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Research article

Quantized $\mathcal{H}_\infty$ synchronization for Hopfield networks subject to time-variable delay

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Abstract: This paper is dedicated to the study of $\mathcal{H}_\infty$ synchronization for Hopfield networks (HNs) with time-variable delay via quantized control. The delay function is assumed to be continuous and bounded, yet not necessarily differentiable. A delay-dependent sufficient condition is derived using a linear combination of Lyapunov functional candidates along with two matrix inequalities to ensure the $\mathcal{H}_\infty$ synchronization of drive-response HNs. Based on this, an alternative sufficient condition characterized by reduced nonlinearities is further established by employing decoupling techniques. With fixed bounds on the time delay and the $\mathcal{H}_\infty$ disturbance attenuation level, an algorithm is then proposed to compute the minimum required gain for the quantized controller. Finally, a numerical example demonstrates the effectiveness of the proposed $\mathcal{H}_\infty$ synchronization results.

Keywords: Hopfield network; quantized control; $\mathcal{H}_\infty$ synchronization; time-variable delay

1. Introduction

Hopfield network (HN), designed by Hopfield in 1982 [1], is a single-layer, fully connected recurrent neural network (RNN) subject to undirected connections between neurons [2]. Compared with other RNN models, HNs are relatively easy to implement in hardware and offer efficient computational capabilities. Notably, they are famous for their powerful associative memory ability. Owing to these significant advantages, HNs have been successfully applied in a variety of realms, including optimization [3], random number generation [4], and image encryption [5]. Moreover, inherent time delays resulting from the finite switching speed of amplifiers are unavoidable in hardware implementations of RNNs and often play a key role in degrading system performance [6, 7]. Compared to constant delays, time-variable delays are more commonly encountered in real-world scenarios. Consequently, the dynamics analysis issues of delayed HNs, including but not limited to stability, synchronization, and filtering, have gained wide research interest, especially for HNs with time-variable delays [8–10].

In nonlinear science, synchronization refers to the phenomenon that two or more systems with different initial conditions evolve to a common dynamic behavior under the influence of coupling or external forces, which widely occurs in biological neural systems [11]. HNs are inherently highly nonlinear and can exhibit rich chaotic behaviors. The introduction of time-variable delays makes their chaotic dynamics even more complex. As a result, the synchronization study of HNs with time-variable delays has become a significant research focus, and various valid synchronization criteria have been proposed. For example, the automorphic synchronization of quaternion-valued HNs with time-variable and distributed delays was studied in [12], and the existence of automorphic solutions was established by means of the Banach fixed point theorem. The $\mathcal{H}_\infty$ synchronization problem of chaotic HNs subject to time-variable delays was discussed in [13], and a resilient dynamic output-feedback controller was developed. The dissipativity-based synchronization was addressed for discrete-time switched HNs subject to time-variable delays by devising a combined switching paradigm in [14]. The drive-response exponential synchronization for high-order quaternion HNs with time-variable discrete delays was investigated in [9], where a global synchronization criterion was established. In [15], finite-/fixed-time synchronization for diffusive HNs subject to leakage and discrete delays was examined, and several methods for designing negative exponential state-feedback controllers were proposed.

Because of the limited transmission capacity of network channels, network control systems are prone to data dropouts during periods of high load [16, 17]. Therefore, from the control perspective, it is essential to develop an energy-efficient control scheme to reduce channel resource consumption and enhance the efficiency of information transmission. The quantized control (QC), as a perfect candidate, has been proposed and commonly used since it can convert continuous signals into discrete forms taking values in a finite set [18–20]. Nowadays, the synchronization of RNNs under QC scheme has become a prominent research topic, yielding a range of noteworthy results. To name a few, the asymptotic synchronization for fractional-order RNNs was considered in [21], where two different classes of aperiodically intermittent quantized controllers were proposed. The quantized sampled-data synchronization of RNNs with stochastic packet dropouts was investigated, and sufficient conditions for ensuring the stochastic mean-square exponential synchronization were derived via an enhanced looped functional in [22]. A quantized memory-based sampled-data control approach was developed to address the synchronization problem of switching stochastic RNNs in [23]. However, to date, the $\mathcal{H}_\infty$ synchronization for HNs under QC has not gained adequate research attention yet, particularly when time-variable delays are also taken into account. It is, therefore, the main motivation for this paper.

Building on prior analysis, this work endeavors to investigate the $\mathcal{H}_\infty$ quantized synchronization for HNs with time-variable delay. The delay function here is assumed to be continuous and bounded, yet not necessarily differentiable. By constructing a linear combination of Lyapunov functional candidates and employing inequality techniques, two sufficient conditions are established to assure that the norm from external disturbances to the synchronization error remains below a prescribed bound, referred to as the $\mathcal{H}_\infty$ disturbance attenuation level (HDAL). With fixed bounds on the time delay and the HDAL, an algorithm is then proposed to compute the minimum required gain for the quantized controller. The remainder is organized as follows: Section 2 introduces the HN model, the QC law, synchronization error system (SES), and two needed matrix inequalities. Section 3 proposes two sufficient conditions for ensuring the $\mathcal{H}_\infty$ synchronization of drive-response HNs and details the algorithm for determining the minimum required gain for the quantized controller. Section 4 provides a numerical example to illustrate the applicability and superiority of the $\mathcal{H}_\infty$ synchronization results. Section 5 summarizes the conclusions.

