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Research article

Sigmoidal dynamics of macro-financial leverage

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Abstract: Logistic sigmoids due to their flexibility seem to be natural candidates for modelling macrofinancial leverage behavior. The sigmoidal leverage transition towards its stationary value, which was driven by the yield spreads, could have replicated the dynamics of macrofinancial assets, debt and capital. The leverage transition, in its turn, has been a major factor in better balancing macrofinancial liabilities and assets. The sigmoidal leverage trajectories including their inflections and different phases were identified by a nonlinear transition function providing information necessary for steering the process towards its stable state. Solving the stationary Kolmogorov-Fokker-Plank logistic equation revealed that random leverage realizations might follow the gamma distribution. Parameters of its stationary probability density function, as well as the expected and the modal leverage, were dependent on the process variance and the yield spreads. Thus, the stochastic leverage behaviour reproduced a sequence of stylized phases similar to the observed in the US Treasuries market meltdown in 2020. In particular, larger yield spreads and smaller modal leverage signalled a "defensive" market response to sudden volatility increases. In addition, it was shown that the logistic leverage modelling could be helpful in the analysis of debt and money dynamics including some consequences of "minting a one trillion dollars coin".

Keywords: sigmoid; macrodebt; financial leverage; logistic equation; gamma distribution

JEL Codes: C63, E44, G17

1. Introduction

Logistic sigmoids that replicated volatile dynamics of debt and capital accumulation were proposed to model macro-financial leverage behavior. Such a proposal, at first glance, may seem unexpected, since the logistic ODE (ordinary differential equation) has been used traditionally in modeling the dynamics of human and biological populations while macro-financial leverage, or asset-to-equity ratio, is an indicator of a completely different nature.

Meanwhile, logistic sigmoidal curves are widely used nowadays in various fields of science and technology. In addition to their numerous applications to the studies of demographic and environmental processes, they model signal transmissions, neural networks and artificial intelligence, epidemics and learning processes. Logistic models are successfully used in research on energy and mining processes, and the "Bass diffusion" model is a popular instrument in studying management and actuarial problems. The original logistic ODE of F. Verhulst has a number of interesting generalizations, including models of Richards (1959), Blumberg (1968), and Tsoularis (2001). The family of sigmoids has been replenished recently with hyperbolastic and oscillobolastic curves (Tabatabai et al., 2012), which are solutions to nonlinear differential equations. In short, the literature on various sigmoidal models is vast, stimulating, and its review goes far beyond the scope of this paper. The competent review of a somewhat narrow field of logistic models can be found, for example, in the aforementioned work of Tsoularis (2001).

Economists are well aware of probabilistic logit models (Rodriguez, 2012), as well as of logistic mappings that generate chaotic behavior (Sornette, 2006). These promising avenues of research are, however, beyond the scope of this paper since the logistic ODE of a macro-financial leverage does not produce chaotic behavior while, on the other hand, the leverage stochastic dynamics follow gamma distribution.

The paper was focused on the analysis of economic fundamentals of a macro-financial leverage behavior which was modeled by logistic sigmoids. This simple analytical tool allowed the revealing of several important features in the leverage dynamic by relating them to volatility, macro-financial yields and their spreads. It should be stressed that studying transition process of leverage was especially relevant to macrofinancial analysis when the global financial assets, four-fifths of which were formed by debt instruments, became almost four times as large as the world GDP (Institute of International Finance, 2021; World Bank, 2021).

Economic prerequisites of the proposed model were based on several important observations and results that are instrumental in the modeling of a leverage transition process. It could be recalled that financial leverage, as a relative debt indicator, was being used since time immemorial[1]. The leverage equation could be viewed as one of the consequences of the Modigliani-Miller theorem (Anderson, 2014; Holmstrom, 2015). The leverage/rates of return conjugacy that appeared in a macro-financial system dynamic appeared to be closely related to the J. Geanakoplos concept of a financial "promise" (1999, 2010) which he developed in a context of the Arrow-Debreu model.

The leverage impact upon the behavior of market agents had been thoroughly studied in the Adrian (2005) and Adrian et al. (2018). They established and thoroughly analyzed asymmetric statistical relations between changes in financial assets and leverage. They showed, in particular, that

[1] Surprisingly, a precise description of the duality between interest rates and collateral ratios could be found in W. Shakespeare's masterpiece *The Merchant of Venice*. Namely, a loan extended by Shylock to Bassanio was outlined as interest-free, but its redemption was guaranteed by a pound of Antonio's flesh which served as the loan collateral.

households reacted to larger assets by reducing leverage while the reaction of financial corporations was exactly the opposite. These observed features of macro-financial behavior were used in the model by the former with the aggregate of lenders, and the latter—with the aggregate of borrowers.

Modeling relationships between volatility and financial leverage in the context of a risk-free debt could be traced back to the works of Black (1976) and Christie (1982). The modern approach to studying general links between market beliefs, financial risks and macroeconomic variables could be found, for example, in (Kozlowski et al., 2015). Modeling the above relationships by stochastic differential equations (SDE) revealed some important mutual effects of leverage and volatility. Thus, the analysis of a gamma probability density function was helpful in discovering the straightforward causal links between (increasing) market turbulence and (decreasing) modal leverage.

In particular, the unexpected increases in volatility, by damaging standard market beliefs, exposed structural problems which were not manifested during a time of normal market functioning. The Basel Bank's study of the meltdown in March 2020 of US Treasuries market (Schrimp et al., 2020) revealed a sequence of distinctive phases being formed in the evolvement of that unprecedented disruption. A very similar phase structure was generated by the gamma probability density function due to large fluctuations in the random leverage variance. In our opinion, it is an evidence of the model applicability to solving complex economic problems.

