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Research article

Asian option pricing under sub-fractional vasicek model

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Abstract: This paper investigates the pricing formula for geometric Asian options where the underlying asset is driven by the sub-fractional Brownian motion with interest rate satisfying the sub-fractional Vasicek model. By applying the sub-fractional Itô formula, the Black-Scholes (B-S) type Partial Differential Equations (PDE) to Asian geometric average option is derived by Delta hedging principle. Moreover, the explicit pricing formula for Asian options is obtained through converting the PDE to the Cauchy problem. Numerical experiments are conducted to test the impact of the stock price, the Hurst index, the speed of interest rate adjustment, and the volatilities and their correlation for the Asian option and the interest rate model, respectively. The results show that the main parameters such as Hurst index have a significant influence on the price of Asian options.

Keywords: sub-fractional Brownian motion; vasicek model; zero-coupon bond; Asian option pricing

JEL Codes: C15, C32, C63

1. Introduction

Since the emerge of Black-Scholes framework in the 1970s, option pricing has always been one of the hot issues in financial research. Classic Black-Scholes framework heavily relies on unrealistic assumptions, and only works for plain vanilla options, like European options. However, plain vanilla options are susceptible to possible spot manipulation at maturity. As a natural and reasonable extension, exotic options, especially the path-dependence options, attract the attention of both partitioner and researchers. Asian options, generally described as options whose payoff depends on the average price of the underlying asset during a pre-specified period within an option's lifetime and pre-specified observation frequency, is one of the most-used hedging tools in the over-the-counter (OTC) marketplace (Zhang, 1997).

Theoretically, there are two types of Asian options, arithmetic Asian options and geometric Asian options. The key difference between them is that when the underlying asset prices are log-normally distributed, the geometric Asian options follow the same distribution while the arithmetic Asian options do not. In real-world practice, the arithmetic Asian option is the only type traded in the market, whose closed-form expression is very difficult to obtain. The pricing process of arithmetic Asian options involves the price of geometric Asian options as a control variate, due to the fact that the closed-form solution of geometric Asian options is relatively easy to obtain. Therefore, it is worth studying the pricing dynamics of geometric Asian options. In recent years, there are some research results regarding geometric Asian options, such as Sander (2019),Yao and Li (2018) and Gen and Zhou (2018). However, up to the authors' best knowledge, for most existing literature, the underlying asset price is modelled by a stochastic process driven by standard Brownian motion, which is a martingale with the Markov property. The Markovian setting is equivalent to the weak form of efficient market hypothesis (Hull, 2003), which is very ideal and unrealistic. According to Greene and Fielitz (1977) and Lo (1991), there is empirical evidence showing that the asset pricing dynamics involve past information. As a result, it is natural to introduce fractional Brownian motion (fBm), a stochastic process which preserves long-dependence and self-similarity properties. Mandelbrot and Van Ness (1968) firstly used fBm to model financial asset pricing dynamics, Xiao and Zhou (2008) and Liu et al. (2008) use fractional Brownian motion to describe the evolution process of financial asset prices, and more research results can be found in Duncan et al. (2000) and Hu and Øksendal (2003) and references there in. Since fractional Brownian motion has stationary increment, which can't match the virtue of financial data, a more general Gaussian process, the sub-fractional Brownian motion (sfBm), has been introduced to model the dynamics of financial assets. According to Bojdecki et al. (2004), the sub-fractional Brownian motion not only preserves the long-dependence and self-similarity properties, but also has a faster degeneration. In Tudor (2008), it has been shown that the sub-fractional Brownian motion can be a good choice to describe the volatility. Although, both fractional Brownian motion and sub-fractional Brownian motion are neither Markovian nor martingale, and there are arbitrages in such market models, which was first proved by Rogers (1997). However, it still worth discussing in the pricing formula of financial derivatives under such environment, since such arbitrages can be excluded from these models by restricting the class of trading strategies, more details can be found in Cheridito (2003) and Xiao et al. (2021).

The classical B-S model assumes that the interest rate is a constant, but in the real financial market, the interest rate preserves the term structure. Especially in the long run, the interest rate fluctuates within a certain range under the influence of both macro and micro economic factors, that is, the interest rate has the phenomenon of "mean reversion". To characterize the stochastic behavior of interest rates, the Vasicek model has been proposed and widely used in the study of stochastic interest rates. Cajueiro and Tabak (2007) found that there is a long-term dependence of stochastic interest rates. Zhou and Li (2014), Huang et al. (2012) studied the option pricing formulae under the fractional Vasicek rate model. Tudor (2008) found that sub-fractional Brownian motion had stronger memory than fractional Brownian motion and could better describe the long-term dependence of financial assets. Guo and Zhang (2017) used the sub-fractional Vasicek model to price European options and obtained the pricing formula of European options. See Ji et al. (2022), Yang et al. (2022) and Wang et al. (2021) for more recent progress.

Enlightened by these existing results, in this paper underlying asset is assumed to follow the sub-fractional Brownian motion and the interest rates satisfy the sub-fractional Vasicek model, then the B-S type PDE corresponding to Asian geometric average option pricing formula is derived, and a close-form of such formula can be obtained. Such a formula will play an essential role when a more precise control variate price is required in the pricing of arithmetic Asian options.

The rest of this paper is organized as follows: In Section 2, we introduce some necessary preliminary knowledge about option pricing and the zero-coupon bond pricing model. In Section 3, we investigate the Asian geometric average option pricing model, and give the analytical solution and the pricing formula for Asian geometric average options where the underlying asset is driven by the sub-fractional Brownian motion with interest rate satisfying the sub-fractional Vasicek model. In Section 4, we discusses the influence of Hurst index, stock price and zero-coupon bond price on Asian option price. Some further implications will be added in the last section as well.

2. Preliminaries

Let $(\Omega, \mathcal{F}_t, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying usual conditions.

