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Research article

A new construction of probabilistic Hermite polynomials with their certain applications

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Abstract: Our aim of this paper was to introduce a new construction of probabilistic Hermite polynomials based on moment generating functions. By using this generating function, we derived several new relations and formulas among the aforementioned polynomials and other types of probabilistic special number sequences and polynomials, such as the probabilistic Stirling numbers of the second kind, probabilistic Bernoulli polynomials of higher order, probabilistic Bernstein polynomials, and probabilistic Euler polynomials of higher order. By selecting special random variables, including Poisson, Uniform, Gamma, Geometric, Exponential, and Normal random variables, we showed that the generating function of probabilistic Hermite polynomials yields distinct and unique generating functions, which lead to new relations among other types of special numbers and polynomials, as presented in the application section of this paper.

Keywords: random variable; moment generating functions; probabilistic Hermite polynomials; probabilistic Stirling numbers of the second kind; probabilistic Bernoulli polynomials of higher order; probabilistic Bernstein polynomials

1. Introduction

Let us begin with introducing the following standard notation:

$$\mathbb{N} := \{1,2,3, \ldots\},$$

and

$$\mathbb{N}_0 := \{0,1,2,3, \ldots\} = \mathbb{N} \cup \{0\}.$$

Additionally, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers.

Adell and Lekuona [1] introduced the probabilistic Stirling numbers of the second kind by defining this sequence in terms of suitably chosen random variables Y for each case. Subsequently, Adell [2] proposed a further generalization of the Stirling numbers of the second kind by incorporating complex-valued random variables satisfying certain integrability conditions. Further developments were made by Gomaa and Magar [3], who studied a new generating function for generalized Fubini-type polynomials defined with respect to a random variable. Through this generating function, they derived various identities and relations, and investigated several applications and related properties. T. Kim and D. S. Kim also gave probabilistic extensions for degenerate Bell polynomials [4], Bernoulli and Euler polynomials [5], degenerate Stirling polynomials of the second kind [6], related probabilistic special polynomial identities [7], degenerate Laguerre polynomials [8], degenerate poly-Bell polynomials [9], degenerate Bernstein polynomials [10], fully degenerate Bernoulli and degenerate Euler polynomials [11], bivariate Bell polynomials [12], poly-Bernoulli numbers [13], Dowling polynomials [14], and degenerate Hermite polynomials [15]. Their approach was based on replacing the classical exponential function in the generating functions of special numbers and polynomials with a *moment generating function (mgf)*.

The generating function of the classical Hermite polynomials is defined as follows:

$$\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = e^{2xt-t^2}. \tag{1.1}$$

By Eq (1.1), one can see that

$$H_n(x) = n! \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k}{k!\,(n-2k)!} (2x)^{n-2k},$$

where [.] is the Gauss' symbol [16]. For several special values of n, one has

$H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2$, $H_3(x) = 8x^2 - 12x$, $H_4(x) = 16x^4 - 48x^2 + 12$, $H_5(x) = 32x^5 - 160x^3 + 120x$, $H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120, \ldots$

The probabilists' Hermite polynomials can be expressed by the following generating function:

$$\sum_{n=0}^{\infty} He_n(x) \frac{t^n}{n!} = e^{xt-\frac{t^2}{2}}, \tag{1.2}$$

which are solutions of the following differential equation in case $y = He_n(x)$:

$$\left(e^{-\frac{x^2}{2}}y'\right)' + \lambda e^{-\frac{1}{2}x^2}y = 0,$$

where λ is a constant with the boundary conditions [17, 18]. Motivated by this classical setting, in the next section, we introduce the probabilistic extension of the classical Hermite polynomials.

Let Y be denoted a random variable. $\mathrm{E}[e^{tY}] < \infty$ for $|t| < r$. Then, it follows that $\mathrm{E}[Y^n] < \infty$ for all $n = 1,2,\ldots$, and

$$E[e^{tY}] = \sum_{n=0}^{\infty} E[Y^n]\frac{t^n}{n!}, \quad (|t| < r; r \in \mathbb{R}^+),$$

which is also known as a *mgf* of a random variable Y.

Let $\{Y_j\}_{j=1}^{k} = \{Y_1, Y_2, \cdots, Y_k\}$ be a sequence of mutually independent choices of Y and denoted by

$$S_k = Y_1 + Y_2 + \cdots + Y_k, \quad (k \in \mathbb{N})$$

with the initial assumption $S_0 := 0$, see [3, 19, 20].

Mathematically, the Stirling numbers of the second kind are defined by the generating function:

$$\sum_{n=k}^{\infty} S(n,k)\frac{t^n}{n!} = \frac{(e^t-1)^k}{k!}, \text{ see } [1,2]. \tag{1.3}$$

Adell and Lekuona [1] provided the probabilistic extension of this formula as:

$$\sum_{n=k}^{\infty} S_Y(n,k)\frac{t^n}{n!} := \sum_{n=k}^{\infty} \left\{ n \atop k \right\}_Y \frac{t^n}{n!} = \frac{(E[e^{Yt}]-1)^k}{k!}. \tag{1.4}$$

It is evident that the special case $Y = 1$ yields the Stirling numbers of the second kind. Based on this observation, we now present the following remark.

Remark 1. By replacing the exponential function e^t with the moment generating function $\mathrm{E}[e^{Yt}]$, one can obtain a probabilistic extension of the aforementioned special functions associated with a random variable Y.

