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Research article

Quantum option pricing and data analysis

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Abstract: The paper proposes to treat financial models using techniques of quantum mechanics. The methodology relies on the Dirac matrix formalism and the Feynman path integral approach. This leads us to reexamine in this framework the classical option pricing models of Cox-Ross-Rubinstein and Black-Scholes. Moreover, financial data are classified with respect to the spectrum of a certain observable and then analyzed to identify price jumps using supervised machine learning tools.

Keywords: option pricing; quantum binomial model; quantum mechanics; machine learning; data analysis
JEL Codes: C02, C10, C53, G12

1. Introduction

As pointed out in Claessens et al. (2014), global financial crises underline the importance of innovative modelling approaches to financial markets. The quantum mechanics approach suggests an alternative way to describe the unpredictable stock market behaviour (see e.g., Baaquie (2009)).

This paper is motivated by the quantum mechanics approach to the actuarial modelling and risk analysis as initiated in Tamturk and Utev (2018) and developed in Lefèvre et al. (2018), Tamturk and Utev (2019). We modify and adapt their methods to derive and analyze a quantum-type financial modelling. Specifically, the Schrödinger, Heisenberg, Feynman, Dirac approaches in quantum mechanics (see, e.g., Griffiths and Schroeter (2018); Parthasarathy (2012); Plenio (2002)) will be applied to the well-known option pricing models of Black and Scholes (1973) and Cox et al. (1979).

Moreover, motivated by the data analysis of the quantum reserve process proposed in Lefèvre et al. (2018), we model the financial data as eigenvalues of certain 1 or 2-step observable operators. These data then are analyzed to identify price jumps using supervised machine learning tools such as k-fold cross-validation techniques (see, e.g., Bishop (2006); Hastie et al. (2009); Wittek (2014)).

Option pricing models using quantum techniques discussed, for example, in Baaquie (2004, 2014); Bouchaud and Potters (2003); Haven (2002) are often based on the Schrödinger wave function with Hamiltonian operator H and are mainly oriented to continuous-time markets. For discrete-time markets as considered here, following Chen (2001, 2004), we choose the discrete-time formalism and analyze the quantum version of the Cox-Ross-Rubinstein binomial model. Then, we establish the limit of the spectral measures providing the convergence to the geometric Brownian motion model. We also identify the limit of the N-step non-self adjoint bond market as a planar Brownian motion.

The paper is organized as follows. Section 2 deals with heterogeneous quantum binomial markets. In Section 3, two convergence properties to continuous-time quantum markets are obtained. In Section 4, discrete and continuous-time quantum mechanics techniques are applied to the problem of option pricing. Section 5 is devoted to an analysis of stock market data.

2. Quantum modelling in finance

The motivation to our models come from the quantum mechanics approach to insurance proposed in Tamturk and Utev (2018, 2019), the insurance claim data analysis via quantum tools introduced in Lefèvre et al. (2018) and the non-traditional financial modelling initiated in Ma and Utev (2012) and developed in Karadeniz and Utev (2015, 2018). The quantum modelling approach is partly inspired by Baaquie (2004); Chen (2001, 2004), and Parthasarathy (2012).

2.1. Share price operators

We begin by outlining some standard arguments for a quantum-type modelling (see e.g., Lefèvre et al. (2018); Parthasarathy (2012)). An observable is a linear operator (matrix) on a certain Hilbert space. The quantum product of two independent observables A, B is implemented by the tensor product of the observables $A \otimes B$. So, $\ln(A \otimes B)$ acts as a quantum sum of two independent observables; in particular, $A \otimes A$ is the quantum product of two independent identical observables.

The 1-step quantum geometric random walk is defined as a 2×2 matrix A with eigenvalues e^u, e^d. Thus, $A^{\otimes N}$ $(N \geq 0)$ models the N-step geometric random walk.

2.1.1. Quantum binomial model

The quantum type modification of the classical Cox-Ross-Rubinstein model is originated in Chen (2001, 2004). The dynamics per period is defined by two moves: e^u (the share price goes up) and e^d (the share price goes down) with $d < 0 < u$.

The quantum binomial model over N periods is then represented by the share price operator

$$H_{S_N} = S_0 H^{\otimes N}, \tag{1}$$

where the main 1-step observable $H = A$ is a 2×2 self-adjoint (hermitian) matrix with eigenvalues e^u, e^d and representation

$$A = U^* D U = U^* \begin{pmatrix} e^u & 0 \\ 0 & e^d \end{pmatrix} U, \tag{2}$$

where U is a 2×2 unitary matrix.

In the sequel, the quantum binomial model discussed will be heterogeneous with share price operator

$$H_{S_N} = S_0 H_1 \otimes \ldots \otimes H_N. \quad (3)$$

2.1.2. Quantum actuarial-type model

The motivation to this relatively new financial model comes from Lefèvre et al. (2018) quantum mechanics approach to non-life insurance and the Lamplighter group approximation to the financial modelling suggested in Ma and Utev (2012). Based on the approach to the financial data analysis developed in Ma and Utev (2012); Karadeniz and Utev (2015, 2018) (see also references therein), we treat the data as having big jumps, say e^u, small jumps, say e^d or no jump. Moreover, the financial stock is observed at fixed times Δk but the number of jumps occurred during the time period $((k-1)\Delta, k\Delta]$ is not observed. For simplicity and following Lefèvre et al. (2018), we then assume there are at most two jumps per period.

In this circumstance, the main observable operator is given, similarly to the insurance case Lefèvre et al. (2018), by

$$H_3 = S_0\Big(P_0 \otimes I_4 + P_1 \otimes (A \otimes I_2) + P_2 \otimes (A^{\otimes 2})\Big), \quad (4)$$

where I_n is a $n \times n$ identity matrix, A is the 2×2 matrix representing the 1-step geometric move operator (2) with eigenvalues e^u, e^d, and P_0, P_1, P_2 are 3×3 matrix projection operators corresponding to the $0, 1, 2$ claim occurrence operators and defined by

$$P_i = W^* D_{i+1|3} W, \quad i = 0, 1, 2, \quad (5)$$

where $D_{i+1|3}$ is a 3×3 diagonal matrix which has all its elements equal to 0 except the $(i+1, i+1)$-th with value 1, and W is a 3×3 unitary matrix.