Definition 2.1. Let $H \in (0, 1)$ be the Hurst index. The sub-fractional Brownian motion $\xi^H = \left\{\xi^H_t : t \in \mathbb{R}\right\}$ with the Hurst index H is a continuous Gaussian process satisfying that

1. $\mathrm{E}[\xi^H_t] = 0$,
2. $\mathrm{E}[\xi^H_t \xi^H_s] = t^{2H} + s^{2H} - \frac{1}{2}\left(|t + s|^{2H} + |t - s|^{2H}\right), s, t \in \mathbb{R}$,

where $E = E_p$ denotes the expectation under the probability measure P .

In case $H = 1/2$, ξ^H_t is a standard Brownian motion. The sub-fractional Brownian motion is a modified fractional Brownian motion, which has the properties of self-similarity and long memory. More details can be found in Tudor (2008),Guo and Zhang (2017), Bojdecki et al. (2004) and references therein.

Definition 2.2. Rao (2016): Assumes that A_t is the path-dependent variable of Asian options, which represents the average value of stock price on $[0, t]$. It is a geometric average, that is,

$$A_t = \exp\left\{\frac{1}{t}\int_0^t \ln S_\tau d\tau\right\}.$$

The Asian option corresponding to this path is called the Asian geometric average option. If at the maturity time t, the Asian option is executed at the fixed strike price K, the return of Asian geometric average call option can be expressed as $(A_t - K)^+$ and the return of Asian geometric average put option can be expressed as $(K - A_t)^+$.

Lemma 2.3. *Yan et al. (2011): Suppose $\{Y(t)\}$ is a sub-fractional* Itô *process on probability space $(\Omega, \mathcal{F}, P)$, which is given by $\mathrm{d}Y(t) = r\mathrm{d}t + \sigma \mathrm{d}\xi^H_t$. Let $f(t, Y(t)) \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R} \to \mathbb{R})$, and that $f(t, Y(t))$, $\int_0^t \frac{\partial f}{\partial \tau}(\tau, Y(\tau))\mathrm{d}\tau$, $\int_0^t \frac{\partial^2 f}{\partial Y^2}(\tau, Y(\tau))d\tau$, $\int_0^t \frac{\partial^2 f}{\partial Y^2}(\tau, Y(\tau))\tau^{2H-1}d\tau$ belong to the $L^2(P)$ space, then we have the following sub-fractional* Itô *formula,*

$$\begin{aligned} f(t, Y(t)) =& f(0, 0) + \int_0^t [\frac{\partial f}{\partial \tau}(\tau, Y(\tau)) + \sigma^2 H(2 - 2^{2H-1})\frac{\partial^2 f}{\partial Y^2}(\tau, Y(\tau))\tau^{2H-1} \\ &+ r\frac{\partial f}{\partial Y}(\tau, Y(\tau))]\mathrm{d}\tau + \int_0^t \sigma\frac{\partial f}{\partial Y}(\tau, Y(\tau))\mathrm{d}\xi^H_t. \end{aligned} \tag{1}$$

The proof of Lemma 2.3 can be found in Yan et al. (2011).

Before proceeding to our main results, the term structure of interest rate should be discussed. Suppose that interest rate r_t satisfies the following stochastic differential equation under risk neutral measure Q (Guo and Zhang, 2017)

$$\mathrm{d}r_t = a(\theta - r_t)\mathrm{d}t + \sigma \mathrm{d}\xi_t^H, \tag{2}$$

where a, θ, σ are constants, $\left\{\xi_t^H; t > 0\right\}$ is sub-fractional Brownian motion, a is the speed of interest rate adjustment, θ is the long-term interest rate, σ is the coefficient of influence on interest rates. This equation model is the so-called sub-fractional Vasicek model.

The sub-fractional Vasicek model is a modified form and an extension of the fractional Vasicek model. In this paper, we use the sub-fractional Brownian motion in place of the fractional Brownian motion, simultaneously, we use the sub-fractional Vasicek model instead of the fractional Vasicek model.

We give the corresponding assumptions as follows.

1. Financial markets are complete and frictionless;
2. Short selling is allowed;
3. Underlying assets pay no dividend;
4. There is no transaction cost and it is tax-free;
5. The expected return rate of the risk-free portfolio is equal to the risk-free interest rate;

Suppose there are two kinds of continuous free-trade assets in the market, one is the risk-free asset, such as treasury bills, and the other is the risky asset, for instant stocks. The stock price, S_t, satisfies the following equation

$$\mathrm{d}S_t = r_t S_t \mathrm{d}t + \sigma_1 S_t \mathrm{d}\xi_1^H(t), \tag{3}$$

while r_t , the risk-free interest rate, satisfies the following sub-fractional Vasicek model

$$\mathrm{d}r_t = a(\theta - r_t)\mathrm{d}t + \sigma_2 \mathrm{d}\xi_2^H(t), \tag{4}$$

where σ_1 and σ_2 are constants, and $\left\{\xi_1^H(t); t > 0\right\}$ and $\left\{\xi_2^H(t); t > 0\right\}$ are two correlated sub-fractional Brownian motions such that

$$\mathrm{d}\xi_1^H(t)\mathrm{d}\xi_2^H(t) = \rho(2 - 2^{2H-1})\mathrm{d}t^{2H}.$$

Since the interest rate is non-exchangeable, a zero-coupon bond, as its carrier, plays a unique role in the study of option pricing when the interest rate is of stochastic nature. In this paper, we denote the face value of the zero-coupon bond at time t as $F(r_t, t; T)$, and the face value of the zero-coupon rate at maturity is 1 dollar, i.e. $F(r_T, T; T) = 1$.

An application of Delta hedging principle, together with Lemma 2.3, yields that price of zero-coupon bond $F(r_t, t; T)$ satisfies the following partial differential equation:

$$\begin{cases} \dfrac{\partial F}{\partial t} + a\left(\hat{\theta} - r_t\right)\dfrac{\partial F}{\partial r_t} + (2 - 2^{2H-1})H\sigma_2^2 t^{2H-1}\dfrac{\partial^2 F}{\partial r_t^2} - r_t F = 0, \\ F\left(r_T, T; T\right) = 1, \end{cases} \tag{5}$$

where $\hat{\theta} = \theta - \frac{\lambda}{a}\sigma_2$, λ is the price of the financial market with interest rate risk. According to Guo and Zhang (2017), Equation (5) has unique explicit solution

$$F(r_t, t; T) = e^{-r_t B(t,T) - A(t,T)}, \tag{6}$$

where

$$\begin{cases} A(t,T) = \hat{\theta}(T-t) - \hat{\theta}B(t,T) - (2 - 2^{2H-1})H \int_t^T \sigma_2^2 s^{2H-1} B^2(t,T)\mathrm{d}s, \\ B(t,T) = \frac{1}{a}(1 - e^{-a(T-t)}). \end{cases}$$

When $T - t \ll 1$,we can get $A(t,T) \sim 0$,$B(t,T) \sim T - t$, and the zero-coupon bond price is $F(r_t, t; T) = e^{-r_t(T-t)}$.

