$$

Let F_1 and F_2 be two generalized Weibull discount functions with the same parameter k, given by:

$$F(t) = \frac{1}{(1+kt^{\alpha_1})^{\beta_1}}$$

and

$$F(t) = \frac{1}{(1+kt^{\alpha_2})^{\beta_2}}.$$

If $\delta_1 \geq \delta_2$, simple algebra shows:

$$\alpha_1\beta_1 t^{-\alpha_2} + k\alpha_1\beta_1 \geq \alpha_2\beta_2 t^{-\alpha_1} + k\alpha_2\beta_2.$$

Letting $t \to +\infty$, one has $\alpha_1\beta_1 \geq \alpha_2\beta_2$. If, moreover, $\alpha_1 \geq \alpha_2$, the former inequality holds for every $t \geq 1$. However, for every $0 < t < 1$, taking into account that $t^{-\alpha_1}$ and $t^{-\alpha_2}$ converge to infinity, when $t \to 0^+$, and that $t^{-\alpha_1}$ tends to $+\infty$ more quickly than $t^{-\alpha_2}$, it can be stated that there exists t_0 in $]0,1[$ such that

$$\delta_1(t) < \delta_2(t), \text{ for every } 0 < t \leq t_0,$$

and

$$\delta_1(t) \geq \delta_2(t), \text{ for every } t > t_0,$$

Obviously, this is a contradiction whereby it can deduced that $\alpha_1 < \alpha_2$.

Therefore, we can enunciate the following proposition.

Proposition 5. *Let F_1 and F_2 be two generalized Weibull discount functions. If $F_1 \preceq_{dr} F_2$, then $\alpha_1\beta_1 \geq \alpha_2\beta_2$ and $\alpha_1 < \alpha_2$.*

The following example confirms the reasoning included in Example 7.

Example 8. *Let us consider two Weibull discount functions such that* $\alpha_1 = 1$, $\beta_1 = 0.8$, $\alpha_2 = 0.75$, *and* $\beta_2 = 1$. *In this case, one has* $\alpha_1\beta_1 \geq \alpha_2\beta_2$ *and* $\alpha_1 \geq \alpha_2$. *Observe, in effect, that there exists* t_0 *in* $]0, 1]$ *such that* δ_1 *(in green) and* δ_2 *(in black) satisfy the inequalities included in Example 7 (see Figure 3).*