The share price operator over N periods is then defined as

$$H_{S_N} = S_0 H_3^{\otimes N} = S_0\Big(P_0 \otimes I_4 + P_1 \otimes (A \otimes I_2) + P_2 \otimes (A^{\otimes 2})\Big)^{\otimes N}. \quad (6)$$

Furthermore, when constructing the density operator, we consider the following two cases.

Maxwell-Boltzmann statistics. In this case, the jump sizes are i.i.d. with a two-point distribution. More precisely, there are $0, 1, 2$ jumps with probabilities $\delta_0, \delta_1, \delta_2$ given via a Poisson process, and each jump size has two possible values e^d, e^u with probabilities q, p (see also later in Section 4.1).

Bose-Einstein statistics. In this case, the claim sizes are dependent but remain independent of the claim occurrences.

2.1.3. Quantum trinomial model

This model makes a bridge between the traditional binomial model and the actuarial type model. In this case, the dynamics per period is defined by three moves: no change, down and up. The trinomial type financial modelling is a well established topic in finance (see e.g., Boyle (1986); Tian (1993); Leisen and Reimer (1996)).

Now, the 1-step observable operator $H_2 = B$ is the 3×3 self-adjoint matrix with eigenvalues $e^u, 1, e^d$ corresponding to these three moves and is given by

$$H_2 = B = U^* DU = U^* \begin{pmatrix} e^u & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^d \end{pmatrix} U, \tag{7}$$

where U is now a 3×3 unitary matrix.

Then, the share price operator over N periods is defined by

$$H_{S_N} = S_0 H_2^{\otimes N} = S_0 (U^*)^{\otimes N} D^{\otimes N} U^{\otimes N}. \tag{8}$$

Remarks. Notice that the heterogeneous versions of the trinomial model and the actuarial type model are available with the representation similar to (3). However, they are not treated in this paper because the main purpose for considering these two models is the non-standard data analysis (to be presented in Section 5). Although the data analysis of time dependent models is a fascinating topic, it is out of the scope of this paper.

2.2. *Quantum binomial price*

As mentioned in above, the quantum binomial model (1) is originated in Chen (2001, 2004). Our presentation is somewhat different and based on the algebraic tensor product properties. In addition, since heterogeneity is an important topic in option pricing (see e.g., Benninga and Mayshar (2000)), we treat a slightly more general time dependent or heterogeneous quantum binomial model (3).

A quantum state ρ is defined as a positive self-adjoint operator with trace $tr(\rho) = 1$. We recall the following properties.

Lemma 2.1. *Let A, B, C, D be self-adjoint operators, U a unitary matrix, f a function of observables and ρ a quantum state. Then,*

$$\begin{aligned} & tr(ABC) = tr(CAB), \qquad tr(A \otimes B) = tr(A)tr(B)\ , \\ & (A \otimes B)(C \otimes D) = (AC) \otimes (BD), \quad (A \otimes B)^* = A^* \otimes B^*, \\ & A, B \geq 0 \to A \otimes B \geq 0, \quad A = U^* DU \to f(A) = U^* f(D)U, \\ & U\rho U^* = \textit{quantum state}. \end{aligned} \tag{9}$$

For simplicity, we choose a quantum state ρ as a tensor product, i.e.

$$\rho = \rho_1 \otimes \ldots \otimes \rho_N, \tag{10}$$

where each ρ_i is a self-adjoint non-negative 2×2 matrix such that $tr(\rho_i) = 1$. From Lemma 2.1, we see that ρ is a proper quantum state.

The risk-neutral world of the quantum Black-Scholes model consists of self-adjoint non-negative 2×2 matrices ρ_i that satisfy

$$tr(\rho_i H_i) = 1 + r_i, \quad i = 1, \ldots, N, \tag{11}$$

where r_i is the risk-free interest rate for the period i.

Let us define

$$H_i = U_i^* D_i U_i, \quad \text{and let} \quad \tilde{\rho}_i = U_i \rho_i U_i^*, \quad i = 1, \ldots, N. \tag{12}$$

Note that the $\tilde{\rho}_i$ are also quantum states. In addition, $\tilde{\rho}_i$ has non-negative diagonal elements $q_u^{(i)}, q_d^{(i)}$ and it can be shown to have the representation

$$\tilde{\rho}_i = \begin{pmatrix} q_{u_i} & x \\ \bar{x} & q_{d_i} \end{pmatrix}, \tag{13}$$

where

$$q_{u_i} = \frac{1 + r_i - d_i}{u_i - d_i}, \quad q_{d_i} = 1 - q_{u_i}. \tag{14}$$

The transformed operator $\tilde{\rho}$ is then obtained from Lemma 2.1 (third property) as

$$\begin{aligned} \widetilde{\rho} &= U(\rho_1 \otimes ... \otimes \rho_N)U^* = (U_1 \otimes \ldots \otimes U_N)(\rho_1 \otimes ... \otimes \rho_N)(U_1^* \otimes \ldots \otimes U_N^*) \\ &= (U_1 \rho_1 U_1^*) \otimes \ldots \otimes (U_N \rho_N U_N^*) = \tilde{\rho}_1 \otimes \ldots \otimes \tilde{\rho}_N, \end{aligned} \tag{15}$$

after using (12), and $\tilde{\rho}$ is again a quantum state. For the classical probability case, that is when all matrices are commutative, this transform is the form of the change of measure.

Moreover, the quantum no arbitrage condition (11) is satisfied for the transformed density, i.e.

$$tr(\tilde{\rho}_i D_i) = 1 + r_i, \quad i = 1, \ldots, N. \tag{16}$$

2.3. *Quantum binomial option pricing*

Consider a payoff function f and a discount factor

$$d_N = 1/(1 + r_1) \ldots (1 + r_N). \tag{17}$$

From (3), the price of the general option without arbitrage $OP(f(H_{S_N}))$ for the N-step quantum binomial model is then defined by

$$OP(f(H_{S_N})) = d_N\, tr(\rho f(H_{S_N})) = d_N\, tr(\rho_1 \otimes ... \otimes \rho_N f(S_0 H_1 \otimes ... \otimes H_N)). \tag{18}$$

Applying properties given in Lemma 2.1, we then obtain

$$\begin{aligned} OP(f(H_{S_N})) &= \frac{tr(\rho_1 \otimes ... \otimes \rho_N f(S_0 H_1 \otimes ... \otimes H_N))}{(1 + r_1) \ldots (1 + r_N)} \\ &= \frac{tr(\rho_1 \otimes ... \otimes \rho_N U^* f(S_0 D_1 \otimes ... \otimes D_N) U)}{(1 + r_1) \ldots (1 + r_N)} \\ &= \frac{tr([U(\rho_1 \otimes ... \otimes \rho_N)U^*] f(S_0 D_1 \otimes ... \otimes D_N))}{(1 + r_1) \ldots (1 + r_N)} \\ &= \frac{tr(\tilde{\rho}_1 \otimes \ldots \otimes \tilde{\rho}_N f(S_0 D_1 \otimes ... \otimes D_N))}{(1 + r_1) \ldots (1 + r_N)}, \end{aligned} \tag{19}$$

after using (15). Since $\tilde{\rho}$ is a tensor product and $f(S_0 D_1 \otimes ... \otimes D_N)$ is a diagonal matrix, we deduce the option pricing formula (20) below.

