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Research article

Mild solutions and logarithmic decay for the time-fractional Navier-Stokes equations with Caputo-Hadamard derivative

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Abstract: This paper is devoted to the study of mild solutions for the time-fractional Navier-Stokes equations, where the time derivative is interpreted in the Caputo-Hadamard sense. The Caputo-Hadamard derivative, which involves a logarithmic kernel, is particularly suitable for describing anomalous diffusion processes with ultra-slow dynamics. The main contribution of this work is threefold. First, by reformulating the equations as an abstract Cauchy problem and employing the Mittag-Leffler operator representations, we derived an integral formulation of mild solutions. Second, using Banach fixed point theorem with suitable decay estimates of the linear semigroup generated by the Stokes operator, we established the existence and uniqueness of both global (for sufficiently small initial data) and local (for arbitrarily large initial data) mild solutions in critical Lebesgue spaces. Third, we provided explicit logarithmic decay estimates for these solutions, which reflect the ultra-slow dissipation mechanism induced by the Caputo-Hadamard derivative.

Keywords: Caputo-Hadamard derivative; Navier-Stokes equations; mild solution; existence and uniqueness; logarithmic decay

1. Introduction

Fractional calculus utilizes non-local operators to characterize the memory and hereditary properties of systems, providing a precise modeling tool for complex processes such as anomalous diffusion and viscoelastic mechanics [1–6]. Consequently, fractional differential equations hold significant application value in science and engineering. In recent years, numerical methods for nonlinear fractional differential equations, with the existence and decay properties of their solutions, have emerged as topics of considerable theoretical significance and practical potential (see [7–11]). Existence analysis lays the foundation for the well-posedness of the models, while decay reveals the stable behavior of systems over long time scales. Both aspects are essential for understanding the dynamic evolution of practical systems and have thus attracted considerable scholarly interest.

The fractional Navier-Stokes equations generalize the classical derivative operator to a fractional derivative operator. Their key advantage lies in the ability to simultaneously characterize solutions corresponding to different fractional-order derivative values. This generalization significantly expands the applicability of the model, demonstrating substantial potential for practical applications in areas such as turbulence analysis, flow in heterogeneous media, and viscoelastic mechanics [12–15]. Over the past two decades, research on fractional Navier-Stokes equations has achieved significant progress. For instance, El-Shahed and Salem [16] investigated the analytic solutions of such equations. Zhou and Peng [17] established the existence and uniqueness of global and local mild solutions. Wang and Liang [18] examined the existence, decay, and regularity of mild solutions for functional Navier-Stokes differential inclusions with time-fractional derivatives. Carvalho-Neto and Planas [19] addressed the existence and uniqueness of mild solutions. Tang and Yu [20] studied steady suitable weak solutions to the fractional Navier-Stokes system in $\mathbb{R}^3$. Furthermore, research on numerical analysis of fractional Navier-Stokes equations has made substantial advances (see references [21–24] and their citations for details). Moreover, advances in numerical methods for complex nonlocal models, such as the high-order explicit hybrid algorithm for the nonlocal ternary viscous Cahn–Hilliard model proposed by Weng et al. [25] and the energy dissipation and maximum bound principle preserving scheme for the nonlocal ternary Allen–Cahn model developed by Zhai et al. [26], provide valuable guidance for further extending the numerical analysis of fractional Navier-Stokes equations to more complex systems.

Hadamard fractional calculus, including the Hadamard integral/derivative and the Caputo-Hadamard derivative, originated in 1892 [3]. However, its potential value in describing slow dynamics has led to renewed and significant research interest only in recent years. This framework is particularly well–suited for modeling phenomena exhibiting ultra-slow anomalous diffusion [27], which arises from stochastic processes and is characterized by logarithmic growth of the mean square displacement, expressed as $\langle x(t)^2 \rangle \propto \log^{\alpha} t$, $\alpha \in (0,1)$. Li et al. [28] studied the regularity and logarithmic decay of solutions to the Caputo-Hadamard fractional diffusion equation. Li and Li investigated the blow-up and global existence of solutions to a semilinear time-space fractional diffusion equation in [29, 30], analyzing the asymptotic behaviors of solutions to time-space fractional partial differential equations involving the Caputo-Hadamard derivative. Wang and Sun [31] considered the existence and uniqueness of mild solutions to Hadamard-type fractional Fokker-Planck equations. Furthermore, the Hadamard derivative has been successfully utilized by Cai et al. to predict the COVID-19 dynamics of the Omicron variant [32].

In this paper, we investigate the fractional Navier-Stokes equations with fractional order $\alpha \in (0,1)$, given by

$$\begin{cases} {}_{CH}\mathrm{D}^{\alpha}_{a,t}\mathbf{u} - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p, & \text{in} \quad \mathbb{R}^d \times (a,T), \\ \nabla\cdot\mathbf{u} = 0, & \text{in} \quad \mathbb{R}^d \times (a,T), \\ \mathbf{u}(\mathbf{x},a) = \mathbf{u}_a(\mathbf{x}), & \text{in} \quad \mathbb{R}^d, \end{cases} \tag{1.1}$$

where $0 < a < T \le \infty$, $\mathbf{u} = \mathbf{u}(\mathbf{x},t)$ denotes the velocity vector field, $p = p(\mathbf{x},t)$ represents the pressure field, and $\nu > 0$ is the kinematic viscosity. The system is supplemented with the initial condition $\mathbf{u}_a(\mathbf{x})$, which is a given divergence-free vector field. The temporal fractional derivative is characterized by the α-th order Caputo-Hadamard operator ${}_{CH}\mathrm{D}^{\alpha}_{a,t}$ (see Definition 3). The Caputo-Hadamard derivative is defined by a logarithmic kernel $(\log t - \log\tau)^{-\alpha}$, which makes it fundamentally different from the more common Caputo derivative (power-law kernel $(t-\tau)^{-\alpha}$). The logarithmic memory kernel is known to model ultra-slow diffusion processes, where the mean square displacement grows only logarithmically in time. Such behavior has been observed in certain complex systems, e.g., diffusion in heterogeneous porous media with strong trapping effects, relaxation in viscoelastic polymers with stretched-exponential or logarithmic creep, and low-frequency dispersion in turbulent flows [28,29]. The Caputo-Hadamard fractional derivative naturally arises in these contexts because it preserves the physical requirement that the initial condition be given in the usual (integer-order) sense, while the logarithmic convolution introduces an extremely weak singularity, leading to solutions that decay polynomially in the logarithm of time.

In [33], Zhu et al. studied the existence and uniqueness of mild solutions for the time-fractional Navier-Stokes equations with a ψ-Caputo derivative, and proved that under appropriate assumptions on the initial conditions, as $\alpha \to 1^-$, the mild solution of the time-fractional equation converges to the mild solution of its classical counterpart. The time Caputo-Hadamard derivative considered in this paper is a special case of this framework, namely when $\psi(t) = \log t$. We observe that studies on the time-fractional Navier-Stokes equations have primarily focused on the Caputo derivative, while research involving the Caputo-Hadamard derivative remains relatively limited. This underserved area motivates this work. Here, we first introduce the concepts of global and local mild solutions for the corresponding system. Subsequently, we establish the existence and uniqueness of such mild solutions to Eq (1.1). Moreover, we provide estimates for the logarithmic decay of these solutions, thereby revealing the ultra-slow dissipation mechanism induced by the Caputo-Hadamard operator.

The paper is organized as follows: In Section 2, we present the definition of Hadamard fractional calculus, introduce the definition of a mild solution to Eq (1.1), and discuss the properties of Mittag-Leffler operators. In Section 3, we investigate the existence, uniqueness, and logarithmic decay of mild solutions. In the last section, we provide concluding remarks.

2. Preliminaries

In this section, we introduce the necessary notations, definitions, and key preliminary results. Throughout this paper, scalars are written in regular font, while vectors are represented in boldface. For notational convenience, we denote $v(\mathbf{x},t) = v(t)$.

Let $d \geq 1$ and $1 \leq p \leq \infty$. The Lebesgue space $L^p(\mathbb{R}^d)$ consists of all measurable functions $v\colon \mathbb{R}^d \to \mathbb{R}$ with finite norm $\|v\|_{L^p(\mathbb{R}^d)}$, defined by

$$\|v\|_{L^p(\mathbb{R}^d)} := \begin{cases} \left(\displaystyle\int_{\mathbb{R}^d} |v(\mathbf{x})|^p d\mathbf{x}\right)^{1/p}, & \text{for } 1 \leq p < \infty, \\ \operatorname*{ess\,sup}_{\mathbf{x}\in\mathbb{R}^d} |v(\mathbf{x})|, & \text{for } p = \infty. \end{cases}$$

A fundamental subspace is the set of divergence free vector fields

$$L^p_0(\mathbb{R}^d) := \{\mathbf{v} \in L^p(\mathbb{R}^d) : \nabla \cdot \mathbf{v} = 0 \text{ in } \mathbb{R}^d\}.$$

For Banach spaces S and X, we introduce the following function spaces: The space of continuous functions $C(S;X)$; the space of continuous bounded functions $C_b(S;X)$, equipped with the norm

$\|f\|_{C_b(S;X)} = \sup_{s\in S}\|f(s)\|_X$; the Bochner–Lebesgue space $L^p(S;X)$ of X-valued functions, which for $1 \le p < \infty$ is endowed with the norm

$$\|u\|_{L^p(S;X)} := \left(\int_S \|u(s)\|_X^p d\mu(s)\right)^{1/p},$$

and for $p = \infty$ with the essential supremum norm $\|u\|_{L^\infty(S;X)} = \operatorname{ess\,sup}_{s\in S}\|u(s)\|_X$ (with the underlying measure μ understood); and the Sobolev space $W^{k,p}(S;X)$, which is the subspace of $L^p(S;X)$ such that all weak derivatives $D^\alpha u$ of order $|\alpha| \le k$ also lie in $L^p(S;X)$. For $1 \le p < \infty$, the norm on $W^{k,p}(S;X)$ is given by

$$\|u\|_{W^{k,p}(S;X)} := \left(\sum_{|\alpha|\le k}\|D^\alpha u\|_{L^p(S;X)}^p\right)^{1/p},$$

with the natural modification for $p = \infty$.