3. Main result

Theorem 3.1. *Assume that stock price S_t satisfies Equation (3), interest rate r_t satisfies Equation (4), denote $V(t, r_t, S_t, A_t)$ as the Asian geometric average call option at time t ($0 \le t \le T$), with fixed strike pricing K and maturity T, then $V(t, r_t, S_t, A_t)$ satisfies the partial differential equation*

$$\begin{aligned} &\frac{\partial V}{\partial t} + (2 - 2^{2H-1})H\sigma_2^2 t^{2H-1}\frac{\partial^2 V}{\partial r_t^2} + (2 - 2^{2H-1})H\sigma_1^2 S_t^2 t^{2H-1}\frac{\partial^2 V}{\partial S_t^2} + a(\hat{\theta} - r_t)\frac{\partial V}{\partial r_t} \\ &+ 2(2 - 2^{2H-1})H\rho\sigma_1\sigma_2 S_t t^{2H-1}\frac{\partial^2 V}{\partial r_t \partial S_t} + \frac{(\ln S_t - \ln A_t)}{t}A_t\frac{\partial V}{\partial A_t} + r_t\frac{\partial V}{\partial S_t}S_t - r_t V = 0, \end{aligned}$$

with terminal condition

$$V(T, r_T, S_T, A_T) = (A_T - K)^+,$$

where $A_t = \exp\left\{\frac{1}{t}\int_0^t \ln S_\tau d\tau\right\}$ denotes the geometric mean of stock price on $[0,t]$ which is equivalent to the path of Asian geometric options.

Proof. According to Delta hedging principle, a portfolio is constructed with a long position of one unit Asian option V, and a short position of Δ_1 unit S_t and Δ_2 zero-coupon bonds F. Define the value of portfolio at the t as

$$\Pi_t = V - \Delta_1 S_t - \Delta_2 F_t, \tag{7}$$

then the change of the portfolio in the time interval dt is

$$\mathrm{d}\Pi_t = \mathrm{d}V - \Delta_1 \mathrm{d}S_t - \Delta_2 \mathrm{d}F_t. \tag{8}$$

Differentiating A_t with respect to t yields

$$\mathrm{d}A_t = A_t(-\frac{1}{t^2}\int_0^t \ln S_\tau \mathrm{d}\tau + \frac{1}{t}\ln S_t) = A_t\frac{\ln S_t - \ln A_t}{t}\mathrm{d}t. \tag{9}$$

According to Lemma 2.3, we can get

$$\begin{aligned}\mathrm{d}V(t,r_t,S_t,A_t) = \Bigg[\frac{\partial V}{\partial t} + \frac{(\ln S_t - \ln A_t)}{t}A_t\frac{\partial V}{\partial A_t} + (2-2^{2H-1})H\sigma_1^2S_t^2t^{2H-1}\frac{\partial^2 V}{\partial S_t^2} \\ + (2-2^{2H-1})H\sigma_2^2t^{2H-1}\frac{\partial^2 V}{\partial r_t^2} \\ +2(2-2^{2H-1})H\rho\sigma_1\sigma_2S_tt^{2H-1}\frac{\partial^2 V}{\partial r_t\partial S_t}\Bigg]\mathrm{d}t + \frac{\partial V}{\partial r_t}\mathrm{d}r_t + \frac{\partial V}{\partial S_t}\mathrm{d}S_t,\end{aligned} \tag{10}$$

and

$$\mathrm{d}F(t,r_t) = \left[\frac{\partial F}{\partial t} + (2-2^{2H-1})H\sigma_2^2t^{2H-1}\frac{\partial^2 F}{\partial r_t^2}\right]\mathrm{d}t + \frac{\partial F}{\partial r_t}\mathrm{d}r_t. \tag{11}$$

Applying Equations (10) and (11) over the time interval dt , we can get

$$\begin{aligned}\mathrm{d}\Pi_t &= \mathrm{d}V - \Delta_1\mathrm{d}S_t - \Delta_2\mathrm{d}F \\ &= \Bigg[\frac{\partial V}{\partial t} + \frac{(\ln S_t - \ln A_t)}{t}A_t\frac{\partial V}{\partial A_t} + (2-2^{2H-1})H\sigma_1^2S_t^2t^{2H-1}\frac{\partial^2 V}{\partial S_t^2} \\ &+ (2-2^{2H-1})H\sigma_2^2t^{2H-1}\frac{\partial^2 V}{\partial r_t^2} \\ &+2(2-2^{2H-1})H\rho\sigma_1\sigma_2S_tt^{2H-1}\frac{\partial^2 V}{\partial r_t\partial S_t}\Bigg]\mathrm{d}t + \left(\frac{\partial V}{\partial S_t} - \Delta_1\right)\mathrm{d}S_t + \left(\frac{\partial V}{\partial r_t} - \Delta_2\frac{\partial F}{\partial r_t}\right)\mathrm{d}r_t \\ &- \Delta_2\left(\frac{\partial F}{\partial t} + (2-2^{2H-1})H\sigma_2^2t^{2H-1}\frac{\partial^2 F}{\partial r_t^2}\right)\mathrm{d}t.\end{aligned}$$