Theorem 2.2. *For the heterogeneous quantum binomial model,*

$$OP(f(H_{S_N})) = \frac{\sum_{\sigma} f(S_0 y_\sigma) q_\sigma}{(1+r_1)\dots(1+r_N)}, \tag{20}$$

where the index σ denotes any feasible path y_σ of the form

$$y_\sigma = e^{\sigma_1}\dots e^{\sigma_N} \ \ with \ \ \sigma_i \in \{u_i, d_i\}, \tag{21}$$

which occurs with the probability $q_\sigma = q_{\sigma_1}\dots q_{\sigma_N}$ where $q_{\sigma_i} \in \{q_{u_i}, q_{d_i}\}$.

In particular, for the homogeneous case (the quantum Cox-Ross-Rubinstein model) where for all i, $r_i = r, u_i = u, d_i = d$ with $q_{u_i} = q_u, q_{d_i} = q_d$, the formula (20) reduces to

$$OP(f(H_{S_N})) = \frac{1}{(1+r)^N}\sum_{n=0}^{N} f(S_0 e^{un} e^{d(N-n)}) q_d^n q_u^{N-n}. \tag{22}$$

Non-self adjoint quantum binomial market. Assume that H_i are invertible 2×2 matrices with two different eigenvalues e^d and e^u but no more self-adjoint, in general. In this case the transformed density $\tilde{\rho}_i$ are no more proper states, in general. However, they still have same diagonal elements q_{u_i} and q_{d_i} and so the transformed matrix $\tilde{\rho}$ defined in (15) again has same diagonal elements as in the self-adjoint case, but is not a proper state, in general. By inspecting the proof, we see that Theorem 2.2 holds true in this case as well.

3. Convergence to continuous-time markets

In this section, we consider the homogeneous quantum binomial model, and we discuss two examples on the convergence to continuous-time markets, namely the Black-Scholes model and the planar Brownian motion.

3.1. Convergence to the Black-Scholes model

Theorem 3.1. *Let μ_N be the measure of the eigenvalues of $H^{\otimes N}$ with respect to the quantum state $\rho^{\otimes N}$. Suppose that $r = \lambda/N$ and $u = -d = \sigma N^{-1/2}$. Then, as $N \to \infty$,*

$$\int f(S_0 x) d\mu_N(x) \to \int f(S_0 x) d\mu(x), \tag{23}$$

where μ is a lognormal distribution (i.e. $\mu(x) = P(e^{a+\sigma Z} \le x)$ for suitable constants a, σ and Z a standard normal variable).

Proof. We begin with the representation via the spectral measure μ_N. Observe that

$$Ef(H_{S_N}) = tr(\rho^{\otimes N} f(S_0 H^{\otimes N})) = \int f(S_0 x)\mu_N(dx), \tag{24}$$

where μ_N is the measure of the eigenvalues λ_σ of $H^{\otimes N}$ with respect to the quantum probability $tr(\rho^{\otimes N} H^{\otimes N})$. Since $\lambda_\sigma = e^{\sigma_1}\dots e^{\sigma_N}$ with $\sigma_i \in \{u_i, d_i\}$, we get

$$\mu_N(x) = \sum_{\sigma:\lambda_\sigma \le x} q_\sigma = \sum_{\sigma: e^{\sigma_1}\dots e^{\sigma_N} \le x} q_{\sigma_1}\dots q_{\sigma_N}, \tag{25}$$

where $q_{\sigma_i} \in \{q_u, q_d\}$ respectively.

Let us move on to the weak convergence desired. We can write that

$$q_{\sigma_1} \dots q_{\sigma_N} = P(Y_1 = e^{\sigma_1}) \dots P(Y_N = e^{\sigma_N}) = P(Y_1 = e^{\sigma_1}, \dots, Y_N = e^{\sigma_N}), \tag{26}$$

where the Y_i are i.i.d. variables with $P(Y_i = e^u) = q_u, P(Y_i = e^d) = q_d$. Thus,

$$\mu_N(x) = P(Y_1 \dots Y_N \leq x) = P(\ln Y_1 + \dots + \ln Y_N \leq \ln x) = P(T_N \leq \ln x), \tag{27}$$

with $T_N \equiv \ln Y_1 + \dots + \ln Y_N$. As $N \to \infty$, we obtain from the central limit theorem that

$$\mu_N(x) = P(T_N \leq \ln x) \to P(e^{a+\sigma Z} \leq x) = \mu(x), \tag{28}$$

for certain constants a, b and Z a standard normal. This gives the limit result (23). $\diamond$

Remarks. (i) The limiting measure μ corresponds to that of the geometric Brownian motion $S_t = S_0 e^{(\rho-\sigma^2/2)t+\sigma B_t}$ when $S_0 = 1$, $t = 1$ and $\rho - \sigma^2/2 = a$.

(ii) It would be interesting to compare the technique with the semi-classical approximation, such as expanding the action around the classical path (see (48) in Contreras et al. (2010)). Another interesting question is to analyse connection with arbitrage as discussed in Haven (2002) and Contreras et al. (2010).