Definition 1. *Let $f(t)$ be a given function and $\alpha > 0$. The Hadamard fractional integral of order α is defined as*

$${}_H\mathrm{D}_{a,t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{f(\tau)}{\tau}d\tau, \quad t > a > 0,$$

where $\Gamma(\cdot)$ denotes the Euler Gamma function.

Definition 2. *For a given function $f(t)$ and $\alpha \in (n-1, n)$ with $n \in \mathbb{Z}^+$, the Hadamard fractional derivative of order α is defined as*

$${}_H\mathrm{D}_{a,t}^{\alpha} f(t) = \delta^n\left({}_H\mathrm{D}_{a,t}^{-(n-\alpha)} f(t)\right) = \frac{1}{\Gamma(n-\alpha)}\delta^n\int_a^t \left(\log\frac{t}{\tau}\right)^{n-\alpha-1}\frac{f(\tau)}{\tau}d\tau,$$

for $t > a > 0$, where $\delta^n f(t) = \left(t\frac{d}{dt}\right)^n f(t)$.

Definition 3 ([34]). *For a given function $f(t)$ and $\alpha \in (n-1, n)$ with $n \in \mathbb{Z}^+$, the Caputo-Hadamard fractional derivative of order α is defined as*

$${}_{CH}\mathrm{D}_{a,t}^{\alpha} f(t) = {}_H\mathrm{D}_{a,t}^{-(n-\alpha)}[\delta^n f(t)] = \frac{1}{\Gamma(n-\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{n-\alpha-1}\frac{\delta^n f(\tau)}{\tau}d\tau, \tag{2.1}$$

for $t > a > 0$.

Denote by R_j the j-th Riesz transform, defined as $R_j = \partial_j(-\Delta)^{-1/2}$, with Fourier transform characterization

$$\mathcal{F}(R_j g)(\xi) = i\frac{\xi_j}{|\xi|}\mathcal{F}g(\xi), \quad \xi \in \mathbb{R}^d,$$

where $\mathcal{F}$ denotes the Fourier transform. The Helmholtz-Leray projector $P\colon L^p(\mathbb{R}^d) \to L_0^p(\mathbb{R}^d)$ is given componentwise by

$$(P\mathbf{f})_j = \sum_{k=1}^d(\delta_{jk} + R_jR_k)f_k = f_j + \sum_{k=1}^d R_jR_kf_k, \quad j = 1, \ldots, d,$$

where $\mathbf{f} = (f_1, \ldots, f_d)$.

Let $A_p : \mathcal{D}(A_p) \subset L^p_0(\mathbb{R}^d) \to L^p_0(\mathbb{R}^d)$ denote the Stokes operator. Define the nonlinear operators

$$F(\mathbf{u}, \mathbf{v}) := -P((\mathbf{u} \cdot \nabla)\mathbf{v}) \quad \text{and} \quad F(\mathbf{u}) := -P((\mathbf{u} \cdot \nabla)\mathbf{u}).$$

For exponents $p_1, p_2, p_3 \in (1, \infty)$ satisfying

$$\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2},$$

there exists $C > 0$, such that for all $\mathbf{v}_1 \in L^{p_1}(\mathbb{R}^d)$ and $\mathbf{v}_2 \in W^{1,p_2}(\mathbb{R}^d)$,

$$\|F(\mathbf{v}_1, \mathbf{v}_2)\|_{L^{p_3}(\mathbb{R}^d)} \leq C\|\mathbf{v}_1\|_{L^{p_1}(\mathbb{R}^d)}\|\nabla \mathbf{v}_2\|_{L^{p_2}(\mathbb{R}^d)}. \tag{2.2}$$

A detailed proof is available in [35].

Applying the Helmholtz-Leray projector P to the problem (1.1) eliminates the pressure term, enabling Eq (1.1) to be reformulated in the following abstract form

$${}_{CH}\mathrm{D}^{\alpha}_{a,t}\mathbf{u} = -A_p\mathbf{u} + F(\mathbf{u}), \quad t > a > 0, \tag{2.3}$$

subject to the initial condition $\mathbf{u}(\mathbf{x}, a) = \mathbf{u}_a(\mathbf{x}) \in L^p_0(\mathbb{R}^d)$.

Following the solution procedure for linear fractional differential equations presented in [28, Example 2.1], we obtain the solution $\mathbf{u}(\mathbf{x}, t)$ to Eq (2.3) in the form

$$\begin{aligned}\mathbf{u}(\mathbf{x}, t) &= E_\alpha(-(\log t - \log a)^\alpha A_p)\mathbf{u}_a \\ &\quad + \int_a^t (\log t - \log s)^{\alpha-1} E_{\alpha,\alpha}(-(\log t - \log s)^\alpha A_p)F(\mathbf{u})\,\frac{ds}{s},\end{aligned} \tag{2.4}$$

where $E_{\alpha,\beta}(z)$ denotes the two-parameter Mittag-Leffler function, defined as

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha > 0,\ \beta \in \mathbb{C},\ z \in \mathbb{C}.$$

Here, $E_\alpha(z)$ is used as a shorthand notation for $E_{\alpha,1}(z)$.

We now give the definition of the mild solution to problem (1.1).

Definition 4. *For $\alpha \in (0, 1)$ and $1 < p < \infty$.*

- *(i) A continuous function $\mathbf{u}(t) : [a, \infty) \to L^p_0(\mathbb{R}^d)$ is called a global mild solution to problem* (1.1) *in $L^p_0(\mathbb{R}^d)$ if it satisfies Eq* (2.4) *for all $t \in [a, \infty)$.*
- *(ii) A continuous function $\mathbf{u}(t) : [a, T] \to L^p_0(\mathbb{R}^d)$ is called a local mild solution to problem* (1.1) *in $L^p_0(\mathbb{R}^d)$ if there exists $T > a$ such that Eq* (2.4) *holds for all $t \in [a, T]$.*

Lemma 1. *(See [19]) Let $1 < p_1 \leq p_2 < \infty$ and $\mathbf{v} \in L^{p_1}_0(\mathbb{R}^d)$. Then there exists a constant $C(p_1, p_2, d) > 0$, such that for all $t > a > 0$*

$$\|e^{-(\log t - \log a)A_{p_1}}\mathbf{v}\|_{L^{p_2}(\mathbb{R}^d)} \leq C(\log t - \log a)^{-\frac{d}{2}\left(\frac{1}{p_1} - \frac{1}{p_2}\right)}\|\mathbf{v}\|_{L^{p_1}(\mathbb{R}^d)}, \tag{2.5}$$

$$\|\nabla e^{-(\log t - \log a)A_{p_1}}\mathbf{v}\|_{L^{p_2}(\mathbb{R}^d)} \leq C(\log t - \log a)^{-\frac{1}{2}-\frac{d}{2}\left(\frac{1}{p_1} - \frac{1}{p_2}\right)}\|\mathbf{v}\|_{L^{p_1}(\mathbb{R}^d)}. \tag{2.6}$$

Lemma 2. *Let* $\alpha \in (0,1)$ *and real numbers* p_1, p_2 *satisfy* $1 < p_1 \le p_2 < \infty$ *with* $p_2 d/(2p_2 + d) < p_1$. *The Mittag-Leffler operators* $\{E_\alpha(-(\log t - \log a)^\alpha A)\colon t \ge a\}$ *and* $\{E_{\alpha,\alpha}(-(\log t - \log a)^\alpha A)\colon t \ge a\}$ *are respectively defined by*

$$E_\alpha(-(\log t - \log a)^\alpha A) = \int_a^\infty M_\alpha(\log s - \log a) e^{-(\log s - \log a)(\log t - \log a)^\alpha A} \frac{ds}{s}, \tag{2.7}$$

and

$$\begin{aligned} &E_{\alpha,\alpha}(-(\log t - \log a)^\alpha A) \\ &= \int_a^\infty \alpha(\log s - \log a) M_\alpha(\log s - \log a) e^{-(\log s - \log a)(\log t - \log a)^\alpha A} \frac{ds}{s}, \end{aligned} \tag{2.8}$$

where the Mainardi function $M_\alpha : \mathbb{C} \to \mathbb{C}$ *is given by*

$$M_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(1 - \alpha(1+n))}.$$

For any $\mathbf{v} \in L_0^{p_1}(\mathbb{R}^d)$, *there exists a positive constant* $C(p_1, p_2, d, \alpha)$, *such that for all* $t > a > 0$,

$$\|E_\alpha(-(\log t - \log a)^\alpha A_{p_1})\mathbf{v}\|_{L^{p_2}(\mathbb{R}^d)} \le C(\log t - \log a)^{-\alpha d(1/p_1 - 1/p_2)/2} \|\mathbf{v}\|_{L^{p_1}(\mathbb{R}^d)}, \tag{2.9}$$

and

$$\|E_{\alpha,\alpha}(-(\log t - \log a)^\alpha A_{p_1})\mathbf{v}\|_{L^{p_2}(\mathbb{R}^d)} \le C(\log t - \log a)^{-\alpha d(1/p_1 - 1/p_2)/2} \|\mathbf{v}\|_{L^{p_1}(\mathbb{R}^d)}. \tag{2.10}$$