Investigations on probabilistic versions and/or representations involving the Stirling numbers of the second kind [1, 2], type 2 Bernoulli $B_{n,Y}(x)$ and Euler polynomials $E_{n,Y}(x)$ [21] by

$$\sum_{n=0}^{\infty} B_{n,Y}(x)\frac{t^n}{n!} = \frac{t}{E\left[e^{\frac{Y}{2}t}\right] - \left[e^{-\frac{Y}{2}t}\right]}e^{xt}$$

and

$$\sum_{n=0}^{\infty} E_{n,Y}(x)\frac{t^n}{n!} = \frac{2}{E\left[e^{\frac{Y}{2}t}\right] + \left[e^{-\frac{Y}{2}t}\right]} e^{xt},$$

generalized Fubini Apostol-type polynomials [3], degenerate Bell polynomials [4] by

$$\sum_{n=0}^{\infty} \phi_{n,\lambda}^{Y}(x)\frac{t^n}{n!} = e^{x\left(E\left[e_{\lambda}^{Y}(t)\right]-1\right)},$$

Appell polynomials [22], Adomian polynomials [23], Bernstein polynomials [24] and q-Bernstein polynomials [25] by

$$\sum_{n=k}^{\infty} B_{k,n}^{Y}(x)\frac{t^n}{n!} = \frac{\left([x]_q t\right)^k}{k!}\left(E\left[e^{Yt}\right]\right)^{[1-x]_q},$$

where

$$[x]_q = \frac{q^x - 1}{q - 1} \qquad (q \neq 1)$$

and

$$\lim_{q\to 1}[x]_q = x,$$

have been studied extensively. These works have provided significant motivation for this paper.

Table 1. Generating functions for several special numbers and polynomials.

Generating functions		Polynomials	Cited references
$\sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!} = \left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt}$	(1.5)	Bernoulli polynomials of higher order α, $B_n^{(\alpha)}(x)$	[19]
$\sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!} = \frac{2}{e^t+1} e^{xt}$	(1.6)	Euler polynomials	[5, 19, 21]
$\sum_{n=0}^{\infty} F_n^{(\alpha)}(x\vert u)\frac{t^n}{n!} = \left(\frac{1-u}{e^t-u}\right)^{\alpha} e^{xt}$ $(u \in \mathbb{C} - \{1\})$	(1.7)	Frobenius-Euler polynomials of order α	[24, 25]
$\sum_{n=0}^{\infty} \phi_n(x)\frac{t^n}{n!} = e^{x(e^t-1)}$	(1.8)	Bell polynomials, $\phi_n(x)$	[7, 12, 20, 26]
$\sum_{n=k}^{\infty} \left\{ {n \atop k} \right\}\frac{t^n}{n!} = \frac{(e^t-1)^k}{k!}$	(1.9)	Stirling numbers of the second kind, $S(n,k) \coloneqq \left\{ {n \atop k} \right\}$	[1, 2]
$\sum_{n=k}^{\infty} B_{k,n}(x)\frac{t^n}{n!} = \frac{(tx)^k}{k!} e^{(1-x)t}$	(1.10)	Bernstein polynomials, $B_{k,n}(x);\ k = 0,1,2,\ldots,n$	[24, 25]
$\sum_{n=0}^{\infty} C_n\frac{t^n}{n!} = \frac{1-\sqrt{1-4t}}{2t}$	(1.11)	Catalan numbers, $C_n = \frac{1}{n+1}\binom{2n}{n}$	[27, 28]
$\sum_{n=0}^{\infty} (x)_n\frac{t^n}{n!} = (1+t)^x$	(1.12)	Falling factorial, $(x)_n$	[7, 11, 24, 25]

We summarize Table 1, as follows: By Eqs (1.6)–(1.8), we have

$$B_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k} B_k^{(\alpha)} x^{n-k},$$

$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} E_k\, x^{n-k},$$

and

$$F_n^{(\alpha)}(x|u) = \sum_{k=0}^{n} \binom{n}{k} F_k^{(\alpha)}(u) x^{n-k},$$

where $B_k^{(\alpha)}$, E_k and $F_k^{(\alpha)}(u)$ are Bernoulli numbers of higher order α, Euler numbers, and Frobenius-Euler numbers of order α, respectively. By Eqs (1.9) and (1.10), the link between Bell polynomials and Stirling numbers of the second kind is as follows:

$$\phi_n(x) = \sum_{k=0}^{n} {n \brace k} x^k.$$

By Eq (1.11), Bernstein polynomials have the following explicit formula:

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad k = 0,1,2, \dots, n,$$

and we list a few Bernstein polynomials:

$$B_{0,0}(x) = 1, \qquad B_{0,2}(x) = (1-x)^2,$$

$$B_{0,1}(x) = 1 - x, \qquad B_{1,2}(x) = 2x(1-x),$$

$$B_{1,1}(x) = x, \qquad B_{2,2}(x) = x^2.$$

From Eq (1.12), we have

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

It follows that $C_0 = 1,\ C_1 = 1,\ C_2 = 2,\ C_3 = 5,\ C_4 = 14, \dots$

The distributions, which are used in this paper, are listed in Table 2 (see also [29–32]):

Table 2. A list of distributions with moment generating functions and notations.

Distribution	Notation	Moment generating function	
Poisson distribution	$Y \sim Poisson(\alpha)$	$\mathrm{E}[e^{tY}] = e^{\alpha(e^{t}-1)}$	(1.13)
Uniform distribution	$Y \sim U(0,1)$	$\mathrm{E}[e^{Yt}] = \frac{e^{t}-1}{t}$	(1.14)
Geometric distribution	$Y \sim Geo(p)$	$\mathrm{E}[e^{Yt}] = \frac{pe^{t}}{1-(1-p)e^{t}}$	(1.15)
Normal distribution	$Y \sim N(0,1)$	$\mathrm{E}[e^{Yt}] = e^{\frac{1}{2}t^{2}}$	(1.16)
Exponential distribution	$Y \sim E(\alpha)$	$\mathrm{E}[e^{Yt}] = \frac{1}{1-\alpha t}, \left(t < \frac{1}{\alpha}\right)$	(1.17)
Gamma distribution	$Y \sim \Gamma(\alpha,\beta)$	$\mathrm{E}[e^{Yt}] = \frac{\alpha}{\Gamma(\beta)}(\alpha x)^{\beta-1}e^{-\alpha x}, (x > 0)$	(1.18)