Theoretical provisions were synthesized by developing and refining the model of stochastic dynamics of a macro-financial leverage (Smirnov, 2018), and could be summarized as follows. Mutually adaptive behavior of lenders and borrowers formed foundations of their beliefs, and it was modeled by the logistic ODE and SDE parameterized by yield spreads and variance of a macro-financial leverage. The leverage transition process generated by its arbitrary initial value transformed it into a stationary ratio of yield spreads, which served as the process attractor. The quadratic transition (pass-through) function, together with the "bell-shaped" leverage derivative, provided an information about the different phases in the leverage development. In particular, they were instrumental in the correspondence between leverage phases and its trajectory inflection. Increasing the market spreads gave rise to the reduction of a financial leverage while the "deleveraging" effects could be considered as kind of "belief-scarring" market reactions. The maximal value of a transition function could be associated, under some conditions, with the greatest confidence of creditors in the repayment of their funds.

Macro-financial leverage followed the logistic SDE in a stochastic version of the model. Solving the stationary Kolmogorov-Fokker-Planck equation allowed the recovering probability density function of a gamma distribution, parameters of which were estimated via yield spreads and leverage variance. The model showed that the increasing leverage turbulence entailed decreases in the modal leverage, and this fact could be used in the elaboration of "defensive" debt market actions. The model also demonstrated its ability to alleviate some of the monetary consequences of "minting a one trillion dollar coin" relevant for the debt market management (Grey, 2020–2021). Some aspects of the model generalization were discussed in the final part of the paper.

The model simulations and graphs were performed in *Mathematica 10.3* program, and were based on information regarding the global financial system assembled from different sources including author's estimations. The numerical results are of an illustrative character.

The rest of the paper is organized as follows. Section 2 sets out economic context for the deterministic logistic model. Section 3 discusses the model behavior and effects of changing the yield spreads. Sections 4 and 5 contain a summary of the stochastic model as well as results of the debt market simulation. Some aspects of the model usage and its generalization are discussed in Section 6, and Section 7 contains concluding remarks.

2. Logistic model of financial leverage

This section discusses the logistic leverage model under the assumption of a zero variance, $\sigma^2 = 0$, and a unitary heterogeneity, $\theta = 1$, of the macro-financial system. Thus, its economic meaning could be extended to both versions, stochastic and θ—generalized, which are considered later.

2.1. Macro-financial balances, rates and leverage

A macro-financial system is composed of assets, debts and capital, which levels and flows are interconnected in the model. At each moment, the states (system levels) are given by the following balance of assets and liabilities:

$$A(t) = D(t) + E(t) \tag{1}$$

where $A(t)$ is the value of macro-financial assets, $D(t)$ is the value of a total debt, and $E(t)$ is the value of capital. In economic terms, $E(t) \leq A(t)$, but the complete macrodebt retirement, $D(t) = 0$, would mean the collapse of a modern financial system.

In turn, the balance of financial flows is given by the relationship between differentials of corresponding variables:

$$dA(t) = D(t) + E(t) \tag{2}$$

Thus, the structure of a macro-financial system is represented by an indicator of leverage, $l(t) = A(t)/E(t)$, and by the rates of return on assets, $\mu = dA(t)/A(t)$, on the capital, $\rho = dE(t)/E(t)$, as well as by the cost of borrowing, $r = D(t)/E(t)$. In the economic sense, the rates would satisfy the inequalities: $r \leq \mu \leq \rho$, while the leverage takes only positive values for $l \geq 1$.

Rate differences are treated as financial spreads, or net relative returns to the aggregate lender, $a = (\mu - r)$, and to the aggregate borrower, $c = (\rho - r)$. Spreads are assumed positive, and due to debt aggregation, "short" and "long" rates of return do not differ. Negative spreads, which are characteristic for high-frequency critical processes, are beyond the scope of this paper.

The balances of states (1) and flows (2) for given rates of return being reduced to the scalar Equation:

$$\mu l = r(l - 1) + \rho \tag{3}$$

Equation 3 could be treated simultaneously. Equation 3 reflects the balance between macro-financial assets and liabilities normalized per unit of capital. It formalized the duality of leverage and rates of return as in the well-known microfinancial equation:

$$\rho = r + (\mu - r)l \tag{3'}$$

which appeared as the consequence of the Modigliani-Miller theorem. It explained the usage of a leverage in microfinancial transactions, in which the expected return on equity is a function of leverage. Empirically, the size of microfinancial leverage varied widely for different institutions. For example, the infamous bank of *Leman Brothers* was leveraged by a factor of 40.

In contrast, the macrofinancial leverage in Equation 3 is an observed variable given the spread and the rates of return. Equation 3, though it is identical formally to Equation 3', defines the behaviour a macro-financial market. The assets and liabilities of the latter are to be balanced at some level of a macrofinancial leverage thus making the aggregated spreads independent, to a large extent, of individual actions even of the largest market participants. Given the expected yield spreads the maximal value of global leverage was about 6.4 at the peak of the Great Financial Crisis 2007–2008.

Obviously, Equation 3 is satisfied for the leverage value:

$$\mu l = r(l-1) + \rho \tag{4}$$

where economic meaning of parameter b will be explained in section 2.2. The leverage value K is, in essence, a kind of "target setting" for the evolving macro-financial system. It balances returns on assets, debt and equity:

$$\mu K = r(K-1) + \rho \tag{5}$$

Implying the balance between macrofinancial assets and liabilities in absolute terms it sets a condition for the normal debt repayment that could be violated (in a deterministic process) only by the occurrence of external factors. The expected spreads, on the other hand, do not reflect the impact of all factors affecting actual (or initial) financial leverage, $l_0 = A_0/E_0$. Thus, the leverage transition could be viewed as generated by a discrepancy between actual and "expected" leverage values that ends up at the stationary value, $K = c/a$. At the stationary leverage the macro-financial assets and liabilities are balanced, and studying the process of transition requires the modelling of leverage dynamics.

