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Research article

Financial market disruption and investor awareness: the case of implied volatility skew

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Abstract: The crash of 1987 is considered one of the most significant events in the history of financial markets due to the severity and swiftness of market declines worldwide. In the aftermath of the crash, a permanent change in options market occurred; implied volatility skew started appearing in options markets worldwide. In this article, we argue that the emergence of the implied volatility skew can be understood as arising from increased investor awareness about the stock price process and its implications for delta hedging. Delta-hedging aims to eliminate the directional risk associated with price movements in the underlying asset. Before the crash, investors were unaware of the proposition that "a delta-hedged portfolio is risky". That is, they implicitly believed in the proposition that "a delta-hedged portfolio is risk-free". The crash caused "portfolio insurance delta-hedges" to fail spectacularly. The resulting visceral shock drove home the lesson that "a delta-hedged portfolio is risky", thus, increasing investor awareness. We show that this sudden realization that a delta-hedged portfolio is risky is sufficient to generate the implied volatility skew and is equivalent to replacing the risk-free rate with a higher rate in the European call option formula. It follows that investor awareness (beyond asymmetric information) is an important consideration that matters for financial market behavior.

Keywords: partial awareness; restricted awareness; black scholes model; analogy making; generalized principle of no-Arbitrage; implied volatility skew; implied volatility smile; portfolio insurance delta-hedge

JEL Codes: G13, G12

1. Introduction

On Monday, October 19, 1987, stock markets in the US (on Tuesday, October 20, in a variety of other markets worldwide), along with the corresponding futures and options markets, crashed; with the S&P 500 index falling more than 20%. To date, in percentage terms, this is the largest ever one day drop in the value of the index. The crash of 1987 is considered one of the most significant events in the history of financial markets due to the severity and swiftness of market declines worldwide. In the aftermath of the crash, a permanent change in options market occurred; implied volatility skew started appearing in options markets worldwide (see Rubinstein (1994), Jackwerth (2000), and Siddiqi (2019) among others). Before the crash, implied volatility when plotted against strike/spot is almost a straight line, consistent with the Black Scholes model, see Rubinstein (1994). After the crash, for index options, implied volatility starts falling monotonically as strike/spot rises. That is, the implied volatility skew appeared. What caused this sudden and permanent appearance of the skew? As noted in Jackwerth (2000), it is difficult to attribute this change in behavior of option prices entirely to the knowledge that highly liquid financial markets can crash spectacularly. Such an attribution requires that investors expect a repeat of the 1987 crash at least once every four years, even when a repeat once every eight years seems too pessimistic. Perhaps, the crash not only imparted knowledge that risks are greater than previously thought, it also caused a change in the mental processes that investors use to value options.

Before the crash, a popular market practice was to engage in a strategy known as "portfolio insurance". The strategy involved creating a "synthetic put" to protect equity portfolios by creating a floor below which the value must not fall. However, the creation of a "synthetic put" requires a key assumption dating back to the celebrated derivation of the Black Scholes option pricing formula (Black and Scholes, (1973); Merton, 1973). The key assumption is that an option's payoff can be replicated by a portfolio consisting only of the underlying and a risk-free asset and all one needs to do is shuffle money between the two assets in a specified way as the price of the underlying changes. This assumption known as "dynamic replication" is equivalent to saying that "a delta-hedged portfolio is risk-free". The idea of "dynamic replication" or equivalently the idea that "a delta hedged portfolio is risk-free" was converted into a product popularly known as "portfolio insurance", by Leland O'Brien Rubinstein Associates in the early 1980s. The product was replicated in various forms by many other players. "Portfolio insurance" was so popular by the time of the crash, that the Brady Commission report (1988) lists it as one of the factors causing the crash. For further details regarding portfolio insurance and its popularity before the crash, see Mackenzie (2004).