Let $\Delta_1 = \dfrac{\partial V}{\partial S_t}$ and $\Delta_2 = \dfrac{\partial V/\partial r_t}{\partial F/\partial r_t}$, we have

$$\begin{aligned}\mathrm{d}\Pi_t &= \mathrm{d}V - \Delta_1\mathrm{d}S_t - \Delta_2\mathrm{d}F \\ &= \Bigg[\frac{\partial V}{\partial t} + \frac{(\ln S_t - \ln A_t)}{t}A_t\frac{\partial V}{\partial A_t} + (2-2^{2H-1})H\sigma_1^2S_t^2t^{2H-1}\frac{\partial^2 V}{\partial S_t^2} \\ &+ (2-2^{2H-1})H\sigma_2^2t^{2H-1}\frac{\partial^2 V}{\partial r_t^2} \\ &+2(2-2^{2H-1})H\rho\sigma_1\sigma_2S_tt^{2H-1}\frac{\partial^2 V}{\partial r_t\partial S_t}\Bigg]\mathrm{d}t - \Delta_2\left(r_tF - a(\hat{\theta}-r_t)\frac{\partial F}{\partial r_t}\right)\mathrm{d}t.\end{aligned} \tag{12}$$

A simple application of Delta hedging principle yields

$$\mathrm{d}\Pi_t = r_t\Pi_t\mathrm{d}t = r_t\left(V - \Delta_1S_t - \Delta_2F\right)\mathrm{d}t, \tag{13}$$

By combining (12) and (13), we obtain that

$$\begin{aligned}&\frac{\partial V}{\partial t} + (2-2^{2H-1})H\sigma_2^2t^{2H-1}\frac{\partial^2 V}{\partial r_t^2} + (2-2^{2H-1})H\sigma_1^2S_t^2t^{2H-1}\frac{\partial^2 V}{\partial S_t^2} + a(\hat{\theta}-r_t)\frac{\partial V}{\partial r_t} \\ &+ 2(2-2^{2H-1})H\rho\sigma_1\sigma_2S_tt^{2H-1}\frac{\partial^2 V}{\partial r_t\partial S_t} + \frac{(\ln S_t - \ln A_t)}{t}A_t\frac{\partial V}{\partial A_t} + r_t\frac{\partial V}{\partial S_t}S_t - r_tV = 0,\end{aligned} \tag{14}$$

with $V(T,r_T,S_T,A_T) = (A_T-K)^+$. □

Theorem 3.2. *Assume that stock price S_t satisfies Equation (3), interest rate r_t satisfies Equation (4), then for fixed strike pricing K and maturity date T, the value of Asian geometric average call option $V(t, r_t, S_t, A_t)$ is given by*

$$V(t, r_t, S_t, A_t) = F^{\frac{aT+1}{aT}} A_t^{\frac{t}{T}} \left(\frac{S_t}{F^{\frac{1}{aB(t,T)}}} \right)^{\frac{T-t}{T}} e^{L_1} N(d_1) - KFe^{L_2} N(d_2),$$

where

$$\begin{aligned}
\alpha_1(t) &= \frac{a\hat{\theta}}{T}\left[\frac{e^{-a(T-t)}-1}{a^2}+\frac{T-t}{a}\right]+(2-2^{2H-1})Ht^{2H-1}\left[-\sigma_1^2\frac{T-t}{T}\right.\\
&\left.+2\sigma_2^2\frac{e^{-a(T-t)}-1}{aT}\left(\frac{e^{-a(T-t)}-1}{a^2}+\frac{T-t}{a}\right)+2\rho\sigma_1\sigma_2\frac{e^{-a(T-t)}-1}{a}\frac{T-t}{T}\right],\\
\alpha_2(t) &= (2-2^{2H-1})Ht^{2H-1}\left[\sigma_1^2\left(\frac{T-t}{T}\right)^2+\frac{\sigma_2^2}{T^2}\left(\frac{e^{-a(T-t)}-1}{a^2}+\frac{T-t}{a}\right)^2\right.\\
&\left.+2\rho\sigma_1\sigma_2\left(\frac{e^{-a(T-t)}-1}{a^2}+\frac{T-t}{a}\right)\frac{T-t}{T^2}\right],\\
\alpha_3(t) &= (2-2^{2H-1})Ht^{2H-1}\sigma_2^2\left(\frac{e^{-a(T-t)}-1}{a^2}\right)^2+\hat{\theta}\left(e^{-a(T-t)}-1\right),
\end{aligned}$$

$$\begin{aligned}
L_1 &= \frac{A(t,T)\left[B(t,T)(aT+1)-(T-t)\right]}{aTB(t,T)}+\int_t^T\left[\alpha_1(s)+\alpha_2(s)+\alpha_3(s)\right]\mathrm{d}s,\\
L_2 &= A(t,T)+\int_t^T\alpha_3(s)\mathrm{d}s,\\
d_1 &= \frac{\frac{1}{T}\left[t\ln A_t+(T-t)\ln S_t+\frac{(\ln F+A(t,T))(B(t,T)-T+t)}{aB(t,T)}\right]+\int_t^T\left[\alpha_1(s)+2\alpha_2(s)\right]\mathrm{d}s-\ln K}{\sqrt{2\int_t^T\alpha_2(s)\mathrm{d}s}},\\
d_2 &= d_1-\sqrt{2\int_t^T\alpha_2(s)\mathrm{d}s},\\
N(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x e^{-\frac{t^2}{2}}\mathrm{d}t.
\end{aligned}$$

Proof. Consider

$$y = \frac{1}{T}\left[t\ln A_t+(T-t)\ln S_t+r_t\left(\frac{e^{-a(T-t)}-1}{a^2}+\frac{T-t}{a}\right)\right],$$

with

$$\hat{V}(y,t) = \frac{V}{\exp\left(\frac{1}{a}\left(e^{-a(T-t)}-1\right)r_t\right)}.$$