Proof. For $\mathbf{v} \in L_0^{p_1}$, applying Lemma 1 yields the existence of a constant $C(p_1, p_2, d, \alpha) > 0$, such that

$$\begin{aligned} &\|E_\alpha(-(\log t - \log a)^\alpha A_{p_1})\mathbf{v}\|_{L^{p_2}(\mathbb{R}^d)} \\ &\le \int_a^\infty M_\alpha(\log s - \log a) \|e^{-(\log s - \log a)(\log t - \log a)^\alpha A_{p_1}} \mathbf{v}\|_{L^{p_2}(\mathbb{R}^d)} \frac{ds}{s} \\ &\le C\left(\int_0^\infty M_\alpha(s) s^{-(d/p_1 - d/p_2)/2} ds\right) (\log t - \log a)^{-\alpha(d/p_1 - d/p_2)/2} \|\mathbf{v}\|_{L^{p_1}(\mathbb{R}^d)} \\ &\le C(\log t - \log a)^{-\alpha(d/p_1 - d/p_2)/2} \|\mathbf{v}\|_{L^{p_1}(\mathbb{R}^d)}, \quad t > a > 0, \end{aligned}$$

where we have utilized the property of the Mainardi function [19]

$$\int_0^\infty s^p M_\alpha(s) ds = \frac{\Gamma(p+1)}{\Gamma(\alpha p + 1)}, \quad \alpha \in (0,1), \quad -1 < p < \infty. \tag{2.11}$$

Similarly, for the second term, we have

$$\begin{aligned} &\|E_{\alpha,\alpha}(-(\log t - \log a)^\alpha A_{p_1})\mathbf{v}\|_{L^{p_2}(\mathbb{R}^d)} \\ &\le \int_a^\infty \alpha(\log s - \log a) M_\alpha(\log s - \log a) \|e^{-(\log s - \log a)(\log t - \log a)^\alpha A_{p_1}} \mathbf{v}\|_{L^{p_2}(\mathbb{R}^d)} \frac{ds}{s} \\ &\le C\alpha \int_a^\infty (\log s - \log a)^{1 - \frac{1}{2}\left(\frac{d}{p_1} - \frac{d}{p_2}\right)} M_\alpha(\log s - \log a) \\ &\qquad \times (\log t - \log a)^{-\frac{\alpha}{2}\left(\frac{d}{p_1} - \frac{d}{p_2}\right)} \|\mathbf{v}\|_{L^{p_1}(\mathbb{R}^d)} \frac{ds}{s} \quad \text{(Using Lemma 1)} \\ &= C\alpha \left(\int_0^\infty M_\alpha(s) s^{1 - \frac{1}{2}\left(\frac{d}{p_1} - \frac{d}{p_2}\right)} ds\right) (\log t - \log a)^{-\frac{\alpha}{2}\left(\frac{d}{p_1} - \frac{d}{p_2}\right)} \|\mathbf{v}\|_{L^{p_1}(\mathbb{R}^d)} \quad \text{(Using (2.11))} \\ &\le C(\log t - \log a)^{-\frac{\alpha}{2}\left(\frac{d}{p_1} - \frac{d}{p_2}\right)} \|\mathbf{v}\|_{L^{p_1}(\mathbb{R}^d)}, \qquad t > a > 0. \end{aligned}$$

The proof is complete.

Lemma 3. *Let* $\alpha \in (0,1)$ *and let* $p_1, p_2 > 1$ *satisfy*

$$\frac{p_2 d}{p_2 + d} < p_1 \leq p_2 < \infty.$$

Then for any $\mathbf{v} \in L_0^{p_1}(\mathbb{R}^d)$*, there exists a constant* $C(p_1, p_2, d, \alpha) > 0$*, such that for all* $t > a > 0$*,*

$$\left\|\nabla E_\alpha(-(\log t - \log a)^\alpha A_{p_1})\mathbf{v}\right\|_{L^{p_2}(\mathbb{R}^d)} \leq C(\log t - \log a)^{-\alpha(1+d/p_1-d/p_2)/2}\|\mathbf{v}\|_{L^{p_1}(\mathbb{R}^d)},$$

and

$$\left\|\nabla E_{\alpha,\alpha}(-(\log t - \log a)^\alpha A_{p_1})\mathbf{v}\right\|_{L^{p_2}} \leq C(\log t - \log a)^{-\alpha(1+d/p_1-d/p_2)/2}\|\mathbf{v}\|_{L^{p_1}}.$$

Proof. According to definitions Eqs (2.7) and (2.8), and by Lemma 1, it follows that for any $\mathbf{v} \in L_0^{p_1}(\mathbb{R}^d)$, one has

$$\begin{aligned}
&\|\nabla E_\alpha(-(\log t - \log a)^\alpha A_{p_1})\mathbf{v}\|_{L^{p_2}(\mathbb{R}^d)} \\
&\quad\leq \int_a^\infty M_\alpha(\log s - \log a)\|\nabla e^{-(\log s - \log a)(\log t - \log a)^\alpha A_{p_1}}\mathbf{v}\|_{L^{p_2}(\mathbb{R}^d)}\frac{ds}{s} \\
&\quad\leq \left(C\int_a^\infty M_\alpha(s)s^{-(1+d/p_1-d/p_2)/2}\,ds\right)(\log t - \log a)^{-\alpha(1+d/p_1-d/p_2)/2}\|\mathbf{v}\|_{L^{p_1}(\mathbb{R}^d)} \\
&\quad\leq C(\log t - \log a)^{-\alpha(1+d/p_1-d/p_2)/2}\|\mathbf{v}\|_{L^{p_1}(\mathbb{R}^d)},
\end{aligned}$$

and similarly,

$$\begin{aligned}
&\|\nabla E_{\alpha,\alpha}(-(\log t - \log a)^\alpha A_{p_1})\mathbf{v}\|_{L^{p_2}(\mathbb{R}^d)} \\
&\quad\leq \int_a^\infty \alpha s M_\alpha(s)\|\nabla e^{-s(\log t - \log a)^\alpha A_{p_1}}\mathbf{v}\|_{L^{p_2}(\mathbb{R}^d)}\,ds \\
&\quad\leq \left(\alpha C\int_0^\infty M_\alpha(s)s^{1-(1+d/p_1-d/p_2)/2}\,ds\right)(\log t - \log a)^{-\alpha(1+d/p_1-d/p_2)/2}\|\mathbf{v}\|_{L^{p_1}(\mathbb{R}^d)} \\
&\quad\leq C(\log t - \log a)^{-\alpha(1+d/p_1-d/p_2)/2}\|\mathbf{v}\|_{L^{p_1}(\mathbb{R}^d)}.
\end{aligned}$$

Therefore, the proof is complete.

Lemma 4. *Let* $\alpha \in (0,1)$ *and* $T \in (0,\infty]$ *be given. For both cases below, let* $\rho_1, \rho_2, \rho_3 > 0$ *satisfy* $\rho_3 \leq \rho_1 + \rho_2$ *and define the exponent* $\gamma = 1 - \frac{\rho_1+\rho_2-\rho_3}{2}$.

(i) Suppose $\rho_3 \leq \rho_1 + \rho_2 < \min\{d, 2+\rho_3\}$*. Then there exists* $C(\rho_1, \rho_2, \rho_3, d, \beta) > 0$*, such that*

$$\begin{aligned}
&\left\|\int_a^t (\log t - \log s)^{\alpha-1}E_{\alpha,\alpha}(-(\log t - \log s)^\alpha A_{d/(\rho_1+\rho_2)})F(\mathbf{v}_1(s), \mathbf{v}_2(s))\frac{ds}{s}\right\|_{L^{d/\rho_3}(\mathbb{R}^d)} \\
&\qquad\leq C\int_a^t (\log t - \log s)^{\alpha(1-(\rho_1+\rho_2-\rho_3)/2)-1}\|\mathbf{v}_1(s)\|_{L^{d/\rho_1}}\|\nabla\mathbf{v}_2(s)\|_{L^{d/\rho_2}(\mathbb{R}^d)}\frac{ds}{s} \qquad (2.12)
\end{aligned}$$

holds for all $t \in [a,T]$, $\mathbf{v}_1 \in L^\infty(a,T;L_0^{d/\rho_1}(\mathbb{R}^d))$*, and* $\mathbf{v}_2$ *with* $\nabla\mathbf{v}_2 \in L^\infty(a,T;L_0^{d/\rho_2}(\mathbb{R}^d))$.

(ii) Let $\rho_1, \rho_2, \rho_3 > 0$ *satisfy* $\rho_3 \leq \rho_1 + \rho_2 < \min\{d, 1 + \rho_3\}$. *Then there exists* $C(\rho_1, \rho_2, \rho_3, d, \alpha) > 0$, *such that*

$$\left\| \nabla \int_a^t (\log t - \log s)^{\alpha-1} E_{\alpha,\alpha}(-(\log t - \log s)^{\alpha} A_{d/(\rho_1+\rho_2)}) F(\mathbf{v}_1(s), \mathbf{v}_2(s)) \frac{ds}{s} \right\|_{L^{d/\rho_3}(\mathbb{R}^d)} \leq C \int_a^t (\log t - \log s)^{\alpha(1-(\rho_1+\rho_2-\rho_3))/2-1} \|\mathbf{v}_1(s)\|_{L^{d/\rho_1}(\mathbb{R}^d)} \|\nabla \mathbf{v}_2(s)\|_{L^{d/\rho_2}(\mathbb{R}^d)} \frac{ds}{s}. \tag{2.13}$$

Proof. Set $p_1 = d/(\rho_1 + \rho_2)$ and $p_2 = d/\rho_3$ in Lemma 2 to obtain

$$\left\| \int_a^t (\log t - \log s)^{\alpha-1} E_{\alpha,\alpha}(-(\log t - \log s)^{\alpha} A_{d/(\rho_1+\rho_2)}) F(\mathbf{v}_1(s), \mathbf{v}_2(s)) \frac{ds}{s} \right\|_{L^{d/\rho_3}(\mathbb{R}^d)} \leq C \int_a^t (\log t - \log s)^{\alpha-1} (\log t - \log s)^{-\alpha(\rho_1+\rho_2-\rho_3)/2} \|F(\mathbf{v}_1(s), \mathbf{v}_2(s))\|_{L^{d/\rho_1+\rho_2}(\mathbb{R}^d)} \frac{ds}{s}.$$

By Eq (2.2), this implies

$$\left\| \int_a^t (\log t - \log s)^{\alpha-1} E_{\alpha,\alpha}(-(\log t - \log s)^{\alpha} A_{d/(\rho_1+\rho_2)}) F(\mathbf{v}_1(s), \mathbf{v}_2(s)) \frac{ds}{s} \right\|_{L^{d/\rho_3}(\mathbb{R}^d)} \leq C \int_a^t (\log t - \log s)^{\alpha(1-(\rho_1+\rho_2-\rho_3)/2)-1} \|\mathbf{v}_1(s)\|_{L^{d/\rho_1}(\mathbb{R}^d)} \|\nabla \mathbf{v}_2(s)\|_{L^{d/\rho_2}(\mathbb{R}^d)} \frac{ds}{s}.$$

This completes (i).