(iii) The Cauchy transform is an alternative approach to deal with quantum probabilities (see Mudakkar and Utev (2013)). In particular, the convergence of spectral measures is reduced to the convergence of the Cauchy transforms. We recall that the Cauchy transform for the measure μ is defined by

$$S_\mu(z) = \int_R \frac{\mu(dt)}{t-z}, \quad \text{where} \quad \mu((a,b)) = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_a^b \Im S_\mu(x + i\epsilon) dx, \tag{29}$$

for all $z \in C \backslash R = \{z \in C : \Im z \neq 0\}$ and open intervals (a, b) with $\mu(\{a, b\}) = 0$. The goal is then to show that $S_{\mu_n}(z) \to S_\mu(z)$. However, the common approach of the moment expansion does not work in this case since

$$\int \frac{d\mu(x)}{z-x} = \sum_{k=0}^{\infty} \frac{1}{z^{k+1}} \int x^k d\mu(x) = \sum_{k=0}^{\infty} \frac{1}{z^{k+1}} E(e^{k(a+\sigma Z)}) = \sum_{k=0}^{\infty} \frac{1}{z^{k+1}} e^{ka+k^2\sigma^2/2} = \infty. \tag{30}$$

3.2. Convergence to the planar Brownian motion

In this part, by treating the relatively simple bond market, we show that non-commutative markets provide a richer class of financial models (see also Haven (2002); Contreras et al. (2010); Herscovich (2016)).

Bond market. Assume that returns are non-risky, that is the outcomes are equal. In the quantum setup, we suppose that matrix H has a single eigenvalue, e^u say.

Self-adjoint quantum bond market. Let $\rho^{\otimes N}$ denote a quantum state defined as before. Assume that in addition operator H is self-adjoint. Then $H = e^u I_2$ and

$$H_{S_N} = S_0 H^{\otimes N} = S_0 e^{uN} I_2^{\otimes N} \quad \text{and} \quad f(H_{S_N}) = f(S_0 e^{uN}) I_2^{\otimes N}. \tag{31}$$

In particular, any option claim $f(H_{S_N})$ is commutative with the state $\rho^{\otimes N}$ which implies that the *self-adjoint quantum bond market is commutative*, that is equivalent to the simple classical probability financial market with two non-risky assets. The no-arbitrage condition becomes global to give $e^u = tr(\rho H) = 1 + r$ since $tr(\rho) = 1$ which restates that under no-arbitrage the non-risky returns are equal. Moreover, the option price for $f(H_{S_N})$ is provided by

$$OP(f(H_{S_N})) = \frac{tr(\rho^{\otimes N} f(S_0 H^{\otimes N}))}{(1+r)^N} = e^{-uN} f(S_0 e^{uN}). \tag{32}$$

Non self-adjoint quantum bond market. Now, assume that H is no longer self-adjoint, for example a product of two noncommutative self adjoint observables. Thus, *non self-adjoint quantum bond market is noncommutative*, in general. In this case, we show that the limit is sensitive to the density state ρ and the representation of H. For simplicity, we assume that the basic observable H and the state ρ are defined by

$$H_u = \begin{pmatrix} e^u & 1 & 0 \\ 0 & e^u & 1 \\ 0 & 0 & e^u \end{pmatrix}, \quad \rho = \begin{pmatrix} \rho_{11} & y & \delta \\ y & \rho_{22} & 0 \\ \delta & 0 & \rho_{33} \end{pmatrix}, \tag{33}$$

i.e., H_u is now a 3×3 Jordan matrix. The bond price process is then defined by

$$H_{S_N} = S_0 H_u^{\otimes N}. \tag{34}$$

To satisfy the no-arbitrage condition, we ask that

$$tr(\rho H_u) = 1 + r, \quad \text{thus} \quad y = 1 + r - e^u. \tag{35}$$

Theorem 3.2. *Suppose that $r = \lambda/N$, $u = a/N$ and $\delta = -\Delta/N$ with $\Delta \geq 0$. Then, for any positive integer k, as $N \to \infty$,*

$$OP([H_{S_N}]^k) \to e^{-\lambda} S_0^k E[e^{k(\lambda+\Delta/2+iB_\Delta)}], \tag{36}$$

regardless of a, where i is the imaginary unit and B_Δ is a Brownian motion at time Δ.

Proof. Notice that

$$H_u^k = \begin{pmatrix} e^{ku} & e^{(k-1)u}k & e^{(k-2)u}k(k-1)/2 \\ 0 & e^{ku} & e^{(k-1)u}k \\ 0 & 0 & e^{ku} \end{pmatrix}. \tag{37}$$

From the definition of H_{S_N} and ρ, we then get

$$\begin{aligned} OP([H_{S_N}]^k) &= OP([S_0(H^{\otimes N})]^k) = (1+r)^{-N}\, tr(\rho^{\otimes N}[S_0(H^{\otimes N})]^k) \\ &= (1+r)^{-N}\, tr(S_0^k \rho^{\otimes N}[(H^k)^{\otimes N}]) = S_0^k (1+r)^{-N}\, tr([\rho H^k]^{\otimes N}) \\ &= S_0^k (1+r)^{-N} (\rho_{11} e^{uk} + \rho_{22} e^{uk} + \rho_{33} e^{uk} + y e^{u(k-1)} k + \delta e^{(k-2)u} k(k-1)/2)^N \\ &= S_0^k (1+r)^{-N} (e^{uk} + y e^{u(k-1)} k + \delta e^{(k-2)u} k(k-1)/2)^N \\ &= S_0^k (1+r)^{-N} [1 + ak/N + (\lambda - a)k/N - \Delta k(k-1)/2N + O(1/N^2)]^N \\ &\to S_0^k e^{-\lambda}\, e^{(\lambda+\Delta/2)k - k^2\Delta/2} \end{aligned} \tag{38}$$

after using the assumptions made on r, u, δ. This provides the limit result (36). $\diamond$

Remarks. (i) This representation is useful in computing the European call option $\mathrm{OP}([H_{S_N} - K)_+])$ via the Fourier techniques combined with the Monte Carlo approximation. For the distribution of this process, we view R^2 as the complex plane and the planar Brownian motion $\tilde{B}_t = (B_1(t), B_2(t))$ is then interpreted as a complex-valued Brownian motion.

(ii) Although observables are traditionally considered to be self-adjoint, the non self-adjoint data is modelled by considering an observable such as $H = AB$, i.e. the product of two self-adjoint matrices A, B where A and B may represent the non-trade and trade time changes, respectively.

(iii) Similarly to the convergence to the Black-Scholes model, the limit does not depend on the shift parameter a. However, now the limit depends on the mysterious characteristic Δ.

4. Quantum mechanics in finance

In this section, we will apply some methods of quantum mechanics to finance. Discrete and continuous-time markets must be treated separately because of different stochastic behaviors. To simplify the presentation, we assume that the interest rate is 0 and that the risky processes for share prices are martingales.

4.1. Discrete-time quantum approach

The use of Dirac-Feynman quantum mechanics techniques for insurance risk modelling was initiated in Tamturk and Utev (2018) and then developed in Lefèvre et al. (2018); Tamturk and Utev (2019). We adapt this approach to the problem of option pricing in finance, in particular for the pricing of path-dependent options.

