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*Research article*

# A nonlinear optimal control approach to stabilization of a macroeconomic development model

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**Abstract:** A nonlinear optimal (H-infinity) control approach is proposed for the problem of stabilization of the dynamics of a macroeconomic development model that is known as the Grossman-Helpman model of endogenous product cycles. The dynamics of the macroeconomic development model is divided in two parts. The first one describes economic activities in a developed country and the second part describes variation of economic activities in a country under development which tries to modify its production so as to serve the needs of the developed country. The article shows that through control of the macroeconomic model of the developed country, one can finally control the dynamics of the economy in the country under development. The control method through which this is achieved is the nonlinear H-infinity control. The macroeconomic model for the country under development undergoes approximate linearization round a temporary operating point. This is defined at each time instant by the present value of the system's state vector and the last value of the control input vector that was exerted on it. The linearization is based on Taylor series expansion and the computation of the associated Jacobian matrices. For the linearized model an H-infinity feedback controller is computed. The controller's gain is calculated by solving an algebraic Riccati equation at each iteration of the control method. The asymptotic stability of the control approach is proven through Lyapunov analysis. This assures that the state variables of the macroeconomic model of the country under development will finally converge to the designated reference values.

**Keywords:** macroeconomic development models; Grossman-Helpman model; endogenous growth; nonlinear optimal control; H-infinity control; approximate linearization; Jacobian matrices; Riccati equation; asymptotic stability
**JEL classification numbers**: C61, E00

## 1. Introduction

It is necessary to move progressively from ad-hoc methods and empirical knowledge about the management of financial systems, into methods of proven performance that assure that such systems will behave in accordance to given specifications (Rigatos, 2017; Platen and Health, 2006; Harvey and Koopman, 2009). Stability and stabilization in financial systems and particularly in macroeconomic models remains a primary objective of research in financial engineering (Barnett and He, 1999, 2001a, b, 2002, 2008; Barnett and Duzhak, 2008; Zhang, 2005). With the use of systems theory approaches and optimization methods it is possible to modify the dynamics of financial systems. Actually, one can compute exogenous inputs that steer the financial system to a desirable final state (Barnett and He, 1998; Blueschke et al., 2013; Blueshke-Nikolaeva et al., 2012).

The present article demonstrates that it is possible to achieve control and stabilization of macroeconomic development models, such as the Grossman-Helpman model (Mondal, 2008; Mondal and Gupta, 2009; Barnett and Ghosh, 2013, 2014; Guarini, 2011). This is achieved through the application of exogenous control inputs. The Grossman-Helpman model considers two interacting business entities. The first one describes economic activities in a developed country and the second part describes variation of economic activities in a country under development which tries to modify its production so as to serve the needs of the developed country (Sasaki et al., 2013; Shimizu et al., 2009; Baldwin and Robert-Nicoud, 2008; Hirose and Yamamoto, 2007; Baldwin et al., 2005). The article shows that through control of the macroeconomic model of the developed country, one can finally control the dynamics of the economy in the country under development.

The macroeconomic model describes the wide-gap case of the Grossman-Helpman model, that is the case in which the wages rate at the peripheral country (South) is smaller than the wages rate of the developed country (North) (Mondal, 2008; Mondal and Gupta, 2009). The state vector of this model comprises three state variables.The first state variable signifies the rate of change of the number of products developed by the two countries (North and South) divided by the total number of products. The second state variable signifies the number of products developed in North divided by the total number of products developed by the two countries. The third state variable signifies the rate of change of the products developed in South over the number of products developed in North.

To solve the control and stabilization problem for this macroeconomic system, the Grossman-Helpman undergoes linearization around a local operating point (equilibrium) which are redefined at each iteration of the control algorithm (Rigatos and Siano, 2015; Rigatos et al., 2015). The equilibrium consists of the present value of the development model state vector and the last value of the control input vector that was exerted on it. The linearization procedure is based on Taylor series expansion and on the computation of the Jacobian matrices of the macroeconomic model (Rigatos and Tzafestas, 2007; Basseville and Nikiforov, 1993; Toussaint et al., 2000; Rigatos and Zhang, 2009). The linearization error due to truncation of higher order terms in the Taylor series expansion is considered to be a perturbation which is compensated by the robustness of the control algorithm.

For the approximately linearized macroeconomic model an H-infinity feedback controller is designed. The H-infinity controller provides solution to the optimal control problem for the considered development model under uncertainty and external disturbances. It also represents the solution to a min-max differential game in which the control inputs they to minimize a quadratic cost functional associated with the state vector error of the macroeconomic model, while the perturbation inputs try to maximize it. The computation of the H-infinity controller's gain relies on the repetitive solution of an algebraic Riccati equation, taking place at each iteration of the control algorithm (Rigatos, 2011, 2013, 2015, 2017). The stability features of the control method are confirmed through Lyapunov analysis. Under moderate conditions it is shown that the macroeconomic model is globally asymptotically stable. This assures that if suitable control is exerted in the economy of the North then the state variables of the macroeconomic model of the South will finally converge to the designated reference values.