Apply Lemma 3 with $p_1 = d/(\rho_1 + \rho_2)$, $p_2 = d/\rho_3$ to bound

$$\begin{aligned} &\left\| \nabla \int_a^t (\log t - \log s)^{\alpha-1} E_{\alpha,\alpha}(-(\log t - \log s)^{\alpha} A_{d/(\rho_1+\rho_2)}) F(\mathbf{v}_1(s), \mathbf{v}_2(s)) \frac{ds}{s} \right\|_{L^{d/\rho_3}(\mathbb{R}^d)} \\ &\leq C \int_a^t (\log t - \log s)^{\alpha-1} (\log t - \log s)^{-\alpha(1+\rho_1+\rho_2-\rho_3)/2} \|F(\mathbf{v}_1(s), \mathbf{v}_2(s))\|_{L^{d/(\rho_1+\rho_2)}(\mathbb{R}^d)} \frac{ds}{s} \\ &\leq C \int_a^t (\log t - \log s)^{\alpha(1-(\rho_1+\rho_2-\rho_3))/2-1} \|\mathbf{v}_1(s)\|_{L^{d/\rho_1}(\mathbb{R}^d)} \|\nabla \mathbf{v}_2(s)\|_{L^{d/\rho_2}(\mathbb{R}^d)} \frac{ds}{s}. \end{aligned}$$

This proves (ii).

Lemma 5. *For any* $\alpha, \theta \in (0, 1)$ *and* $\mathbf{v} \in L_0^d(\mathbb{R}^d)$, *the following properties hold*

$$(\log t - \log a)^{\alpha(1-\theta)/2} E_{\alpha}(-(\log t - \log a)^{\alpha} A_d)\mathbf{v} \in C_b([a, \infty); L_0^{d/\theta}(\mathbb{R}^d)),$$

$$(\log t - \log a)^{\alpha/2} \nabla E_{\alpha}(-(\log t - \log a)^{\alpha} A_d)\mathbf{v} \in C_b([a, \infty); L_0^{d}(\mathbb{R}^d)).$$

Furthermore, both expressions vanish at t = a.

Proof. Using Lemma 1 with $p_1 = d$ and $p_2 = d/\theta$, we obtain the following estimate

$$(\log t - \log a)^{(1-\theta)/2} \left\| e^{-(\log t - \log a)A_d} \mathbf{v} \right\|_{L^{d/\theta}(\mathbb{R}^d)} \leq C \|\mathbf{v}\|_{L^d(\mathbb{R}^d)}, \quad t > a > 0.$$

Let $\varepsilon > 0$. Since Eq (2.11) holds, for any $\epsilon > 0$, there exist sufficiently small constants $0 < \delta_1 < \delta_2$, such that the following inequalities hold:

$$2C\|\mathbf{v}\|_{L^d(\mathbb{R}^d)} \int_0^{\delta_1} s^{-(1-\theta)/2} M_\alpha(s)\, ds < \varepsilon/3, \tag{2.14}$$

$$2C\|\mathbf{v}\|_{L^d(\mathbb{R}^d)} \int_{\delta_2}^{\infty} s^{-(1-\theta)/2} M_\alpha(s)\, ds < \varepsilon/3. \tag{2.15}$$

Fix $t_0 \geq a$. Since $t^{(1-\theta)/2} e^{-tA_d}\mathbf{v} \in C([0,\infty); L_0^{d/\theta}(\mathbb{R}^d))$ (cf. [35, Eq (2.7)]) and $s(\log t - \log a)^\alpha$ is uniformly continuous for $\delta_1 \leq s \leq \delta_2$, there exists $\delta > 0$, satisfying

$$\begin{aligned}\int_{\delta_1}^{\delta_2} s^{-(1-\theta)/2} M_\alpha(s) \Big\| &(s(\log t - \log a)^\alpha)^{(1-\theta)/2} e^{-s(\log t - \log a)^\alpha A_d}\mathbf{v} \\ &- (s(\log t_0 - \log a)^\alpha)^{(1-\theta)/2} e^{-s(\log t_0 - \log a)^\alpha A_d}\mathbf{v}\Big\|_{L^{d/\theta}(\mathbb{R}^d)} ds < \varepsilon/3\end{aligned}$$

for all $t \geq a$ with $0 < |t - t_0| < \delta$.

By applying the definition of the Mittag-Leffler function and utilizing Eq (2.5), we deduce that

$$\begin{aligned}&\Big\|(\log t - \log a)^{\alpha(1-\theta)/2} E_\alpha(-(\log t - \log a)^\alpha A_d)\mathbf{v} \\ &- (\log t_0 - \log a)^{\alpha(1-\theta)/2} E_\alpha(-(\log t_0 - \log a)^\beta A_d)\mathbf{v}\Big\|_{L^{d/\theta}(\mathbb{R}^d)} \\ &\leq \int_a^\infty (\log s - \log a)^{-(1-\theta)/2} M_\alpha(\log s - \log a) \Big\|((\log s - \log a)(\log t - \log a)^\alpha)^{(1-\theta)/2} \\ &\quad \times e^{-(\log s - \log a)(\log t - \log a)^\alpha A_d}\mathbf{v} - ((\log s - \log a)(\log t_0 - \log a)^\alpha)^{(1-\theta)/2} \\ &\quad \times e^{-(\log s - \log a)(\log t_0 - \log a)^\alpha A_d}\mathbf{v}\Big\|_{L^{d/\theta}(\mathbb{R}^d)} \frac{ds}{s} \\ &\leq 2C\|\mathbf{v}\|_{L^d(\mathbb{R}^d)} \left(\int_0^{\delta_1} + \int_{\delta_2}^\infty\right) s^{-(1-\theta)/2} M_\alpha(s) ds \\ &\quad + \int_{\delta_1}^{\delta_2} s^{-(1-\theta)/2} M_\alpha(s) \Big\|(s(\log t - \log a)^\alpha)^{(1-\theta)/2} e^{-s(\log t - \log a)^\alpha A_d}\mathbf{v} \\ &\quad - (s(\log t_0 - \log a)^\alpha)^{(1-\theta)/2} e^{-s(\log t_0 - \log a)^\alpha A_d}\mathbf{v}\Big\|_{L^{d/\theta}(\mathbb{R}^d)} ds < \varepsilon.\end{aligned}$$

The divergence-free property is inherent. Combining this with the above results shows that

$$(\log t - \log a)^{\alpha(1-\theta)/2} E_\alpha(-(\log t - \log a)^\alpha A_d)\mathbf{v} \in C_b([a,\infty); L_0^{d/\theta}(\mathbb{R}^d)).$$

To verify the vanishing property at $t = a$, we set $r_1 = r_2 = N$ in Lemma 2, and obtain

$$\begin{aligned}&\lim_{t\to a^+} (\log t - \log a)^{\alpha(1-\theta)/2} \|E_\alpha(-(\log t - \log a)^\alpha A_d)\mathbf{v}\|_{L^d(\mathbb{R}^d)} \\ &\qquad \leq C \lim_{t\to a^+} (\log t - \log a)^{\alpha(1-\theta)/2} \|\mathbf{v}\|_{L^d(\mathbb{R}^d)} = 0.\end{aligned}$$

Similarly, applying Lemma 1 with $p_1 = p_2 = d$ and noting that

$$(\log t - \log a)^{1/2} \nabla e^{-(\log t - \log a)A_d}\mathbf{v} \in C_b([a,\infty); L_0^d(\mathbb{R}^d))$$

with

$$\lim_{t\to a^+}(\log t-\log a)^{1/2}\|\nabla e^{-(\log t-\log a)A_d}\mathbf{v}\|_{L^d(\mathbb{R}^d)}=0,$$

we establish the continuity and boundedness of $(\log t-\log a)^{\alpha/2}\nabla E_\alpha(-(\log t-\log a)^\alpha A_d)\mathbf{v}$ in $L^d_0(\mathbb{R}^d)$. The estimate

$$\begin{aligned}&(\log t-\log a)^{\alpha/2}\|\nabla E_\alpha(-(\log t-\log a)^\alpha A_d)\mathbf{v}\|_{L^d(\mathbb{R}^d)}\\&\qquad\le\int_0^\infty M_\alpha(s)s^{-1/2}\|[s(\log t-\log a)^\alpha]^{1/2}\nabla e^{-s(\log t-\log a)^\alpha A_d}\mathbf{v}\|_{L^d(\mathbb{R}^d)}ds,\end{aligned}$$

combined with the Dominated Convergence Theorem, implies $\lim_{t\to a^+}(\log t-\log a)^{\alpha/2}\nabla E_\alpha(-(\log t-\log a)^\alpha A_d)\mathbf{v}=0$.