2.2. The logistic leverage ODE

It is rather obvious that changes (instantaneous) in leverage were related to its levels and the rates of return:

$$\dot{l} = d/dt[A(t)/E(t)] = (\mu - \rho)l(t) \tag{6}$$

Yet, the linear ODE 6 does not satisfy the economic essence of a leverage dynamic, for example, given the expected rates of return, it has a zero stationary value which is meaningless.

Therefore, it seems reasonable to assume that the market beliefs and behavior could be structured in a more general differential equation:

$$\dot{l} = f[l(t)] = l(t)g[l(t)] \tag{7}$$

where changes (instantaneous) in the leverage $\dot{l} = dl(t)/dt$were formed by interactions between its level $l(t)$ and the relative growth rate$g[l(t)]$. The transition function$f[l(t)]$ was assumed linearly homogeneous with respect to the leverage, the stationary value of which corresponded to a zero growth rate, $g(l^*) = 0$. The scale of leverage and its growth rate in Equation 7 differ, empirically, by about an order of magnitude.

Model 7 becomes operational after expanding function $g[l(t)]$ into the Taylor series near the stationary value, l^*:

$$g(l) \cong g(l^*) + g^{'}(l^*)(l - l^*) = -g^{'}(l^*)l^* + g^{'}(l^*)l \tag{8}$$

And estimating its coefficients. Using the series expansion eliminated disadvantages of the linear model 6 in two steps. First, notice that the rate of returns on equity follows Equation 3 and the difference:

$$\mu - \rho[l(t)] = a - al(t) \tag{9}$$

Equation 9 would transform ODE 7 into a one-parameter logistic equation. This step corresponds to transcritical bifurcation: the zero stationary leverage becomes unstable and a new, stable steady leverage appears. Second, in accordance with empirical observations and economic essence, stationary leverage has to be associated with some point $K > 1$ where macro-financial assets and liabilities are balanced. This requirement is satisfied due to the following condition[2]:

$$a - \frac{a}{K} l(t) = a - bl(t) \tag{10}$$

where parameter b is defined by Equation 4. Thus, coefficients in the Taylor series Equation 8 could be calculated *ipso facto* as:

$$-g'(l^*)l^* = a \ and \ g'(l^*) = -b \tag{11}$$

Taking these equalities into account, the macro-financial ODE 7 becomes the logistic leverage Equation:

$$dl(t) = [al(t) - bl^2(t)]dt, l(0) = l_0 \tag{12}$$

According to Equation 12, the leverage transition process is determined by interactions between its trend (the first component in RHS) and stabilizing feedbacks (the second component). The lenders' yield spread a sets the intensity of the leverage growth while feedbacks were defined by a combination of lenders' and borrowers' spreads, $b = a^2/c$. According to model 12 the discrepancy between initial and stationary leverage was generated by misbalanced increments in the value of assets and liabilities, and given the expected market structure, the leverage transition ends up at its stationary value, K. Since the stationary leverage corresponded to the intensities ratio it balanced macro-financial assets and liabilities in relative terms.

2.3. *The leverage transition process*

The exact correspondence between the debt service (in absolute terms) and leverage will be discussed in section 6.1. Different phases occurring in the leverage dynamics following logistic ODE 12 are given by its non-linear transition function:

$$f(l) = al - bl^2 = al(1 - l/K) \tag{13}$$

Phases in the leverage transition are illustrated in Table 1, where signs (+, –) indicate positive or negative values of the respective functions.

Table 1. Phases of the transient process.

	$1 \le l < c/2a$	$\hat{l} = c/2a$	$c/2a < l < K$	$l^* = K$	$l > K$
$f(l)$	+	max	+	0	–
$f'(l)$	+	0	–	–	–
$g(l)$	+	+	+	0	–

As seen in Table 1, the debt growth outstrips capital increases for small values of leverage, then reaches its maximum and decreases approaching a stationary value. The market activity could be measured by the Jacobian:

[2] Estimations of the global financial leverage showed that the feedback parameter is of order of magnitude smaller than the leverage growth parameter, which confirm the logistic model empirically.

$$f^{'}(l) = a - 2bl \tag{14}$$

which indicates the stable balanced macro-financial market. However, the most active market at the leverage:

$$\hat{l} = a/2b \equiv c/2a \tag{15}$$

Equation 15 is unstable due to mismatched macro-financial assets and liabilities.

3. Calculating sigmoidal trajectories

3.1. Leverage and capital intensity

In a broad economic context, leverage dynamics could be associated with an indicator of capital intensity,$w(t) = E(t)/A(t)$. The behavior of the latter was modeled by a linear ODE while the model of a leverage dynamic is given by a nonlinear equation. The conjugated leverage and capital intensity processes could be expressed by the following relation:

$$l(t) = w^{-1}(t) \tag{16}$$

which allows for solving the logistic ODE 12 as a special case of the Bernoulli differential Equation. Thus, Equation 16 being substituted into Equation 12 reduces the latter to a linear ODE:

$$\dot{w} = -aw(t) + b; w(0) = w_0 \tag{17}$$

which is solved for the following capital intensity trajectory:

$$w(t) = [w_0 - b/a]\, exp[-at] + b/a \tag{18}$$

According to Equation 18, the capital intensity indicator goes on as a process of relaxation from some initial to its stationary value. In its turn, Equation 18 is easily transformed into a conjugate trajectory of the leverage sigmoidal dynamics:

$$l(t) = K(1 + (\frac{K}{l_0} - 1)\, exp[-at])^{-1} \tag{19}$$

3.2. The leverage trajectories analysis

The "bell-shaped" derivative of sigmoid 19 reaches its maximum at $l_0 = 0.5K$, which corresponds to the trajectory inflection. In addition, the phase classification in Table 1 could be supplemented by three different types of the leverage trajectories which are discerned by a following multiplier:

$$(\frac{K}{l_0} - 1)\begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 1 \tag{20}$$