2. Preliminaries

Throughout, $\mathbb{R}^m$ denotes the m-dimensional Euclidean space, and $\mathbb{R}^{m_1\times m_2}$ represents the family of $m_1 \times m_2$ real matrices. For a matrix X, the notation X^T denotes its transpose, and $\mathscr{S}(X)$ refers to the sum of X and X^T. If $U = U^T$, then $U > 0$ (respectively, $U \geq 0$) indicates that U is a positive-definite (respectively, positive semi-definite) matrix. The set of all real symmetric positive-definite matrices of order n is denoted by $\mathbb{S}^n_+$. The symbol $*$ is used to represent the block induced by symmetry in a matrix. Furthermore, $diag\{\cdots\}$ denotes a diagonal matrix, $col\{\cdot\}$ a column vector, and I (respectively, 0) refers to the unity (respectively, zero) matrix of appropriate dimensions.

Consistent with studies such as [24–26], the HN model considered here incorporates a time-variable delay. Its dynamic model is given by

$$\dot{r}(t) = A_0 r(t) + A_1 r(t-\lambda(t)) + B_0 f(r(t)) + B_1 g(r(t-\lambda(t))) + \mathcal{J}, \tag{2.1}$$

where $r(t) \in \mathbb{R}^n$ represents the neuron state vector; $f(\cdot) \in \mathbb{R}^n$ and $g(\cdot) \in \mathbb{R}^n$ represent activation functions; A_0, A_1, B_0, and B_1 are known real matrices of appropriate dimensions; $\mathcal{J}$ stands for a constant input vector; and the time-variable delay $\lambda(t)$ is continuous and satisfies

$$0 \leq \lambda_1 \leq \lambda(t) \leq \lambda_2,\ \lambda_{12} \triangleq \lambda_2 - \lambda_1, \tag{2.2}$$

where λ_1 and λ_2 are given constants. Notice that delay function $\lambda(t)$ considered here can degenerate into the constant delay studied in [27, 28] when $\lambda_1 = \lambda_2$. Moreover, it is not required to be differentiable, thereby relaxing the differentiability requirement imposed in [29–32].

The delayed HNs in Eq (2.1) is regarded as the drive system. Then the response HN under consideration is given as

$$\begin{aligned}\dot{\hat{r}}(t) = {} & A_0\hat{r}(t) + A_1\hat{r}(t-\lambda(t)) + B_0 f(\hat{r}(t)) + B_1 g(\hat{r}(t-\lambda(t))) \\ & + \mathcal{J} + Eu(t) + F\omega(t),\end{aligned} \tag{2.3}$$

where $\hat{r}(t) \in \mathbb{R}^n$ is state vector of the response HN; $u(t) \in \mathbb{R}^n$ represents the control input; $\omega(t) \in \mathbb{R}^k$ denotes the disturbance belonging to $\mathcal{L}_2[0,\infty)$, the space of square-integrable vector functions on $[0,\infty)$; and E and F are given real matrices of appropriate dimensions.

Define $e(\cdot) = \hat{r}(\cdot) - r(\cdot)$. The following system can be established from HNs in Eqs (2.1) and (2.3):

$$\begin{aligned}\dot{e}(t) = {} & A_0 e(t) + A_1 e(t-\lambda(t)) + B_0\bar{f}(e(t)) \\ & + B_1\bar{g}(e(t-\lambda(t))) + Eu(t) + F\omega(t),\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}\bar{f}(e(t)) &= f(\hat{r}(t)) - f(r(t)), \\ \bar{g}(e(t-\lambda(t))) &= g(\hat{r}(t-\lambda(t))) - g(r(t-\lambda(t))).\end{aligned}$$

In this paper, a quantized controller is employed, which is designed as

$$u(t) = kq(e(t)), \tag{2.5}$$

where k is the control gain, and $q(\cdot) : \mathbb{R}^n \to \Upsilon^n$ is a quantizer defined as $q(x) = [q_1(x_1), q_2(x_2), \cdots, q_n(x_n)]^T$, in which $\Upsilon = \{\pm\beta_l, \beta_l = \varrho^l\beta_0\} \cup \{0\}$ with $\beta_0 > 0$ and $0 < \varrho < 1$ ([33, 34]). For any $x_j \in \mathbb{R}$, $j = 1, 2, \cdots, n$, $q_j(x_j)$ is given by

$$q_j(x_j) = \begin{cases} \beta_l, & \beta_l/(1+\beta) < x_j \le \beta_l/(1-\beta), \\ 0, & x_j = 0, \\ -q_j(-x_j), & x_j < 0, \end{cases}$$

where $\beta = (1-\varrho)/(1+\varrho)$. According to the sector bound method [35], $q(x)$ can be expressed as

$$q(x) = (1+\Delta)x, \ \Delta \in [-\beta, \beta]. \tag{2.6}$$

Combining HN in Eq (2.4) with Eq (2.6), one gets

$$\begin{aligned} \dot{e}(t) = {} & A_0 e(t) + A_1 e(t-\lambda(t)) + B_0 \bar{f}(e(t)) \\ & + B_1 \bar{g}(e(t-\lambda(t))) + Ek(1+\Delta)e(t) + F\omega(t). \end{aligned} \tag{2.7}$$

The following definition (partially adapted from [25]), assumption, and lemmas are required to obtain the main results.

Definition 1. *The drive HN in Eq (2.1) and the response HN in Eq (2.3) are said to be $\mathcal{H}_\infty$ synchronized if the following two conditions hold:*

(i) The SES in Eq (2.7) satisfies the HDAL; that is,

$$\int_0^\infty e^T(s)We(s)ds \le \gamma^2 \int_0^\infty \omega^T(s)\omega(s)ds \tag{2.8}$$

for a given constant $\gamma > 0$ and a given positive definite symmetric matrix W, under the zero initial condition;

(ii) The SES in Eq (2.7) is globally asymptotically stable (GAS) when $\omega(t) \equiv 0$.