Similarly, applying Lemma 1 with $p_1=p_2=d$ and noting that

$$(\log t-\log a)^{1/2}\nabla e^{-(\log t-\log a)A_d}\mathbf{v}\in C_b([a,\infty);L^d_0(\mathbb{R}^d))$$

with

$$\lim_{t\to a^+}(\log t-\log a)^{1/2}\|\nabla e^{-(\log t-\log a)A_d}\mathbf{v}\|_{L^d(\mathbb{R}^d)}=0,$$

we establish the continuity and boundedness of $(\log t-\log a)^{\alpha/2}\nabla E_\alpha(-(\log t-\log a)^\alpha A_d)\mathbf{v}$ in $L^d_0(\mathbb{R}^d)$. The estimate

$$\begin{aligned}&(\log t-\log a)^{\alpha/2}\|\nabla E_\alpha(-(\log t-\log a)^\alpha A_d)\mathbf{v}\|_{L^d(\mathbb{R}^d)}\\&\qquad\le\int_0^\infty M_\alpha(s)s^{-1/2}\|[s(\log t-\log a)^\alpha]^{1/2}\nabla e^{-s(\log t-\log a)^\alpha A_d}\mathbf{v}\|_{L^d(\mathbb{R}^d)}ds,\end{aligned}$$

combined with the Dominated Convergence Theorem, implies

$$\lim_{t\to a^+}(\log t-\log a)^{\alpha/2}\nabla E_\alpha(-(\log t-\log a)^\alpha A_d)\mathbf{v}=0. \tag{2.16}$$

Indeed, set $\varepsilon=\log t-\log a>0$. Then the estimate becomes

$$\varepsilon^{\alpha/2}\left\|\nabla E_\alpha(-\varepsilon^\alpha A_d)\mathbf{v}\right\|_{L^d(\mathbb{R}^d)}\le\varepsilon^{\alpha/2}\int_0^\infty M_\alpha(s)\left\|\nabla e^{-s\varepsilon^\alpha A_d}\mathbf{v}\right\|_{L^d(\mathbb{R}^d)}ds.$$

Define $I(\varepsilon):=\int_0^\infty M_\alpha(s)\|\nabla e^{-s\varepsilon^\alpha A_d}\mathbf{v}\|_{L^d(\mathbb{R}^d)}ds$, so that

$$\varepsilon^{\alpha/2}\|\nabla E_\alpha(-\varepsilon^\alpha A_d)\mathbf{v}\|_{L^d(\mathbb{R}^d)}\le\varepsilon^{\alpha/2}I(\varepsilon). \tag{2.17}$$

As $\varepsilon\to0^+$, for each fixed $s>0$ the strong continuity of the semigroup yields $e^{-s\varepsilon^\alpha A_d}\to I$ and hence $\nabla e^{-s\varepsilon^\alpha A_d}\mathbf{v}\to\nabla\mathbf{v}$ in $L^d(\mathbb{R}^d)$. Therefore, the integrand converges pointwise

$$M_\alpha(s)\|\nabla e^{-s\varepsilon^\alpha A_d}\mathbf{v}\|_{L^d(\mathbb{R}^d)}\to M_\alpha(s)\|\nabla\mathbf{v}\|_{L^d(\mathbb{R}^d)},\quad\forall s>0.$$

To apply the Dominated Convergence Theorem, we need an ε-independent integrable dominating function. By the smoothing properties of the semigroup generated by A_d, there exists a constant $C>0$, such that $\|\nabla e^{-s\varepsilon^{\alpha}A_d}\mathbf{v}\|_{L^d(\mathbb{R}^d)} \leq C\|\mathbf{v}\|_{L^d(\mathbb{R}^d)}$ for all $s, \varepsilon > 0$. Moreover, $\int_0^\infty M_\alpha(s)ds < \infty$. Hence, $g(s) := C\|\mathbf{v}\|_{L^d(\mathbb{R}^d)}M_\alpha(s)$ is an integrable dominating function

$$M_\alpha(s)\|\nabla e^{-s\varepsilon^{\alpha}A_d}\mathbf{v}\|_{L^d(\mathbb{R}^d)} \leq g(s) \quad \text{a.e. } s > 0,\ \forall \varepsilon > 0.$$

By the Dominated Convergence Theorem,

$$\lim_{\varepsilon\to 0^+} I(\varepsilon) = \int_0^\infty M_\alpha(s)\|\nabla\mathbf{v}\|_{L^d(\mathbb{R}^d)}ds = \|\nabla\mathbf{v}\|_{L^d(\mathbb{R}^d)}\int_0^\infty M_\alpha(s)ds < \infty.$$

Thus, $I(\varepsilon)$ is bounded for small ε, say $I(\varepsilon) \leq K$. Substituting into Eq (2.17) gives

$$\varepsilon^{\alpha/2}\|\nabla E_\alpha(-\varepsilon^\alpha A_d)\mathbf{v}\|_{L^d(\mathbb{R}^d)} \leq K\varepsilon^{\alpha/2} \to 0 \quad (\varepsilon \to 0^+).$$

Since $\varepsilon = \log t - \log a \to 0^+$ if $t \to a^+$, we conclude Eq (2.16). This completes the proof.

Lemma 6. *Let* $\alpha, \theta \in (0, 1)$ *and* $T \in (0, \infty]$. *For functions* $\mathbf{v}_1$, $\mathbf{v}_2$ *satisfying* $(\log t - \log a)^{\alpha(1-\theta)/2}\mathbf{v}_1 \in C_b([a, T]; L_0^{d/\theta}(\mathbb{R}^d))$ *and* $(\log t - \log a)^{\alpha/2}\nabla\mathbf{v}_2 \in C_b([a, T]; L_0^d(\mathbb{R}^d))$, *the following properties hold*

$$(\log t - \log a)^{\alpha(1-\theta)/2}\int_a^t (\log t - \log s)^{\alpha-1}E_{\alpha,\alpha}\left(-(\log t - \log s)^\alpha A_{d/(1+\theta)}\right)F(\mathbf{v}_1(s), \mathbf{v}_2(s))\frac{ds}{s} \in C_b([a, T]; L_0^{d/\theta}(\mathbb{R}^d)), \quad (2.18)$$

$$(\log t - \log a)^{\alpha/2}\nabla\int_a^t (\log t - \log s)^{\alpha-1}E_{\alpha,\alpha}\left(-(\log t - \log s)^\alpha A_{d/(1+\theta)}\right)F(\mathbf{v}_1(s), \mathbf{v}_2(s))\frac{ds}{s} \in C_b([a, T]; L_0^d(\mathbb{R}^d)). \quad (2.19)$$

Both expressions vanish at the initial time t = a.

Proof. For a fixed $t_0 > a > 0$, we proceed to establish the case for $t > t_0$; the scenario $t < t_0$ follows similarly. First, observe that

$$\begin{aligned}
&\Big\|(\log t - \log a)^{\alpha(1-\theta)/2}\int_a^t (\log t - \log s)^{\alpha-1}E_{\alpha,\alpha}\big(-(\log t - \log s)^\alpha A_{d/(1+\theta)}\big)F(\mathbf{v}_1(s), \mathbf{v}_2(s))\frac{ds}{s}\\
&\quad - (\log t_0 - \log a)^{\alpha(1-\theta)/2}\int_a^{t_0} (\log t_0 - \log s)^{\alpha-1}E_{\alpha,\alpha}\left(-(\log t_0 - \log s)^\alpha A_{d/(1+\theta)}\right)\\
&\quad \times F(\mathbf{v}_1(s), \mathbf{v}_2(s))\frac{ds}{s}\Big\|_{L^{d/\theta}(\mathbb{R}^d)}\\
&\leq (\log t - \log a)^{\alpha(1-\theta)/2}\int_{t_0}^t (\log t - \log s)^{\alpha-1}\Big\|E_{\alpha,\alpha}\left(-(\log t - \log s)^\alpha A_{d/(1+\theta)}\right)\\
&\quad \times F(\mathbf{v}_1(s), \mathbf{v}_2(s))\Big\|_{L^{N/\theta}(\mathbb{R}^d)}\frac{ds}{s}\\
&\quad + \int_a^{t_0}\Big\|\Big[(\log t - \log s)^{\alpha-1}(\log t - \log a)^{\alpha(1-\theta)/2}E_{\alpha,\alpha}\left(-(\log t - \log s)^\alpha A_{d/(1+\theta)}\right)\\
&\quad - (\log t_0 - \log s)^{\alpha-1}(\log t_0 - \log a)^{\alpha(1-\theta)/2}E_{\alpha,\alpha}\left(-(\log t_0 - \log s)^\alpha A_{d/(1+\theta)}\right)\Big]\\
&\quad \times F(\mathbf{v}_1(s), \mathbf{v}_2(s))\Big\|_{L^{d/\theta}(\mathbb{R}^d)}\frac{ds}{s}\\
&=: \Theta_1(t) + \Theta_2(t),
\end{aligned} \quad (2.20)$$

which will be estimated separately.