Thus, inequality$(K/l_0 - 1) > 1$, or, equivalently, $c > 2l_0 a$, shows smaller net returns of lenders comparing to that of borrowers. This phase of "lenders oppression" might take place as a consequence of liquidity overflows ensuring large sales of new debts. The market activity reaches its maximal value for the equality, $(K/l_0 - 1) = 1$, or $c = 2l_0 a$, which, however, is short-lived and unstable[3]. In its turn, inequality,$(K/l_0 - 1) < 1$ or$c < 2l_0 a$, corresponds to the unwinding market activity when costs of borrowing are increased thereby "oppressing" borrowers. The macro-financial market as a whole is not balanced along these trajectories since assets and liabilities are equal only at a stationary leverage.

3.3. Changes in spreads and market reactions

The proposed model claims that changes in spreads and leverage are interrelated. This claim is reflected, in particular, in the stationary leverage definition, which makes it very different from the logistic models of populations where parameters are assumed to be completely independent. Within the model itself, macro-financial leverage is an inertial characteristic of a market while changes in the rates of return belong to the high-frequency part of a spectrum. Therefore, we can consider the effects of a variable lenders' spread on the stationary leverage value, albeit with proviso of a logical reversibility.

Let the change in the yield spread of lenders be characterized by an intensity factor $\pm\delta$ which is proportional to the leverage value. Then Equation 12 takes the following form:

$$dl(t) = [al(t) - bl^2(t)]dt \pm \delta l(t)dt \tag{21}$$

The stationary leverage to Equation 21 is reduced or increased accordingly:

$$K_1 = K\frac{a}{a \pm \delta} \tag{22}$$

Due to increases (decreases) in the lenders' spread, $a_1 = a \pm \delta$. For example, the decrease in the stationary leverage from $K = 4.14$ to $K_1 = 3$ is illustrated in Fig. 1 as a consequence of spread $a = 0.029$ expansion by $\delta = 0.014$. It could be noted that the logistic Equation (21) has a form of the leverage "control", but we rather refrain from its statement because appropriate mechanisms of macro-financial regulation are not evident in a competitive economy.

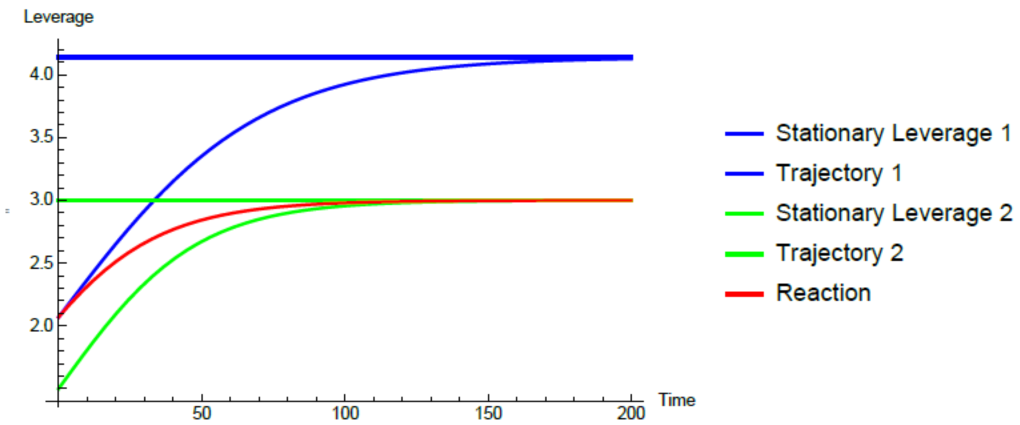

Figure 1. Spreads changes and the leverage correction.

[3] This state of a system is examined in Section 6.2 under specific condition, namely, of equality between expected rates of return for lenders and borrowers.

Unfavorable events, which influence lenders' spreads and force the market to react, are of particular interest. Intuitively, it could be stated that the markets "protective" mechanisms would reduce, as a rule, the value of a stationary leverage thus making "deleveraging effects" especially pronounced.

4. Stochastic leverage dynamics

The deterministic model 12 while explaining mechanisms of macro-financial leverage dynamics has rather limited practical applications. The reasons are obvious: spreads of borrowers and lenders do not reflect the impact of all factors affecting assets, debt and capital. Since the actual (observable) leverage transition did not occur at zero variance, its approximation by a deterministic trajectory would not be expected as relevant, as a rule. More realistic results of the model application to real processes could be achieved by introducing an additional parameter of the leverage dynamics. A natural candidate for this role is a variance of leverage (or its volatility), which reflects a cumulative impact of exogenous factors driving its stochastic dynamics.

General beliefs about the market behavior could be structured in the first-order SDE with a multiplicative factor of random fluctuations:

$$dl(t) = l(t)\{[a - bl(t)]dt + \sigma dZ(t)\} \tag{23}$$

where σ is the leverage volatility parameter, and $Z(t) = \int_0^t dZ(u)$is the standard Brownian motion. It was in agreement with reasoning of (Kozlowski et al., 2015), but, obviously, it was not the only approach to stochastic modeling of logistic dynamics (Mao, 2001; Dennis et al, 2003).

The advantage of the SDE 23 representation is that it generalized the deterministic ODE 12 in the most natural way, thus preserving its economic interpretation entirely. An important addition to Equation 12 consisted of a random term,$\sigma l(t)dZ(t)$, which accumulated the impact of the exogenous factors. A family of random realizations of a macro-financial leverage, as well as its deterministic dynamics, is shown in Figure 2 for $a = 0.029$, $\sigma^2 = 0.016$and $l_0 = 4.5$.