The generating functions of probabilistic extensions of several polynomials have been studied and investigated by several mathematicians. For example, Soni et al. [20] introduced new families of Bernoulli and Euler polynomials based on a random variable Y as follows:

$$\sum_{n=0}^{\infty} \beta_{n}^{Y}(x)\frac{t^{n}}{n!} = \frac{t}{E[e^{Yt}]-1}(E[e^{Yt}])^{x} \quad \text{and} \quad \sum_{n=0}^{\infty} \mathcal{E}_{n}^{Y}(x)\frac{t^{n}}{n!} = \frac{2}{E[e^{Yt}]+1}(E[e^{Yt}])^{x}, \tag{1.19}$$

where $\beta_{n}^{Y}(x)$ and $\mathcal{E}_{n}^{Y}(x)$ are called probabilistic Bernoulli and probabilistic Euler polynomials, respectively. In the case $Y = 1$, $\beta_{n}^{Y=1}(x) := B_{n}(x)$ and $\mathcal{E}_{n}^{Y=1}(x) \coloneqq E_{n}(x)$ turn out to be well known (classical or ordinary) Bernoulli and Euler polynomials. Additionally, at $x = 0$ in Eq (1.3), they are called the probabilistic Bernoulli numbers and probabilistic Euler numbers. In the same viewpoint, one can consider the probabilistic Bernoulli polynomials of higher order (or order α) associated with Y as follows:

$$\left(\frac{t}{E[e^{Yt}]-1}\right)^{\alpha}(E[e^{Yt}])^{x} = \sum_{n=0}^{\infty} \beta_{n,Y}^{(\alpha)}(x)\frac{t^{n}}{n!}. \tag{1.20}$$

Karagenc et al. [24] gave a new family of Bernstein polynomials called probabilistic Bernstein polynomials by means of the following Taylor series expansion at $t = 0$:

$$\sum_{n=k}^{\infty} B_{k,n}^{Y}(x)\frac{t^{n}}{n!} = \frac{(xt)^{k}}{k!}(E[e^{Yt}])^{1-x}. \tag{1.21}$$

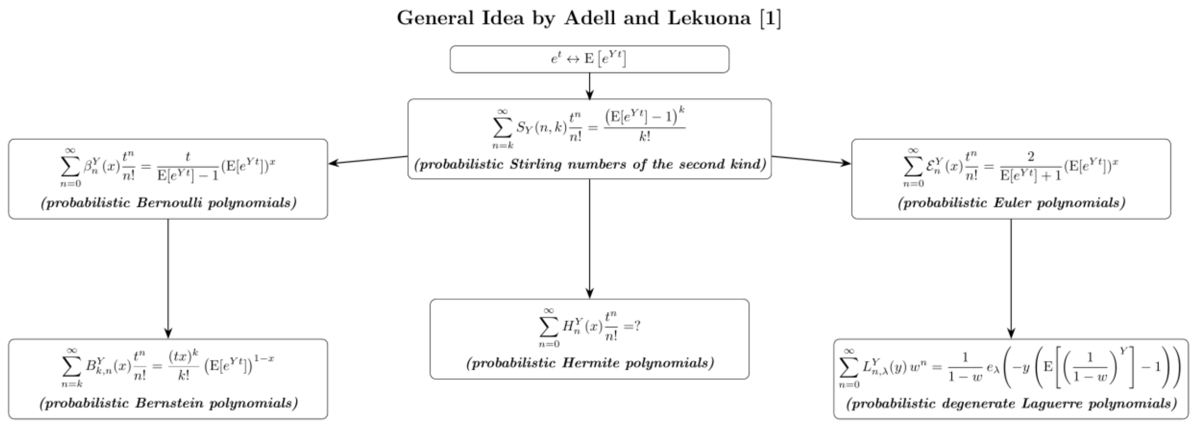
From $Y = 1$ in Eq (1.15), we see that $B_{k,n}^{Y=1}(x) := B_{k,n}(x)$.

The outline of this paper is as follows: In Section 1, we review definitions of special functions, notations, and parameters that will be used in the paper. In Section 2, we introduce a new generating function of Hermite polynomials derived from probabilistic perspective. By using this generating function, we delve into some new identities including probabilistic Stirling numbers of the second kind, probabilistic Bernstein polynomials, and Bernoulli polynomials of higher order in terms of probabilistic Hermite polynomials, and derivative property of probabilistic Hermite polynomials. Section 3 contains a useful application of the probabilistic Hermite polynomials to obtain new formulas between other type special numbers and polynomial sequences given in the paper by picking a suitable random variables such as Poisson, Geometric, Exponential, Normal, Gamma, and Uniform. Finally, in

Section 4, we provide a conclusion and observation remarks, including a discussion of possible directions for future research.

2. The probabilistic extension of Hermite polynomials and related number sequences and polynomials

In this section, we introduce the probabilistic Hermite polynomials with their fundamental properties, relevant applications, and results related to probabilistic number sequences and distributions. As mentioned in the introduction, the probabilistic version of Stirling numbers of the second kind was initially constructed by Adell and Lekuona [1], followed by T. Kim and D. S. Kim [5], L. Luo et al. [8], P. Xue et al. [9], L. Chen et al. [21], and A. Karagenç et al. [24], who developed probabilistic versions of the Bernoulli and Euler polynomials, type 2 Bernoulli and Euler polynomials, Bernstein polynomials, degenerate Laguerre polynomials, and degenerate poly-Bell polynomials. Motivated by these works, we focus on the probabilistic extension of the Hermite polynomials. The main idea of our study is summarized in the following diagram:



Motivated by the question posed in the diagram above, namely "What is the generating function of the probabilistic Hermite polynomials?", we now provide the following definition as a response.

Definition 2.1. Let $H_n^Y(x)$ be probabilistic Hermite polynomials. Then, the generating function of probabilistic Hermite polynomials is defined by

$$\sum_{n=0}^{\infty} H_n^Y(x)\frac{t^n}{n!} = e^{-t^2}(E[e^{Yt}])^{2x}. \tag{2.1}$$

Remark 2.1. By taking $Y = 1$ in Definition 2.1, it turns out to be the classical Hermite polynomials given in Eq (1.1).