Let $E = \exp\left(\frac{1}{a}\left(e^{-a(T-t)}-1\right)r_t\right)$, and recall the price solutions of zero-coupon bonds (5), we get

$$F = Ee^{-A(t,T)}.$$

A substitution of r_t, A_t, S_t yields that

$$V(r_t, A_t, S_t, t) = E\hat{V}(y,t) = Fe^{A(t,T)}\hat{V}(y,t).$$

A direct calculation yields that

$$\frac{\partial V}{\partial t} = E\left\{\frac{\partial \hat{V}}{\partial t} + \left[\frac{\ln A_t - \ln S_t}{T} + \frac{r_t}{aT}\left(e^{-a(T-t)} - 1\right)\right]\frac{\partial \hat{V}}{\partial y} + \hat{V}e^{-a(T-t)}r_t\right\},$$

$$\frac{\partial V}{\partial A_t} = E\frac{\partial \hat{V}}{\partial y}\frac{t}{TA_t}, \qquad \frac{\partial V}{\partial S_t} = \frac{E}{S_t}\frac{\partial \hat{V}}{\partial y}\frac{T-t}{T},$$

$$\frac{\partial^2 V}{\partial S_t^2} = \frac{E}{S_t^2}\left[\left(\frac{T-t}{T}\right)^2\frac{\partial^2 \hat{V}}{\partial y^2} - \frac{\partial \hat{V}}{\partial y}\frac{T-t}{T}\right],$$

$$\frac{\partial V}{\partial r_t} = E\left\{\frac{1}{T}\left[\frac{e^{-a(T-t)} - 1}{a^2} + \frac{T-t}{a}\right]\frac{\partial \hat{V}}{\partial y} + \hat{V}\frac{1}{a}\left(e^{-a(T-t)} - 1\right)\right\},$$

$$\frac{\partial^2 V}{\partial S_t \partial r_t} = \frac{E}{aS_t}\left(e^{-a(T-t)} - 1\right)\frac{\partial \hat{V}}{\partial y}\frac{T-t}{T} + \frac{E}{S_t}\frac{T-t}{T^2}\left(\frac{e^{-a(T-t)} - 1}{a^2} + \frac{T-t}{a}\right)\frac{\partial^2 \hat{V}}{\partial y^2},$$

$$\begin{aligned}\frac{\partial^2 V}{\partial r_t^2} &= 2\frac{e^{-a(T-t)} - 1}{a}E\left[\frac{1}{T}\left(\frac{e^{-a(T-t)} - 1}{a^2} + \frac{T-t}{a}\right)\frac{\partial \hat{V}}{\partial y}\right] + E\hat{V}\left[\frac{e^{-a(T-t)} - 1}{a}\right]^2 \\ &+ \frac{1}{T^2}\left(\frac{e^{-a(T-t)} - 1}{a^2} + \frac{T-t}{a}\right)^2 E\frac{\partial^2 \hat{V}}{\partial y^2}.\end{aligned}$$

Substituting the previous Equation into (14), we obtain that

$$\frac{\partial \hat{V}}{\partial t} + \alpha_1(t)\frac{\partial \hat{V}}{\partial y} + \alpha_2(t)\frac{\partial^2 V}{\partial y^2} + \alpha_3(t)\hat{V} = 0, \tag{15}$$

where

$$\alpha_1(t) = \frac{a\hat{\theta}}{T}\left[\frac{e^{-a(T-t)}-1}{a^2} + \frac{T-t}{a}\right] + (2-2^{2H-1})Ht^{2H-1}\left[-\sigma_1^2\frac{T-t}{T}\right.$$

$$\left. +2\sigma_2^2\frac{e^{-a(T-t)}-1}{aT}\left(\frac{e^{-a(T-t)}-1}{a^2}+\frac{T-t}{a}\right) + 2\rho\sigma_1\sigma_2\frac{e^{-a(T-t)}-1}{a}\frac{T-t}{T}\right],$$

$$\alpha_2(t) = (2-2^{2H-1})Ht^{2H-1}\left[\sigma_1^2\left(\frac{T-t}{T}\right)^2 + \frac{\sigma_2^2}{T^2}\left(\frac{e^{-a(T-t)}-1}{a^2}+\frac{T-t}{a}\right)^2\right.$$

$$\left. +2\rho\sigma_1\sigma_2\left(\frac{e^{-a(T-t)}-1}{a^2}+\frac{T-t}{a}\right)\frac{T-t}{T^2}\right],$$

$$\alpha_3(t) = (2-2^{2H-1})Ht^{2H-1}\sigma_2^2\left(\frac{e^{-a(T-t)}-1}{a^2}\right)^2 + \hat{\theta}\left(e^{-a(T-t)}-1\right).$$

At maturity date T, $V(T, r_T, S_T, A_T) = (A_T - K)^+$ can be rewritten as

$$\hat{V}(T, y) = (e^y - K)^+.$$

For Equation (15), it can be transformed into a heat equation by substitution

$$\tau = \gamma(t),\ \eta = y + \alpha(t),\ U(\eta, \tau) = \hat{V}(y, t)e^{\beta(t)},$$

where $\alpha(t), \beta(t), \gamma(t)$ are undetermined functions.
By calculation,

$$\frac{\partial \hat{V}}{\partial t} = e^{-\beta(t)}\left[\alpha'(t)\frac{\partial U}{\partial \eta} + \gamma'(t)\frac{\partial U}{\partial \tau} - \beta'(t)U\right],$$

$$\frac{\partial \hat{V}}{\partial y} = e^{-\beta(t)}\frac{\partial U}{\partial \eta},$$

$$\frac{\partial^2 \hat{V}}{\partial y^2} = e^{-\beta(t)}\frac{\partial^2 U}{\partial \eta^2}.$$

Substituting the previous Equations into (15) yields

$$(\alpha'(t) + \alpha_1(t))\frac{\partial U}{\partial \eta} + \gamma'(t)\frac{\partial U}{\partial \tau} + \alpha_2(t)\frac{\partial^2 U}{\partial \eta^2} + (\alpha_3(t) - \beta'(t))\,U = 0. \tag{16}$$

By letting

$$\alpha'(t) + \alpha_1(t) = 0,\ \gamma'(t) + \alpha_2(t) = 0,\ \alpha_3(t) - \beta'(t) = 0,$$

According to the terminal conditions $\alpha(T) = \beta(T) = \gamma(T) = 0$, we can get

$$\alpha(t) = \int_t^T \alpha_1(s)\mathrm{d}s, \gamma(t) = \int_t^T \alpha_2(s)\mathrm{d}s, \beta(t) = -\int_t^T \alpha_3(s)\mathrm{d}s,$$