In this article, we argue that before the crash, investors were unaware of the proposition that "a delta-hedged portfolio is risky". That is, they implicitly believed in the proposition that "a delta-hedged portfolio is risk-free". The crash caused "portfolio insurance delta-hedges" to fail spectacularly. The resulting visceral shock drove home the lesson that "a delta-hedged portfolio is risky", thus, increasing investor awareness. Before the crash, the belief that a delta-hedged portfolio is risk-free led to options being priced based on no-arbitrage considerations. Principle of no arbitrage says that assets with identical state-wise payoffs should have the same price, or equivalently, assets with identical state-wise payoffs should have identical state-wise returns. So, a delta-hedged portfolio, if it has the same state-wise payoffs as a risk-free asset, should offer the same state-wise returns as the risk-free asset. What if a delta-hedged portfolio does not have state-wise payoffs that are identical to a risk-free asset or any other asset for the matter? Experimental evidence suggests that when people cannot apply the principle of no-arbitrage to value options because they cannot find another asset with identical state-wise payoffs, they rely on a weaker version of the principle, which can be termed the generalized principle of no-arbitrage or analogy making (Siddiqi, 2019, 2012). The generalized principle of no-arbitrage or analogy making says, assets with similar state-wise payoffs should have the same state-wise returns on average, or equivalently, assets with similar state-wise payoffs should have the same expected return. The cognitive foundations of this experimentally observed rule are provided by the notion of mental accounting (Thaler, 1980; Thaler, 1999), and discussion in Rockenbach (2004), and categorization theories of cognitive science (Henderson and Peterson, 1992). See Siddiqi (2019) for details. The prices determined by the generalized principle of no-arbitrage are potentially arbitrage-free if an equivalent martingale measure exists. Existence of a risk neutral measure or an equivalent martingale measure is both necessary and sufficient for prices to be arbitrage-free. See Harrison and Kreps (1979). We show that the model developed in this article, permits an equivalent martingale measure, hence prices are potentially arbitrage-free.

A call option is widely believed to be a surrogate for the underlying stock as it pays more when the stock pays more and it pays less when the stock pays less. We follow Siddiqi (2019) in taking the similarity between a call option and the underlying as given and apply the generalized principle of no-arbitrage or analogy making to value options.

Li (2008) uses the term partial awareness to describe a situation in which one is unaware of a proposition but not of its negation, which is implicitly assumed to be true. So, in Li (2008)'s terminology, investors had partial awareness before the crash as they were unaware of the proposition "a delta-hedged portfolio is risky". They implicitly assumed that the proposition "a delta-hedged portfolio is riskless" is true. Even though, it seems natural to characterize states of nature in terms of propositions, it is often useful to refer to the state space directly. Quiggin (2016) points out that there is a mapping between the characterizations in terms of propositions (syntactic representation of unawareness) to the more usual semantic interpretation in which one describes the state space directly. In our case, the semantic interpretation is that, before the crash, investors were unaware of the states in which the delta-hedged portfolio is risky. Such a semantic concept of unawareness is called restricted awareness. Grant and Quiggin (2013), and Halpern and Rego (2008) extend the notion of restricted awareness to include sub-game perfect and sequential equilibria in interactive settings.

Quiggin (2016) proposes an extension of the notion of unawareness to stochastic processes and defines restricted awareness as a situation in which one is unaware of at least one state in the discrete stochastic process. For example, let's say the true process is trinomial; however, one is only aware of two states. A person with such restricted awareness may create a delta-hedged portfolio which would be risk-free in the two states he is aware of, but if the third state, which he is unaware of, is realized, the portfolio will lose value. That is, he would falsely believe that the delta-hedged portfolio is risk-free, whereas in reality, the delta-hedged portfolio is risky. Note, if the third state occurs with a small enough frequency, then one can stay oblivious of it for a considerable time period.

Overall, a key contribution of this article is to show that the notion of unawareness potentially has important implications for financial market behavior. This article studies the implications of changes in investor awareness for option pricing; however, one can potentially study major financial crises as arising out of sudden changes in investor awareness. Quiggin (2012) makes this point quite forcefully and lays out a challenge for finance research to explore the implications of changes in awareness. This article is an initial tentative attempt at responding to this challenge.