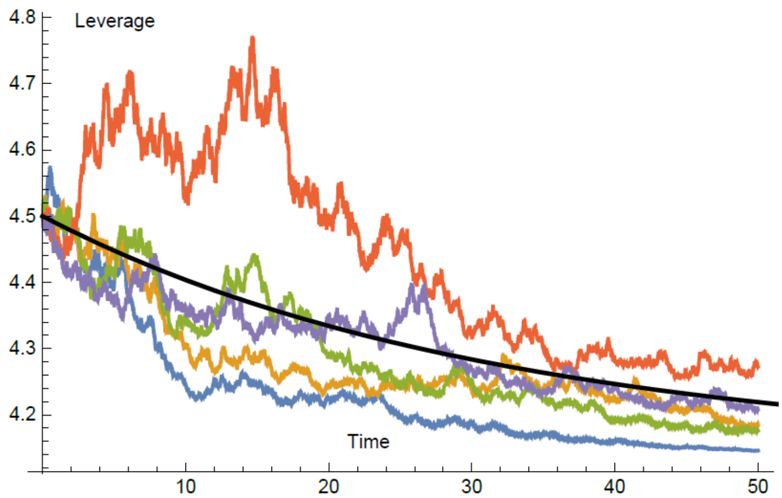

Figure 2. Macrofinancial leverage realizations.

4.1. Stationary probability distribution

It is known that the SDE 23 has an analytical solution (Sciadas, 2010), but its probability density function$p[l(t),t]$ conveys all the information necessary for our purposes. Conveniently, a simpler stationary density function $p(l)$ could be found by solving the Kolmogorov-Fokker-Planck ODE:

$$-\frac{d}{dl}[al(1-\frac{l}{K})p(l)]+\frac{d^2}{dl^2}[\sigma^2 l^2 p(l)]=0 \tag{24}$$

Omitting mathematical subtleties (Gardiner, 1994), we can say that Equation 24, after being reduced to a linear first-order ODE, could be solved with regard to the following PDF:

$$p(l;\alpha,\beta)=\frac{\beta^\alpha}{\Gamma(\alpha)}l^{\alpha-1}\,exp[-\beta l] \tag{25}$$

where the constant of integration is found from the condition $\int_0^\infty l^{\alpha-1}\,exp[-\beta l]dl=1$. Equation 25 is the probability density of the gamma distribution with parameters α,β which exists in the interval of observable variance values, $0<\sigma^2<2a$.

The following properties of the gamma distribution are important for the logistic model of a stochastic leverage, and should be taken into account. First, solving the stationary KFP Equation 24 provides parameters of the gamma distribution:

$$\alpha=\frac{2a}{\sigma^2}-1;\ \beta=\frac{2b}{\sigma^2} \tag{26}$$

which require datasets about spreads and variance only for their calculation. This feature stresses the stochastic model relevance to its valid deterministic analogue.

Second, the stochastic leverage model is extremely sensitive to volatility (variance) since the latter accumulated general information about this random macro-financial process. Thus, if the variance increased due to higher market turbulence then, for $\sigma^2\to 2a$ or $\alpha\to 1$, gamma distribution would be transformed into a one-parametric exponential distribution (Walck, 1996). According to the model assumptions, an existing debt market would collapse if leverage trajectories were concentrated in a small neighborhood to the right of the origin. On the other hand, when variance tends to zero then gamma distribution degenerates into the Dirac delta distribution for which the mass is concentrated at the stationary leverage, K (Pascuali, 2001).

The observable variance of a macro-financial leverage usually satisfies inequality,$0<\sigma^2<a$, meaning that its random realizations oscillate around the most probable, modal value, l_m. The gamma-distributed leverage satisfies the following inequalities:

$$l_m<\langle l\rangle<K \tag{27}$$

where $\langle l\rangle=\alpha/\beta$is the expected value of a macro-financial leverage. Due to relations (26) a functional relationship exists between the modal leverage, l_m , and the parameter of variance:

$$l_m=\frac{\alpha-1}{\beta}=K(1-\frac{\sigma^2}{a}) \tag{28}$$

Relations 28 are illustrated in Figure 3 for $a=0.029$,$\sigma^2=0.016$ and $K=4.14$. According to 28, an increase (decrease) in the variance decreases (increases) the modal leverage. It is rather obvious, because lenders confronting market uncertainty would tighten, as a rule, their bids for liquidity while investors tend to behave more cautiously. These actions, in aggregate, are expected to result in reduced amounts of total borrowing accompanied by a smaller steady leverage[4].

The expansion to the right of the upper variance value is noticeable in Figure 3 In a sense, increasing volatility compensates for the influence of an expanding lenders' spread. This effect follows on from Equation 28 whereby changing market beliefs specify a "defensive" market reaction to the growing uncertainty. A similar market behavior, with a profound impact on the global financial system, was observed in the world's largest government bonds market in March 2020.

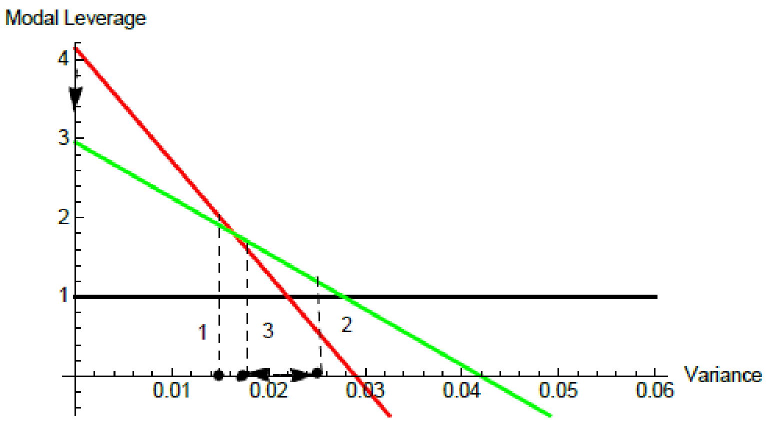

Figure 3. Changes in the modal leverage.