In the Dirac formalism, bra-ket notation is a standard way of describing quantum states. Consider a class of $n \times n$ matrices treated as C^* algebra. A column vector x corresponds to a ket-vector $|x>$. An associated bra-vector $<x|$ is a row vector defined as its Hermitian conjugate. The usual inner product is denoted by $<x|y>$, while the outer product $|x><y|$ is the operator/matrix defined by

$$|x><y||z>=<y|z>|x> \quad (abc = bca \text{ rule}). \tag{39}$$

In the Feynman path integral methods, the transition probability $P(x_j \to x_{j+1})$ is computed as the propagator $<x_j|A_{j+1}|x_{j+1}>$ when A_j is a Markovian operator. Thus, the typical path is written as $|x_0>\to|x_1>\to\ldots\to|x_n>$, and its probability is given by

$$P(|x_0>\to|x_1>\to|x_2\to\ldots\to|x_n>) =<x_0|A_1|x_1><x_1|A_2|x_2>\ldots<x_{n-1}|A_n|x_n>. \tag{40}$$

The main ingredient is then the path calculation formula that calculates the probability $P(x_0 \to x_n)$ via the sum of the probabilities on all the appropriate paths, i.e.

$$\begin{aligned} P(x_0 \to x_n) &= <x_0|A_1A_2\ldots A_n|x_n> \\ &= \sum_{x_1,\ldots,x_{n-1}} <x_0|A_1|x_1><x_1|A_2|x_2>\ldots<x_{n-1}|A_n|x_n>. \end{aligned} \tag{41}$$

It remains to define a suitable propagator for discrete time. For simplicity, we take the operators A_j all equal to A and the time intervals Δt_i all equal to Δt. In a similar way to e.g. Baaquie (2004);

Tamturk and Utev (2018), we assume that the operator A is defined via an Hamiltonian operator H such as $A = e^{-\Delta t\, H}$, where $-H$ is a Markovian generator called Markovian Hamiltonian. Thus, $P(x_i \rightarrow x_{i+1}) =< x_i|e^{-\Delta t\, H}|x_{i+1} >$ which can be computed applying the Fourier transform to the momentum space (e.g., Griffiths and Schroeter (2018); Tamturk and Utev (2018)). Specifically, let $|p>$ be a basis in that space, and write $< x|p >= e^{ipx}$ and $< p|x >= e^{-ipx}$. Then, we get

$$\begin{aligned} < x_i|e^{-\Delta t\, H}|x_{i+1} > &= \int_0^{2\pi} \frac{d\alpha}{2\pi} < x_i|e^{-\Delta t\, H}|\alpha >< \alpha|x_{i+1} > \\ &= \int_0^{2\pi} \frac{d\alpha}{2\pi} < x_i|\alpha >< \alpha|x_{i+1} > e^{-\Delta t\, K_\alpha} \\ &= \frac{1}{2\pi} \int_0^{2\pi} (e^{ix_i\alpha} e^{-ix_{i+1}\alpha}) e^{-\Delta t\, K_\alpha} d\alpha, \end{aligned} \tag{42}$$

where $\{|\alpha >, K_\alpha\}$ is the set of eigenvectors and eigenvalues of the Hamiltonian operator H (i.e., $H|\alpha >= K_\alpha|\alpha >$). Therefore, we deduce from (41), (42) that

$$\begin{aligned} P(x_0 \rightarrow x_n) &= \sum_{x_1,\ldots,x_{n-1}} < x_0|e^{-\Delta t H}|x_1 >< x_1|e^{-\Delta t H}|x_2 > \ldots < x_{n-1}|e^{-\Delta t H}|x_n > \\ &= \sum_{x_1,\ldots,x_{n-1}} \frac{1}{(2\pi)^n} \int_0^{2\pi}\int_0^{2\pi}\ldots\int_0^{2\pi} [\prod_{i=0}^{n-1} (e^{ix_i\alpha_i} e^{-ix_{i+1}\alpha_i}) e^{-\Delta t\, K_{\alpha_i}}]\, d\alpha_0 d\alpha_1 \ldots d\alpha_{n-1}. \end{aligned} \tag{43}$$

For a more detailed overview of the theory, we refer the reader to the books by Feynman and Hibbs (2010); Griffiths and Schroeter (2018); Parthasarathy (2012) and Plenio (2002).

Option pricing formula. Consider a claim of the form $C = f(S_0, S_1, ..., S_N)$, using the notation of Section 2. From (43), we obtain for the corresponding option price

$$OP(C) = d_N \sum_{S_0, S_1, \ldots, S_N} f(S_0, S_1, ..., S_N) < S_0|A_1|S_1 > \ldots < S_{N-1}|A_N|S_N > . \tag{44}$$

The discrete-time approach followed to derive (43) can then be easily applied to the current formula (44).

A modified Cox-Ross-Rubinstein model. Consider a discrete-time market in which, during each i-th time interval Δt, the share price S_i can

(.) have 1 jump giving $S_{i+1} = S_i e^u, S_i e^d$ with probabilities $p, q = 1 - p$, or

(.) have 2 jumps giving $S_{i+1} = S_i e^{u+d}, S_i e^{2u}, S_i e^{2d}$ with probabilities $2pq, p^2, q^2$, or

(.) remain the same giving $S_{i+1} = S_i$,

where d, u are integers with $d < 0 < u$. Furthermore, the possible jumps arrive according to a Poisson process of parameter λ so that

(.) $\delta_1 \equiv P[N(\Delta t) = 1] = e^{-\lambda\Delta t}(\lambda\Delta t)$,

(.) $\delta_2 \equiv P[N(\Delta t) = 2] = e^{-\lambda\Delta t}(\lambda\Delta t)^2/2$,

(.) $\delta_0 \equiv P[N(\Delta t) = 0] \approx 1 - \delta_1 - \delta_2$.