Assumption 1. *The activation functions $f(\cdot)$ and $g(\cdot)$ are globally Lipschitz continuous. That is, for any $\rho_1, \rho_2 \in \mathbb{R}^n$, there exist Lipschitz constants l_f and l_g, such that*

$$\|f(\rho_1) - f(\rho_2)\| \le l_f\|\rho_1 - \rho_2\|, \tag{2.9}$$

$$\|g(\rho_1) - g(\rho_2)\| \le l_g\|\rho_1 - \rho_2\|. \tag{2.10}$$

Such an assumption is commonly adopted in the analysis and synthesis of RNNs due to its simplicity and suitability for linear matrix inequality-based or M-matrix-based designs; see, e.g., [36–39]. Numerous widely-used activation functions satisfy this assumption, including the tanh, logistic, and saturated linear functions.

Lemma 1. *(see [40]) Given a matrix $Q \in \mathbb{S}^n_+$, then for any differentiable function h: $[p, q] \to \mathbb{R}^n$, it holds that*

$$\int_p^q \dot{h}^T(s)Q\dot{h}(s)ds \ge \frac{1}{q-p}\Omega^T diag\{Q, 3Q, 5Q\}\Omega,$$

where

$$\Omega = \begin{bmatrix} h(q) - h(p) \\ h(q) + h(p) - \frac{2}{q-p}\int_p^q h(s)ds \\ h(q) - h(p) - \frac{6}{q-p}\int_p^q \delta(s)h(s)ds \end{bmatrix},$$

$$\delta(s) = 2(\frac{s-p}{q-p}) - 1.$$

Lemma 2. *(see [41]) Consider a parameter-dependent matrix $\Theta(\alpha)$ in $\mathbb{S}^m$ satisfying*

$$\Theta(\alpha) \leq (1-\alpha)\Theta(0) + \alpha\Theta(1)$$

for all α in $[0, 1]$. Suppose that there are matrices Q in $\mathbb{S}_+^n$, Γ in $\mathbb{R}^{2n\times m}$ with rank$(\Gamma) = 2n$, and N_1, N_2 in $\mathbb{R}^{m\times n}$ such that

$$\begin{bmatrix} \Theta(\alpha) - \Gamma^T \mathcal{Q}(\alpha)\Gamma - \mathcal{S}\left(\Gamma^T \begin{bmatrix} (1-\alpha)N_1^T \\ \alpha N_2^T \end{bmatrix}\right) & * \\ \alpha N_1^T + (1-\alpha)N_2^T & -Q \end{bmatrix} < 0$$

holds for $\alpha = \{0, 1\}$, where

$$\mathcal{Q}(\alpha) = \begin{bmatrix} (2-\alpha)Q & 0 \\ 0 & (1+\alpha)Q \end{bmatrix}.$$

Then,

$$\Theta(\alpha) - \Sigma(\alpha) < 0, \forall\alpha \in (0, 1)$$

holds, where

$$\Sigma(\alpha) = \Gamma^T \begin{bmatrix} \frac{1}{\alpha}Q & 0 \\ 0 & \frac{1}{1-\alpha}Q \end{bmatrix}\Gamma.$$

Lemma 3. *(Schur complement [42]) Given matrices U_a, U_b, and U_c of appropriate dimensions,*

$$\begin{bmatrix} U_a & U_b \\ * & U_c \end{bmatrix} < 0$$

holds if and only if

$$U_c < 0 \text{ and } U_a - U_b U_c^{-1} U_b^T < 0.$$

3. Main results

For the simplicity of presentation, the following notations are introduced:

$$\pi_i = \begin{bmatrix} 0_{n\times(i-1)n} & I_n & 0_{n\times(16-i)n} \end{bmatrix}, \quad i = 1, \ldots, 16,$$

$$\xi_0(t) = \begin{bmatrix} e^T(t) & e^T(t-\lambda_1) & e^T(t-\lambda(t)) & e^T(t-\lambda_2) \end{bmatrix}^T,$$

$$\xi_1(t) = \frac{1}{\lambda_1}\begin{bmatrix} \int_{-\lambda_1}^0 e_t^T(s)ds & \int_{-\lambda_1}^0 \delta_1(s)e_t^T(s)ds \end{bmatrix}^T,$$

$$\xi_2(t) = \frac{1}{\lambda(t)-\lambda_1}\begin{bmatrix} \int_{-\lambda(t)}^{-\lambda_1} e_t^T(s)ds & \int_{-\lambda(t)}^{-\lambda_1} \delta_2(s)e_t^T(s)ds \end{bmatrix}^T,$$

$$\xi_3(t) = \frac{1}{\lambda_2 - \lambda(t)} \left[\int_{-\lambda_2}^{-\lambda(t)} e_t^T(s)ds \ \ \int_{-\lambda_2}^{-\lambda(t)} \delta_3(s) e_t^T(s)ds \ \right]^T,$$
$$\xi_4(t) = (\lambda(t) - \lambda_1)\,\xi_2(t), \quad \xi_5(t) = (\lambda_2 - \lambda(t))\,\xi_3(t),$$
$$\xi_6(t) = \left[\ \int_{-\lambda_2}^{-\lambda_1} e_t^T(s)ds \quad \lambda_{12} \int_{-\lambda_2}^{-\lambda_1} \delta_4(s) e_t^T(s)ds \ \right]^T,$$
$$\xi_7(t) = \left[\ \bar{f}^T(r(t)) \quad \bar{g}^T(r(t-\lambda(t))) \ \right]^T,$$
$$\xi(t) = col\{\xi_0(t), \xi_1(t), \xi_2(t), \xi_3(t), \xi_4(t), \xi_5(t), \xi_7(t)\},$$
$$\delta_1(s) = 2\frac{s+\lambda_1}{\lambda_1} - 1, \quad \delta_2(s) = 2\frac{s+\lambda(t)}{\lambda(t)-\lambda_1} - 1,$$
$$\delta_3(s) = 2\frac{s+\lambda_2}{\lambda_2-\lambda(t)} - 1, \quad \delta_4(s) = 2\frac{s+\lambda_2}{\lambda_{12}} - 1,$$
$$e_t(s) = e(t+s), \ \alpha = \frac{\lambda(t)-\lambda_1}{\lambda_{12}}.$$