Combining Lemma 2 with Eq (2.2) yields

$$\left\|E_{\alpha,\alpha}\left(-(\log\tau-\log a)^{\alpha}A_{d/(1+\theta)}\right)F(\mathbf{v}_1(s),\mathbf{v}_2(s))\right\|_{L^{d/\theta}(\mathbb{R}^d)}$$
$$\leq C(\log\tau-\log a)^{-\alpha/2}\|\mathbf{v}_1(s)\|_{L^{d/\theta}(\mathbb{R}^d)}\|\nabla\mathbf{v}_2(s)\|_{L^d(\mathbb{R}^d)},\quad \tau,s\in[a,T].$$

By the regularity assumptions, there exists a constant $C'>0$, such that

$$\max\left\{\sup_{s\in[a,T]}(\log s-\log a)^{\alpha(1-\theta)/2}\|\mathbf{v}_1(s)\|_{L^{d/\theta}(\mathbb{R}^d)},\ \sup_{s\in[a,T]}(\log s-\log a)^{\alpha/2}\|\nabla\mathbf{v}_2(s)\|_{L^d(\mathbb{R}^d)}\right\}\leq C'.$$

For arbitrary $\varepsilon>0$, properties of the Beta function guarantee $\delta>0$, such that $0<t-t_0<\delta$ implies

$$CC'^2\int_{\frac{\log t_0-\log a}{\log t-\log a}}^{1}(1-s)^{\alpha/2-1}s^{-\alpha(2-\theta)/2}ds<\varepsilon/2.$$

Consequently, we obtain

$$\begin{aligned}\Theta_1(t)&\leq C(\log t-\log a)^{\alpha(1-\theta)/2}\int_{t_0}^{t}(\log t-\log s)^{(\alpha/2)-1}\|\mathbf{v}_1(s)\|_{L^{d/\theta}(\mathbb{R}^d)}\|\nabla\mathbf{v}_2(s)\|_{L^d(\mathbb{R}^d)}\frac{ds}{s}\\&\leq CC'^2\int_{\frac{\log t_0-\log a}{\log t-\log a}}^{1}(1-s)^{\alpha/2-1}s^{-\alpha(2-\theta)/2}ds<\varepsilon/2.\end{aligned}\tag{2.21}$$

Since the Mittag-Leffler operators are analytic in time, for $\delta>0$ sufficiently small and $0<t-t_0<\delta$, we have

$$\begin{aligned}\Theta_2(t)=&\left\|\int_a^{t_0}\Big[(\log t-\log s)^{\alpha-1}(\log t-\log a)^{\alpha(1-\theta)/2}E_{\alpha,\alpha}\left(-(\log t-\log s)^{\alpha}A_{d/(1+\theta)}\right)\right.\\&\times F(\mathbf{v}_1(s),\mathbf{v}_2(s))-(\log t_0-\log s)^{\alpha-1}(\log t_0-\log a)^{\alpha(1-\theta)/2}\\&\left.\times E_{\alpha,\alpha}\big(-(\log t_0-\log s)^{\alpha}A_{d/(1+\theta)}\big)F(\mathbf{v}_1(s),\mathbf{v}_2(s))\Big]\right\|_{L^{d/\theta}(\mathbb{R}^d)}\frac{ds}{s}\\&<\varepsilon/2.\end{aligned}$$

Combining this with Eqs (2.20) and (2.21) yields

$$\begin{aligned}&\left\|(\log t-\log a)^{\alpha(1-\theta)/2}\int_a^t(\log t-\log s)^{\alpha-1}E_{\alpha,\alpha}\big(-(\log t-\log s)^{\alpha}A_{d/(1+\theta)}\big)F(\mathbf{v}_1(s),\mathbf{v}_2(s))\frac{ds}{s}\right.\\&-(\log t_0-\log a)^{\alpha(1-\theta)/2}\int_a^{t_0}(\log t_0-\log s)^{\alpha-1}E_{\alpha,\alpha}\left(-(\log t_0-\log s)^{\alpha}A_{d/(1+\theta)}\right)\\&\left.\times F(\mathbf{v}_1(s),\mathbf{v}_2(s))\frac{ds}{s}\right\|_{L^{d/\theta}(\mathbb{R}^d)}<\varepsilon\end{aligned}$$

for $0<t-t_0<\delta$.

Next, we verify that Eq (2.18) vanishes at $t=a$. Applying Lemma 4 with $\rho_1=\theta,\rho_2=1$, and $\rho_3=1$ yields the estimate

$$\begin{aligned}&(\log t-\log a)^{\alpha(1-\theta)/2}\left\|\int_a^t(\log t-\log s)^{\alpha-1}E_{\alpha,\alpha}\left(-(\log t-\log s)^{\alpha}A_{d/(1+\theta)}\right)F(\mathbf{v}_1(s),\mathbf{v}_2(s))\frac{ds}{s}\right\|_{L^d(\mathbb{R}^d)}\\&\leq C(\log t-\log a)^{\alpha(1-\theta)/2},\quad t>a>0.\end{aligned}$$

The proof for Eq (2.19) follows similarly. This completes the proof.

3. The major results

To prove the existence and uniqueness of solutions to Eq (1.1), we first introduce the following fixed point lemma.

Lemma 7. *(See [36]) Consider a Banach space* $(X, \|\cdot\|_X)$ *and a bilinear operator* $\mathcal{H} : X \times X \to X$. *Suppose there exists* $L > 0$, *satisfying*

$$\|\mathcal{H}(u, v)\|_X \le L\|u\|_X\|v\|_X \quad \text{for all } u, v \in X.$$

Then for any initial element $x_0 \in X$ *with* $\|x_0\|_X < \frac{1}{4L}$, *the equation*

$$x = x_0 + \mathcal{H}(x, x)$$

admits a unique solution $x \in X$.

Define the bilinear operator $\mathcal{B}(\mathbf{v}_1(t), \mathbf{v}_2(t))$ as

$$\mathcal{B}(\mathbf{v}_1(t), \mathbf{v}_2(t)) := \int_a^t (\log t - \log s)^{\alpha-1} E_{\alpha,\alpha}(-(\log t - \log s)^{\alpha} A_p) F(\mathbf{v}_1(s), \mathbf{v}_2(s)) \frac{ds}{s}.$$

Based on Eq (2.4), the solution to problem (1.1) can be expressed as

$$\mathbf{u}(\mathbf{x}, t) = E_{\alpha}(-(\log t - \log a)^{\alpha} A_p)\mathbf{u}_a + \mathcal{B}(\mathbf{u}(\mathbf{x}, t), \mathbf{u}(\mathbf{x}, t)).$$

3.1. Global mild solution ($T = \infty$)

In this subsection, we focus on investigating the existence and uniqueness of global mild solutions to Eq (1.1), as well as their logarithmic decay properties.

For $\alpha \in (0, 1)$ and $2 \le N < q < \infty$, define the Banach space $\mathcal{X}_q^{\alpha}$ as the set of continuous functions $\mathbf{v}(t) : [a, \infty) \to L_0^d(\mathbb{R}^d)$, satisfying

$$(\log t - \log a)^{\alpha[1-(d/q)]/2}\mathbf{v}(t) \in C_b([a, \infty); L_0^q(\mathbb{R}^d)) \quad \text{with} \quad \lim_{t \to a^+} (\log t - \log a)^{\alpha[1-(d/q)]/2}\mathbf{v}(t) = \mathbf{0},$$

and

$$(\log t - \log a)^{\alpha/2}\nabla\mathbf{v}(t) \in C_b([a, \infty); L_0^d(\mathbb{R}^d)) \quad \text{with} \quad \lim_{t \to a^+} (\log t - \log a)^{\alpha/2}\nabla\mathbf{v}(t) = \mathbf{0}.$$

The associated norm is given by

$$\begin{aligned} \|\mathbf{v}\|_{\mathcal{X}_q^{\alpha}} := & \sup_{t \ge a} \|\mathbf{v}(t)\|_{L^d(\mathbb{R}^d)} + \sup_{t \ge a} (\log t - \log a)^{\alpha[1-(d/q)]/2}\|\mathbf{v}(t)\|_{L^q(\mathbb{R}^d)} \\ & + \sup_{t \ge a} (\log t - \log a)^{\alpha/2}\|\nabla\mathbf{v}(t)\|_{L^d(\mathbb{R}^d)}. \end{aligned} \tag{3.1}$$

Lemma 8. *Let* $\mathbf{u}_a \in L_0^d(\mathbb{R}^d)$, $\alpha \in (0, 1)$, *and q be a real number such that* $2 \le d < q < \infty$. *Then, for all* $t \ge a$, *the operator* $E_{\alpha}(-(\log t - \log a)^{\alpha} A_d)\mathbf{u}_a \in \mathcal{X}_q^{\alpha}$, *and there exists a positive constant* $M(d, q, \alpha)$, *such that*

$$\|E_{\alpha}(-(\log t - \log a)^{\alpha} A_d)\mathbf{u}_a\|_{\mathcal{X}_q^{\alpha}} \le M\|\mathbf{u}_a\|_{L^d(\mathbb{R}^d)}.$$

Proof. By Lemmas 2 and 5 with $\theta = d/q$, we obtain $E_\alpha(-(\log t - \log a)^\alpha A_d)\mathbf{u}_a \in C_b([a,\infty); L_0^d(\mathbb{R}^d)) \cap \mathcal{X}_q^\alpha$.

By selecting $p_1 = p_2 = d$ in Lemma 2, we obtain $\|E_\alpha(-(\log t - \log a)^\alpha A_d)\mathbf{u}_a\|_{L^d(\mathbb{R}^d)} \leq C_2\|\mathbf{u}_a\|_{L^d(\mathbb{R}^d)}$. Setting $p_1 = d$ and $p_2 = q$ in the same lemma yields

$$(\log t - \log a)^{\alpha[1-(d/q)]/2}\|E_\alpha(-(\log t - \log a)^\alpha A_d)\mathbf{u}_a\|_{L^q(\mathbb{R}^d)} \leq C\|\mathbf{u}_a\|_{L^d(\mathbb{R}^d)}.$$

Similarly, choosing $p_1 = p_2 = d$ in Lemma 3 gives

$$(\log t - \log a)^{\alpha/2}\|\nabla E_\alpha(-(\log t - \log a)^\alpha A_d)\mathbf{u}_a\|_{L^d(\mathbb{R}^d)} \leq C\|\mathbf{u}_a\|_{L^d(\mathbb{R}^d)}.$$

These bounds, combined with Eq (3.1), complete the proof.

Lemma 9. *For $\alpha \in (0,1)$ and $2 \leq d < q < \infty$, the bilinear operator $\mathcal{B} : \mathcal{X}_q^\alpha \times \mathcal{X}_q^\alpha \to \mathcal{X}_q^\alpha$ is continuous.*

Proof. Let $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{X}_\alpha^q$. Setting $\theta = d/q$ in Lemma 6 yields that $(\log t - \log a)^{\alpha(1-d/q)/2}\mathcal{B}(\mathbf{v}_1, \mathbf{v}_2) \in C_b([a,\infty); L_0^q(\mathbb{R}^d))$ and $(\log t - \log a)^{\alpha/2}\nabla\mathcal{B}(\mathbf{v}_1, \mathbf{v}_2) \in C_b([a,\infty); L_0^d(\mathbb{R}^d))$, with both expressions vanishing as $t \to a^+$.