Define $x_i = \ln(S_i)$. The transition probabilities $P(S_i \rightarrow S_{i+1})$ are equivalent to $P(x_i \rightarrow x_{i+1})$. Set $\Delta_t = 1$, say. We can now apply (42) with $A = e^{-H}$ where the set $\{|\alpha >, K_\alpha\}$ is found by solving the Schrödinger equation $e^{-H}|\alpha >= e^{-K_\alpha}|\alpha >$ in which

$$e^{-H}|\alpha > (x) = E_x(e^{i\alpha x_{i+1}}) = e^{i\alpha x}(\delta_0 + e^{i\alpha u} p\delta_1 + e^{i\alpha d} q\delta_1 + e^{i\alpha 2u} p^2\delta_2 + e^{i\alpha x 2d} q^2\delta_2 + e^{i\alpha(u+d)} 2pq\delta_2). \tag{45}$$

Notice that the Hamiltonian H is not Markovian, but (42) is still applicable to the discrete times $k\Delta t$. Moreover, by construction, the martingale probabilities $p = q_u$ and $q = q_d$ for the Cox-Ross-Rubinstein model (without interest) yield a martingale in the present situation too since we have

$$\begin{aligned} E(S_{i+1}|S_i) &= S_i[\delta_0 + \delta_1(q_u e^u + q_d e^d) + \delta_2(e^{2u} q_u^2 + e^{2d} q_d^2 + e^{q+d} 2q_u q_d)] \\ &= S_i(\delta_0 + \delta_1 + \delta_2) \ = \ S_i. \end{aligned} \tag{46}$$

We have illustrated numerically the option pricing results obtained for the model. The tables and figures are however too large to be included here.

4.2. Continuous-time quantum approach

This short part is mostly a review of the application of quantum mechanics approach to the continuous time markets and closely follows Baaquie (2004).

The continuous time formalism is based on the Fourier transform of tempered distributions on the basis $|p>$ in the momentum space. Consider a risky asset price S_t that evolves in function of an Hamiltonian operator H. First, the method is applied to compute the pricing kernel

$$p(x, \tau; x') =< x|e^{-\tau H}|x' >= \int_{-\infty}^{\infty} \frac{dp}{2\pi} < x|e^{-\tau H}|p >< p|x' > . \tag{47}$$

Then, the option price at time t for the claim $Q \equiv Q(S_T)$, $T > t$, given $S_t = x$ is defined by

$$OP(Q(S_T)|S_t = x) =< x|e^{-(T-t)H}|Q > . \tag{48}$$

Black-Scholes model. In this classical approach, the stock price S_t follows a geometric Brownian motion, i.e. $S_t = S_0 e^{\tilde{B}_t}$ where $\tilde{B}_t = \mu t + \sigma B_t$ (μ is the drift, σ the volatility and B_t a standard Brownian motion).

We apply the quantum mechanics approach. Motivated by Baaquie (2004), we work with $\tilde{B}_t$ rather than with S_t. The corresponding Hamiltonian for the Brownian motion is $Hf = -(1/2)f'' - \mu f'$ (computed, for example, via the Itô formula). The Brownian motion kernel is then defined from (47) by the normal density function

$$p(u, \tau; u') = f_{N(\tau\mu, \tau\sigma^2)}(u' - u) = \frac{1}{\sqrt{2\pi\tau\sigma^2}} \exp[-(1/2\tau\sigma^2)(u' - u - \tau\mu)^2]. \tag{49}$$

To obtain the option price (48), we set $x = S_0 e^u$ and $Q(x) = Q(S_0 e^u) \equiv g(u)$. Then, the Feynman-Kac formula yields

$$\begin{aligned} OP(Q(S_T)|S_t = x) =< u|e^{-(T-t)H}|g > &= \int_{-\infty}^{\infty} < u|e^{-(T-t)H}|u' > g(u')du' \\ &= \int_{-\infty}^{\infty} p(u, T - t; u')Q(S_0 e^{u'})du'. \end{aligned} \tag{50}$$

Via path integrals. To find the pricing kernel $p(x, \tau; x') =< x|e^{-\tau H}|x' >$ for $\tau = T - t$, an alternative method consist in using path integral methods. For this, we discretize the time in N intervals of length

Δ and consider the $x_i = x(t_i)$ where $t_i = i\Delta$. We then proceed as in the situation in discrete-time. The pricing kernel for $(x, x') = (x_0, x_N)$ becomes

$$p(x, N\Delta; x') = \int \int \ldots \int dx_1 dx_2 \ldots dx_{N-1} \prod_{i=1}^{N} < x_{i-1}|e^{-\Delta H}|x_i > . \tag{51}$$

Applications to non-life insurance. Consider Feynman's modification $\tilde{H}$ of the Brownian motion Hamiltonian H by adding the potential V, i.e. for the Hamiltonian $\tilde{H} = H + V$. For example, choose a ruin level B and take $V(x) = +\infty$ for $x < B$. Then, the path calculation formula (43) allows us to compute ruin probabilities when $B = 0$ (Tamturk and Utev (2018, 2019)) and exotic options with barriers when $B > 0$.

Finally, let us mention that numerically, the binomial model formula (20) and the path integral approach (51) were found to give results close to the Black-Scholes formula, even for relatively small values of N (of order 40).

5. Analysis of stock market data

We are going to analyse a generated set of financial data by choosing two different quantum models described in Section 2.1. First, we consider the quantum actuarial-type model (see Section 2.1.3 and in 4.1) with the N-step observable defined in (6). Then, we consider the quantum trinomial model (see Section 2.1.2) with the N-step observable defined in (8). Note that matching 1-step for the actuarial case to two steps for the trinomial model is natural, since in the actuarial case, we choose at most two jumps.

5.1. Methodology for data analysis

We apply supervised machine learning methods such as developed e.g. in the books by Bishop (2006); Hastie et al. (2009) and Wittek (2014).

Overall approach. The dataset is supposed to come from a non-ordered class of randomly perturbed observables. Each data is the observation of an eigenvalue λ of the observable perturbed by i.i.d. error terms. The observable is the 1-step operator for the actuarial-type model and the 2-step oparator for the trinomial model. Matching 1-step in the actuarial case to 2-step in the trinomial case is natural since there are at most two jumps in the actuarial case considered.

First, the data are classified in classes G_λ with respect to the eigenvalues λ. Then, the probabilities p_λ are estimated by maximum likelihood using Maxwell-Boltzmann or Bose-Einstein statistics. Finally, the λ are estimated via the weighted L_1-norm risk error function.

Let us explain in more detail for the actuarial-type model, for example.

Step 1. An initial $(u = u_0, u = d_0)$ is chosen randomly.

Step 2. For a given (u, d), the data is classified and labeled against the eigenvalues of the observable using a nearest neighbor algorithm. This leads to the classes G_λ.

Step 3. For the same (u, d), the estimates $\hat{p}$ and $\hat{q}$ are obtained by maximizing the likelihood function $L(p, q)$.