The structure of the article is as follows: in Section 2 the dynamics of the Grossman-Helpman model is analyzed. In Section 3 an approximate linearization of the macroeconomic model is performed using Taylor series expansion and the computation of Jacobian matrices. In Section 4 the H-infinity feedback control problem for the macroeconomic model is formulated and the H-infinity feedback controller is computed through the repetitive solution of an algebraic Riccari equation. In Section 5 the stability properties of the H-infinity control loop are analyzed with the use of the Lyapunov method. Global asymptotic stability is finally proven. In Section 6 the performance of the H-infinity control scheme in the stabilization and control of the macroeconomic model is further demonstrated through simulation experiments. Finally, in Section 7 concluding remarks are stated.

## 2. Dynamics of the Grossman-Helpman model

The Grossman-Helpman model in the wide-gap case, that is when the wages rate at the peripheral country (South) remains lower than the wages rate at the developed country (North), is described by the following set of differentia; equations (Mondal, 2008)

$$\dot{g} = (\tfrac{L_N}{a_N} - g)[\tilde{\rho} + m + g - \tfrac{1-a}{a}(\tfrac{L_N}{a_N} - g)\tfrac{1}{\xi}] \tag{1}$$

$$\dot{\xi} = g - (g + m)\xi \tag{2}$$

$$\dot{m} = \tfrac{1-\xi}{\xi}(\tfrac{L_s}{a_s} - m\tfrac{\xi}{1-\xi})[\tilde{\rho} + m\tfrac{\xi}{1-\xi} - \tfrac{1-a}{a}(\tfrac{L_s}{a_s}] - m\tfrac{\xi}{1-\xi}) - \tfrac{m}{\xi(1-\xi)}[g - (g + m)\xi] \tag{3}$$

Next, by defining the state variables $x_1 = g = \frac{\dot{n}}{n}$ which signifies the rate of change of the number of products developed by the two countries (North and South) divided by the total number of products, $x_2 = \xi = \frac{n_N}{n}$ which signifies the number of products developed in North divided by the total number of products developed by the two countries and $x_3 = m = \frac{\dot{n}_s}{n_N}$ which signifies the rate of change of the products developed in South over the number of products developed in North. By considering an exogenous control input $u$ (control policy in North) which is included in the first row of the state-space model

$$\dot{x}_1 = (\tfrac{L_N}{a_N} - x_1)[\tilde{\rho} + x_3 + x_1 - \tfrac{1-a}{a}(\tfrac{L_N}{a_N} - x_1)\tfrac{1}{x_2}] + u \tag{4}$$

$$\dot{x}_2 = x_1 - (x_1 + x_3)x_2 \tag{5}$$

$$\begin{aligned}\dot{x}_3 = \tfrac{1-x_2}{x_2}(\tfrac{L_s}{a_s} - x_3\tfrac{x_2}{1-x_2})[\tilde{\rho} + x_3\tfrac{x_2}{1-x_2} - \tfrac{1-a}{a}(\tfrac{L_s}{a_s} - x_3\tfrac{x_2}{1-x_2})]- \\ -\tfrac{x_3}{x_2(1-x_2)}[x_1 - (x_1 + x_3)x_2]\end{aligned} \tag{6}$$

The rest of the parameters appearing in the macroeconomic model of Equation (4) to Equation (6) are defined as follows (Mondal, 2008): $L_N$ is the level of employment in North, $a_N$ is the labor requirement per unit in the R&D sector in North, $L_s$ is the level of employement in South, $a_s$ is the labor requirement per unit in the R&D sector in South, $a$ is parameter related to the elasticity of demand $a = \frac{\epsilon-1}{\epsilon}$, and $\tilde{\rho}$ is the rate of time preference in North which can be interpreted as the consistency to time deadlines in North for accomplishing payments and investments.

Next, Equation (6) and function $q(x_2) = \frac{1-x_2}{x_2}$ is defined. Thus, Equation (6)is written as

$$\dot{x}_3 = \frac{1}{q(x_2)}(\frac{L_s}{a_s} - x_3q(x_2))[\tilde{\rho} + x_3q(x_2) - \frac{(1-a)}{a}(\frac{L_s}{a_s} - x_3q(x_2))] - \frac{x_3^2}{1-x_2} + \frac{x3}{x_2}x_1 \tag{7}$$

or equivalently

$$\dot{x}_3 = (\frac{L_s}{a_s}\frac{1}{q(x_2)} - x_3)[\tilde{\rho} + x_3q(x_2) - \frac{(1-a)}{a}(\frac{L_s}{a_s} - x_3q(x_2))] - \frac{x_3^2}{1-x_2} + \frac{x3}{x_2}x_1 \tag{8}$$