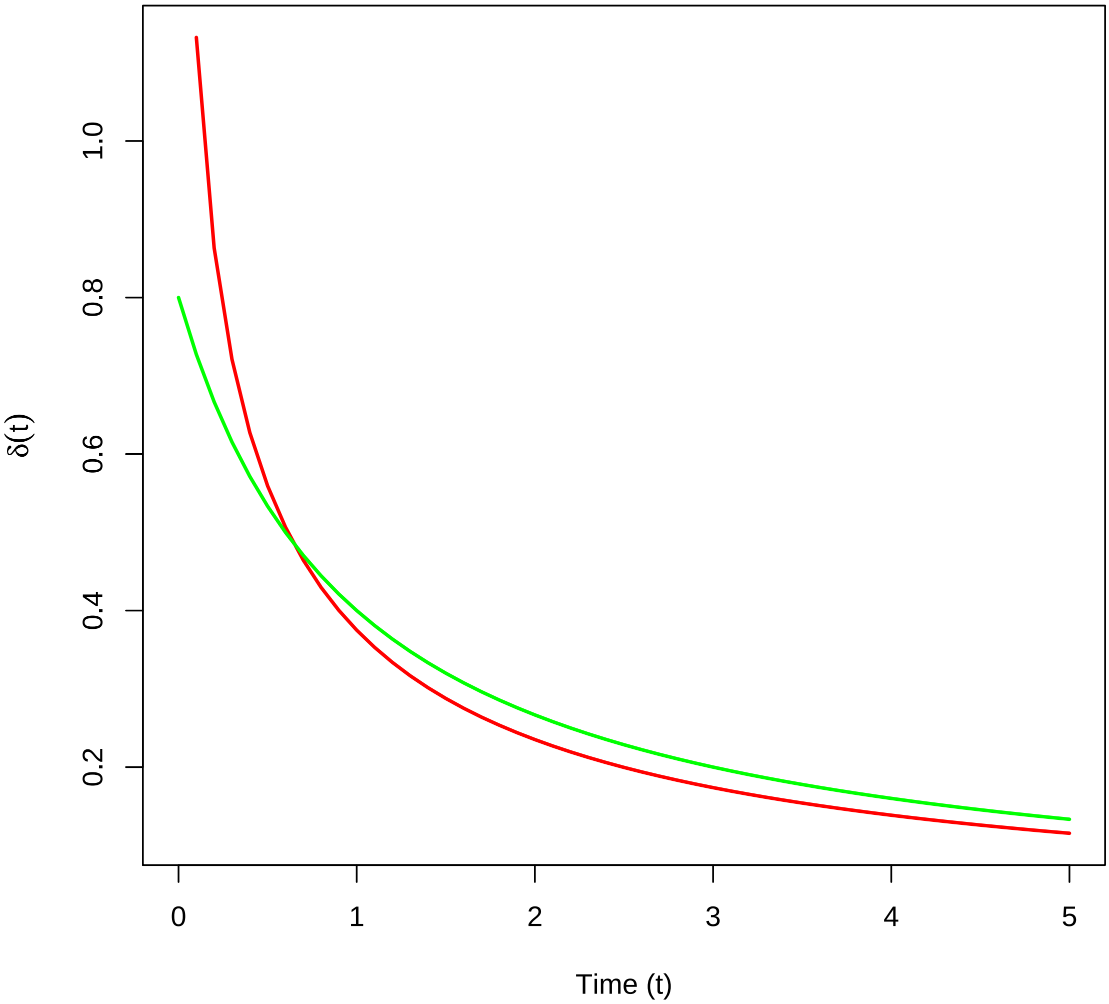

Figure 3. Instantaneous discount rates of Example 8. Source: Own elaboration.

Let us introduce another definition related to the dominance between discount functions for which we are going to follow Sarabia (2008). In effect, consider the so-called 1-moment distribution of the discount $D(t)$, denoted as $D_{(1)}(t)$, which is defined as the following ratio:

$$D_{(1)}(t) := \frac{\int_0^t z f(z)\mathrm{d}z}{\int_0^{+\infty} z f(z)\mathrm{d}z},$$

where the denominator (the mathematical expectation of D) is assumed to be finite. Each $D_{(1)}(t)$

represents the proportion of the total cumulative discounting times until t. Thus, the *Lorenz curve* corresponding to the discounting distribution D can be described as the set of points:

$$(D(t), D_{(1)}(t)),$$

defined in the unit square, where t ranges from 0 to $+\infty$, completed, if necessary, by linear interpolation. Thus, an expression for the Lorenz curve can be constructed in the following way:

$$L(p) := D_{(1)}(D^{-1}(p)),\ \ p \in [0, 1],$$

where closed expressions for $D_{(1)}$ and D^{-1} are needed.

Taking into account that the mathematical expectation of D can be calculated as

$$\int_0^1 D^{-1}(z)\mathrm{d}z,$$

the Lorenz curve corresponding to the discounting distribution D can be written as (Gastwirth, 1971):

$$L(p) = \frac{\int_0^p D^{-1}(z)\mathrm{d}z}{\int_0^1 D^{-1}(z)\mathrm{d}z},\ \ 0 \leq p \leq 1.$$

Now, the following definition can be introduced (Lubrano, 2017).

Definition 8. *The discount function F_1 with discount $D_1 := 1 - F_1$ is said to dominate, according to the Lorenz ordering, the discount function F_2 with discount $D_2 := 1 - F_2$ (denoted by $F_1 \succeq_L F_2$) if $L_1(p) \geq L_2(p)$, for every $p \in [0, 1]$, and $L_1(p) \neq L_2(p)$.*

Proposition 6. *Let F_1 and F_2 be two generalized Weibull discount functions. If $F_1 \preceq_{dr} F_2$, then $F_1 \nsucc_L F_2$.*

Proof. In effect, if $F_1 \preceq_{dr} F_2$, by Proposition 5, $\alpha_1\beta_1 \geq \alpha_2\beta_2$ and $\alpha_1 < \alpha_2$. Observe that these two inequalities are the opposite of the inequalities pointed out by Sarabia (2008) to reach Lorenz ordering. Therefore, $F_1 \nsucc_L F_2$, as required.