Step 4. The (u, d) is updated by minimizing a weighted L_1-norm risk error function $F(u, d) \equiv F(\lambda)$ defined by

$$F(\lambda) = \|\beta - \lambda\| = \sum_{\lambda} p_\lambda \sum_{\beta_i \in G_\lambda} |\beta_i - \lambda|. \tag{52}$$

Step 5. The loop of steps 2 to 4 is repeated until the relative error becomes smaller than a selected difference M, i.e. when

$$|F(u_{i+1}, d_{i+1}) - F(u_i, d_i)| < M. \tag{53}$$

$k-$fold cross-validation. To reduce the risk of error, we use a $k-$fold cross-validation strategy. The dataset is randomly divided into k subsets of equal size. One of the subsets is chosen as the training set and the others as test sets. The process is repeated k times, each subset constituting a training element. At each iteration, steps 1 to 5 above are applied to the training data and the results obtained are then checked in the test data. Finally, the estimates used are an average of those obtained on the k iterations.

Numerical example. As a simple illustration, we will consider the following dataset

$$V = \{86, 63, 35, 52, 41, 8, 24, 12, 19, 24, 42, 5, 91, 95, 50, 49, 34, 91, 37, 11\}. \tag{54}$$

5.2. Data analysis via the actuarial-type model

From the assumptions of the model, the 1-step observable H has the eigenvalues

$$\{\lambda\} = \{1, e^d, e^u, e^{d+u}, e^{2d}, e^{2u}\}, \tag{55}$$

with probabilities respectively given by

$$\{p_\lambda\} = \{\delta_0, q\delta_1, p\delta_1, p_{d+u}\delta_2, p_{2d}\delta_2, p_{2u}\delta_2\}. \tag{56}$$

For the probabilities p_{d+u}, q_{2d}, p_{2u}, we consider two possible statistics often used in the analysis of quantum observables.

Maxwell-Boltzmann independence. This case yields the binomial model since

$$p_{d+u} = 2pq, \; p_{2d} = q^2, \; p_{2u} = p^2. \tag{57}$$

Bose-Einstein dependence. In this case, corresponds to probabilities

$$p_{d+u} = Cpq, \; p_{2d} = Cq^2, \; p_{2u} = Cp^2, \quad \text{where } C(pq + q^2 + p^2) = 1. \tag{58}$$

As pointed out in Lefèvre et al. (2018), both statistics admit a formal construction, via the proper choice of the density operator ρ for the number of occurrences and the density projection operators ρ_λ, such that

$$tr(\rho \otimes \rho_\lambda) = p_\lambda. \tag{59}$$

Likelihood functions. Denote by $\#x$ the number of x observed in the data set. For the Maxwell-Boltzmann statistics, the likelihood is defined by the probabilities p, q, δ_0,δ_1 and δ_2 such that

$$L(p, q) = L(p, q, \delta_0, \delta_1, \delta_2) = (\delta_0)^{\#0}(q\delta_1)^{\#d}(p\delta_1)^{\#u}(2pq\delta_2)^{\#u+d}(q^2\delta_2)^{\#2d}(p^2\delta_2)^{\#2u}. \tag{60}$$

For the Bose-Einstein statistics, the likelihood function is modified as

$$L(p,q) = L(p,q,\delta_0,\delta_1,\delta_2) = (\delta_0)^{\#0}(q\delta_1)^{\#d}(p\delta_1)^{\#u}(Cpq\delta_2)^{\#u+d}(Cq^2\delta_2)^{\#2d}(Cp^2\delta_2)^{\#2u}. \tag{61}$$

Risk functions. From (55), the scaled share price spectrum of H_{S_1} is given by $S_0\{\lambda\} = \{S_0, S_0e^u, S_0e^d, S_0e^{u+d}, S_0e^{2d}, S_0e^{2u}\}$. Thus, using the risk function (52), we get for the Maxwell-Boltzmann case

$$\begin{aligned} F_1(u,d) = {} & \delta_0 \sum_{\beta_i \in G_0} |\beta_i - S_0| + q\delta_1 \sum_{\beta_i \in G_d} |\beta_i - S_0e^d| p\delta_1 \sum_{\beta_i \in G_u} |\beta_i - S_0e^u| \\ & + 2pq\delta_2 \sum_{\beta_i \in G_{u+d}} |\beta_i - S_0e^{u+d}| + q^2\delta_2 \sum_{\beta_i \in G_{2d}} |\beta_i - S_0e^{2d}| + p^2\delta_2 \sum_{\beta_i \in G_{2u}} |\beta_i - S_0e^{2u}|, \end{aligned} \tag{62}$$

and for the Bose-Einstein case,

$$\begin{aligned} F_2(u,d) = {} & \delta_0 \sum_{\beta_i \in G_0} |\beta_i - S_0| + q\delta_1 \sum_{\beta_i \in G_d} |\beta_i - S_0e^d| + p\delta_1 \sum_{\beta_i \in G_u} |\beta_i - S_0e^u| \\ & + Cpq\delta_2 \sum_{\beta_i \in G_{u+d}} |\beta_i - S_0e^{u+d}| + Cq^2\delta_2 \sum_{\beta_i \in G_{2d}} |\beta_i - S_0e^{2d}| + Cp^2\delta_2 \sum_{\beta_i \in G_{2u}} |\beta_i - S_0e^{2u}|. \end{aligned} \tag{63}$$

Numerical illustration. The data (54) is treated as the eigenvalues of the 1-step observable H defined in (6) for $N = 1$. Let us assume that the Poisson process rate is 1 and the length of time is $\Delta t = 1$.

We estimate the values d, u and the probabilities q, p by applying the algorithm of Section 5.1 and using the Maxwell-Boltzmann statistics. Choose, for instance, $S_0 = 70$, $(u_0, d_0) = (0.8, -0.1)$ and $M = 0.000001$. The results obtained from (60), (62) are presented in Table 1.

Table 1. Estimation for $N = 1$ in the Maxwell-Boltzmann case.