As known, there are two types of Hermite polynomials in the literature. One of them (see Eq (1.1)) is used by physicists, and the other is used by probabilists (see Eq (1.2)). In this work, we extend the physicists' Hermite polynomials to the probabilistic framework by making use of the moment generating function, following the idea of Adell and Lekuona as in Definition 2.1.

Theorem 2.1. Let Y be a random variable. Then, the following explicit identity holds:

$$\sum_{k=0}^{n}(2x)_k\left\{{n \atop k}\right\}_Y=\sum_{j=0}^{[n/2]}\frac{n!}{j!\,(n-2j)!}H_{n-2j}^{Y}(x),$$

where $[x]$ denotes the greatest integer less than or equal to x.

Proof. Based on Eq (2.1), we first consider the following alternative form:

$$e^{t^2}\sum_{n=0}^{\infty}H_n^Y(x)\frac{t^n}{n!}=(E[e^{Yt}])^{2x}.$$

For the left-hand side, we begin with the series expansion of e^{t^2}, and then apply the Cauchy product to obtain

$$\left(\sum_{n=0}^{\infty}\frac{t^{2n}}{n!}\right)\left(\sum_{n=0}^{\infty}H_n^Y(x)\frac{t^n}{n!}\right)=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{[n/2]}\frac{H_{n-2j}^Y(x)}{j!\,(n-2j)!}\right)t^n.$$

For the right-hand side, we observe that

$$\begin{aligned}\sum_{k=0}^{\infty}\binom{2x}{k}(E[e^{Yt}]-1)^k&=\sum_{k=0}^{\infty}(2x)_k\frac{(E[e^{Yt}]-1)^k}{k!}\\&=\sum_{k=0}^{\infty}(2x)_k\sum_{n=k}^{\infty}\left\{{n \atop k}\right\}_Y\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left\{{n \atop k}\right\}_Y(2x)_k\right)\frac{t^n}{n!}.\end{aligned}$$

Thus, comparing the coefficients of t^n on both sides completes the proof of the theorem.

The following theorem states that the expected value of the nth power of the sum of the first m random variables $Y_1, Y_2, \dots, Y_m$ can be expressed in terms of the probabilistic Hermite polynomials.

Theorem 2.2. Let Y be a random variable with $m \in \mathbb{N}$. Then, we have:

$$E[S_m^n]=\sum_{j=0}^{[n/2]}\frac{n!}{(n-2j)!\,j!}H_{n-2j}^{Y}\left(\frac{m}{2}\right),$$

where

$$S_m=\sum_{k=1}^{m}Y_k.$$

Proof. At the value $x = \frac{m}{2}$, $(m \in \mathbb{N})$, in Eq (2.1), we first consider

$$e^{t^2} \sum_{n=0}^{\infty} H_n^Y\left(\frac{m}{2}\right)\frac{t^n}{n!} = (E[e^{Yt}])^m,$$

and by employing the series expansion and applying Cauchy product, we obtain

$$\left(\sum_{j=0}^{\infty} \frac{t^{2j}}{j!}\right)\left(\sum_{k=0}^{\infty} H_k^Y\left(\frac{m}{2}\right)\frac{t^k}{k!}\right) = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{[n/2]} \frac{H_{n-2j}^Y\left(\frac{m}{2}\right)}{j!\,(n-2j)!}\right)t^n.$$

On the other hand,

$$(E[e^{Yt}])^m = \sum_{n=0}^{\infty} E[S_m^n]\frac{t^n}{n!}.$$

Thus, by equating the coefficients of $\frac{t^n}{n!}$, we arrive at the desired result.

The following theorem states that the derivative of the probabilistic Hermite polynomials can be expressed as a binary-summation involving the products of probabilistic Stirling numbers of the second kind and probabilistic Hermite polynomials.

Theorem 2.3. Let $n \in \mathbb{N}$. Then, we have

$$\frac{d}{dx} H_{n+1}^Y(x) = 2\sum_{l=0}^{n}\binom{n+1}{l+1}\sum_{k=1}^{l+1}(-1)^{k-1}\,(k-1)!\left\{ {l+1 \atop k} \right\}_Y H_{n-l}^Y(x).$$

Proof. From Eq (2.1), we have

$$\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty} H_n^Y(x)\frac{t^n}{n!}\right) = \frac{\partial}{\partial x}\left(e^{-t^2}(E[e^{Yt}])^{2x}\right).$$

Taking the partial derivative of both sides of the equation reveals a relationship between the right-hand side and the probabilistic Stirling numbers of the second kind:

$$\begin{aligned}
&= 2\left(\sum_{n=0}^{\infty} H_n^Y(x)\frac{t^n}{n!}\right)\left(\sum_{k=1}^{\infty}(-1)^{k-1}(k-1)!\sum_{n=k}^{\infty}\left\{ {n \atop k} \right\}_Y \frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{k=1}^{l+1} 2(-1)^{k-1}(k-1)!\,H_{n-l}^Y(x)\left\{ {l+1 \atop k} \right\}_Y\right)\frac{t^{n+l}}{(n\text{-}l)!(l\text{+}1)!}.
\end{aligned}$$

Therefore, it follows that

$$\frac{d}{dx}H_n^Y(x) = 2\sum_{l=0}^{n}\sum_{k=1}^{l+1}(-1)^{k-1}(k-1)!\,H_{n-l}^y(x)\begin{Bmatrix} l+1 \\ k \end{Bmatrix}_Y \frac{(n+1)!}{(n-l)!\,(l+1)!}.$$

Thus, the proof of the theorem is complete.

We now present the following theorem, which expresses the probabilistic Hermite polynomials in terms of probabilistic Bernstein polynomials.