The notion of restricted awareness when applied to stochastic processes implies that one is not aware of the true stochastic process. Throughout this article, we assume that if one is unaware, then he is unaware that he is unaware. From this point forward, we suppress explicitly mentioning this assumption, for clarity. Section 2 sets up the basic model in discrete time to bring out the economic intuition of partial or restricted awareness in the context of option pricing. In the continuous limit, if the Black Scholes model represents strongly restricted awareness, then the corresponding analogy-based formula (assuming full awareness) is derived in section 2. The formula can be considered a generalization of Merton's jump diffusion formula (Merton, 1976). If the Black Scholes model represents weakly restricted awareness, then the corresponding analogy-based formula (assuming full-awareness) is derived in section 3. That formula, first derived in Siddiqi (2019), can be considered a generalization of the Black Scholes model. The implied volatility skew is generated in both cases; hence, the sudden appearance of the skew after the crash of 1987 can be understood as the consequence of growing awareness induced by the crash. Section 4 compares strongly restricted awareness with weakly restricted awareness and shows that the implied volatility skew as well as the implied volatility smile is generated in the first case, whereas, in the second case, only the skew is explained. Hence, it is argued, that for options on individual stocks, assuming strongly restricted awareness is better, whereas for index options, the weaker form seems like a natural choice. Section 5 concludes with suggestions for future research.

2. A discrete-time model of growing awareness

In this section, we explore the implications of growing awareness in discrete time. This brings out the economic intuition and clarifies the valuation of options by providing a comparison of analogy making and no-arbitrage pricing. The basic set-up of the model is the same as in Amin (1993), which is a generalization of Cox et al. (1979) binomial model (CRR model). Assume that trade occurs only on discrete dates indexed by 0, 1, 2, 3, 4,……T. Initially, there are only two assets. One is a riskless bond that pays every period meaning that if B dollars are invested at time i, the payoff at time i+1 is $\dot{r}B$. The second asset is a risky stock. As in Amin (1993), we assume that the stock price at date i can only take values from an exogenously specified set given by $S_j(i)$ where $j\epsilon\{-\infty,\dots..,-2,-1,0,1,2,\dots..,\infty\}$. The variable j is an index for state and the variable i is an index for time. In this set-up, the state-space for a two-period model is shown in Appendix Fig. A.1. The transition probabilities in this state-space are represented by Q.

In each time period, the stock price can undergo either of the two mutually exclusive changes. Most of the time, the price changes correspond to a state change of one unit. That is, if at time i, the state is $S_j(i)$, then at time i+1, it changes either to $S_{j+1}(i)$ or $S_{j-1}(i)$. Such changes, termed local price changes, correspond to the binomial changes assumed in CRR model. On rare occasions, the state changes by more than one unit. Such non-local changes are referred to as jumps. We assume that, in case of a jump, the stock price can jump to any of the non-adjacent states. So, the structure of the state-space is that of jumps super-imposed on the binomial model of CRR. For simplicity, we assume that there are no dividends.

Assume that a new asset, which is a call option on the stock, $C_j(i)$, is introduced, with maturity at τ. Without loss of generality, assume that j=0. Consider the following portfolio:

$$V(i) = S_0(i)x - C_0(i) \tag{1}$$

where $x = \frac{C_{+1}(i+1)-C_{-1}(i+1)}{S_{+1}(i+1)-S_{-1}(i+1)}$.

Table 1. The state space for stock price dynamics over two time periods.

Time 0	Time 1	Time 2
	$S_{+2}(1)$	$S_{+2}(2)$
	$S_{+1}(1)$	$S_{+1}(2)$
$S_0(0)$	$S_0(1)$	$S_0(2)$
	$S_{-1}(1)$	$S_{-1}(2)$
	$S_{-2}(1)$	$S_{-2}(2)$

The portfolio in (1) is called the delta-hedged portfolio because such a portfolio gives the same value if either of the adjacent states is realized in the next period. That is, conditional on local price changes in the underlying stock, the portfolio is risk-free. In what follows, for ease of reading, we suppress the subscripts and/or time index, wherever doing so is unambiguous.

If only local price changes happen, then, in the next period:

$$V_{+1} = V_l(i+1) = S_{+1}(i+1)x - C_{+1}(i+1) \tag{2}$$

$$V_{-1} = V_l(i+1) = S_{-1}(i+1)x - C_{-1}(i+1) \tag{3}$$

Define the single period capital gain return on the underlying stock as follows:

$\Delta_k = \frac{S_{+k}(i+1)}{S(i)}$ where k=……, −2, −1 ,0, 1, 2,……..