5. Simulating the T-bond market meltdown

The gamma probability density Equation 25 was used to simulate phases in the US Treasuries market meltdown, which were structured in the BIS publication (Schrimp et al., 2020). A brief overview of the March-April 2020 events on it is presented as follows.

5.1. Overview of the US Treasury bond market

Traders in Treasuries, mainly hedge funds, under normal conditions used to take out large loans in order to profit from the yield spreads between bonds and their respective futures. But severe market turbulence (Figure 4) in early March due to the announcement of the Covid-19 pandemic had triggered the fall in bond values, which was more rapid than that of the futures. Such a discrepancy resulted in significant losses for investors in "relative value of assets" since they operated with high leverage, and it induced the so called "rush for dollar cash". When the market solvency was threatened, lenders demanded repayment of their loans, thus forcing hedge funds to sell bonds further *en masse*.

Market dealers holding the excess stocks of debt instruments were able only to widen the bid-ask spreads, which could not, obviously, prevent a collapse in bond prices. Such an outcome had never previously happened in the world's most liquid, deepest and supposedly risk-free Treasuries market.

[4] In the logistic model (23) the volatility parameter is defined for risky borrowings while the Black/Christie hypothesis implied existence of a risk-free debt. Therefore, leaving aside some controversial issues (Hasanholzic and Lo, 2011) formula (28) does not contradict to their hypothesis.

Moreover, such an event had been considered absolutely impossible until it took place (FT, July 29, 2021). Yet, several days later rapid and massive (about $670 billion) purchases by the Federal Reserve System were the only means powerful enough to reduce the excessive risks, and the market was eventually calmed down.

5.2. Simulation of market behavior

According to the above review, the bond market functioning could be structured as a sequence of phases generated by fluctuations in the process volatility. The phases consisted of sharp risk increases; the market meltdown due to abruptly changed beliefs and stereotypes; its "defensive" response; and subsequent normalization due to massive central bank purchases. This sequence of events was simulated on stylized information about variance, yield spreads and modal leverage. The results of the model simulation are presented in Table 2.

The Treasuries market initial position was denoted as phase 1. The unexpected doubling of the market variance was caused by the announcement of the Covid-19 pandemic. It marked phase 2, in which ensuing panic triggered, in the "rush for dollar cash", spasmodic sales of all financial assets, including supposedly riskless government bonds. The market, by increasing lenders' spreads to 0.042 in phase 3, produced the effect of "deleveraging" that manifested itself in a stationary leverage decrease to 3.0 units. Finally, the pre-crisis market variance was restored to its pre-crisis level of 0.0164 thus reassuring investors and normalizing the market in phase 4.

Table 2. Phases of the debt market turbulence.

	Phase I	Phase II	Phase III	Phase IV
Variance, σ^2	0.0144	0.0256	0.0256	0.0164
Spread, a	0.029	0.029	0.042	0.042
Spread, c	0.124	0.124	0.124	0.124
Parameter, b	0.007	0.007	0.014	0.014
Steady leverage, K	4.14	4.14	3.0	3.0
Expected leverage, $\langle l \rangle$	3.21	2.38	2.05	2.376
Modal leverage, l_m	2.15	0.5	1.15	1.8
Parameter, α	3.028	1.266	2.181	4.122
Parameter, β	0.943	0.53	1.11	1.335

The above phases were reproduced in Figure 4 by a family of gamma distributed probability density functions estimated for appropriate parameters given in Table 2.

The most important of analyzed factors—changes in volatility during the Treasuries market meltdown—reflected disruptions in market beliefs having been exposed to structural imbalances. These complicated relations do not usually manifest themselves under normal circumstances though in a crisis they become major drivers of disruptive events. Their fairly accurate, though simplified, reproduction indicated the logistic model flexibility and its applicability to the analysis of the actual debt market performance.

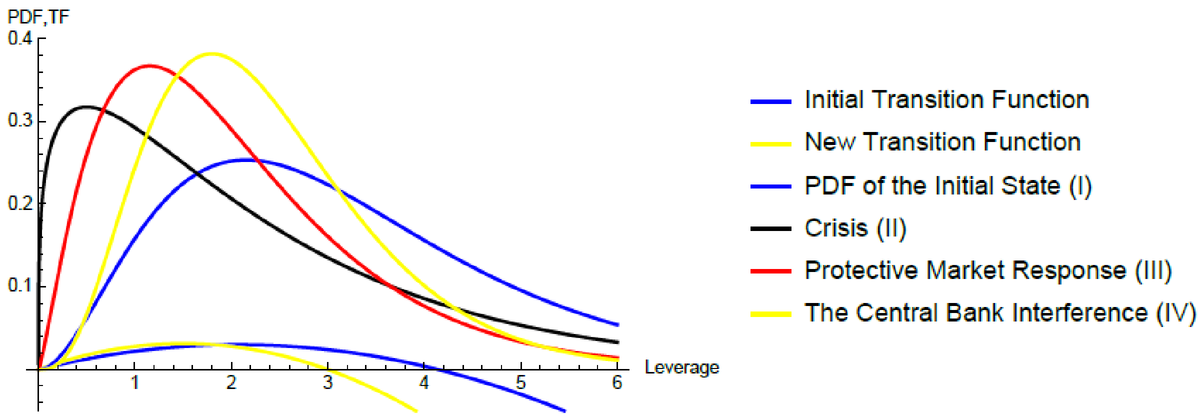

Figure 4. Phases of critical leverage dynamics.

6. The model discussion

This section is devoted to three interrelated problems of the logistic leverage model (12) interpretation and its generalization.