Theorem 2.4. Let Y be a random variable and $x \neq \frac{1}{2}$. Then, the following explicit identity holds:

$$H_n^Y(x) = \sum_{k=0}^{[n/2]}\left(-\frac{1}{2}\right)_k \frac{2^{2k}B_{2k,n}^Y(1-2x)}{(2x-1)^{2k}}.$$

Proof. Based on Eq (2.1), we have the following alternative form:

$$\sum_{n=0}^{\infty} H_n^Y\left(\frac{1-x}{2}\right)\frac{t^n}{n!} = e^{-t^2}(E[e^{Yt}])^{1-x}.$$

Using Eq (1.15), we compute

$$\sum_{n=0}^{\infty} H_n^Y\left(\frac{1-x}{2}\right)\frac{t^n}{n!} = \sum_{k=0}^{\infty}(-1)^k\frac{t^{2k}}{k!}x^{2k}\,(\mathrm{E}[e^{Yt}])^{1-x}\frac{(2k)!}{(2k)!}\frac{1}{x^{2k}}$$
$$= \sum_{k=0}^{\infty}\frac{(-1)^k(2k)!}{x^{2k}k!}\sum_{n=2k}^{\infty}B_{2k,n}^Y(x)\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{[n/2]}\frac{(-1)^k(2k)!}{x^{2k}k!}B_{2k,n}^Y(x)\right)\frac{t^n}{n!}.$$

From these straightforward calculations, proof of the theorem is complete.

Remark 2.2. In the case $Y = 1$ in Theorem 2.4, we recover the well-known explicit identity of the classical Hermite polynomials, as follows:

$$H_n(x) = n!\sum_{k=0}^{[n/2]}\frac{(-1)^k}{k!\,(n-2k)!}(2x)^{n-2k}.$$

For more details, see [15].

Theorem 2.5. Let Y be a random variable. Then, the following explicit identity holds:

$$H_n^Y\left(\frac{x}{2}\right) = \sum_{l=0}^{[n/2]}\sum_{k=0}^{n-2l}\frac{\alpha!\,(-1)^l n!}{(n-2l-k)!\,(k+l)!\,l!}\begin{Bmatrix} k+l \\ l \end{Bmatrix}_Y B_{n-2l-k,Y}^{(\alpha)}(x).$$

Proof. Starting from Eq (2.1), we first consider

$$\sum_{n=0}^{\infty} H_n^Y\left(\frac{x}{2}\right)\frac{t^n}{n!} = \left(\frac{t^\alpha}{(E[e^{Yt}]-1)^\alpha}\right)\left(\frac{(E[e^{Yt}]-1)^\alpha}{t^\alpha}\right)e^{-t^2}(E[e^{Yt}])^x.$$

Using Eq (1.14), we compute as follows:

$$\begin{aligned}\sum_{n=0}^{\infty} H_n^Y\left(\frac{x}{2}\right)\frac{t^n}{n!} &= \left(\frac{t^\alpha}{(E[e^{Yt}]-1)^\alpha}(E[e^{Yt}])^x\right)\left(\frac{e^{-t^2}}{t^\alpha}\alpha!\right)\left(\sum_{n=\alpha}^{\infty}\begin{Bmatrix} n \\ \alpha \end{Bmatrix}_Y \frac{t^n}{n!}\right)\\ &= \left(\sum_{n=0}^{\infty} B_{n,Y}^{(\alpha)}(x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(-1)^n\frac{t^{2n}}{n!\,t^\alpha}\alpha!\right)\left(\sum_{n=0}^{\infty}\begin{Bmatrix} n+\alpha \\ \alpha \end{Bmatrix}_Y \frac{t^n t^\alpha}{(n+\alpha)!}\right)\\ &= \left(\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{\alpha!}{(n-k)(k+\alpha)!}B_{n-k,Y}^{(\alpha)}(x)\begin{Bmatrix} k+\alpha \\ \alpha \end{Bmatrix}_Y\right)t^n\right)\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}t^{2n}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{[n/2]}\sum_{k=0}^{n-2l}\frac{(-1)^l}{l!}\frac{\alpha!}{(n-2l-k)!\,(k+l)!}B_{n-2l-k,Y}^{(\alpha)}(x)\begin{Bmatrix} k+\alpha \\ \alpha \end{Bmatrix}_Y\right)t^n.\end{aligned}$$

Matching the coefficients of t^n completes the proof.

In a similar manner, we present the following theorem without proof.

Theorem 2.6. Let Y be a random variable. Then, the following explicit identity holds:

$$H_n^Y\left(\frac{x}{2}\right) = \sum_{l=0}^{[n/2]}\sum_{j=0}^{n-2l}\binom{n-2j}{l}\binom{\alpha}{m}\mathcal{E}_{n-2l-j,Y}^{(\alpha)}(x)\frac{E[S_m^j]}{2^\alpha}.$$

Remark 2.7. When $Y = 1$, all the results obtained in this part of this paper reduce to those for the classical Hermite polynomials.

In the forthcoming section, by selecting suitable random variables, we investigate new identities, relations, and formulas involving probabilistic Hermite polynomials and various special polynomials, such as Bell polynomials, Frobenius-Euler polynomials of higher order, Bernoulli polynomials of higher order, the power function x^n, and Stirling numbers of the second kind.

3. Case studies

In this section, we begin with the following corollaries, which are derived from [23, Theorems 3.1, 3.4, and 3.6, respectively], by substituting $B_{2k,n}^Y(x)$ into Theorem 2.4.

Corollary 3.1. We have

$$H_n^Y(x) = \sum_{k=0}^{[n/2]} \binom{n}{2k}\left(-\frac{1}{2}\right)_k \left(\frac{2x}{2x-1}\right)^{2k} \phi_{n-2k}\big(\alpha(1-x)\big),$$

where $\phi_n(x)$ denotes the Bell polynomial and $Y \sim Poisson(\alpha)$.