With the above definition, substituting the value of x in either (2) or (3) results in the same value of:

$$V_{\pm}(i+1) = \frac{C_{+}(i+1)\Delta_{-1} - C_{-}(i+1)\Delta_{+1}}{\Delta_{+1} - \Delta_{-1}} \tag{4}$$

If only local price changes are allowed, then the portfolio in (1) takes the value shown in (4) in the next period. That is, the delta-hedged portfolio is locally risk free; however, it is not globally risk free as jumps can also happen.

Now, we can specify the dynamics of awareness. Initially, assume that investors are only aware of local price changes. That is, they are only aware of a binomial sub-lattice in the whole state space. In syntactic representation, they are unaware of the proposition, “the delta-hedged portfolio in (1) is risky”. So, they believe that the proposition “the delta-hedged portfolio is risk-free” is true. In semantic representation, if the current state is j, they are unaware of the following states (and states that can only be reached from these states): $S_{j+f}(i+1)$ with $f \neq \pm 1$.

What are the implications of this belief for the price dynamics of the call option? If the delta-hedged portfolio is believed to be risk-free then according to the principle of no-arbitrage (assets with identical state-wise payoffs should have identical state-wise returns), it should offer the same return as the risk-free bond. That is:

$$V_{\pm}(1+i) = \dot{r}V \tag{5}$$

Substituting (1) and (5) in (4) and simplifying leads to:

$$\dot{r}\left[\frac{C_{+}(i+1) - C_{-}(i+1)}{\Delta_{+1} - \Delta_{-1}}\right] - \frac{C_{+}(i+1)\Delta_{-1} - C_{-}(i+1)\Delta_{+1}}{\Delta_{+1} - \Delta_{-1}} = \dot{r}C(i) \tag{6}$$

Rearranging (6):

$$\left(\frac{\dot{r} - \Delta_{-1}}{\Delta_{+1} - \Delta_{-1}}\right) C_{+}(i+1) + \left(\frac{\Delta_{+1} - \dot{r}}{\Delta_{+1} - \Delta_{-1}}\right) C_{-}(i+1) = \dot{r} C(i) \tag{7}$$

In (7), the terms in brackets in front of $C_{+}(i+1)$ and $C_{-}(i+1)$ are risk-neutral probabilities.

As local price changes and jumps are assumed to be mutually exclusive[1], the two are distinguished ex-post. We assume that once a jump is observed, investors become aware of the full state space. That is, they become aware that apart from local price changes, jumps can also happen. The awareness of full state space implies that investors are no longer unaware of the proposition, "the delta-hedged portfolio is risky". The delta-hedged portfolio can no longer be considered risk free by fully aware investors.

Consider the value of the delta-hedged portfolio in case of a jump. Define Y as the one period capital gain return on the stock, in case of a jump. That is, in case of a jump, the next period stock price is $S(i)Y$. The corresponding state induced by the jump is denoted by y. Hence, the value of the delta-hedged portfolio conditional on the jump is:

$$V(i+1) jump = V_y(i+1) = S(i)Yx - C_y(i+1) \tag{8}$$

The delta-hedged portfolio is no longer risk free. In the case of local price changes, its value is risk free and is given by (4), and in the case of a jump, its value is risky and is given by (8). Assume that the true probability (under Q) of there being a jump is γ. Define the expectations operator with respect to the distribution of Y as E_γ. The expected value of the delta-hedged portfolio can now be written as:

$$E[V(i+1)] = \gamma E_\gamma\left[V_y(i+1)\right] + (1-\gamma) V_{\pm}(i+1) \tag{9}$$

As the delta-hedged portfolio can no longer be considered identical to the risk-free asset, the principle of no-arbitrage cannot be applied to determine a unique price for the call option. A call option is similar to the underlying stock, so in accordance with the principle of analogy making/generalized principle of no-arbitrage, it should offer the same expected return as the underlying. It follows that the delta-hedged portfolio should also offer the same expected return as the underlying stock. Proposition 1 shows the recursive pricing equation that the call option must satisfy under analogy making.

Proposition 1: If analogy making determines the price of the call option, then the following recursive pricing equation must be satisfied:

$$(1-\gamma)\left\{ C_{+1}(i+1)\left[\frac{\frac{r+\delta-\gamma E_\gamma[Y]}{1-\gamma} - \Delta_{-1}}{\Delta_{+1} - \Delta_{-1}}\right] + C_{-1}(i+1)\left[\frac{\Delta_{+1} - \frac{r+\delta-\gamma E_\gamma[Y]}{1-\gamma}}{\Delta_{+1} - \Delta_{-1}}\right]\right\} \tag{10}$$
$$+\gamma E_\gamma\left[C_y(i+1)\right] = (r+\delta) C(i)$$

where δ is the risk-premium on the underlying stock.