Given (u,d)	Maximum likelihood	Optimum (p,q)	Optimum (u,d)	Risk error $F(u,d)$	$\lvert F(u_i,d_i) - F(u_{i+1},d_{i+1})\rvert$
(0.8,-0,1)	4.6063e-14	(0.01,0.99)	(0.3,-0.4)	43.3107	43.3107
(0.3,-0.4)	2.2681e-18	(0.16,0.84)	(0.3,-0.5)	22.8734	20.4373
(0.3,-0.5)	5.1730e-19	(0.23,0.77)	(0.3,-0.6)	15.8654	7.0080
(0.3,-0.6)	1.7631e-18	(0.24,0.76)	(0.3,-0.7)	14.4741	1.3913
(0.3,-0.7)	1.2479e-19	(0.29,0.71)	(0.3,-0.7)	11.1976	3.2765
(0.3,-0.7)	1.2479e-19	(0.29,0.71)	(0.3,-0.7)	11.1976	0

We then also apply a 4-foldcross-validation procedure. Let V_i, $1 \le i \le 4$, before randomly chosen subsets of V such that $\cup_{i=1}^4 V_i = V$. For each iteration, $\{V \setminus V_i\}$ and V_i are treated as the training set and the test set, respectively. First, the maximum likelihood estimation and the risk error computation are executed for the training data. Then, the obtained estimates are implemented in the test data. This gives the results of Table 2.

Table 2. Using a 4-fold cross-validation strategy.

Training & Test data	Training data Optimum (u,d)	Training data Risk error $F(u,d)$	Test data Optimum (u,d)	Test data Risk error $F(u,d)$	Total risk error function
$\{V \setminus V_1\}, V_1$	(0.2,-0.4)	9.5462	(0.2,-0.5)	0.0842	9.6304
$\{V \setminus V_2\}, V_2$	(0.2,-0.3)	11.1223	(0.2,-0.1)	0.1928	11.3151
$\{V \setminus V_3\}, V_3$	(0.2,-0.3)	8.0269	(0.3,-0.4)	0.2042	8.2311
$\{V \setminus V_4\}, V_4$	(0.2,-0.6)	9.2466	(0.4,-0.7)	0.0629	9.3095

We observe that the risk errors are small for the test data but are larger for the training data. Thus, the total risk errors are significant. Similar numerical calculations have also been performed under the Bose-Einstein assumption.

5.3. *Data analysis via the trinomial model*

The spectrum of $H_2^{\otimes N}$ is given by

$$\{\lambda\} = \{e^{(N-i-j)u+id}, 0 \le j \le N, 0 \le i \le N-j\}. \tag{64}$$

Thus, for the case $N = 2$, the set of observables is exactly the same as (55). However, the associated probabilities differ from (56) and are equal to

$$\{p_\lambda\} = \{p_2^2, 2p_2p_3, 2p_1p_2, 2p_1p_3, p_3^2, p_1^2\}. \tag{65}$$

Likelihood and risk functions. Using the Maxwell-Boltzmann statistics, the likelihood is defined by the probabilities p_1, p_2, p_3 as

$$L(p_1,p_2,p_3) = (p_2^2)^{\#0}(2p_2p_3)^{\#d}(2p_1p_2)^{\#u}(2p_1p_3)^{\#d+u}(p_3^2)^{\#2d}(p_1^2)^{\#2u}. \tag{66}$$

Note that for the Cox-Ross-Rubinstein model, $p_2 = 0$ so that the likelihood is simplified to $L(p_1,p_3) = (2p_1p_3)^{\#d+u}(p_3^2)^{\#2d}(p_1^2)^{\#2u}$.

The corresponding risk function is then defined by

$$\begin{aligned} F_3(u,d) = {} & p_2^2 \sum_{\beta_i \in G_0} |\beta_i - S_0| + 2p_2p_3 \sum_{\beta_i \in G_d} |\beta_i - S_0e^d| + 2p_1p_2 \sum_{\beta_i \in G_u} |\beta_i - S_0e^u| \\ & + 2p_1p_3 \sum_{\beta_i \in G_{d+u}} |\beta_i - S_0e^{u+d}| + p_3^2 \sum_{\beta_i \in G_{2d}} |\beta_i - S_0e^{2d}| + p_1^2 \sum_{\beta_i \in G_{2u}} |\beta_i - S_0e^{2u}|. \end{aligned} \tag{67}$$

Numerical illustration. We process the data (54) again with $S_0 = 70$, $(u_0,d_0) = (0.8,-0.1)$ and $M = 0.000001$. The results obtained using the algorithm of Section 5.1 with the functions (66), (67) are given in Table 3.

Table 3. Estimation for $N = 2$ in the Maxwell-Boltzmann case.

Given (u,d)	Maximum likelihood	Optimum (p_1, p_2, p_3)	Optimum (u,d)	Risk error $F(u,d)$	$\lvert F(u_i,d_i) - F(u_{i+1},d_{i+1})\rvert$
(0.8,-0.1)	7.3122e-10	(0.01,0.22,0.77)	(0.5,-0.4)	0.0321	0.0321
(0.5,-0.4)	2.2847e-11	(0.1,0.18,0.72)	(0.8,-0.5)	0.3296	0.2975
(0.8,-0.5)	7.4285e-12	(0.1,0.23,0.67)	(0.8,-0.5)	0.3302	0.0006
(0.8,-0.5)	7.4285e-12	(0.1,0.23,0.67)	(0.8,-0.5)	0.3302	0

Then, we apply a 4-fold cross-validation procedure as previously done. The obtained results are shown in Table 4. Note that this method allows to reduce the risk errors.

Table 4. Using a 4-fold cross-validation strategy.

	Training data		Test data		
Training & Test data	Optimum (u,d)	Risk error $F(u,d)$	Optimum (u,d)	Risk error $F(u,d)$	Total risk error function
$\{V \setminus V_1\}, V_1$	(0.3,-0.1)	0.1093	(0.3,-0.2)	0.0860	0.1953
$\{V \setminus V_2\}, V_2$	(0.3,-0.1)	0.1843	(0.7,-0.6)	0.0057	0.1900
$\{V \setminus V_3\}, V_3$	(0.8,-0.5)	0.1591	(0.8,-0.6)	0.0138	0.1729
$\{V \setminus V_4\}, V_4$	(0.7,-0.4)	0.2575	(0.8,-0.6)	0.0721	0.3296

The observable operators of the quantum actuarial-type and trinomial models give us different results as expected. Which model to choose? One possible approach might be to consider a mixture of quantum models via the mixture of Hamiltonians (see, e.g., Wittek (2014)).

6. Conclusion

Several quantum type financial models are constructed that benefit from the physical interpretation of the unpredictable stock market behaviour and associated dependences. The models provide a general physical type framework for pricing of derivatives and a possibility to construct quantum trading strategies. Moreover, it is revealed that certain quantum type models are applied both in actuarial and financial sciences.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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