Corollary 3.2. We have

$$H_n^Y\left(\frac{1-x}{2}\right) = \sum_{k=0}^{[n/2]} \frac{(-1)^{n-k}(n)_{2k}}{k!} F_{n-2k}^{(1-x)}(q),$$

where $F_{n-2k}^{(1-x)}(q)$ denotes the Frobenius-Euler polynomials of higher order with respect to q and $Y \sim Geo(p)$ with $p + q = 1$.

Corollary 3.3. We have

$$H_n^Y\left(\frac{1-x}{2}\right) = \sum_{k=0}^{[n/2]} \frac{(-1)^{k}(n)_{2k}}{k!} B_{n-2k}^{(x-1)},$$

where $B_{n-2k}^{(x-1)}$ denotes the Bernoulli polynomials of higher order and $Y \sim U(0,1)$.

Theorem 3.1. Let $Y \sim U(0,1)$. The connection between probabilistic Hermite polynomials and probabilistic Bernoulli polynomials is given by

$$H_n^Y(x) = n! \sum_{j=0}^{[n/2]} \frac{(-1)^j}{(n-2j)!\,j!} B_{n-2j}^{(-2x)}.$$

Proof. Let Y be a uniform random variable on the interval (0,1). Then, the *mgf* of Y is given by

$$E[e^{Yt}] = \frac{e^t - 1}{t}.$$

Substituting this *mgf* into Eq (1.8), we obtain

$$\sum_{n=0}^{\infty} H_n^Y(x)\frac{t^n}{n!} = \left(\frac{t}{e^t-1}\right)^{-2x} e^{-t^2} = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{[n/2]} \frac{B_{n-2j}^{(-2x)}}{(n-2j)!\,j!}(-1)^j n!\right)\frac{t^n}{n!}.$$

It follows by comparing coefficients $\frac{t^n}{n!}$ on both sides of Eq (1.8). Thus, the proof of the theorem is complete.

Remark 3.1. Corollary 3.3 is directly derived from [24, Theorem 3.6] by substituting $B_{2k,n}^{Y}(x)$ into Theorem 2.4. Similarly, Theorem 3.1 is obtained from Eq (2.1) by taking $Y \sim U(0,1)$. Therefore, Corollary 3.3 coincides with Theorem 3.1.

Theorem 3.2. Let $Y \sim \Gamma(1,1)$. Then, we have

$$H_n^Y(x) = n! \sum_{j=0}^{[n/2]} \frac{(-1)^j}{(n-2j)!\,j!} (2x+n-2j-1)_{n-2j}.$$

Proof. Combining Eqs (1.12) and (2.1), it is straightforward to see that

$$\begin{aligned}\sum_{n=0}^{\infty} H_n^Y(x) \frac{t^n}{n!} &= (E[e^{Yt}])^{2x} e^{-t^2} \\ &= \frac{1}{(1-t)^{2x}} e^{-t^2} \\ &= \left(\sum_{n=0}^{\infty} \frac{(2x+n-1)_n}{n!} t^n\right)\left(\sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{n!}\right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{[n/2]} \frac{(2x+n-2j-1)_{n-2j}}{(n-2j)!\,j!} (-1)^j n!\right) \frac{t^n}{n!}.\end{aligned}$$

By comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result.

Theorem 3.3. Let $Y \sim E(\alpha)$. Then, we have

$$H_n^Y(x) = n! \sum_{j=0}^{[n/2]} \frac{(-1)^j}{(n-2j)!\,j!} \alpha^{n-2j} (2x+n-2j-1)_{n-2j}.$$

Proof. The proof is omitted since it follows the same steps.

Remark 3.2. In the case $\alpha = 1$ in Theorem 3.3, the results reduce to that of Theorem 3.2.

Theorem 3.4. Let $Y \sim N(0,1)$. Then we have

$$H_{2n}^Y(x+1) = (n+1)!\, C_n\, x^n,$$

where C_n denotes the *n*th Catalan number.

Proof. By combining Eq (1.10) to Eq (2.1), we have

$$\sum_{n=0}^{\infty} H_n^Y(x+1) \frac{t^n}{n!} = (E[e^{Yt}])^{2x+2} e^{-t^2} = e^{(x+1)\,t^2} e^{-t^2} = e^{xt^2} = \sum_{n=0}^{\infty} x^n \frac{t^{2n}}{n!}.$$

Thus, by comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we arrive at the desired result.

From Theorem 3.4, we immediately obtain the following corollary.

Corollary 3.4. Let $Y \sim N(0,1)$. Then,

$$H_{2n+1}^{Y}(x) = 0.$$

Theorem 3.5. Let $Y \sim Poisson\ (\alpha)$. The probabilistic Hermite polynomials can be represented in terms of the Bell polynomials, as follows:

$$H_n^Y(x) = n! \sum_{j=0}^{[n/2]} \frac{(-1)^j}{(n-2j)!\, j!} \phi_{n-2j}(2x\alpha).$$

Proof. By Eqs (1.8) and (2.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} H_n^Y(x) \frac{t^n}{n!} &= e^{-t^2} e^{2x\alpha(e^t-1)} \\ &= \left(\sum_{n=0}^{\infty} \phi_n(2x\alpha) \frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty} (-1)^m \frac{t^{2m}}{m!}\right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{[n/2]} \frac{\phi_{n-2j}(2x\alpha)}{(n-2j)!\, j!} (-1)^j \right) t^n .\end{aligned}$$

Thus, the proof is completed by matching the coefficients of t^n on both sides.

The following theorem appears to be an inverse relation of Theorem 3.5.

Theorem 3.6. Let $Y \sim Poisson\ (\alpha)$. Then, we have

$$\phi_n(2x\alpha) = \sum_{k=0}^{[n/2]} \frac{n!}{k!\,(n-2k)!} H_{n-2k}^Y(x).$$

Proof. By Eqs (1.8) and (2.1), we get

$$\begin{aligned} e^{t^2} \sum_{n=0}^{\infty} H_n^Y(x) \frac{t^n}{n!} &= e^{2x\alpha(e^t-1)} = \sum_{n=0}^{\infty} \phi_n(2x\alpha) \frac{t^n}{n!} = \left(\sum_{n=0}^{\infty} \frac{t^{2n}}{n!}\right)\left(\sum_{n=0}^{\infty} H_n^Y(x) \frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{[n/2]} \frac{n!}{(n-2k)!\, k!} H_{n-2k}^Y(x) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{[n/2]} \frac{n!\, H_{n-2k}^Y(x)}{k!\,(n-2k)!} \right) \frac{t^n}{n!}. \end{aligned}$$

When both sides of the equations $\frac{t^n}{n!}$ are matched, we conclude the proof.