Proof: Analogy making implies that $(r+\delta)V(i) = E[V(i+1)]$. Substituting (4) and (8) in (9) and collecting terms together leads to (10).

(10) differs from the corresponding recursive pricing equation in Amin (1993) due to the presence of δ, which is the risk premium on the underlying stock. If the delta-hedged portfolio in (1) is

[1]Whether local changes and jumps are mutually exclusive or not does not matter in the continuous limit.

considered identical to a riskless asset, which corresponds to unawareness of a part of the state space, then according to the principle of no-arbitrage, it should offer the risk-free return. In that case the pricing equation for the call option is given in (7). (7) can be obtained from (10) by making γ and δ equal to zero. If the delta-hedged portfolio in (1) is considered risky, which corresponds to full awareness of the state space, then the principle of no-arbitrage cannot be applied. However, the generalized principle of no-arbitrage, which is based on analogy making, and that says, "Assets with similar state-wise payoffs should offer the same expected returns", can be applied. Application of that principle results in the recursive pricing equation shown in (10).

The generalized principle of no-arbitrage or the principle of analogy making allows for an arbitrage-free price for the call option. To see this, one just needs to realize that the existence of the risk neutral measure or the equivalent martingale measure is both necessary and sufficient for prices to be arbitrage-free. See Harrison and Kreps (1979). One can simply multiply payoffs with the corresponding risk neutral probabilities to get the price of an asset times the risk-free rate. Proposition 2 shows the equivalent martingale measure associated with the analogy model developed here.

Proposition 2: The equivalent martingale measure or the risk neutral pricing measure associated with the analogy model is given by:

Risk neutral probability of a +1 change in state: $(1-\gamma_n)$.

Risk neutral probability of a −1 change in state: $(1-\gamma_n)(1-q)$.

Total risk neutral probability of any other change in state or jump: γ_n.

where $q=\frac{\frac{r+\delta-\gamma E_Y[Y]}{1-\gamma}-\Delta_{-1}}{\Delta_{+1}-\Delta_{-1}}$, and $\gamma_n=\frac{(1-\gamma)\dot{r}-S(i)(\dot{r}+\delta-\gamma E_Y)}{(1-\gamma)E_Y[Y]-S(i)(\dot{r}+\delta-\gamma E_Y)}$.

Proof: By the definition of equivalent martingale measure, the following equations must hold:

$$(1-\gamma_n)\{C_{+1}(1+i)q+C_{-1}(1+i)(1-q)\}+\gamma_n E_Y[C_Y(i+1)]=\dot{r}C(i) \tag{11}$$

$$(1-\gamma_n)\{S_{+1}(1+i)q+S_{-1}(1+i)(1-q)\}+\gamma_n E_Y[S_Y(i+1)]=\dot{r}S(i) \tag{12}$$

Substituting the values of q and γ_n in the above equations and simplifying shows that the L.H.S is equal to $\dot{r}C(i)$ and $\dot{r}S(i)$ respectively.

Proposition 2 shows that the risk neutral probability of a jump occurring is different from the actual probability of the jump. Analogy making implies that the jump risk is priced causing a deviation between the actual and risk neutral probabilities.

2.1. Strongly Restricted Awareness in the Continuous Limit

In general, there are infinitely many specifications of the discrete state space described earlier that lead to the jump diffusion stochastic process in the continuous limit. As one example, see Amin (1993). For technical proofs of convergence of associated value functions, see Kushner and DeMasi (1978).

In the continuous limit, the discrete stochastic process described earlier, converges to a jump diffusion process. However, if investors are only aware of local price changes, then they think that the true process is geometric Brownian motion in the continuous limit. Hence, they get the type of the stochastic process wrong. We refer to such unawareness as strongly restricted awareness to distinguish it from a situation in which the type is known but the true parameter values are not known (weakly restricted awareness described in section 3).