This section focuses on the $\mathcal{H}_\infty$ synchronization analysis for HNs in Eqs (2.1) and (2.3). Two sufficient conditions are established by the following theorems:

Theorem 1. *For given scalars $\lambda_2 \geq \lambda_1 \geq 0$, $\gamma > 0$ and given matrices W in $\mathbb{S}_+^n$, suppose that there exist scalars $\varepsilon_f > 0$, $\varepsilon_g > 0$ and matrices $P \in \mathbb{S}_+^{5n}$, R_1, R_2, Q_1, $Q_2 \in \mathbb{S}_+^n$, M_1, $M_2 \in \mathbb{R}^{16n\times 2n}$, N_1, $N_2 \in \mathbb{R}^{16n\times 3n}$, such that for $\alpha = \{0, 1\}$, the following matrix inequality*

$$\begin{bmatrix} \Theta_0(\alpha) - \Theta_1(\alpha) & * & * \\ \alpha N_1^T + (1-\alpha)N_2^T & -Q_2 & * \\ \Theta_2^T(\alpha) & 0 & \Theta_3 \end{bmatrix} < 0 \tag{3.1}$$

holds, where

$$\Theta_0(\alpha) = \mathscr{S}(G_1^T(\alpha)PG_0 + M_1 g_1(\alpha) + M_2 g_2(\alpha)) + \hat{R} + \check{R} - G_2^T Q_1 G_2 + \Pi^T \hat{Q}_{12} \Pi,$$
$$\Theta_1(\alpha) = \Gamma^T Q^0(\alpha)\Gamma + \mathscr{S}\left(\Gamma^T \begin{bmatrix} (1-\alpha)N_1^T \\ \alpha N_2^T \end{bmatrix}\right),$$
$$\Theta_2(\alpha) = G_1^T(\alpha)P\bar{F} + \Pi^T \hat{Q}_{12} F,$$
$$\Theta_3 = F^T \hat{Q}_{12} F - \gamma^2 I,$$
$$g_1(\alpha) = \alpha\lambda_{12} \begin{bmatrix} \pi_7 \\ \pi_8 \end{bmatrix} - \begin{bmatrix} \pi_{11} \\ \pi_{12} \end{bmatrix},$$
$$g_2(\alpha) = (1-\alpha)\lambda_{12} \begin{bmatrix} \pi_9 \\ \pi_{10} \end{bmatrix} - \begin{bmatrix} \pi_{13} \\ \pi_{14} \end{bmatrix}$$

with

$$Q^0(\alpha) = \begin{bmatrix} (2-\alpha)Q_2 & 0 \\ 0 & (1+\alpha)Q_2 \end{bmatrix},$$
$$\hat{R} = diag\{R_1, -R_1 + R_2, 0_n, -R_2, 0_{12n}\},$$
$$\check{R} = diag\{W + \varepsilon_f l_f^2 I, 0_n, \varepsilon_g l_g^2 I, 0_{11n}, -\varepsilon_f I, -\varepsilon_g I\},$$

$$
\begin{aligned}
\hat{Q}_{12} &= \lambda_1^2 Q_1 + \lambda_{12}^2 Q_2, \\
\mathcal{Q}_i &= diag\{Q_i,\ 3Q_i,\ 5Q_i\}, i = 1, 2, \\
G_1(\alpha) &= \begin{bmatrix} \pi_1^T & \lambda_1\pi_5^T & \lambda_1\pi_6^T & \pi_{11}^T + \pi_{13}^T & \hat{G}_1^T(\alpha) \end{bmatrix}^T, \\
\hat{G}_1(\alpha) &= (1-\alpha)\lambda_{12}(\pi_{11} + \pi_{14}) + \alpha\lambda_{12}(\pi_{12} - \pi_{13}), \\
G_0 &= \begin{bmatrix} \Pi^T & \pi_1^T - \pi_2^T & \pi_1^T + \pi_2^T - 2\pi_5^T & \pi_2^T - \pi_4^T & \hat{G}_0^T \end{bmatrix}^T, \\
G_2 &= \begin{bmatrix} \pi_1^T - \pi_2^T & \pi_1^T + \pi_2^T - 2\pi_5^T & \pi_1^T - \pi_2^T - 6\pi_6^T \end{bmatrix}^T, \\
G_3 &= \begin{bmatrix} \pi_2^T - \pi_3^T & \pi_2^T + \pi_3^T - 2\pi_7^T & \pi_2^T - \pi_3^T - 6\pi_8^T \end{bmatrix}^T, \\
G_4 &= \begin{bmatrix} \pi_3^T - \pi_4^T & \pi_3^T + \pi_4^T - 2\pi_9^T & \pi_3^T - \pi_4^T - 6\pi_{10}^T \end{bmatrix}^T, \\
\hat{G}_0 &= \lambda_{12}(\pi_2 + \pi_4) - 2(\pi_{11} + \pi_{13}),\ \Gamma = \begin{bmatrix} G_3^T & G_4^T \end{bmatrix}^T, \\
\Pi &= (A_0 + Ek(1+\Delta)\pi_1 + A_1\pi_3 + B_0\pi_{15} + B_1\pi_{16}, \\
\bar{F} &= \begin{bmatrix} F^T & 0_n & 0_n & 0_n & 0_n \end{bmatrix}^T.
\end{aligned}
$$

Then, the drive-response HNs in Eqs (2.1) and (2.3) are $\mathcal{H}_\infty$ synchronized.