By differentiating Equation (10) once more in time one obtains

$$\begin{aligned}\ddot{x}_3 = (\frac{L_s}{a_s} - \frac{q^{'}(x_2)\dot{x}_2}{q(x_2)} - \dot{x}_3)[\tilde{\rho} + x_3q(x_2) - (\frac{1-a}{a})(\frac{L_s}{a_s} - x_3q(x_2)]+ \\ (\frac{L_s}{a_s}\frac{1}{q(x_2)} - x_2)[\dot{x}_3q(x_2) + x_3q^{'}(x_2)\dot{x}_2 + \frac{(1-a}{a})\dot{x}_3q(x_2) - (\frac{1-a}{a})x_3q^{'}(x_2)\dot{x}_2]- \\ -\frac{\dot{x}_2x_3^2}{(1-x_2)^2} - \frac{2x_3\dot{x}_3}{(1-x_2)} + (-\frac{\dot{x}_2}{x_2^2}x_3 + \frac{1}{x_2}\dot{x}_3)x_1 + (\frac{x_3}{x_2})\dot{x}_1\end{aligned} \tag{9}$$

where $q^{'} = \frac{dq(x_2)}{dx_2}$. By substituting the time derivatives $\dot{x}_1 = f_1(x) + g_1(x)u$, $\dot{x}_2 = f_2(x)$ and $\dot{x}_3 = f_3(x)$, from Equation (4) to Equation (6), into Equation (10) one gets

$$\begin{aligned}\ddot{x}_3 = (\frac{L_s}{a_s} - \frac{q^{'}(x_2)f_2(x)}{q(x_2)} - f_3(x))[\tilde{\rho} + x_3q(x_2) - (\frac{1-a}{a})(\frac{L_s}{a_s} - x_3q(x_2)]+ \\ (\frac{L_s}{a_s}\frac{1}{q(x_2)} - x_2)[f_3(x)q(x_2) + x_3q^{'}(x_2)f_2(x) + \frac{(1-a}{a})f_3(x)q(x_2) - (\frac{1-a}{a})x_3q^{'}(x_2)f_2(x)]- \\ -\frac{f_2(x)x_3^2}{(1-x_2)^2} - \frac{2x_3f_3(x)}{(1-x_2)} + (-\frac{f_2(x)}{x_2^2}x_3 + \frac{1}{x_2}f_3(x))x_1 + (\frac{x_3}{x_2})[f_1(x) + g_1(x)u]\end{aligned} \tag{10}$$

By grouping terms, the previous relation is written as

$$\ddot{x}_3 = -\frac{f_2(x)x_3^2}{(1-x_2)^2} + v(x) \tag{11}$$

where the transformed control input $v(x)$ is given by

$$\begin{aligned}v(x) = (\frac{L_s}{a_s} - \frac{q^{'}(x_2)f_2(x)}{q(x_2)} - f_3(x))[\tilde{\rho} + x_3q(x_2) - (\frac{1-a}{a})(\frac{L_s}{a_s} - x_3q(x_2)]+ \\ (\frac{L_s}{a_s}\frac{1}{q(x_2)} - x_2)[f_3(x)q(x_2) + x_3q^{'}(x_2)f_2(x) + \frac{(1-a}{a})f_3(x)q(x_2) - (\frac{1-a}{a})x_3q^{'}(x_2)f_2(x)]- \\ -\frac{2x_3f_3(x)}{(1-x_2)} + (-\frac{f_2(x)}{x_2^2}x_3 + \frac{1}{x_2}f_3(x))x_1 + (\frac{x_3}{x_2})[f_1(x) + g_1(x)u]\end{aligned} \tag{12}$$

By substituting the relation describing $f_2(x)$ in Equation (11) one obtains

$$\ddot{x}_3 = -\frac{(x_1 - x_1x_2 - x_2x_3)x_3^2}{(1-x_2)^2} + v \tag{13}$$

The previous relation can be also written as

$$\ddot{x}_3 = -\frac{x_1}{(1-x_1)}x_3^2 + \frac{x_2}{(1-x_2)^2}x_3^3 + v \tag{14}$$

which equivalently can be written as

$$\ddot{x}_3 = f_a(x)x_3^2 + f_b(x)x_3^3 + v \tag{15}$$

with $f_a(x) = -\frac{x_1}{(1-x_1)}$ and $f_b(x) = \frac{x_2}{(1-x_2)^2}$. Next, by defining $F(x_3) = f_a(x)x_3^2 + f_b(x)x_3^3$ and $G(x_3) = 1$ the macroeconomic development model is written as

$$\ddot{x}_3 = F(x_3) + G(x)v \tag{16}$$

Next, by defining the new state vector $x = [x_3, \dot{x}_3]$ one obtains the state-space description

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}x_2\\ f_a x_1^2 + f_b x_1^3\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix} v \tag{17}$$

where using the vector fields $f(x)\in R^{2\times 1}$ and $g(x)\in R^{2\times 1}$ one has

$$\dot{x} = f(x) + g(x)v \tag{18}$$

## 3. Approximate linearization of the macroeconomic development model

The approximately linearized macroeconomic model is given by

$$\dot{x} = Ax + Bu + \tilde{d} \tag{19}$$

where $\tilde{d}$ is the modelling error due to truncation of higher order terms in the Taylor series expansion, while matrices $A$ and $B$ are given by

$$\begin{gathered} A = \nabla_x[f(x) + g(x)u]_{|(x^*,u^*)} = [\nabla_x f(x)]_{|(x^*,u^*)} + [\nabla_x g(x)u]_{|(x^*,u^*)} \\ B = \nabla_u[f(x) + g(x)u]_{|(x^*,u^*)} = g(x)_{|(x^*,u^*)} \end{gathered} \tag{20}$$

As noted above, the Jacobians of the state-space model of the system are computed using:

$$\nabla_x f(x) = \begin{pmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2}\\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}\end{pmatrix} \quad \nabla_x g(x) = \begin{pmatrix}\frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2}\\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2}\end{pmatrix} \tag{21}$$