In the continuous case presented here, we make all the assumptions made in Merton (1976) except one. Specifically, Merton (1976) assumes that the jump risk is diversifiable. Here, we assume that the jump risk is systematic and hence must be priced. To price jump risk, we do what we did for the discrete case. That is, we use the generalized principle of no-arbitrage or the principle of analogy making. However, before presenting the results, we highlight the intuition behind the convergence of the discrete model discussed earlier to the differential equation of the continuous process derived in Merton (1976). After that, we deviate from Merton (1976) and apply analogy making.

To see the intuition of this convergence, consider the value of the delta-hedged portfolio in the discrete time model considered earlier with full awareness:

Under a local price change:

$$V(i+1) = S_{+1}(i+1)x - C_{+1}(i+1) = S_{-1}(i+1)x - C_{-1}(i+1) \tag{13}$$

$\Rightarrow [\Delta V]_{local} = \Delta Sx - \Delta C$

Under a jump:

$$V(i+1) = YSx - C_Y(i+1) \tag{14}$$

$\Rightarrow [\Delta V]_{jump} = (Y-1)Sx - (C_Y(i+1) - C(i))$

So, the total change in the value of the portfolio is:

$$[\Delta V]_{total} = [\Delta V]_{local} + [\Delta V]_{jump} \tag{15}$$

As in Merton (1976), assume that the size of the jump does not depend on the time interval; however, the probability of the jump depends on the time interval. This implies that as the time interval goes to zero, the probability of the jump also goes to zero; however, the size of the jump does not go to zero. For local changes, assume that the size of a local change goes to zero, as the time interval goes to zero; however, the probability of a local change tends to a constant. (Note, that in geometric Brownian motion the probability of movement never goes to zero as the probability tends to a constant as $dt \to 0$; however, the size goes to zero as $dt \to 0$). With these assumptions, as $dt \to 0$, (11) can be written as:

$$[dV]_{total} = [dV]_{Brownian} + [dV]_{Poisson} \tag{16}$$

$$[dV]_{Brownian} = [dV]_{Brownian}x - [dC]_{Brownian} \tag{17}$$

To find out the stochastic differential equation for $[dV]_{Poisson}$, consider the process dq where in the interval dt, $dq = 1; with\ probability\ \gamma dt$, and $dq = 0; with\ probability\ (1-\gamma dt)$.

It follows, $E[dq] = \gamma dt$, and $Var[dq] = \gamma dt + O((dt)^2)$.

Suppose a jump occurs in the interval dt with probability γdt. The stochastic differential equation for the jump process of the stock can now be written as:

$$[dS]_{jump} = (Y-1)Sdq \tag{18}$$

It follows (see Kushner and DeMasi (1978)),

$$[dV]_{Poisson} = (Y-1)Sxdq - \big(C(YS,t) - C(S,t)\big)dq \tag{19}$$

From Ito's lemma,

$$[dV]_{Brownian} = (\mu Sdt + \sigma SdW)x - \left\{\left(\frac{\partial C}{\partial t} + \mu S\frac{\partial C}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 S}{\partial C^2}\right)dt + \sigma S\frac{\partial C}{\partial S}dW\right\} \quad (20)$$

where μ and σ are the mean and the standard deviation of the underlying's returns, and $dW = \varphi\sqrt{dt}$. φ is a random draw from a normal distribution with mean 0 and variance 1. Substituting (14) and (15) in (12), realizing that $x = \frac{\partial C}{\partial S}$, and suppressing the subscript 'total' leads to:

$$dV = -\left(\frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 S}{\partial C^2}\right)dt + (Y-1)S\frac{\partial C}{\partial S}dq - \big(C(YS,t) - C(S,t)\big)dq \quad (21)$$

As can be seen in (21), the delta-hedged portfolio is not risk-free due to the appearance of dq.

Under partial awareness, people are unaware of the proposition, "the delta-hedged portfolio is risky". They believe that the proposition, "the delta-hedged portfolio is risk-free", is true. Equivalently, in semantic terms, they have restricted awareness as they are unaware of states in which the stock price can jump. They incorrectly believe that the complete stochastic process is specified by the Brownian component only. That is, they incorrectly believe that the true stochastic differential equation is given by $dV = -\left(\frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 S}{\partial C^2}\right)dt$, whereas the true stochastic partial differential equation is given in (21).