Fix $t_0 \geq a > 0$ and consider $t > t_0 \geq a$. Applying Lemma 4 with $\rho_1 = d/q$, $\rho_2 = 1$, and $\rho_3 = 1$, we obtain the following estimate

$$\begin{aligned}
&\|\mathcal{B}(\mathbf{v}_1(t), \mathbf{v}_2(t)) - \mathcal{B}(\mathbf{v}_1(t_0), \mathbf{v}_2(t_0))\|_{L^d(\mathbb{R}^d)} \\
&\leq C\|\mathbf{v}_1\|_{\mathcal{X}_q^\alpha}\|\mathbf{v}_2\|_{\mathcal{X}_q^\alpha}\int_{t_0}^t (\log t - \log s)^{\alpha[1-(d/2q)]-1}(\log s - \log a)^{-\alpha[1-(d/2q)]}\frac{ds}{s} \\
&\quad + \int_a^{t_0} \Big\|[(\log t - \log s)^{\alpha-1}E_{\alpha,\alpha}(-(\log t - \log s)^\alpha A_d) \\
&\quad - (\log t_0 - \log s)^{\alpha-1}E_{\alpha,\alpha}(-(\log t_0 - \log s)^\alpha A_d)]F(\mathbf{v}_1(s), \mathbf{v}_2(s))\Big\|_{L^d(\mathbb{R}^d)}\frac{ds}{s}.
\end{aligned}$$

Using properties of the Beta function and continuity of Mittag-Leffler operators, for any $\varepsilon > 0$, there exists $\delta > 0$, such that $\|\mathcal{B}(\mathbf{v}_1(t), \mathbf{v}_2(t)) - \mathcal{B}(\mathbf{v}_1(t_0), \mathbf{v}_2(t_0))\|_{L^d(\mathbb{R}^d)} < \varepsilon$ whenever $|t - t_0| < \delta$.

We proceed analogously for $t_0 > t > a$ to show that $B(\mathbf{v}_1(t), \mathbf{v}_2(t)) \in C([a,\infty); L_0^d(\mathbb{R}^d))$. Applying Lemma 4 with $\rho_1 = d/q$, $\rho_2 = 1$, and $\rho_3 = 1$ yields the estimate

$$\begin{aligned}
&\left\|\int_a^t (\log t - \log s)^{\alpha-1}E_{\alpha,\alpha}(-(\log t - \log s)^\alpha A_d)F(\mathbf{v}_1(s), \mathbf{v}_2(s))\frac{ds}{s}\right\|_{L^d(\mathbb{R}^d)} \\
&\leq C\int_a^t (\log t - \log s)^{\alpha[1-(d/2q)]-1}\|\mathbf{v}_1(s)\|_{L^q}\|\nabla\mathbf{v}_2(s)\|_{L^d(\mathbb{R}^d)}\frac{ds}{s} \\
&\leq C\|\mathbf{v}_1\|_{\mathcal{X}_q^\alpha}\|\mathbf{v}_2\|_{\mathcal{X}_q^\alpha}\int_a^t (\log t - \log s)^{\alpha[1-(d/2q)]-1}(\log s - \log a)^{-\alpha[1-(d/2q)]}\frac{ds}{s} \\
&\leq C\|\mathbf{v}_1\|_{\mathcal{X}_q^\alpha}\|\mathbf{v}_2\|_{\mathcal{X}_q^\alpha}
\end{aligned}$$

for all $t \geq a$, establishing $\mathcal{B}(\mathbf{v}_1(t), \mathbf{v}_2(t)) \in C_b([a,\infty); L_0^d(\mathbb{R}^d))$.

Further estimates are obtained through appropriate parameter choices in Lemma 4. Taking $\rho_1 = d/q$, $\rho_2 = 1$, and $\rho_3 = d/q$ gives

$$\begin{aligned}
&(\log t - \log a)^{\alpha[1-(d/q)]/2}\Big\| \int_a^t (\log t - \log s)^{\alpha-1} E_{\alpha,\alpha}(-(\log t - \log s)^{\alpha} A_{d/(\rho_1+\rho_2)}) \\
&\quad \times F(\mathbf{v}_1(s), \mathbf{v}_2(s)) \frac{ds}{s} \Big\|_{L^q(\mathbb{R}^d)} \\
&\le C\|\mathbf{v}_1\|_{\mathcal{X}_q^\alpha}\|\mathbf{v}_2\|_{\mathcal{X}_q^\alpha}(\log t - \log a)^{\alpha[1-(d/q)]/2} \int_a^t (\log t - \log s)^{\alpha/2-1}(\log s - \log a)^{-\alpha[1-(d/2q)]}\frac{ds}{s} \\
&\le C\|\mathbf{v}_1\|_{\mathcal{X}_q^\alpha}\|\mathbf{v}_2\|_{\mathcal{X}_q^\alpha}, \quad t \ge a > 0.
\end{aligned}$$

Similarly, with $\rho_1 = d/q$, $\rho_2 = 1$, and $\rho_3 = 1$, we derive

$$\begin{aligned}
&(\log t - \log a)^{\alpha/2}\left\|\nabla \int_a^t (\log t - \log s)^{\alpha-1} E_{\alpha,\alpha}(-(\log t - \log s)^{\alpha} A_{d/(\rho_1+\rho_2)}) F(\mathbf{v}_1(s), \mathbf{v}_2(s)) \frac{ds}{s}\right\|_{L^d(\mathbb{R}^d)} \\
&\le C\|\mathbf{v}_1\|_{\mathcal{X}_q^\alpha}\|\mathbf{v}_2\|_{\mathcal{X}_q^\alpha}(\log t - \log a)^{\alpha/2} \int_a^t (\log t - \log s)^{\{\alpha[1-(d/q)]/2\}-1}(\log s - \log a)^{-\alpha[1-(d/2q)]}\frac{ds}{s} \\
&\le C\|\mathbf{v}_1\|_{\mathcal{X}_q^\alpha}\|\mathbf{v}_2\|_{\mathcal{X}_q^\alpha}, \quad t \ge a > 0.
\end{aligned}$$

These estimates collectively prove the continuity of the bilinear operator $\mathcal{B}$ in $\mathcal{X}_q^\alpha \times \mathcal{X}_q^\alpha$.

Theorem 1. *Let* $\alpha \in (0, 1)$. *There exists* $\lambda > 0$, *such that for any initial data* $\mathbf{u}_a \in L_0^d(\mathbb{R}^d)$ *with* $\|\mathbf{u}_a\|_{L^d(\mathbb{R}^d)} \le \lambda$, *problem* (1.1) *has a unique global mild solution* $\mathbf{u}(t) : [a, \infty) \to L_0^d(\mathbb{R}^d)$. *Moreover, the solution satisfies the following decay estimates:*

(i) For $2 \le N \le q \le \infty$,

$$(\log t - \log a)^{\alpha[1-(d/q)]/2}\mathbf{u} \in C_b([a, \infty); L_0^q(\mathbb{R}^d)), \tag{3.2}$$

vanishing at $t = a$ *except when* $q = N$, *where* $\mathbf{u}(\mathbf{x}, a) = \mathbf{u}_a(\mathbf{x})$.

(ii) For $2 \le N \le q < \infty$,

$$(\log t - \log a)^{\alpha[1-(d/2q)]}\nabla\mathbf{u} \in C_b([a, \infty); L_0^q(\mathbb{R}^d)), \tag{3.3}$$

vanishing at $t = a$.

Proof. Fix q such that $2 \le d < q < \infty$. Lemma 8 implies the existence of a constant $M_q > 0$, satisfying

$$\left\|E_\alpha(-(\log t - \log a)^\alpha A_d)\mathbf{u}_a\right\|_{\mathcal{X}_q^\alpha} \le M_q\|\mathbf{u}_a\|_{L^d(\mathbb{R}^d)}, \quad \forall t \ge a > 0. \tag{3.4}$$

Additionally, Lemma 9 yields $L_q > 0$, satisfying the estimate

$$\left\|\mathcal{B}(\mathbf{v}_1, \mathbf{v}_2)\right\|_{\mathcal{X}_q^\alpha} \le L_q\|\mathbf{v}_1\|_{\mathcal{X}_q^\alpha}\|\mathbf{v}_2\|_{\mathcal{X}_q^\alpha}, \quad \forall \mathbf{v}_1, \mathbf{v}_2 \in \mathcal{X}_q^\alpha. \tag{3.5}$$

Let $\lambda = \frac{1}{4M_qL_q}$. For any $\mathbf{u}_a \in L_0^d(\mathbb{R}^d)$ with $\|\mathbf{u}_a\|_{L^d(\mathbb{R}^d)} < \lambda$, inequality (3.4) implies

$$\left\|E_\alpha(-(\log t - \log a)^\alpha A_d)\mathbf{u}_a\right\|_{\mathcal{X}_q^\alpha} \le \frac{1}{4L_q}.$$

By Lemma 7, there exists a unique fixed point $\mathbf{u}^* \in \mathcal{X}_q^\alpha$, satisfying

$$\mathbf{u}^*(t) = E_\alpha(-(\log t - \log a)^\alpha A_d)\mathbf{u}_a + \int_a^t (\log t - \log s)^{\alpha-1} E_{\alpha,\alpha}\left(-(\log t - \log s)^\alpha A_d\right) F(\mathbf{u}^*(s))\frac{ds}{s}, \quad t \geq a > 0.$$

Thus, $\mathbf{u}^*$ is the global mild solution of Eq (1.1) in $L_0^d(\mathbb{R}^d)$.