Substituting them back into (16), one gets

$$\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial \eta^2}, \tag{17}$$

while the terminal condition $\hat{V}(T, y) = (e^y - K)^+$ can be converted into

$$U(\eta, 0) = (e^\eta - K)^+.$$

According to the theory of heat Equation Yao and Li (2018), a solution to Equation (17) can be obtained as

$$\begin{aligned} U(\eta, \tau) &= \frac{1}{2\sqrt{\pi\tau}} \int_{\ln K}^{+\infty} (e^z - K) e^{-\frac{(z-\eta)^2}{4\tau}} \mathrm{d}z \\ &= \frac{1}{2\sqrt{\pi\tau}} \int_{\ln K}^{+\infty} e^z e^{-\frac{(z-\eta)^2}{4\tau}} \mathrm{d}z - \frac{K}{2\sqrt{\pi\tau}} \int_{\ln K}^{+\infty} e^{-\frac{(z-\eta)^2}{4\tau}} \mathrm{d}z = I_1 + I_2, \end{aligned}$$

For I_1, let $t = \dfrac{z - \eta - 2\tau}{\sqrt{2\tau}}$, then

$$I_1 = \frac{1}{2\sqrt{\pi\tau}} \int_{\ln K}^{+\infty} e^z e^{-\frac{(z-\eta)^2}{4\tau}} \mathrm{d}z = \frac{1}{\sqrt{2\pi}} e^{\eta+\tau} \int_{\frac{\ln K - \eta - 2\tau}{\sqrt{2\tau}}}^{+\infty} e^{-\frac{t^2}{2}} \mathrm{d}t = e^{\eta+\tau} N\left(\frac{\eta + 2\tau - \ln K}{\sqrt{2\tau}}\right).$$

For I_2, let $\omega = \dfrac{z - \eta}{\sqrt{2\tau}}$, then

$$I_2 = \frac{-K}{2\sqrt{\pi\tau}} \int_{\ln K}^{+\infty} e^{-\frac{(z-\eta)^2}{4\tau}} \mathrm{d}z = \frac{-K}{\sqrt{2\pi}} \int_{\frac{\ln K - \eta}{\sqrt{2\tau}}}^{+\infty} e^{-\frac{\omega^2}{2}} \mathrm{d}\omega = -KN\left(\frac{\eta - \ln K}{\sqrt{2\tau}}\right).$$

By transformation $U(\eta, \tau) = \hat{V}(y, t) e^{\beta(t)}$,

$$\hat{V}(y, t) = U(\eta, \tau) e^{-\beta(t)} = e^{\eta+\tau-\beta(t)} N\left(\frac{\eta + 2\tau - \ln K}{\sqrt{2\tau}}\right) - K e^{-\beta(t)} N\left(\frac{\eta - \ln K}{\sqrt{2\tau}}\right). \tag{18}$$

By substituting the expression of $\eta, \tau, \beta(t)$ into (18), we can get

$$\hat{V}(y, t) = e^{y + \int_t^T [\alpha_1(s) + \alpha_2(s) + \alpha_3(s)]\mathrm{d}s} N(d_1) - K e^{\int_t^T \alpha_3(s)\mathrm{d}s} N(d_2), \tag{19}$$

where,

$$d_1 = \frac{y + \int_t^T \alpha_1(s)\mathrm{d}s + 2\int_t^T \alpha_2(s)\mathrm{d}s - \ln K}{\sqrt{2\int_t^T \alpha_2(s)\mathrm{d}s}},$$

$$d_2 = d_1 - \sqrt{2\int_t^T \alpha_2(s)\mathrm{d}s}.$$

By using substitution of variable conditions, together with Equation (19) , we can get

$$V(t, r_t, S_t, A_t) = F^{\frac{aT+1}{aT}} A_t^{\frac{t}{T}} \left(\frac{S_t}{F^{\frac{1}{aB(t,T)}}} \right)^{\frac{T-t}{T}} e^{L_1} N(d_1) - KFe^{L_2} N(d_2), \tag{20}$$

where

$$L_1 = \frac{A(t,T)\left[B(t,T)(aT+1) - (T-t)\right]}{aTB(t,T)} + \int_t^T \left[\alpha_1(s) + \alpha_2(s) + \alpha_3(s)\right] \mathrm{d}s,$$

$$L_2 = A(t,T) + \int_t^T \alpha_3(s) \mathrm{d}s,$$

$$d_1 = \frac{\frac{1}{T}\left[t \ln A_t + (T-t) \ln S_t + \frac{(\ln F + A(t,T))(B(t,T) - T + t)}{aB(t,T)}\right] + \int_t^T \left[\alpha_1(s) + 2\alpha_2(s)\right] \mathrm{d}s - \ln K}{\sqrt{2 \int_t^T \alpha_2(s) \mathrm{d}s}},$$

$$d_2 = d_1 - \sqrt{2 \int_t^T \alpha_2(s) \mathrm{d}s}.$$

□

Corollary 3.3. *Under the same assumptions in Theorem 3.2. Assume that stock price S_t satisfies Equation (3), interest rate r_t Equation formula (4), denote $V_p(t, r_t, S_t, A_t)$ as the value of Asian geometric average put option at time t ($0 \leq t \leq T$) with fixed strike pricing K and maturity T, then*

$$V_p(t, r_t, S_t, A_t) = KFe^{L_2} N(-d_2) - F^{\frac{aT+1}{aT}} A_t^{\frac{t}{T}} \left(\frac{S_t}{F^{\frac{1}{aB(t,T)}}} \right)^{\frac{T-t}{T}} e^{L_1} N(-d_1).$$

Proof. According to the terminal condition $V_p(T, r_T, S_T, A_T) = (K - A_T)^+$, the value of put option $V_p(t, r_t, S_t, A_t)$ at time t ($0 \leq t \leq T$) can be obtained by solving Equation (14) with the solution method in Theorem 3.2.