Remark 3. *The converse statement does not hold. In effect, if $F_1 \nsucc_L F_2$ and $\alpha_1\beta_1 > \alpha_2\beta_2$, by Sarabia (2008, p. 182), one has $\alpha_1 \leq \alpha_2$, but this does not necessarily imply $\delta_1 > \delta_2$, as can be seen in the following counterexample.*

Example 9. *Let us consider that $\alpha_1 = 2$, $\beta_1 = 2$, and $\alpha_2 = 2.5$, $\beta_2 = 0.7$. In this case, one has $F_1 \nsucc_L F_2$ (see Figure 4), and $\alpha_1\beta_1 > \alpha_2\beta_2$, $\alpha_1 \leq \alpha_2$, but $\delta_1 > \delta_2$ does not hold because there exists t_0, small enough, without satisfying the following inequality:*

$$4t^{-2.5} + 4 \geq 2.45t^{-2} + 2.45.$$

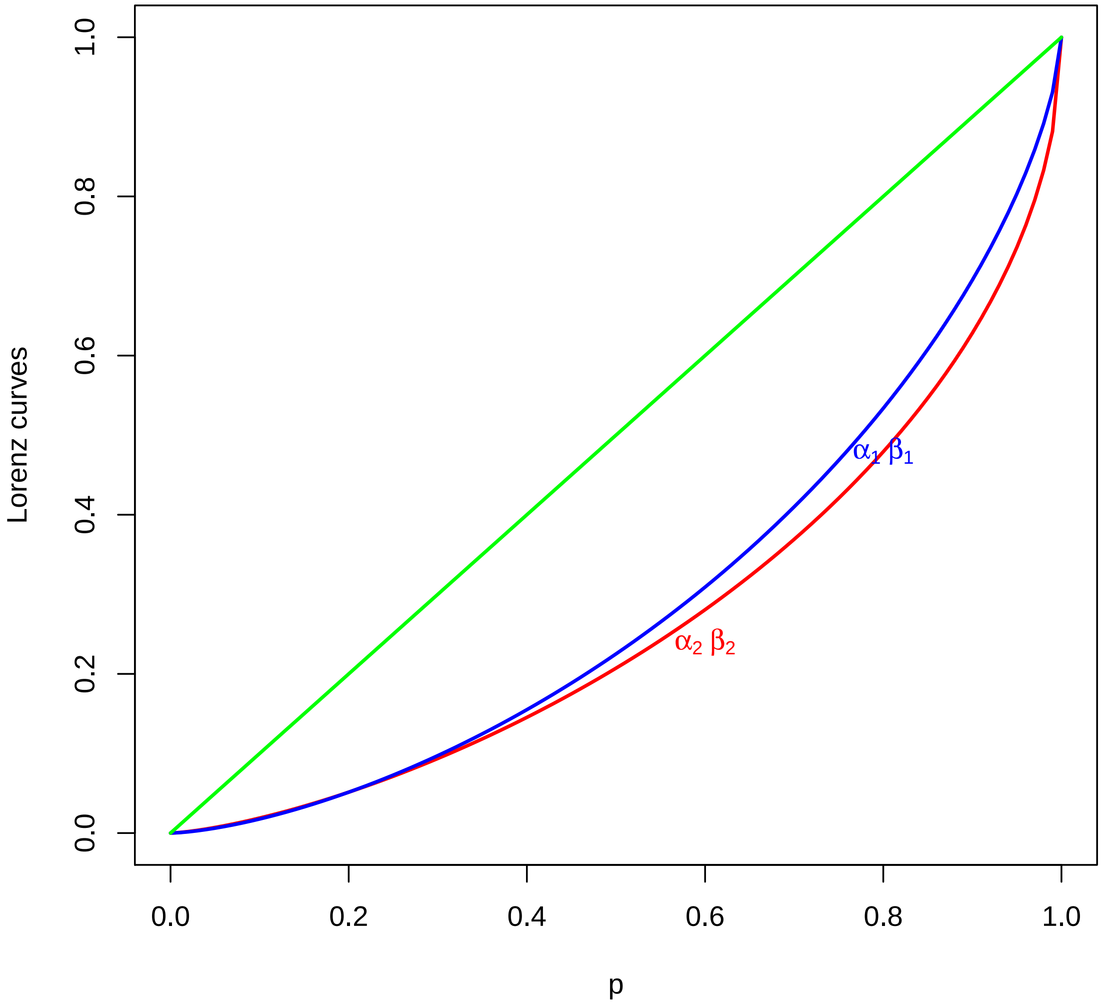

Figure 4. Lorenz curves of Example 9. Source: Own elaboration.