Merton (1976) assumes that the jump risk is diversifiable, hence is not priced. However, this is a rather strong assumption. Here, we do not make that assumption and instead, apply the previously stated generalized principle of no-arbitrage or the principle of analogy to price the option. As discussed earlier, the principle states that assets with similar state-wise payoffs should offer the same returns on average or expected returns. Given that the delta-hedged portfolio in (21) is not risk-free, it cannot be priced at the risk-free rate. The source of risk is the unhedged component in the underlying stock; hence, by applying the analogy principle, it follows that the expected return from the delta-hedged portfolio is the same as the expected return from the underlying. Proposition 3 shows that the European call option formula resulting from the principle of analogy are equivalent to the corresponding formula in Merton (1976) with the risk-free rate being replaced by a higher rate.

Proposition 3: If analogy making determines the price of a European call option, and the distribution of jumps (distribution of Y) is assumed to be log-normal with a mean of one (implying the distribution is symmetric around the current stock price) and variance of ϑ^2, then the price of a European call option is given by Merton's jump diffusion formula with the risk-free rate, r, replaced by a higher rate equal to $\mathrm{r} + \delta$, where δ is the risk-premium on the underlying stock.

Proof: Follows from (16) by realizing that $E[dV] = (r+\delta)dt$ and following the same steps as in the derivation of Merton's jump diffusion formula (Merton 1976).

The analogy jump diffusion formula has a number of advantages over the Merton jump diffusion formula:

1. A key advantage over Merton jump diffusion formula is that the analogy formula does not assume that the jump risk is diversifiable. For index options, the risk clearly cannot be diversified away. The analogy approach provides a convenient (non-utility maximization) way of pricing options in the presence of systematic jump risk, based on the compelling principle of analogy making.
2. Merton jump diffusion formula cannot generate the implied volatility skew (monotonically declining implied volatility as a function of strike/spot) if jumps are assumed to be symmetrically distributed around the current stock price. The analogy formula can generate the skew even when the jumps are assumed to be symmetrically distributed Assuming symmetric distribution of jumps around the current stock price, greatly simplifies the formula and increases applicability.

3. Even if we assume an asymmetric jump distribution around the current stock price, Merton formula, when calibrated with historical data, generates a skew which is a lot less pronounced (steep) than what is empirically observed. See Andersen and Andreasen (2002). The skew generated by the analogy formula is more pronounced (steep)

3. Weakly restricted awareness: unawareness of true parameter values

If people are unaware of some states in a discrete stochastic process, then, in the continuous limit, it leads to two possibilities:

1. They may be unaware of the type of the true stochastic process. This can be termed strongly restricted awareness.
1. They are aware of the type of the true stochastic process but not of the true parameter values. Such awareness can be called weakly restricted awareness.

The first possibility has been explored in the previous section. In the previous section, we showed that partial awareness in which people are unaware of the proposition, "the delta-hedged portfolio is risky", is equivalent to restricted awareness in which people are unaware of some states. In the previous section, we assumed that the distribution of states is such that under partial or restricted awareness, the stochastic process is geometric Brownian motion, whereas the true stochastic process is jump diffusion. While under partial or restricted awareness, the principle of no-arbitrage (assets with identical state-wise payoffs must have identical state-wise returns) can be applied to price a call option, under full awareness, it cannot be applied as the 'identical asset' does not exist anymore. If the generalized principle of no-arbitrage or analogy making (assets with similar state-wise payoffs assets should offer similar state-wise returns on average) is applied, then it leads to a new option pricing formula (Analogy based jump diffusion formula), which can be considered a generalization of Merton's jump diffusion formula. If option prices are determined in accordance with the Analogy formula, and the Black Scholes formula is used to back-out implied volatility, then the implied volatility skew is observed. Hence, the sudden appearance of the skew after the crash of 1987 can be thought of as arising due to an increase in awareness in which people became aware of the proposition, "the delta-hedged portfolio is risky".

In this section, we assume that the distribution of states is such that the true stochastic process is geometric Brownian motion in the continuous limit. Restricted awareness in which people are unaware of at least one state or equivalently partial awareness in which people are unaware of the proposition, "the delta-hedged portfolio is risky" then amounts to people being unaware of the true parameter values. That is, the true type or form of the stochastic process is known, however, the true parameter values are not known.