Proof. Construct the following linear combination of Lyapunov functional candidates:

$$
V(t) = V_1(e_t) + V_2(e_t) + V_3(e_t, \dot{e}_t), \tag{3.2}
$$

where

$$
\begin{aligned}
V_1(e_t) &= [G_1(\alpha)\xi(t)]^T P[G_1(\alpha)\xi(t)], \\
V_2(e_t) &= \int_{t-\lambda_1}^{t} e^T(s)R_1 e(s)ds + \int_{t-\lambda_2}^{t-\lambda_1} e^T(s)R_2 e(s)ds, \\
V_3(e_t, \dot{e}_t) &= \lambda_1 \int_{-\lambda_1}^{0}\int_{t+\theta}^{t} \dot{e}^T(s)Q_1\dot{e}(s)dsd\theta \\
&\quad + \lambda_{12}\int_{-\lambda_2}^{-\lambda_1}\int_{t+\theta}^{t} \dot{e}^T(s)Q_2\dot{e}(s)dsd\theta.
\end{aligned}
$$

Calculating the time derivatives of $V_1(t)$, $V_2(t)$, and $V_3(t)$ along the trajectories of HN in Eq (2.7), respectively, yields that

$$
\begin{aligned}
\dot{V}_1(e_t) &= \xi^T(t)\mathscr{S}(G_1^T(\alpha)PG_0)\xi(t) + 2\xi^T(t)G_1^T(\alpha)P\bar{F}\omega(t), \\
\dot{V}_2(e_t) &= \xi^T(t)\hat{R}\xi(t), \\
\dot{V}_3(e_t, \dot{e}_t) &= [\Pi\xi(t) + F\omega(t)]^T\hat{Q}_{12}[\Pi\xi(t) + F\omega(t)] \\
&\quad - \lambda_1\int_{t-\lambda_1}^{t} \dot{e}^T(s)Q_1\dot{e}(s)ds \\
&\quad - \lambda_{12}\int_{t-\lambda_2}^{t-\lambda_1} \dot{e}^T(s)Q_2\dot{e}(s)ds.
\end{aligned}
$$

On account of Lemma 1, one has

$$-\lambda_1\int_{t-\lambda_1}^{t}\dot{e}^T(s)Q_1\dot{e}(s)ds \leq -\xi^T(t)G_2^TQ_1G_2)\xi(t), \tag{3.3}$$

$$\begin{aligned}-\lambda_{12}\int_{t-\lambda_2}^{t-\lambda_1}\dot{e}^T(s)Q_2\dot{e}(s)ds &= -\lambda_{12}\int_{t-\lambda_2}^{t-\lambda(t)}\dot{e}^T(s)Q_2\dot{e}(s)ds - \lambda_{12}\int_{t-\lambda(t)}^{t-\lambda_1}\dot{e}^T(s)Q_2\dot{e}(s)ds\\ &\leq -\frac{\lambda_{12}}{\lambda_2-\lambda(t)}\xi^T(t)G_4^TQ_2G_4\xi(t) - \frac{\lambda_{12}}{\lambda(t)-\lambda_1}\xi^T(t)G_3^TQ_2G_3\xi(t)\\ &= -\xi^T(t)\Sigma_0(\alpha)\xi(t), \end{aligned}\tag{3.4}$$

where

$$\Sigma_0(\alpha) = \Gamma^T\begin{bmatrix} \frac{1}{\alpha}Q_2 & 0\\ 0 & \frac{1}{1-\alpha}Q_2\end{bmatrix}\Gamma.$$

From the given vectors, it is not difficult to observe that

$$\begin{aligned}(\lambda(t)-\lambda_1)\,\xi_2(t) - \xi_4(t) &= g_1(\alpha)\xi(t) = 0,\\ (\lambda_2-\lambda(t))\,\xi_3(t) - \xi_5(t) &= g_2(\alpha)\xi(t) = 0,\end{aligned}$$

which implies that for any matrices M_1, M_2 in $\mathbb{R}^{16n\times 2n}$, the following equality holds:

$$\xi^T(t)\mathscr{S}\left(M_1g_1(\alpha) + M_2g_2(\alpha)\right)\xi(t) = 0. \tag{3.5}$$

Furthermore, in view of inequalities (2.9) and (2.10), one obtains that

$$\varepsilon_f\bar{f}^T(e(t))\bar{f}(e(t)) \leq \varepsilon_f l_f^2\xi^T(t)\pi_1^T\pi_1\xi(t), \tag{3.6}$$

$$\varepsilon_g\bar{g}^T(e(t-\lambda(t)))\bar{g}(e(t-\lambda(t))) \leq \varepsilon_g l_g^2\xi^T(t)\pi_3^T\pi_3\xi(t). \tag{3.7}$$

Combining inequalities (3.3)–(3.7) with the derivative of $V(t)$ leads to

$$\dot{V}(t) \leq \begin{bmatrix}\xi(t)\\ \omega(t)\end{bmatrix}^T\Omega(\alpha)\begin{bmatrix}\xi(t)\\ \omega(t)\end{bmatrix}, \tag{3.8}$$

where

$$\begin{aligned}\Omega(\alpha) &= \begin{bmatrix}\Omega_1(\alpha) - diag\{W, 0_{15n}\} & \Theta_2(\alpha)\\ * & F^T\hat{Q}_{12}F\end{bmatrix},\\ \Omega_1(\alpha) &= \Theta_0(\alpha) - \Sigma_0(\alpha).\end{aligned}$$