According to the above, for the state-space description of the system given in Equation (18) the linearization procedure through Taylor series expansion leads into the Jacobian matrices

$$A = \nabla_x f = \begin{pmatrix}0 & 1\\ 2f_a x_1 + 3f_b x_1^2 & 0\end{pmatrix} \quad B = g(x) = \begin{pmatrix}0\\ 1\end{pmatrix} \tag{22}$$

## 4. Design of an H-infinity nonlinear feedback controller

### *4.1. Equivalent linearized dynamics of the macroeconomic model*

After linearization round its current operating point, the macroeconomic model is written as

$$\dot{x} = Ax + Bu + d_1 \tag{23}$$

Parameter $d_1$ stands for the linearization error in the macroeconomic model appearing in Equation (23). The reference setpoints for the macroeconomic model's state vector are denoted by $x_d = [x_1^d, x_2^d]$. Tracking of this trajectory is succeeded after applying the control input $u^*$. At every time instant the control input $u^*$ is assumed to differ from the control input $u$ appearing in Equation (23) by an amount equal to $\Delta u$, that is $u^* = u + \Delta u$

$$\dot{x}_d = Ax_d + Bu^* + d_2 \tag{24}$$

The dynamics of the controlled system described in Equation (23) can be also written as

$$\dot{x} = Ax + Bu + Bu^* - Bu^* + d_1 \tag{25}$$

and by denoting $d_3 = -Bu^* + d_1$ as an aggregate disturbance term one obtains

$$\dot{x} = Ax + Bu + Bu^* + d_3 \tag{26}$$

By subtracting Equation (24) from Equation (26) one has

$$\dot{x} - \dot{x}_d = A(x - x_d) + Bu + d_3 - d_2 \tag{27}$$

By denoting the tracking error as $e = x - x_d$ and the aggregate disturbance term as $\tilde{d} = d_3 - d_2$, the tracking error dynamics becomes

$$\dot{e} = Ae + Bu + \tilde{d} \tag{28}$$

The above linearized form of the macroeconomic model can be efficiently controlled after applying an H-infinity feedback control scheme.

*4.2. The nonlinear H-infinity control*

The initial nonlinear model of the macroeconomic model is in the form

$$\dot{x} = \tilde{f}(x, u) \ \ x \in R^n, \ u \in R^m \tag{29}$$

Linearization of the macroeconomic model is performed at each iteration of the control algorithm round its present operating point $(x^*, u^*) = (x(t), u(t - T_s))$, where $T_s$ is the sampling period. The linearized equivalent model of the system is described by

$$\dot{x} = Ax + Bu + L\tilde{d} \ \ x \in R^n, \ u \in R^m, \ \tilde{d} \in R^q \tag{30}$$

where matrices $A$ and $B$ are obtained from the computation of the Jacobians

$$A = \begin{pmatrix} \frac{\partial \tilde{f}_1}{\partial x_1} & \frac{\partial \tilde{f}_1}{\partial x_2} & \cdots & \frac{\partial \tilde{f}_1}{\partial x_n} \\ \frac{\partial \tilde{f}_2}{\partial x_1} & \frac{\partial \tilde{f}_2}{\partial x_2} & \cdots & \frac{\partial \tilde{f}_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \tilde{f}_n}{\partial x_1} & \frac{\partial \tilde{f}_n}{\partial x_2} & \cdots & \frac{\partial \tilde{f}_n}{\partial x_n} \end{pmatrix} |_{(x^*, u^*)} \quad B = \begin{pmatrix} \frac{\partial \tilde{f}_1}{\partial u_1} & \frac{\partial \tilde{f}_1}{\partial u_2} & \cdots & \frac{\partial \tilde{f}_1}{\partial u_m} \\ \frac{\partial \tilde{f}_2}{\partial u_1} & \frac{\partial \tilde{f}_2}{\partial u_2} & \cdots & \frac{\partial \tilde{f}_2}{\partial u_m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \tilde{f}_n}{\partial u_1} & \frac{\partial \tilde{f}_n}{\partial u_2} & \cdots & \frac{\partial \tilde{f}_n}{\partial u_m} \end{pmatrix} |_{(x^*, u^*)} \tag{31}$$

and vector $\tilde{d}$ denotes disturbance terms due to linearization errors. The problem of disturbance rejection for the linearized model that is described by

$$\dot{x} = Ax + Bu + L\tilde{d}$$
$$y = Cx \quad (32)$$

where $x \in R^n$, $u \in R^m$, $\tilde{d} \in R^q$ and $y \in R^p$, cannot be handled efficiently if the classical LQR control scheme is applied. This is because of the existence of the perturbation term $\tilde{d}$. The disturbance term $\tilde{d}$ apart from modeling (parametric) uncertainty and external perturbation terms can also represent noise terms of any distribution.