6.1. Debt service and non-linear leverage dynamic

The transition process of leverage was interpreted in Section 2 as an interaction of growth and feedbacks in a macro-financial market. This interpretation, obviously, is very different from the treatment of a well-known linear model:

$$dD(t) = [rD(t) - M(t)]dt \quad (29)$$

where $M(t)$ stands for the size of periodic payments. If function $D(t)$ corresponds to the total debt value, then the leverage and the relative debt dynamics as given in:

$$d/dt[D(t)/E(t)] \equiv \dot{l} = \dot{D}/E - \rho(l-1) \quad (30)$$

And is asynchronous. It follows, in particular, that the volume of debt per unit of capital would continue to grow at the stationary leverage; otherwise, stationary debt would imply decreasing leverage due to growing equity. Thus, leverage stationarity could be considered as a less restrictive requirement when comparing it to the condition, $\dot{D}(t) = 0$.

It is necessary to stress that the linear ODE of the micro-debt future value 29, or its modification for the present value, is a perfectly consistent model. Calculating a particular micro-debt value according to this equation is, beyond any doubt, a reasonable operation. But a macro-financial analogue to model 29 would require assessments of the nonlinear leverage dynamics, and the explaining of its dependence upon interactions of assets, debt and capital. The linear leverage analogue to model 9, as it seems, “overlooks” the debt inertia given in Equation 30. As a result, the premise of linearity does not match neither macro-debt not macro-leverage dynamics. For example, a linear ODE for leverage corresponding to the Equation 29:

$$dl(t)/dt = -cl(t) + (c - m) \quad (31)$$

where $m(t) = M(t)/E(t)$, does not satisfy the economic content of a leverage transition. In particular, its stationary value is less than one, $l^* = 1 - m/c$, and it is at odds with the essence of macrofinancial leverage and is not supported by the economic evidence.

The debt inertia condition 30 serves as a trigger that would ensure normal functioning of a riskless debt (Treasuries) market. Since stationary leverage K corresponds to positive (absolute) debt increments, the trigger could be "turned on" before the debt ceiling is reached. Consequently, given the expected rates of return and their spreads, a feasible debt amount could be estimated endogenously, and well in advance. It is reasonable to expect, in other words, that the usage of a logistic model could have alleviated the problem of "minting a one trillion dollars coin" (Gray, 2020–2021), which is relevant to the US Treasuries market regulation[5].

6.2. On a feasible debt collateralization

Noteworthy is an interesting result of the logistic leverage dynamic, which takes place under the particular condition of equality between expected spreads of aggregated lenders and borrowers. This condition, although unstable, practically guarantees debt reimbursement on the peak of the market activity, $\hat{l} = 0.5K$. It can be shown along the following line of reasoning.

First, volumes of debt and equity are the same, $E = D$, when the leverage takes on the value $l_2 = 2$, as it follows from the macro-financial balance sheet (1). In other words, if leverage is equal to the positive root of equation, $l = l/(l-1)$, or equivalently:

$$l^2 - 2l = 0 \tag{32}$$

Then the debt outstanding is fully backed by the capital. Note, that this condition is actively used in the modern stock markets as a debt collateralization requirement, whereby investors can receive loans from their brokers/dealers in a proportion 1:1 to their own capital. On the other hand, since the total debt (debt outstanding) is not subject to immediate reimbursement in full, such a macro-requirement seems to be excessive, and it should be modified and weakened.

Second, according to Adrian and Shin (2005), responses of corporations and households to changes in macro-financial leverage are asymmetric. Suppose an aggregated borrower (corporations) estimates its expected rate of return $\rho^e[l(t)]$ from the available information on lenders' yield spread $a = (\mu - r)$. Then their expected return can be represented as a function:

$$\rho^e[l(t)] = r + (\mu - r)l(t) \tag{33}$$

which can be viewed as an indicator of aggregate debt supply. Similarly, if the aggregate of lenders, estimating their expected returns, uses information on the current spread of borrowers, $c = \rho - r$, then the Equation:

$$\mu^e[l(t)] = r + (\rho - r)l^{-1}(t) \tag{34}$$

Could be used as an indicator of aggregate demand for debt[6]. Though in the short term, expected returns of lenders and borrowers do not coincide, in the long run their expectations should be equal,

[5] The informative paper by R. Grey (2020–2021) discussing the upper limit of feasible borrowing does not, however, touch upon the problem of the imminent collapse of the Treasury bond market, which would occur in the event of minting a one trillion dollar coin.

[6] This equation is not used in the short run since its solution is identical to the solution of Equation 3.

$\rho^e[l] = \mu^e[l]$. Consequently, the following equation has to take place:

$$l^2 - c/a = 0 \tag{35}$$

And its solution sets the level of leverage, $\tilde{l} = +\sqrt{c/a}$, at which this particular condition is satisfied (negative root has no economic meaning).

Finally, let us require that the long-term returns for creditors and borrowers should be equal and accompanied by the maximal debt collateralization. Then, the system of simultaneous Equations 32 and 35 could be solved for $\hat{l} = c/2a$. This leverage value, as it was shown earlier, delivers maximum to the transition function of the logistic ODE 12. Thus, it could be concluded that at the point of the maximal market activity (leverage trajectory inflection) the debt outstanding is almost completely collateralized by the capital. The point, $\hat{l} = 3$, in Figure 5 illustrates the above statement.