The set-up of the model here is identical to the one described in the previous section except for the distribution of states. As before, assume a discrete lattice of states. Let $S = S_0$ at $t = 0$. Assume that at $t = \Delta t$, the following three possible state transitions can take place:

$$S_0 \to S_0 + \Delta h \; with \; probability \; p \tag{22}$$

$$S_0 \to S_0 - \Delta h \; with \; probability \; q \tag{23}$$

$$S_0 \to S_0 - \epsilon\Delta h \; with \; probability \; l \tag{24}$$

where $p = q + l = 1$ and $\epsilon > 0$.

Assume further: S follows a Markov process i.e., the probability distribution in the future depends only on where it is now.

Suppose l is very small. Assume that initially people are only aware of an up movement by Δh or down movement by Δh. That is, they are unaware of the down movement by $\epsilon\Delta h$. It is straightforward to note that, in this set-up, people have partial awareness as unawareness of the third state amounts to being unaware of the proposition, "the delta-hedged portfolio is risky". That is, they believe the following proposition to be true, "the delta-hedged portfolio is risk-free".

As awareness grows, people become aware of the true stochastic process. Consequently, they realize that the principle of no-arbitrage cannot be used to price options as the delta-hedged portfolio is no longer identical to the risk-free asset (this is similar to people realizing that the true stock price is trinomial whereas they were incorrectly considering it to be binomial). Instead, the generalized principle of no-arbitrage or analogy making is used. Proposition 4 shows the European call option formula resulting from the principle of analogy is equivalent to the corresponding Black-Scholes formula with the risk-free rate replaced with a higher rate.

Proposition 4: The European call option formula is equivalent to the corresponding Black-Scholes formula with the risk-free rate,r, replaced by a higher rate equal to $\mathrm{r}+\delta$, where δ is the risk-premium on the underlying stock.

Proof: Follows from realizing that the expected return from the delta-hedged portfolio with the principle of analogy is $r+\delta$ and not just r as in the Black-Scholes model.

4. Strongly restricted or weakly restricted awareness

Does the Black Scholes option pricing model represent strongly restricted awareness or weakly restricted awareness? If it represents strongly restricted awareness with the true stochastic process including jumps as described in Merton (1976), then the correct analogy-based formula for a European call option (under full awareness) is equivalent to the corresponding Merton's formula with the risk-free rate replaced by a higher-rate. If it represents weakly restricted awareness, then the relevant analogy-based European call option formula (under full awareness) is the same as the corresponding Black-Scholes formula with the risk-free rate replaced by a higher rate. Strongly restricted awareness, even though more complex, can capture both the implied volatility skew and the implied volatility smile, whereas only a skew can be generated under weakly restricted awareness. In both cases, the skew steepens as the risk-premium on the underling increases.

5. Conclusions

It is interesting to try and model forgetfulness in this set-up. Forgetfulness can be described as an opposite process to growing awareness. Suppose one is initially aware that 'a delta-hedged portfolio is risky' but observes for a considerable length of time that 'a delta-hedged portfolio is risk-free'. He may be tempted to think that the stochastic process has changed, and the only possible states are those in which 'a delta-hedged portfolio is risk-free'. Thinking of awareness in the context of a stochastic process allows sufficient flexibility to model forgetfulness. It is also interesting to compare the value of a signal under full-awareness vs. the value of the same signal under partial awareness. Quiggin (2016) shows that the sum of value of awareness and value of information, appropriately defined, is a constant. As awareness corrects either an undervaluation or an overvaluation, positive and negative information signals have different impacts post-awareness when compared with pre-awareness impacts. Perhaps, this asymmetry in impacts can be exploited to devise an econometric test that would help in identifying events around which awareness changed. Such an econometric test is a natural subject for future research.

It is also interesting to see the notion of changes in awareness from the lens of resource scarcity in the brain. An emerging body of literature aims to push the notion of scarcity inside the human brain. Optimizing on scarce resources available in the external world has long been a defining notion in economics. Acknowledging that the brain resources are finite implies that the brain must first optimize on its own internal resources before optimizing on the resources available in the external world (McKenzie 2018, Siddiqi and Murphy 2021). Relating changes in resource allocation decisions in the brain to changes in investor awareness is an interesting line of inquiry for future research.

Conflict of interest

All author declares no conflicts of interest in this paper.

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