In the $H_\infty$ control approach, a feedback control scheme is designed for trajectory tracking by the system's state vector and simultaneous disturbance rejection, considering that the disturbance affects the system in the worst possible manner. The disturbances' effects are incorporated in the following quadratic cost function:

$$J(t) = \frac{1}{2}\int_0^T [y^T(t)y(t) + ru^T(t)u(t) - \rho^2\tilde{d}^T(t)\tilde{d}(t)]dt, \quad r, \rho > 0 \quad (33)$$

The significance of the negative sign in the cost function's term that is associated with the perturbation variable $\tilde{d}(t)$ is that the disturbance tries to maximize the cost function $J(t)$ while the control signal $u(t)$ tries to minimize it. The physical meaning of the relation given above is that the control signal and the disturbances compete to each other within a min-max differential game. This problem of min-max optimization can be written as

$$min_u max_{\tilde{d}} J(u, \tilde{d}) \quad (34)$$

The objective of the optimization procedure is to compute a control signal $u(t)$ which can compensate for the worst possible disturbance, that is externally imposed to the system. However, the solution to the min-max optimization problem is directly related to the value of the parameter $\rho$. This means that there is an upper bound in the disturbances magnitude that can be annihilated by the control signal.

### *4.3. Computation of the feedback control gains*

For the linearized system given by Equation (32) the cost function of Equation (33) is defined, where the coefficient $r$ determines the penalization of the control input and the weight coefficient $\rho$ determines the reward of the disturbances' effects. It is assumed that (i) The energy that is transferred from the disturbances signal $\tilde{d}(t)$ is bounded, that is $\int_0^\infty \tilde{d}^T(t)\tilde{d}(t)dt < \infty$, (ii) the matrices $[A, B]$ and $[A, L]$ are stabilizable, (iii) the matrix $[A, C]$ is detectable. Then, the optimal feedback control law is given by

$$u(t) = -Kx(t) \quad (35)$$

with $K = \frac{1}{r}B^T P$, where $P$ is a positive semi-definite symmetric matrix which is obtained from the solution of the Riccati equation

$$A^T P + PA + Q - P(\frac{1}{r}BB^T - \frac{1}{2\rho^2}LL^T)P = 0 \quad (36)$$

where $Q$ is also a positive definite symmetric matrix. The worst case disturbance is given by

$$\tilde{d}(t) = \frac{1}{\rho^2} L^T P x(t) \tag{37}$$

The diagram of the considered control loop is depicted in Figure 1.

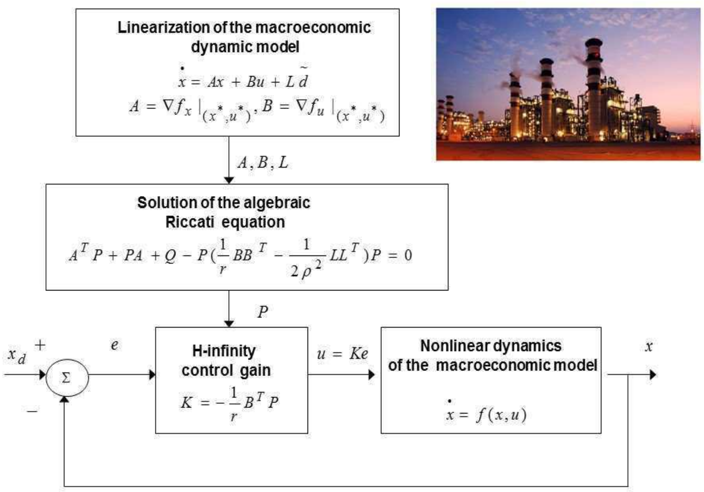



**Figure 1.** Diagram of the control scheme for the macroeconomic model.

### *4.4. The role of Riccati equation coefficients in $H_\infty$ control robustness*

The parameter $\rho$ in Equation (33), is an indication of the closed-loop system robustness. If the values of $\rho > 0$ are excessively decreased with respect to $r$, then the solution of the Riccati equation is no longer a positive definite matrix. Consequently there is a lower bound $\rho_{min}$ of $\rho$ for which the $H_\infty$ control problem has a solution. The acceptable values of $\rho$ lie in the interval $[\rho_{min}, \infty)$. If $\rho_{min}$ is found and used in the design of the $H_\infty$ controller, then the closed-loop system will have elevated robustness. Unlike this, if a value $\rho > \rho_{min}$ is used, then an admissible stabilizing $H_\infty$ controller will be derived but it will be a suboptimal one. The Hamiltonian matrix

$$H = \begin{pmatrix} A & -(\frac{1}{r}BB^T - \frac{1}{\rho^2}LL^T) \\ -Q & -A^T \end{pmatrix} \tag{38}$$

provides a criterion for the existence of a solution of the Riccati equation Equation (36). A necessary condition for the solution of the algebraic Riccati equation to be a positive semi-definite symmetric matrix is that $H$ has no imaginary eigenvalues (Rigatos, 2011).