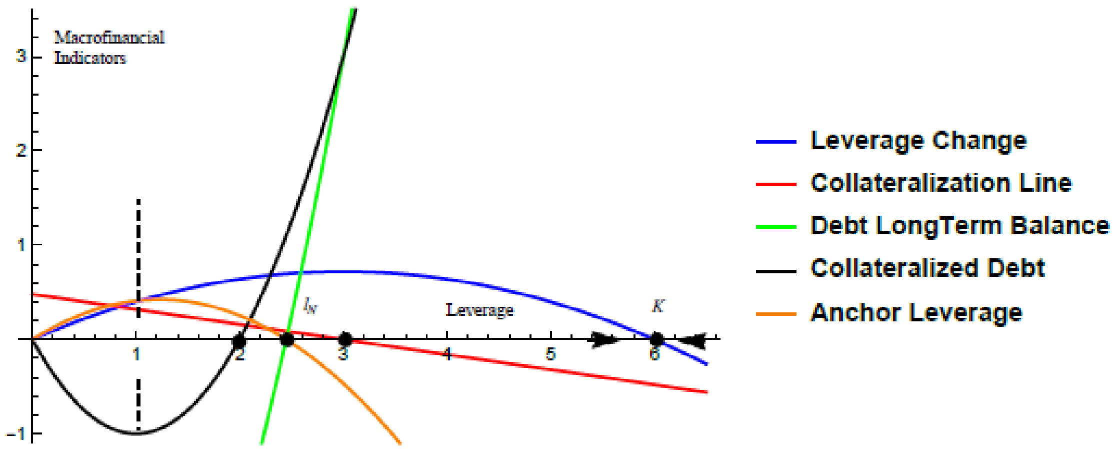

Figure 5. Debt collateralization by the capital.

6.3. *θ-logistic model of the leverage*

Generalizations to the logistic leverage ODE 12 seem to be less urgent than for deterministic population models since the former is not intended to approximate the time series of this indicator. Nevertheless, a reason for the θ-logistic, or the Richards model, could be outlined due to interesting possibilities in studying the heterogeneity of the macro-financial system it would have opened up.

The θ-logistic model of leverage has the following representation:

$$dl(t) = al(t)\left[1 - \left(\frac{l(t)}{K}\right)^{\theta}\right]dt; l(0) = l_0 \tag{36}$$

which is reduced to the basic logistic Equation 12 for $\theta = 1$. Its stochastic version has an analytical distribution density function (Petroni et al., 2004). Equation 36 could be simplified by introducing the characteristic time $\tau = at$of the system, and by the leverage normalization, $x(\tau) = l(\tau)/K$. These transformations reduce Equation 36 to a Bernoulli ODE:

$$dx(\tau) = x(\tau)[1 - x^{\theta}(\tau)]dt \tag{37}$$

In its turn, the nonlinear Equation 37, by analogy with the procedure described in Section 3.1, can be transformed into a linear ODE with regard to a variable,$y(\tau)$, the solution of which would retrieve solving Equation 37:

$$x(\tau) = [({x_0}^{-\theta} - 1)\, exp[-\tau] + 1]^{-\frac{1}{\theta}} \tag{38}$$

Then, at the final stage, the solution of the original Equation 36 could be found as given by:

$$l(t) = K\{(\left(\frac{K}{l_0}\right)^{\theta} - 1)\, exp[-a\theta t] + 1\}^{-\frac{1}{\theta}} \tag{39}$$

Equation 39 is the so-called θ-logistic sigmoid. From an economic point of view, the transformation:

$$y(\tau) = x^{-\theta}(\tau) \tag{40}$$

Plays the key role for its calculation. It generalizes relation 16 for the normalized values of leverage,$y(\tau)$, and capital intensity,$x(\tau)$, both of which were changing along the characteristic time coordinate of a macro-financial system. The economic interpretation of transformations 16 and 40 should be, therefore, similar, which is far from being obvious. It is tempting, for example, to treat parameter θ in Equation 40 as a measure of heterogeneity of a macro-financial system known in economics as a degree of concentration of the system's elements. The factor of heterogeneity would modify the macrofinancial system behavior dramatically, especially in comparison with the classic hypothesis of *laissez-faire*.

As known, modern macro-financial systems consist of a small number of very large financial institutions, and a large number of small and medium-sized banks and other organizations. Suffice to say that the assets of the five largest bank holding companies (BHCs) in the United States account for 64 percent of the commercial banking assets while the entire banking system consists of approximately 3,000 institutions (IMF, 2012). In other words, the modern macro-financial system is a highly heterogeneous conglomerate, which could be treated as an object of fractal nature (Sornette, 2006).

The above considerations, though very general, could justify the need to generalize the basic logistic model 12, and, in any case, thorough economic interpretation of generalized logistic ODE and SDE is a necessary step in developing more sophisticated macro-financial models. Similar conclusions could be made regarding the models of A. Blumberg, S. Turner and co-authors, R. Tsoularis and the softmax models which contain a hierarchy of different exponential functions.

7. Conclusions

In short, the logistic sigmoidal modelling of capital and debt accumulation demonstrated its obvious advantage of flexibility and simplicity. Its economic essence and validity were confirmed by the centuries-old practice of loans provision including that of brokers/dealers in the modern financial markets.

The model application to studying leverage dynamics was illustrated in the paper by several examples. In the general framework it implied the usage of empirical datasets about leverage, volatility, expected yields, rates and spreads. Their provision of feasible leverage realizations could be assessed, and the model, including its probability density function, parameterized. Some structural market imbalances could be isolated due to enhanced adaptation of the model parameters along the leverage transition to its steady state. The quadratic transition function proved to be effective in outlining the leverage phase space. It should be noted that unlike modeling populations its maximum at 0.5*K* did not violate general economic presumptions.

The leverage model demonstrated its realistic assessment of the scale of modern financial processes and their impact upon the global economy. In particular, stationary leverage as a benchmark of feasible borrowing implied the necessity of its adequate collateralization and coherent money issuance. Remember that money in its digital form is defined nowadays as a specific product of social networks (BIS, 2021). Therefore, the widespread usage of sigmoidal models in research on neural networks and artificial intelligence deserves the closest attention of economists and financiers.

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