Define

$$J(t) \triangleq \int_0^t (e^T(s)We(s) - \gamma^2\omega^T(s)\omega(s))ds.$$

Since $V(t) \geq 0$ and inequality (3.8) holds, under the zero initial condition, one has

$$J(\infty) = J(\infty) + \int_0^\infty \dot{V}(s)ds - (V(\infty) - V(0))$$

$$\leq J(\infty) + \int_0^\infty \dot{V}(s)ds$$
$$= \int_0^\infty \begin{bmatrix} \xi(s) \\ \omega(s) \end{bmatrix}^T \hat{\Omega}(\alpha) \begin{bmatrix} \xi(s) \\ \omega(s) \end{bmatrix} ds,$$

where

$$\hat{\Omega}(\alpha) = \begin{bmatrix} \Omega_1(\alpha) & \Theta_2(\alpha) \\ * & \Theta_3 \end{bmatrix}.$$

By Lemma 3 [43], it shows that $\hat{\Omega}(\alpha) < 0$ is equivalent to

$$\Theta_0(\alpha) - \Sigma_0(\alpha) - \Theta_2(\alpha)\Theta_3^{-1}\Theta_2^T(\alpha) < 0. \tag{3.9}$$

The matrix $\Theta_0(\alpha) - \Theta_2(\alpha)\Theta_3^{-1}\Theta_2^T(\alpha)$ is convex for $\alpha \in [0,\ 1]$. Thus, by Lemma 2, inequality (3.9) holds for all $\alpha \in (0, 1)$ if

$$\begin{bmatrix} \Theta_0(\alpha) - \Theta_2(\alpha)\Theta_3^{-1}\Theta_2^T(\alpha) - \Theta_1(\alpha) & * \\ \alpha N_1^T + (1-\alpha)N_2^T & -Q_2 \end{bmatrix} < 0$$

holds for $\alpha = \{0, 1\}$. Using Lemma 3 again, the above inequality can be recast as inequality (3.1). Thus, inequality (3.1) ensures $\hat{\Omega}(\alpha) < 0$, implying $J(\infty) < 0$, and, correspondingly, that inequality (2.8) holds. This means that the HDAL of SES in Eq (2.9) is guaranteed.

In the case when $\omega(t) \equiv 0$, it follows from inequality (3.8) that

$$\dot{V}(t) \leq \xi^T(t)\Big(\Omega_1(\alpha) - diag\{W, 0_{15n}\}\Big)\xi(t),$$

which, together with $\hat{\Omega}(\alpha) < 0$, implies $\dot{V}(t) < 0$ for any $\xi(t) \neq 0$. In the light of the Lyapunov stability theory, the SES in Eq (2.7) with $\omega(t) \equiv 0$ is GAS.

In summary, it is shown that the drive-response HNs in Eqs (2.1) and (2.3) are $\mathcal{H}_\infty$ synchronized in the sense of Definition 1, and the proof is completed.

Clearly, the inequality in Theorem 1 involves certain nonlinear terms. It is time to reduce these terms by employing some decoupling techniques and develop an alternative sufficient condition.

Theorem 2. *For given scalars $\lambda_2 \geq \lambda_1 \geq 0$, $\gamma > 0$, $\beta > 0$ and given matrices W in $\mathbb{S}_+^n$, suppose that there exist scalars $\varepsilon_f > 0$, $\varepsilon_g > 0$, $\varepsilon > 0$ and matrices $P = (P_{ij})_{5n\times5n} \in \mathbb{S}_+^{5n}$, R_1, R_2, Q_1, $Q_2 \in \mathbb{S}_+^n$, M_1, $M_2 \in \mathbb{R}^{16n\times2n}$, N_1, $N_2 \in \mathbb{R}^{16n\times3n}$, such that for $\alpha = \{0, 1\}$, the following condition*

$$\begin{bmatrix} \Lambda_0(\alpha) + \Lambda_1 & * \\ \beta\Lambda_2^T(\alpha) & -\varepsilon I_n \end{bmatrix} < 0 \tag{3.10}$$

holds, where

$$\Lambda_0(\alpha) = \begin{bmatrix} \hat{\Theta}_0(\alpha) - \Theta_1(\alpha) & * & * & * \\ \alpha N_1^T + (1-\alpha)N_2^T & -Q_2 & * & * \\ \hat{\Theta}_2^T(\alpha) & 0 & \Theta_3 & * \\ \hat{Q}_{12}\Pi_1 & 0 & 0 & -\hat{Q}_{12} \end{bmatrix},$$

$$\Lambda_1 = \varepsilon \begin{bmatrix} \pi_1 & 0 & 0 & 0 \end{bmatrix}^T \begin{bmatrix} \pi_1 & 0 & 0 & 0 \end{bmatrix},$$
$$\Lambda_2(\alpha) = G_5(\alpha)k$$

with

$$\begin{aligned}
\hat{\Theta}_0(\alpha) &= \mathscr{S}(G_1^T(\alpha)PG_{01} + M_1 g_1(\alpha) + M_2 g_2(\alpha)) + \hat{R} + \breve{R} - G_2^T Q_1 G_2,\\
\hat{\Theta}_2(\alpha) &= G_1^T(\alpha)P\bar{F} + \Pi_1^T \hat{Q}_{12} F,\\
G_{01} &= \begin{bmatrix} \Pi_1^T & \pi_1^T - \pi_2^T & \pi_1^T + \pi_2^T - 2\pi_5^T & \pi_2^T - \pi_4^T & \hat{G}_0^T \end{bmatrix}^T,\\
\Pi_1 &= (A_0 + Ek)\pi_1 + A_1\pi_3 + B_0\pi_{15} + B_1\pi_{16},\\
\hat{Q}_{12} &= \lambda_1^2 Q_1 + \lambda_{12}^2 Q_2,\\
G_5(\alpha) &= \begin{bmatrix} (G_1^T(\alpha)\hat{P}E)^T & 0 & (F^T\hat{Q}_{12}E)^T & (\hat{Q}_{12}E)^T \end{bmatrix}^T,\\
\hat{P} &= \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} & P_{15} \end{bmatrix}^T.
\end{aligned}$$