Fix $2 \leq d < q < \infty$. Set $\rho_1 = d/p$, $\rho_2 = 1$, and $\rho_3 = d/q$ in Lemma 4 to obtain

$$\begin{aligned}
&(\log t - \log a)^{\alpha[1-(d/q)]/2} \left\| \int_a^t (\log t - \log s)^{\alpha-1} E_{\alpha,\alpha}(-(\log t - \log s)^\alpha A_d) F(\mathbf{u}^*(s))\frac{ds}{s} \right\|_{L^q(\mathbb{R}^d)} \\
&\leq C(\log t - \log a)^{\alpha[1-(d/q)]/2} \int_a^t (\log t - \log s)^{\alpha\{1-[1+(d/p)-(d/q)]/2\}-1} \|\mathbf{u}^*(s)\|_{L^p(\mathbb{R}^d)} \\
&\quad \times \|\nabla \mathbf{u}^*(s)\|_{L^d(\mathbb{R}^d)} \frac{ds}{s} \\
&\leq C\|\mathbf{u}^*\|_{\mathcal{X}_q^\alpha}^2 (\log t - \log a)^{\alpha[1-(d/q)]/2} \int_a^t (\log t - \log s)^{\alpha\{1-[1+(d/p)-(d/q)]/2\}-1} \\
&\quad \times (\log s - \log a)^{-\alpha[1-(d/(2p))]} \frac{ds}{s} \\
&\leq C\|\mathbf{u}^*\|_{\mathcal{X}_q^\alpha}^2, \quad t \geq a > 0.
\end{aligned}$$

The continuity of the nonlinear term follows from computations similar to those in Lemma 9. Since $E_\alpha(-(\log t - \log a)^\alpha A_d)\mathbf{u}_a \in \mathcal{X}_q^\alpha$ by Lemma 8, we obtain that $(\log t - \log a)^{\alpha(1-d/q)/2}\mathbf{u}^* \in C_b([a,\infty); L_0^q(\mathbb{R}^d))$ for $2 \leq N \leq q < \infty$. Furthermore, $(\log t - \log a)^{\alpha(1-d/q)/2}\mathbf{u}^*$ vanishes at $t = a$ when $2 \leq N < q < \infty$, and coincides with $\mathbf{u}_a$ at $t = a$ when $q = N$.

Case $q = \infty$ in Eq (3.2) can be established via the Gagliardo-Nirenberg inequality, which states that

$$\|u\|_{L^\infty(\mathbb{R}^d)} \leq C(d)\, \|u\|_{L^{2d}(\mathbb{R}^d)}^{1/2}\, \|\nabla u\|_{L^{2d}(\mathbb{R}^d)}^{1/2},$$

where the constant $C = C(d) > 0$ is independent of time t. Applying this inequality yields

$$\begin{aligned}
(\log t - \log a)^{\alpha/2}\|\mathbf{u}(t)\|_{L_0^\infty(\mathbb{R}^d)} &\leq C\left(\frac{(\log t - \log a)^\alpha}{4}\|\mathbf{u}(t)\|_{L_0^{2d}(\mathbb{R}^d)}\right)^{1/2} \\
&\quad \times \left(\frac{3(\log t - \log a)^\alpha}{4}\|\nabla \mathbf{u}(t)\|_{L_0^{2d}(\mathbb{R}^d)}\right)^{1/2} < \infty, \qquad t \geq a > 0.
\end{aligned}$$

For the case $q = N$, estimate (3.3) was established previously. To extend this result to the range $N < q < \infty$, we proceed as follows: Selecting $\eta \in (0, d/q)$ and applying Lemma 4 with parameters

$\rho_1 = \eta, \rho_2 = 1$, and $\rho_3 = d/q$, we derive the following bound

$$\begin{aligned}
&(\log t - \log a)^{\alpha[1-(d/2q)]} \left\| \nabla \int_a^t (\log t - \log s)^{\alpha-1} E_{\alpha,\alpha}(-(\log t - \log s)^{\alpha} A_d) F(\mathbf{u}^*(s)) \frac{ds}{s} \right\|_{L^q(\mathbb{R}^d)} \\
&\leq C(\log t - \log a)^{\alpha[1-(d/2q)]} \int_a^t (\log t - \log s)^{-\alpha[\eta-(d/q)]/2-1} \|\mathbf{u}^*(s)\|_{L^{d/\eta}(\mathbb{R}^d)} \|\nabla \mathbf{u}^*(s)\|_{L^d(\mathbb{R}^d)} \frac{ds}{s} \\
&\leq C\|\mathbf{u}^*\|^2_{\mathcal{X}^\alpha_{d/\eta}} (\log t - \log a)^{\alpha[1-(d/2q)]} \int_a^t (\log t - \log s)^{-\alpha[\eta-d/q]/2-1} (\log s - \log a)^{-\alpha(2-\eta)/2} \frac{ds}{s} \\
&\leq C\|\mathbf{u}^*\|^2_{\mathcal{X}^\alpha_{d/\eta}}, \quad t \geq a > 0.
\end{aligned}$$

The continuity of the bilinear operator follows by analogous arguments. Moreover, Lemma 3 combined with the integral representation of $\nabla E_\alpha(-(\log t - \log a)^\alpha A_d)\mathbf{u}_a$ establishes the continuity of the linear component, thereby completing the proof of Eq (3.3).

3.2. *Local mild solution ($T < \infty$)*

In this subsection, we establish the existence and uniqueness of local mild solutions for Eq (1.1) while analyzing their logarithmic decay properties.

Let $2 \leq d < q < \infty$, and define the Banach space $\mathcal{Y}^\alpha_q[T]$ as the set of functions $\mathbf{v}(t) : [a, T] \to L^q_0(\mathbb{R}^d)$, satisfying the following conditions

$$(\log t - \log a)^{\alpha[1-(d/q)]/2} \mathbf{v} \in C_b([a, T]; L^q_0(\mathbb{R}^d)) \quad \text{with} \quad \lim_{t \to a^+} (\log t - \log a)^{\alpha[1-(d/q)]/2} \mathbf{v}(t) = \mathbf{0},$$

and

$$(\log t - \log a)^{\alpha/2} \nabla \mathbf{v} \in C_b([a, T]; L^d_0(\mathbb{R}^d)) \quad \text{with} \quad \lim_{t \to a^+} (\log t - \log a)^{\alpha/2} \nabla \mathbf{v}(t) = \mathbf{0}.$$

The norm for this space is given by

$$\|\mathbf{v}\|_{\mathcal{Y}^\alpha_q[T]} := \sup_{t \in [a,T]} (\log t - \log a)^{\alpha[1-(d/q)]/2} \|\mathbf{v}(t)\|_{L^q(\mathbb{R}^d)} + \sup_{t \in [a,T]} (\log t - \log a)^{\alpha/2} \|\nabla \mathbf{v}(t)\|_{L^d(\mathbb{R}^d)}.$$

Theorem 2. *Let $\alpha \in (0, 1)$. There exists $T^* > 0$ such that for any initial data $\mathbf{u}_a \in L^d_0(\mathbb{R}^d)$, the problem* (1.1) *has a unique local mild solution $\mathbf{u}(t) : [a, \infty) \to L^d_0(\mathbb{R}^d)$. Moreover, the solution satisfies the following decay estimates:*

(i) For $2 \leq N \leq q \leq \infty$,

$$(\log t - \log a)^{\alpha[1-(d/q)]/2} \mathbf{u} \in C_b([a, T^*]; L^q_0(\mathbb{R}^d)), \tag{3.6}$$

vanishing at $t = a$ except when $q = N$, where $\mathbf{u}(\mathbf{x}, a) = \mathbf{u}_a(\mathbf{x})$.

(ii) For $2 \leq N \leq q < \infty$,

$$(\log t - \log a)^{\alpha[1-(d/2q)]} \nabla \mathbf{u} \in C_b([a, T^*]; L^q_0(\mathbb{R}^d)), \tag{3.7}$$

vanishing at $t = a$.

Proof. Fix a value p with $2 \leq d < p < \infty$. As established in the proof of Lemma 9, the bilinear operator $\mathcal{B} : \mathcal{Y}_q^\alpha[T] \times \mathcal{Y}_q^\alpha[T] \to \mathcal{Y}_q^\alpha[T]$ is bounded (and hence continuous) for every $T > a > 0$. Consequently, there exists a constant $L_q > 0$, such that

$$\|\mathcal{B}(\mathbf{v}_1, \mathbf{v}_2)\|_{\mathcal{Y}_q^\alpha[T]} \leq L_q \|\mathbf{v}_1\|_{\mathcal{Y}_q^\alpha[T]} \|\mathbf{v}_2\|_{\mathcal{Y}_q^\alpha[T]}$$

holds for all $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{Y}_q^\alpha[T]$ and every $T > a > 0$.

Given $\mathbf{u}_a \in L_0^d(\mathbb{R}^d)$, set $\theta = d/p$ in Lemma 5 to deduce that for any $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that

$$\|E_\alpha(-(\log t - \log a)^\alpha A_d)\mathbf{u}_a\|_{\mathcal{Y}_q^\alpha[T_\varepsilon]} < \varepsilon, \quad \forall t \in [a, T_\varepsilon].$$

Consequently, we may choose $T^* > 0$ sufficiently small so that

$$\|E_\alpha(-(\log t - \log a)^\alpha A_d)\mathbf{u}_a\|_{\mathcal{Y}_q^\alpha[T^*]} < \frac{1}{4L_q}, \quad \forall t \in [a, T^*].$$

Combining the above inequality with Lemma 7 ensures the existence and uniqueness of a fixed point $\mathbf{u}^* \in \mathcal{Y}_q^\alpha[T^*]$. Following the argument in the proof of Theorem 1, $\mathbf{u}^*$ is concluded to be a local mild solution to (1.1) in $L_0^d(\mathbb{R}^d)$ with the decay properties (3.6) and (3.7) holding on $[a, T^*]$.

4. Conclusions

In this paper, we investigate mild solutions for the time-fractional Navier-Stokes equations with the Caputo-Hadamard derivative. By applying a fixed point theorem, we establish the existence and uniqueness of global and local mild solutions. Moreover, logarithmic decay estimates for these solutions are obtained.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

Author contributions

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