4. Simulations

In this chapter, the impacts of Hurst index H, stock price S_t, interest rate r_t and zero-coupon bond price F on Asian geometric average call option prices are discussed. According to the Asian geometric average call option pricing formula from Theorem 3.2, the constant parameters in the Asian geometric average call option formula of Theorem 3.2 are assumed to be as follows:

$$t = 0, T = 1, a = 0.6, \lambda = 0.3, \theta = 0.8, \sigma_1 = 0.5, \sigma_2 = 0.4, \rho = 0.3.$$

Under such setting of parameters, Table 1 and Table 2 illustrate that the Asian geometric average call option prices at fixed strike prices under different Hurst indices, stock prices and zero-coupon bond prices.

Table 1. Geometric average Asian call option prices under Hurst Index H=0.1 and H=0.11.

parameter	H=0.1					H=0.11				
	F=0.2	F=0.3	F=0.4	F=0.5	F=0.6	F=0.2	F=0.3	F=0.4	F=0.5	F=0.6
S=20	0.106	0.107	0.113	0.127	0.139	0.103	0.104	0.112	0.123	0.129
S=30	0.129	0.172	0.184	0.504	1.023	0.119	0.143	0.154	0.478	0.994
S=40	0.178	0.188	1.000	1.986	2.924	0.148	0.158	0.958	1.941	2.857
S=50	0.277	1.458	2.862	4.330	5.780	0.245	1.414	2.811	4.273	5.716
S=60	1.683	3.384	5.329	7.368	9.414	1.653	3.340	5.274	7.305	9.343
S=70	3.309	5.564	8.237	10.930	13.649	3.282	5.612	8.184	10.867	13.577
S=80	5.073	8.157	11.461	14.880	18.336	5.049	8.120	11.412	14.819	18.266
S=90	6.924	10.818	14.910	19.112	23.359	6.094	10.786	14.865	19.057	23.294
S=100	8.831	13.589	18.518	23.551	28.632	8.816	13.562	18.480	23.503	28.575
S=110	10.776	16.434	22.241	28.143	34.092	10.764	16.413	22.210	28.103	34.034

Table 2. Geometric average Asian call option prices under Hurst Index H=0.6 and H=0.7.

parameter	H=0.6					H=0.7				
	F=0.2	F=0.3	F=0.4	F=0.5	F=0.6	F=0.2	F=0.3	F=0.4	F=0.5	F=0.6
S=20	0.006	0.016	0.105	0.114	0.118	0.003	0.014	0.102	0.110	0.114
S=30	0.029	0.039	0.136	0.254	0.387	0.016	0.028	0.124	0.162	0.278
S=40	0.078	0.088	0.188	0.386	0.489	0.065	0.053	0.175	0.275	0.387
S=50	0.177	0.187	1.056	1.065	2.184	0.164	0.165	0.856	1.020	1.182
S=60	0.583	0.706	1.980	3.478	5.032	0.472	0.605	0.919	2.253	3.676
S=70	1.629	3.013	4.772	6.761	8.848	1.154	2.209	3.670	5.414	7.295
S=80	3.524	5.644	8.052	10.667	13.391	3.079	4.884	6.973	9.296	11.762
S=90	5.492	8.461	11.641	14.992	18.448	5.061	7.738	10.602	13.643	16.812
S=100	7.495	11.382	15.419	19.592	23.857	7.067	10.682	14.416	18.279	22.244
S=110	9.517	14.363	19.315	24.370	29.502	9.085	13.672	18.335	23.088	27.919

From Table 1, and Table 2, the interaction of Hurst index, initial price of the stock, face value of the zero-coupon bond and the price of Asian geometric average call options are illustrated. It is natural to see that for a call-type option, the price of the Asian geometric average call option is positive-related with the initial price of the stock. The same pricing dynamic applies for the relationship between the price of the Asian geometric average call option and face value of the zero-coupon. Keeping the initial price of the stock and the face value of the zero-coupon unchanged, the data from the Table 1 and Table 2 demonstrates that the prices of Asian geometric average call options are negatively-related to the Hurst Index.

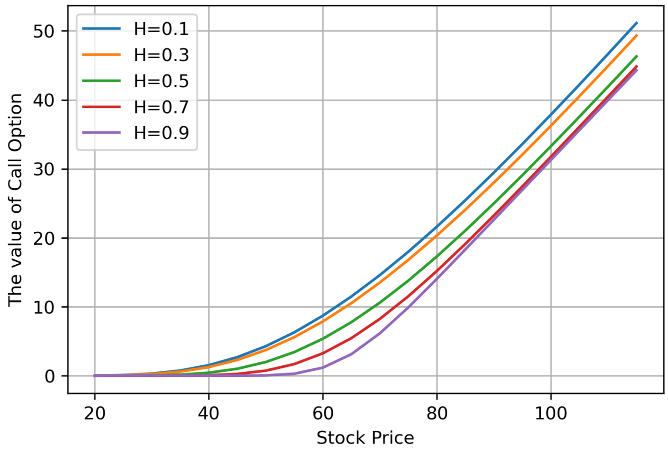

Figure 1. The price of the Asian geometric average call option with different Hurst Index.

For more values of Hurst index, this negative-relation stays consistent, as shown in the Figure 1, where the asset driven by a geometric Brownian motion (GBM) has been employed as a benchmark. Such convergence results validate our model.

Figure 2 illustrates the prices of Asian geometric average call options when the underlying assets are modelled by geometric Brownian motion (GBM), which indicates that when $H = 0.5$, prices of Asian geometric average call options driven by sfBm will converge to the prices of Asian geometric average call option driven by GBM.

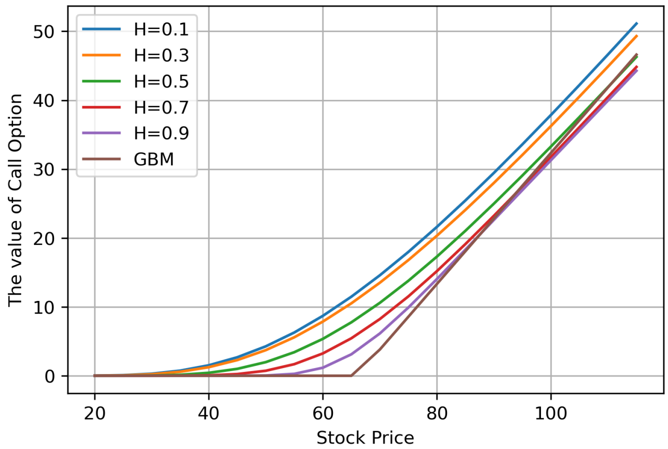

Figure 2. The price of Asian geometric average call option with different Hurst Index and GBM.