Then, the drive-response HNs in Eqs (2.1) and (2.3) are $\mathcal{H}_\infty$ synchronized.

Proof. By utilizing Lemma 3, inequality (3.1) is equivalent to

$$\begin{bmatrix} \bar{\Theta}_0(\alpha) - \Theta_1(\alpha) & * & * & * \\ \alpha N_1^T + (1-\alpha)N_2^T & -Q_2 & * & * \\ \Theta_2^T(\alpha) & 0 & \Theta_3 & * \\ \hat{Q}_{12}\Pi & 0 & 0 & -\hat{Q}_{12} \end{bmatrix} < 0,$$

where

$$\bar{\Theta}_0(\alpha) = \mathscr{S}(G_1^T(\alpha)PG_0 + M_1 g_1(\alpha) + M_2 g_2(\alpha)) + \hat{R} + \breve{R} - G_2^T Q_1 G_2.$$

The above inequality can be recast as

$$\Lambda_0(\alpha) + \Lambda_3(\alpha) < 0, \tag{3.11}$$

where

$$\begin{aligned}
\Lambda_3(\alpha) &= \begin{bmatrix} \mathscr{S}(G_1^T(\alpha)PG_{02}) & 0 & * & * \\ 0 & 0 & 0 & 0 \\ F^T\hat{Q}_{12}\Pi_2 & 0 & 0 & 0 \\ \hat{Q}_{12}\Pi_2 & 0 & 0 & 0 \end{bmatrix},\\
G_{02} &= \begin{bmatrix} \Pi_2^T & 0 & 0 & 0 & 0 \end{bmatrix}^T,\\
\Pi_2 &= Ek\Delta\pi_1.
\end{aligned}$$

According to the inequality $U_a U_b^T + U_b U_a^T \leq \frac{1}{\epsilon} U_a U_a^T + \epsilon U_b U_b^T$ and the fact $\Delta^2 \leq \beta^2$, one gets

$$\Lambda_0(\alpha) + \Lambda_3(\alpha) \leq \Lambda_0(\alpha) + \Lambda_1 + \frac{\beta^2}{\varepsilon}\Lambda_2(\alpha)\Lambda_2^T(\alpha). \tag{3.12}$$

Algorithm 1 Search for the Minimal Gain k^* Ensuring $\mathcal{H}_\infty$ Synchronization

Require: System matrices and scalars required in Theorem 2, initial trial value $k_0 > 0$ (e.g., $k_0 = 10$), tolerance $\delta > 0$ (e.g., $\delta = 10^{-3}$), maximum iteration max_iter

Ensure: Minimal scalar k^* such that inequalities in Theorem 2 are feasible

Initialize: $k_{\text{trial}} = k_0$, $k_{\min} = 0$, $k_{\max} = 1/\delta$, iter = 0
while $|k_{\max} - k_{\min}| > \delta$ **and** iter < max_iter **do**
    Check feasibility of the inequalities in Theorem 2 using $k = k_{\text{trial}}$
    if feasible **then**
        $k_{\max} = k_{\text{trial}},\ k_{\text{trial}} = \frac{1}{2}(k_{\min} + k_{\text{trial}})$
    else
        $k_{\min} = k_{\text{trial}}$
        if $k_{\max} = 1/\delta$ **then**
            $k_{\text{trial}} = 2k_{\text{trial}}$
            if $k_{\text{trial}} > 1/\delta$ **then**
                return "No feasible solution found"
            end if
        else
            $k_{\text{trial}} = \frac{1}{2}(k_{\max} + k_{\min})$
        end if
    end if
    iter = iter + 1
end while
if $|k_{\max} - k_{\min}| \leq \delta$ **and** feasible($k_{\max}$) **then**
    return $k^* = k_{\max}$
else
    return "No feasible solution found"
end if

By Lemma 3 again, inequality (3.10) is equivalent to

$$\Lambda_0(\alpha) + \Lambda_1 + \frac{\beta^2}{\varepsilon}\Lambda_2(\alpha)\Lambda_2^T(\alpha) < 0,$$

which, together with inequality (3.12), implies inequality (3.11). Consequently, inequality (3.10) ensures inequality (3.1), which completes the proof.

Remark 1. *Theorem 2 presents an alternative sufficient condition for ensuring the $\mathcal{H}_\infty$ synchronization of the drive HN in Eq (2.1) and the response HN in Eq (2.3). By employing decoupling techniques (specifically, the Schur complement and bounding the nonlinear term involving Δ), the term $\Pi^T \hat{Q}_{12}\Pi$ in inequality (3.1) is replaced by $\beta\Lambda_2^T(\alpha)$ and $\hat{Q}_{12}\Pi_1$. Consequently, the nonlinearities of the condition is significantly reduced compared to the one of Theorem 1.*

Based on Theorem 2, we develop Algorithm 1 to search for the minimal value of control gain k (denoted as k^*) that ensures the $\mathcal{H}_\infty$ synchronization of the drive-response HNs. Given that the inequalities involved are linear matrix inequalities, the algorithm can be readily implemented using available convex optimization solvers.