## 5. Lyapunov stability analysis

Through Lyapunov stability analysis it will be shown that the proposed nonlinear control scheme assures $H_\infty$ tracking performance for the macroeconomic model, and that in case of bounded disturbance terms asymptotic convergence to the reference setpoints is achieved. The tracking error dynamics for the macroeconomic model is written in the form

$$\dot{e} = Ae + Bu + L\tilde{d} \tag{39}$$

where in the the tracking error dynamics for the macroeconomic model's case $L = I \in R^2$ with $I$ being the identity matrix. Variable $\tilde{d}$ denotes model uncertainties and external disturbances of the financial system's model. The following Lyapunov equation is considered

$$V = \frac{1}{2}e^T Pe \tag{40}$$

where $e = x - x_d$ is the tracking error. By differentiating with respect to time one obtains

$$\begin{gathered} \dot{V} = \frac{1}{2}\dot{e}^T Pe + \frac{1}{2}eP\dot{e} \Rightarrow \\ \dot{V} = \frac{1}{2}[Ae + Bu + L\tilde{d}]^T Pe + \frac{1}{2}e^T P[Ae + Bu + L\tilde{d}] \Rightarrow \end{gathered} \tag{41}$$

$$\begin{gathered} \dot{V} = \frac{1}{2}[e^T A^T + u^T B^T + \tilde{d}^T L^T]Pe+ \\ +\frac{1}{2}e^T P[Ae + Bu + L\tilde{d}] \Rightarrow \end{gathered} \tag{42}$$

$$\begin{gathered} \dot{V} = \frac{1}{2}e^T A^T Pe + \frac{1}{2}u^T B^T Pe + \frac{1}{2}\tilde{d}^T L^T Pe+ \\ \frac{1}{2}e^T PAe + \frac{1}{2}e^T PBu + \frac{1}{2}e^T PL\tilde{d} \end{gathered} \tag{43}$$

The previous equation is rewritten as

$$\begin{gathered} \dot{V} = \frac{1}{2}e^T(A^T P + PA)e + (\frac{1}{2}u^T B^T Pe + \frac{1}{2}e^T PBu)+ \\ +(\frac{1}{2}\tilde{d}^T L^T Pe + \frac{1}{2}e^T PL\tilde{d}) \end{gathered} \tag{44}$$

*Assumption*: For given positive definite matrix $Q$ and coefficients $r$ and $\rho$ there exists a positive definite matrix $P$, which is the solution of the following matrix equation

$$A^T P + PA = -Q + P(\frac{2}{r}BB^T - \frac{1}{\rho^2}LL^T)P \tag{45}$$

Moreover, the following feedback control law is applied to the system

$$u = -\frac{1}{r}B^T Pe \tag{46}$$

By substituting Equation (45) and Equation (46) one obtains

$$\begin{gathered} \dot{V} = \frac{1}{2}e^T[-Q + P(\frac{2}{r}BB^T - \frac{1}{\rho^2}LL^T)P]e+ \\ +e^T PB(-\frac{1}{r}B^T Pe) + e^T PL\tilde{d} \Rightarrow \end{gathered} \tag{47}$$

$$\begin{gathered} \dot{V} = -\frac{1}{2}e^T Qe + \frac{1}{r}e^T PBB^T Pe - \frac{1}{2\rho^2}e^T PLL^T Pe \\ -\frac{1}{r}e^T PBB^T Pe + e^T PL\tilde{d} \end{gathered} \tag{48}$$

which after intermediate operations gives

$$\dot{V} = -\frac{1}{2}e^T Qe - \frac{1}{2\rho^2}e^T PLL^T Pe + e^T PL\tilde{d} \tag{49}$$

or, equivalently

$$\begin{array}{c}\dot{V} = -\frac{1}{2}e^T Qe - \frac{1}{2\rho^2}e^T PLL^T Pe+ \\ +\frac{1}{2}e^T PL\tilde{d} + \frac{1}{2}\tilde{d}^T L^T Pe\end{array} \tag{50}$$

*Lemma*: The following inequality holds

$$\frac{1}{2}e^T PL\tilde{d} + \frac{1}{2}\tilde{d}L^T Pe - \frac{1}{2\rho^2}e^T PLL^T Pe \leq \frac{1}{2}\rho^2\tilde{d}^T\tilde{d} \tag{51}$$

*Proof*: The binomial $(\rho\alpha - \frac{1}{\rho}b)^2$ is considered. Expanding the left part of the above inequality one gets

$$\begin{array}{c}\rho^2 a^2 + \frac{1}{\rho^2}b^2 - 2ab \geq 0 \Rightarrow \frac{1}{2}\rho^2 a^2 + \frac{1}{2\rho^2}b^2 - ab \geq 0 \Rightarrow \\ ab - \frac{1}{2\rho^2}b^2 \leq \frac{1}{2}\rho^2 a^2 \Rightarrow \frac{1}{2}ab + \frac{1}{2}ab - \frac{1}{2\rho^2}b^2 \leq \frac{1}{2}\rho^2 a^2\end{array} \tag{52}$$

The following substitutions are carried out: $a = \tilde{d}$ and $b = e^T PL$ and the previous relation becomes

$$\frac{1}{2}\tilde{d}^T L^T Pe + \frac{1}{2}e^T PL\tilde{d} - \frac{1}{2\rho^2}e^T PLL^T Pe \leq \frac{1}{2}\rho^2\tilde{d}^T\tilde{d} \tag{53}$$

Equation (53) is substituted in Equation (54) and the inequality is enforced, thus giving

$$\dot{V} \leq -\frac{1}{2}e^T Qe + \frac{1}{2}\rho^2\tilde{d}^T\tilde{d} \tag{54}$$