The first $(m+1)$-sum of the expression $\binom{m}{k} H_{m-k}^Y(x)$ has been expressed by the finite binary-sum of the products of two Stirling numbers of the second kinds.

Corollary 3.6. Let $Y\sim Poisson\ (\alpha)$. Then,

$$\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{n!}{(n-2j)!\,j!}H_{n-2j}^{Y}(x)=\sum_{j=0}^{n}\left(\sum_{k=0}^{j}(2x)_k S_2(j,k)S_2(n,j)\right)\alpha^j.$$

Proof. By Eqs (1.8) and (2.1), we have

$$I_1 := e^{t^2}\sum_{n=0}^{\infty}H_n^Y(x)\frac{t^n}{n!}=\left(e^{\alpha(e^t-1)}\right)^{2x} =: I_2$$

$$=\left(e^{\alpha(e^t-1)}-1+1\right)^{2x}$$

$$=\sum_{k=0}^{\infty}\binom{2x}{k}\left(e^{\alpha(e^t-1)}-1\right)^k,$$

$$I_2=\sum_{k=0}^{\infty}(2x)_k\frac{\left(e^{\alpha(e^t-1)}-1\right)^k}{k!}.$$

Considering the definition of the generating function for the Stirling numbers of the second kind yields

$$I_2=\sum_{k=0}^{\infty}(2x)_k\left(\sum_{j=k}^{\infty}S_2(j,k)\,\alpha^j\,\frac{(e^t-1)^j}{j!}\right).$$

One more consideration of the definition of the generating function for Stirling numbers of the second kind gives

$$I_2=\sum_{j=0}^{\infty}\sum_{k=0}^{j}(2x)_k\,S_2(j,k)\alpha^j\sum_{n=j}^{\infty}S_2(n,j)\frac{t^n}{n!},$$

$$I_2=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\sum_{k=0}^{j}(2x)_k S_2(j,k)\alpha^j S_2(n,j)\right)\frac{t^n}{n!}.$$

We now compute I_1 as follows:

$$I_1=\left(\sum_{n=0}^{\infty}\frac{t^{2n}}{n!}\right)\left(\sum_{n=0}^{\infty}H_n^Y(x)\frac{t^n}{n!}\right)=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{[n/2]}\frac{n!}{j!\,(n-2j)!}H_{n-2j}^{Y}(x)\right)\frac{t^n}{n!}.$$

When we equate $I_1=I_2$, we end the proof of the theorem.

4. Graphical representation

In this part, by some special random variables for probabilistic Hermite polynomials, we first find a few polynomials and then sketch the graph to show its behavior.

Choosing Poisson random variable of $\alpha = 2$ in Eq (2.1), we have a few first polynomials as follows:

$$H_0^Y(x) = 1,$$
$$H_1^Y(x) = 4x,$$
$$H_2^Y(x) = 16x^2 + 4x - 2,$$
$$H_3^Y(x) = 64x^3 + 48x^2 - 20x,$$
$$H_4^Y(x) = 256x^4 + 384x^3 - 70x^2 - 44x + 12.$$

Taking random variable Y as Normal random variable in Eq (2.1) yields

$$H_0^Y(x) = 1, H_1^Y(x) = 2x, H_2^Y(x) = 12x^2, H_3^Y(x) = 120x^3, H_4^Y(x) = 1680x^4.$$

Here, we illustrate probabilistic Hermite polynomials over $[-1,1]$ for $n = 0,1,2,3,4$ using Eq (2.1). In Figure 1, the graph of $H_n^Y(x)$ over $[-1,1]$, $Y \sim Poisson\ (2)$ and $n = 0,1,2,3,4$ is sketched, and in Figure 2, a graph of $H_n^Y(x)$ over $[-1,1]$, $Y \sim N(0,1)$ and $n = 0,1,2,3,4$ is plotted.