Last results can be summarized in Table 2 (by assuming that $\alpha_1\beta_1 > \alpha_2\beta_2$).

Table 2. Summarizing the obtained results. Source: Own elaboration.

$\delta_1 \geq \delta_2$	$\delta_1 \ngeq \delta_2$
$\alpha_1 \leq \alpha_2$	$\alpha_1 > \alpha_2$
$F_1 \nsucc_L F_2$	$F_1 \succ_L F_2$

5. Conclusions

This paper has introduced the proposal of a measure of the concentration degree of a discount function. This problem has been solved by introducing the well-known Gini coefficient and the Lorenz curve in the field of finance. This has been possible thanks to the parallelism between a discount function and the distribution function of a continuous random variable.

This characteristic of the discount function has been related to other geometric properties of the discounting curve such as regularity, super-additivity and the finiteness of the area under the curve representing the discount function. Additionally, this manuscript has included some necessary and sufficient conditions when comparing the level of concentration of two discount functions, specifically by analyzing the case of two Weibull discount functions. To the extent of our knowledge, this is the first time that a measure of discounting concentration has been introduced in the field of finance. Undoubtedly, this is the main contribution of this paper.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

We would like to thank the financial support from the Mediterranean Research Center for Economics and Sustainable Development (CIMEDES), University of Almería (Spain).

Conflict of interest

All authors declare no conflicts of interest in this paper.

References

Atkinson T (1970) On the measurement of inequality. *J Econ Theory* 2: 244–263. https://doi.org/10.1016/0022-0531(70)90039-6

Bonetti M, Gigliarano C, Muliere P (2009) The Gini concentration test for survival data. *Lifetime Data Anal* 15: 493–518. https://doi.org/10.1007/s10985-009-9125-5

Caliendo FN, Findley TS (2014) Discount functions and self-control problems. *Econ Lett* 122: 416–419. https://doi.org/10.2139/ssrn.2337093

Calot G (1965) Cours de Statistique Descriptive. Dunod, Paris.

Chambers CP, Echenique F, Miller AD (2023) Decreasing Impatience. *Am Econ J Microecon* 15: 527–551. https://doi.org/10.1257/mic.20210361

Cramér H (1961) *Mathematical Methods of Statistics,* Asia Publishing House, Bombay.

Cruz Rambaud S (2014) A new argument in favor of hyperbolic discounting in very long term project appraisal. *Int J Theoretical Appl Financ* 17: 1450049. https://doi.org/10.1142/S0219024914500496

Cruz Rambaud S, González Fernández I, Ventre V (2018) Modeling the inconsistency in intertemporal choice: The generalized Weibull discount function and its extension. *Ann Financ* 14: 415–426.

Gastwirth JL (1971) A general definition of the Lorenz curve. *Econometrica* 39: 1037–1039. https://doi.org/10.2307/1909675

Kotz S (2006) *Encyclopedia of Statistical Sciences (Second Edition).* Wiley-Interscience, Hoboken (New Jersey).

Lubrano M (2017) The econometrics of inequality and poverty. Lecture 4: Lorenz curves, the Gini coefficient and parametric distributions. Available from: https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.169.4809&rep=rep1&type=pdf.

Lueddeckens S, Saling P, Guenther E (2022) Discounting and life cycle assessment: A distorting measure in assessments, a reasonable instrument for decisions. *Int J Environ Sci Technol* 19: 2961—2972.

Sarabia JM (2008) Parametric Lorenz curves: Models and applications, In: Chotikapanich, D., editor, *Modeling Income Distributions and Lorenz Curves,volume 5 of Economic Studies in Equality, Social Exclusion and Well-Being, Chapter 9* , 167–190. Springer, New York. https://link.springer.com/chapter/10.1007/978-0-387-72796-7_9

Shaked M, Shanthikumar JG (1994) Stochastic Orders and Their Applications, Boston: Academic Press, Inc.

Takeuchi K (2011) Non-parametric test of time consistency: Present bias and future bias. *Game Econ Behav* 71: 456–478. https://doi.org/10.1016/j.geb.2010.05.005

Zhou C, Tang W, Zhao R (2015) Optimal consumer search with prospect utility in hybrid uncertain environment. *J Uncertain Anal Appl* 3, 6. https://doi.org/10.1186/s40467-015-0030-z

AIMS Press