Equation (54) shows that the $H_\infty$ tracking performance criterion is satisfied. The integration of $\dot{V}$ from 0 to $T$ gives

$$\begin{array}{c}\int_0^T \dot{V}(t)dt \leq -\frac{1}{2}\int_0^T ||e||_Q^2 dt + \frac{1}{2}\rho^2\int_0^T ||\tilde{d}||^2 dt \Rightarrow \\ 2V(T) + \int_0^T ||e||_Q^2 dt \leq 2V(0) + \rho^2\int_0^T ||\tilde{d}||^2 dt\end{array} \tag{55}$$

Moreover, if there exists a positive constant $M_d > 0$ such that

$$\int_0^\infty ||\tilde{d}||^2 dt \leq M_d \tag{56}$$

then one gets

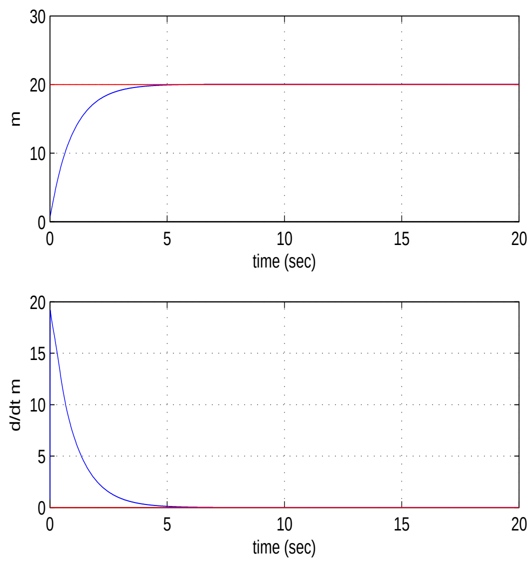
$$\int_0^\infty ||e||_Q^2 dt \leq 2V(0) + \rho^2 M_d \tag{57}$$

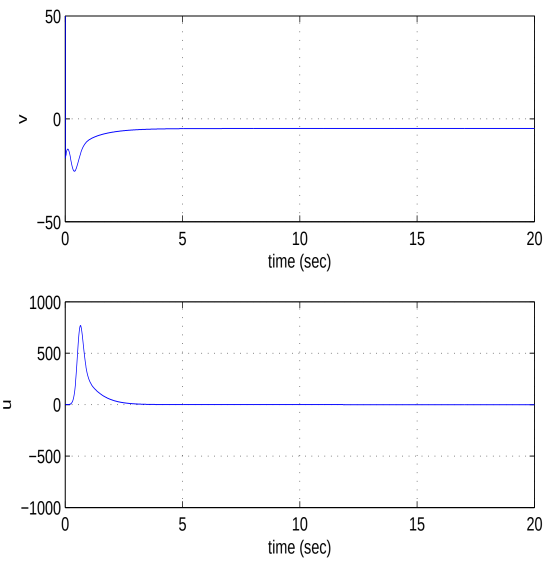
Thus, the integral $\int_0^\infty ||e||_Q^2 dt$ is bounded. Moreover, $V(T)$ is bounded and from the definition of the Lyapunov function $V$ in Equation (40) it becomes clear that $e(t)$ will be also bounded since $e(t) \in \Omega_e = \{e|e^T Pe \leq 2V(0) + \rho^2 M_d\}$. According to the above and with the use of Barbalat's Lemma one obtains $lim_{t\to\infty}e(t) = 0$.

Elaborating on the above, it can be noted that the proof of global asymptotic stability for the control loop of the macroeconomic model is based on Equation (54) and on the application of Barbalat's Lemma. It uses the condition of Equation (56) about the boundedness of the square of the aggregate disturbance and modelling error term $\tilde{d}$ that affects the model. However, as explained above the proof of global asymptotic stability is not restricted by this condition. By selecting the attenuation coefficient $\rho$ to be sufficiently small and in particular to satisfy $\rho^2 < ||e||_Q^2/||\tilde{d}||^2$ one has that the first derivative of the Lyapunov function is upper bounded by 0. Therefore for the i-th time interval it is proven that the Lyapunov function defined in Equation (40) is a decreasing one. This also assures the Lyapunov function of the system defined in Equation (29) will always have a negative first-order derivative.