Figure 3 illustrates the Delta of Asian geometric average call option with respect to time to maturity as well as the price of underlying asset. The Delta is most sensitive when the price of underlying asset is close to the strike price $K = 66$. However, such measurement of Delta is not an optional indicator for the Asian geometric average call option, since it is simultaneous, which is inconsistent with the path-dependent property of Asian geometric average call options.

Figure 4 and 5 illustrate the Gamma and Vega of Asian geometric average call option with respect to time to maturity as well as the price of underlying asset. Both Greeks preserve the properties that they are very sensitive when the price of underlying asset is close to the strike price. Such observations

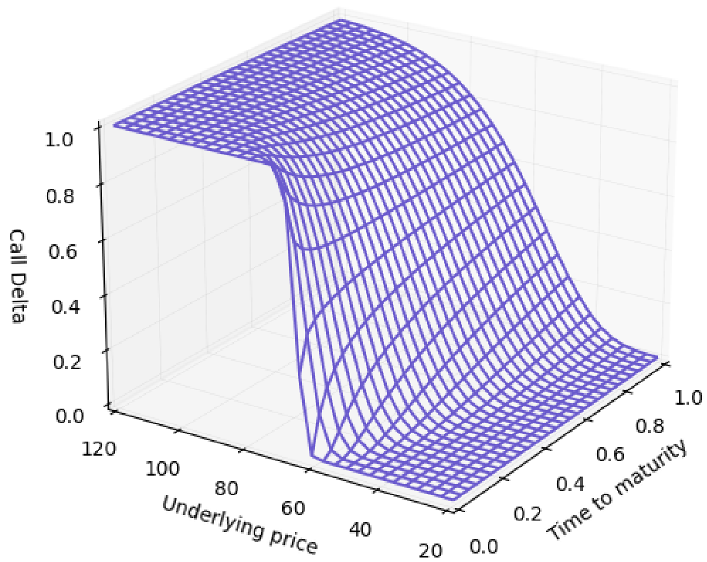

Figure 3. The Delta of Asian geometric average call option option with K = 66.

will lead to a natural conclusion that the existing technique of calculation of Greeks does not apply, and modification of Greeks' calculation should be taken into consideration in the future research.

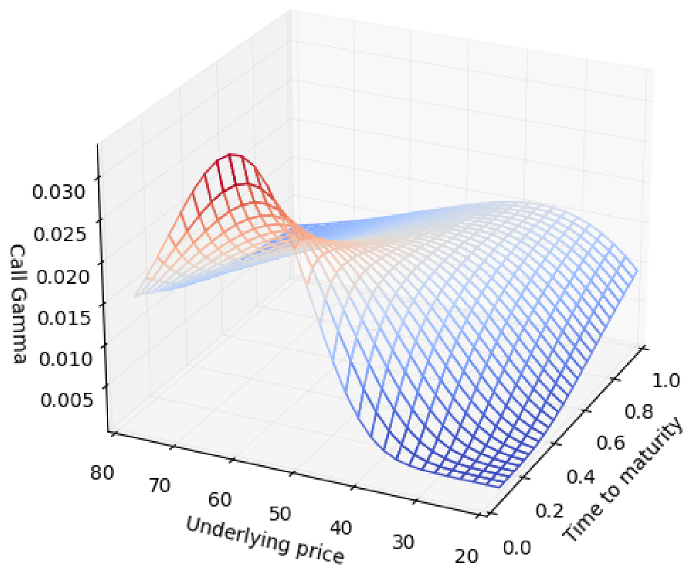

Figure 4. The Gamma of Asian geometric average call option with $K = 66$.

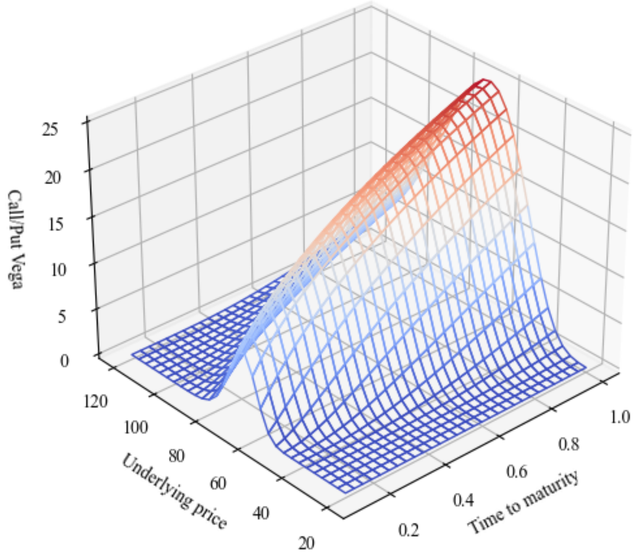

Figure 5. The Vega of Asian geometric average call option with$K = 66$.

5. Conclusions

In this paper, the underlying asset is modelled by sfBm and the interest rate follows the sub-fractional Vasicek model. The explicit pricing formula for the Asian geometric average option has been derived. According to Theorem 3.2, such a result can be seen as an natural extension of the fractional Vasicek model Zhou and Li (2014). The numerical simulation also suggests that when $H = 0.5$, the price of the Asian geometric average call option driven by sfBm converges to the the Asian geometric average call option driven by GBM.

The results of this paper have more implications, in both theoretical and industrial aspects. For the theoretical aspect, Theorem 3.2 can be extended easily to the case where all parameters a, θ, σ are time-dependent, i.e., the corresponding sub-fractional Hull-White model. However, for the sub-fractional CIR model, such result is no longer applicable. For the industrial aspects, our simulation results suggest that the existing calculation technique for Greeks does not work for Asian-type options. More generally speaking, for the path-dependent option, the Greeks could be redefined, i.e., an "average version" rather than a "simultaneous" version. In the future, some restriction will be implemented to remove some trading strategy with arbitrages.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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