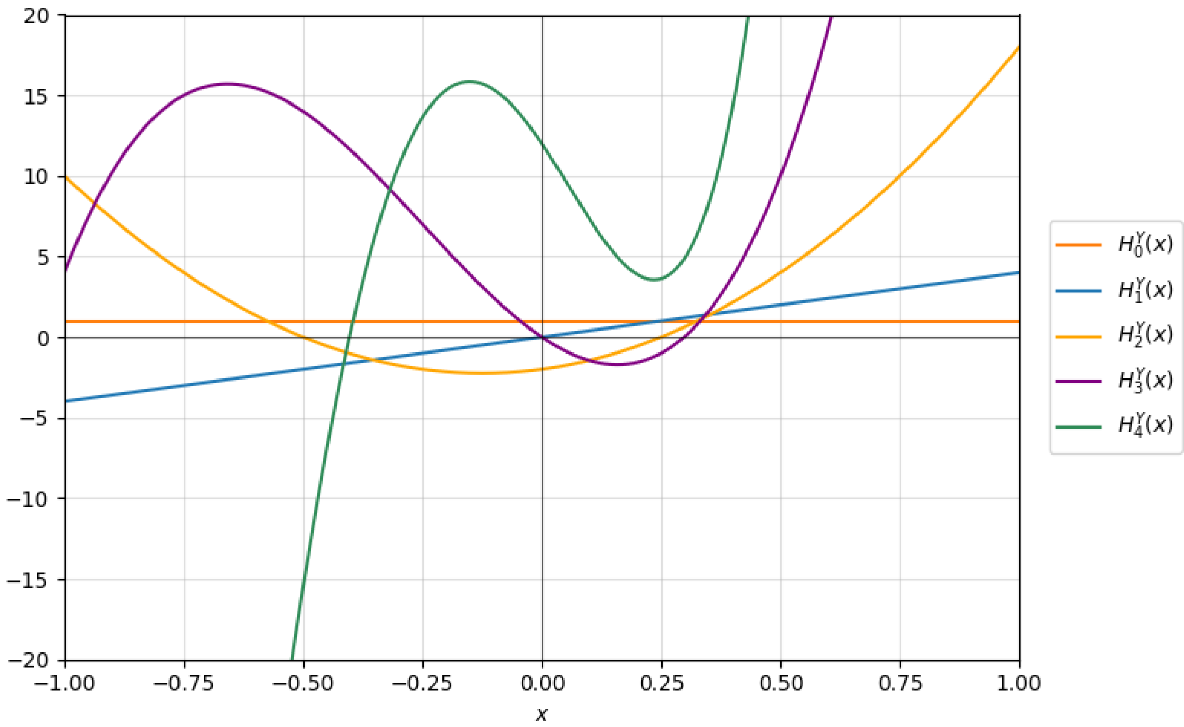


Figure 1. The shapes of $H_n^Y(x)$ over $[-1,1]$, $Y \sim Poisson\ (2)$ and $n = 0,1,2,3,4$.

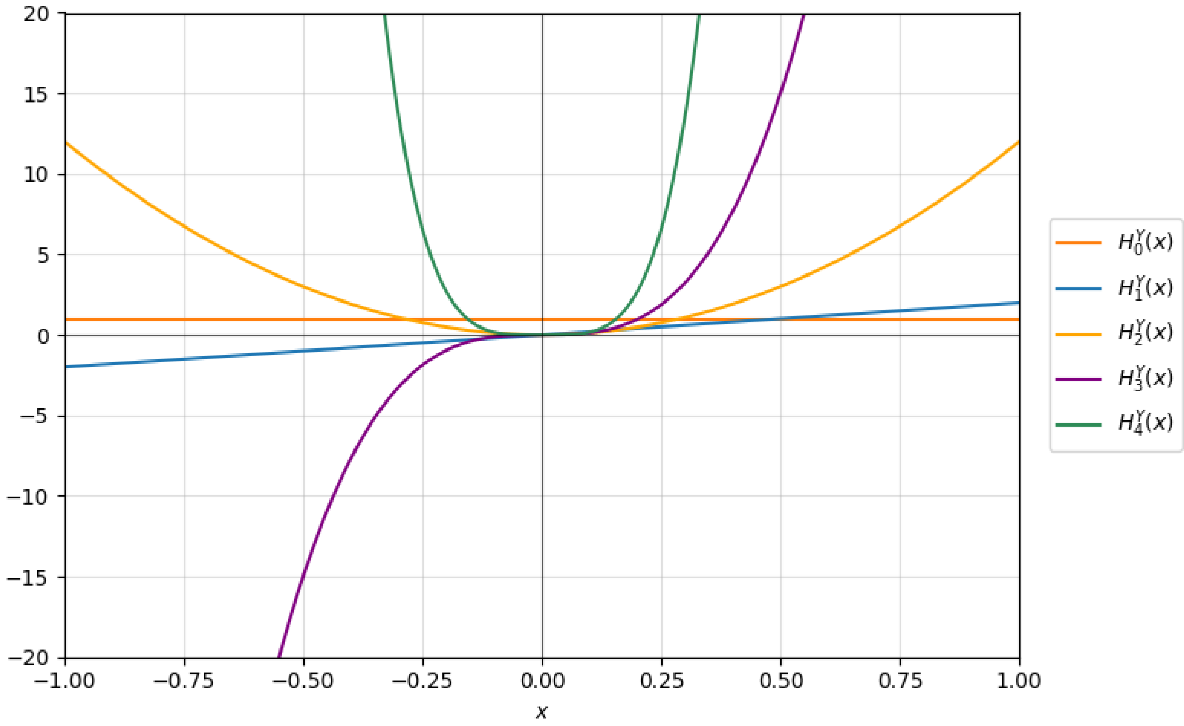


Figure 2. The shapes of $H_n^Y(x)$ over $[-1,1]$, $Y\sim N(0,1)$ and $n=0,1,2,3,4$.

5. Conclusions and observation

In this paper, we have proposed a novel generalization of Hermite polynomials, referred to as probabilistic Hermite polynomials, through the generating function:

$$\sum_{n=0}^{\infty} H_n^Y(x)\frac{t^n}{n!} = e^{-t^2}(E[e^{Yt}])^{2x}.$$

It is straightforward to verify that the construction reduces to the classical Hermite polynomials when the random variable $Y=1$. By employing various random variables such as Poisson, Normal, Geometric, Exponential, Uniform, and Gamma distributions, we derived distinct and novel generating functions. This approach led to new and interesting connections between probabilistic Hermite polynomials, and several special numbers and polynomials discussed throughout the paper.

In [15], T. Kim *et al.* introduced degenerate Hermite polynomials via the generating function:

$$\sum_{n=0}^{\infty} H_{n,\lambda}(x)\frac{t^n}{n!} = e_\lambda^{-1}(t^2)e_\lambda^{x}(2t).$$

Inspired by this, we conclude with the following questions:

Open Question. Is it possible to define probabilistic degenerate Hermite polynomials from a probabilistic perspective? If so, can one construct new relations and closed-form expressions for such polynomials by employing different random variables?

It would also be of interest to further study the structural properties of the generalized Hermite polynomials introduced here, such as orthogonality or pseudo-orthogonality, as well as their possible interpretation within the umbral calculus formalism and probability applications, cf. [33–36]. We believe that these directions may lead to further developments and potential applications in statistics, engineering, and related areas.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare no conflicts of interest in this paper.

Author contributions

All authors of this article contributed equally. All authors have read and approved the final version of the manuscript for publication.

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