## 6. Simulation tests

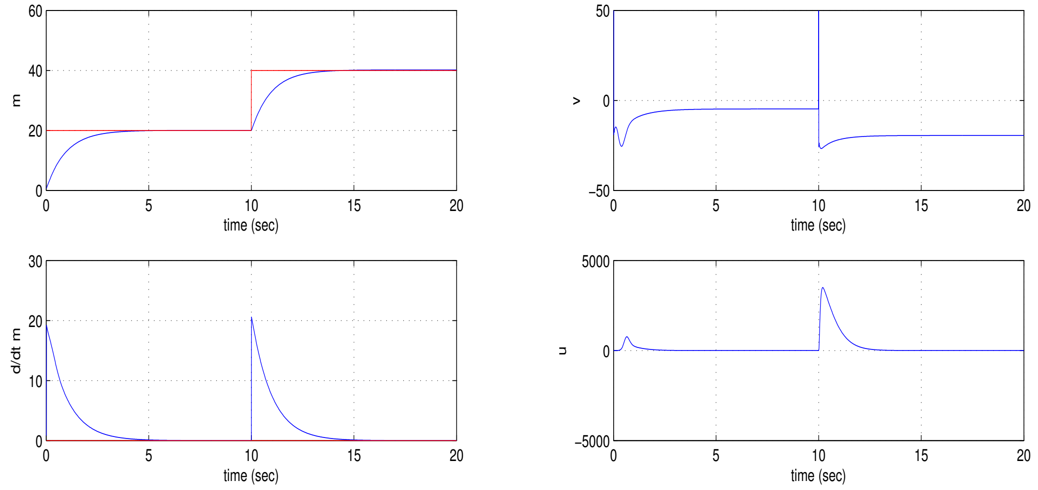
The efficiency of the proposed control scheme for stabilization of the Grossman-Helpman macroeconomic development model is further confirmed through simulation experiments. The computation of the H-infinity controller's feedback gain was repeated at each iteration of the control algorithm and was based on the solution of the algebraic Riccati equation of Equation (45). The obtained results which are depicted if Figure 2 to Figure 4 and confirm the excellent tracking performance of the control method. Actually it is shown that the elements of the state vector of the macroeconomic model converge fast to the reference setpoints and track them with high precision. The variation of the control inputs to the macroeconomic model remained smooth.



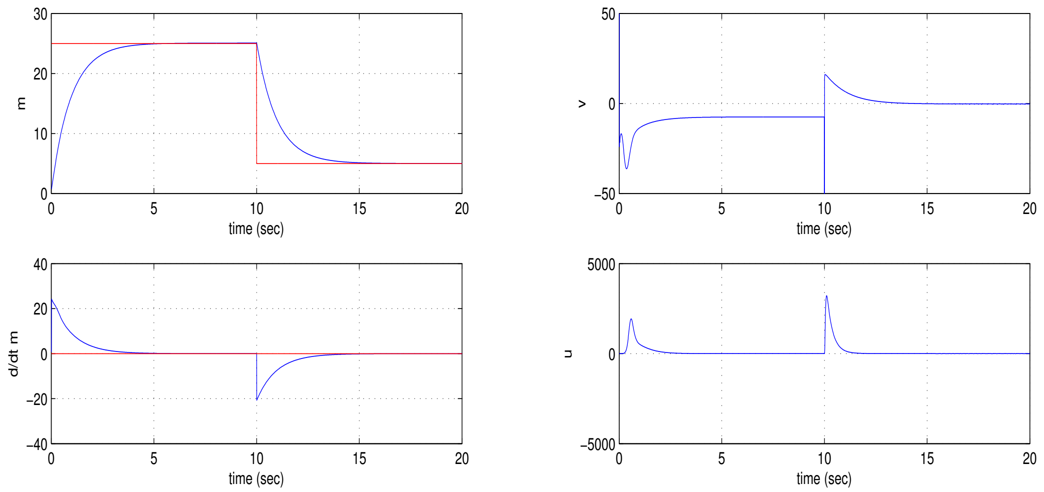

**(a)** **(b)**

**Figure 2.** Setpoint 1: (a) Tracking of the reference setpoints (red line) by state variable $x_3 = m$ (blue line) at the South area (b) Control inputs applied to the macroeconomic model at the North area.




**(a)** **(b)**

**Figure 3.** Setpoint 2: (a) Tracking of the reference setpoints (red line) by state variable $x_3 = m$ (blue line) at the South area (b) Control inputs applied to the macroeconomic model at the North area.




**(a)** **(b)**

**Figure 4.** Setpoint 3: (a) Tracking of the reference setpoints (red line) by state variable $x_3 = m$ (blue line) at the South area (b) Control inputs applied to the macroeconomic model at the North area.

The following features can be attributed to the presented nonlinear H-infinity control scheme (i) despite the strong nonlinearities of the macroeconomic model the control method has an excellent performance, (ii) the computation of the feedback control signal follows an optimal control concept and requires the solution of an algebraic Riccati equation at each iteration of the control algorithm, (iii) the approximate linearization that is induced due to Taylor series expansion round a temporary equilibrium results in modelling error that is compensated by the robustness of the control scheme.

## 7. Conclusions

The article has proposed a nonlinear optimal (H-infinity) control method for the problem of stabilization of the Grossman-Helpman macroeconomic development model. The macroeconomic model has undergone approximate linearization round a temporary equlibrium which is re-computed at each iteration of the control algorithm. The linearization procedure is based on Taylor series expansion and on the computation of the model's Jacobian matrices. For the approximately linearized macroeconomic model an H-infinity feedback controller has been developed.

The feedback gain of the H-infinity controller is re-computed at each time instant through the solution of an algebraic Riccati equation. The stability features of the control scheme are analyzed with the use of the Lyapunov method. Actually, it is shown that under moderate conditions, global asymptotic stability holds. This assures that under suitable control at the finance dynamics of the developed country (North), the growth of the country under development (South) will finally converge to the designated reference values. The article's results show that the developed country, may determine the degree up to which a peripheral economy is going to grow as well as the types of sectors and activities in the peripheral country which are going to remain alive. This primarily excludes randomness from the development dynamics of peripheral economies.

## Conflict of Interest

The authors declare no conflict of interest.

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