4. Numerical simulation

This section presents a numerical example to demonstrate the applicability and superiority of the present theoretical results. Computations and simulations are performed in MATLAB using YALMIP with MOSEK as the optimization solver.

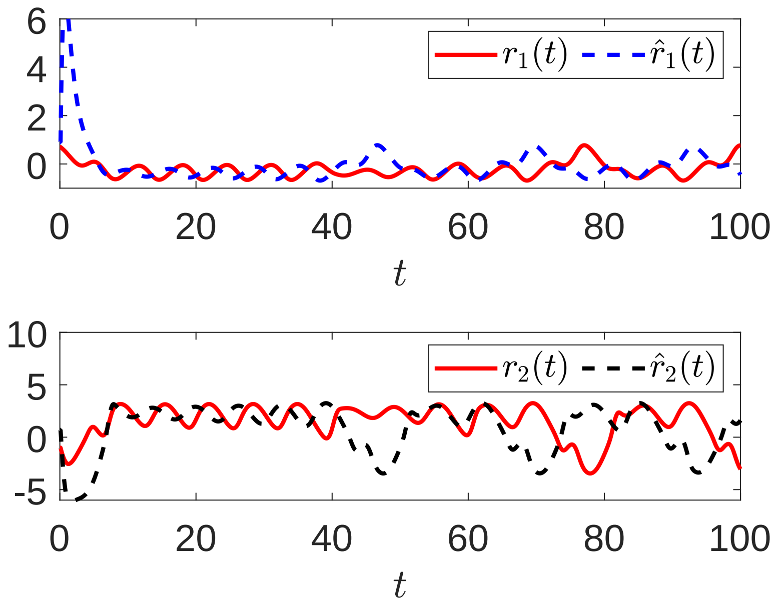

Figure 1. State trajectories of the unforced HNs.

Consider an HN described by

$$\dot{r}(t) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} r(t) + \begin{bmatrix} 2 & -0.1 \\ -5 & 2 \end{bmatrix} tanh(r(t)) + \begin{bmatrix} -1.5 & -0.1 \\ -0.2 & -1.5 \end{bmatrix} tanh(r(t-\lambda(t))), \tag{4.1}$$

where $\lambda(t)$ is taken as $\lambda_2 - \lambda_{12}e^{-t}$. It follows from inequalities (2.9) and (2.10) that $l_f = l_g = 1$. Besides, we set $\omega(t) = [5e^{-2t}, 3e^{-2t}]^T$,

$$E = \begin{bmatrix} -1.2 & 0 \\ 0 & -1.7 \end{bmatrix}, F = \begin{bmatrix} 1.4 & 1 \\ -1.1 & 1 \end{bmatrix}, W = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

The initial conditions of the unforced drive-response HNs are taken as $\varphi(s) = [0.7, -1]^T$ and $\psi(s) = [0.9, 0.7]^T$, respectively, where $s \in [-1, 0]$. The corresponding state trajectories are displayed in Figure 1. The phase-plane trajectories depicted in Figures 2 and 3 exhibit the chaotic behaviors.

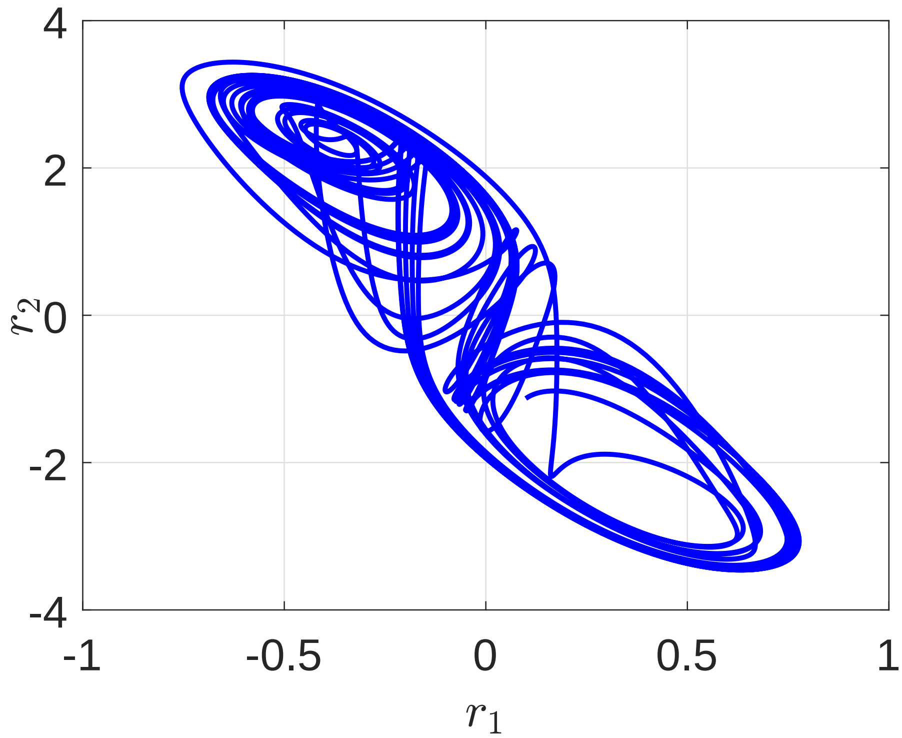

Figure 2. Phase-plane trajectory of the drive HN with the initial condition $[0.7, -1]^T$.

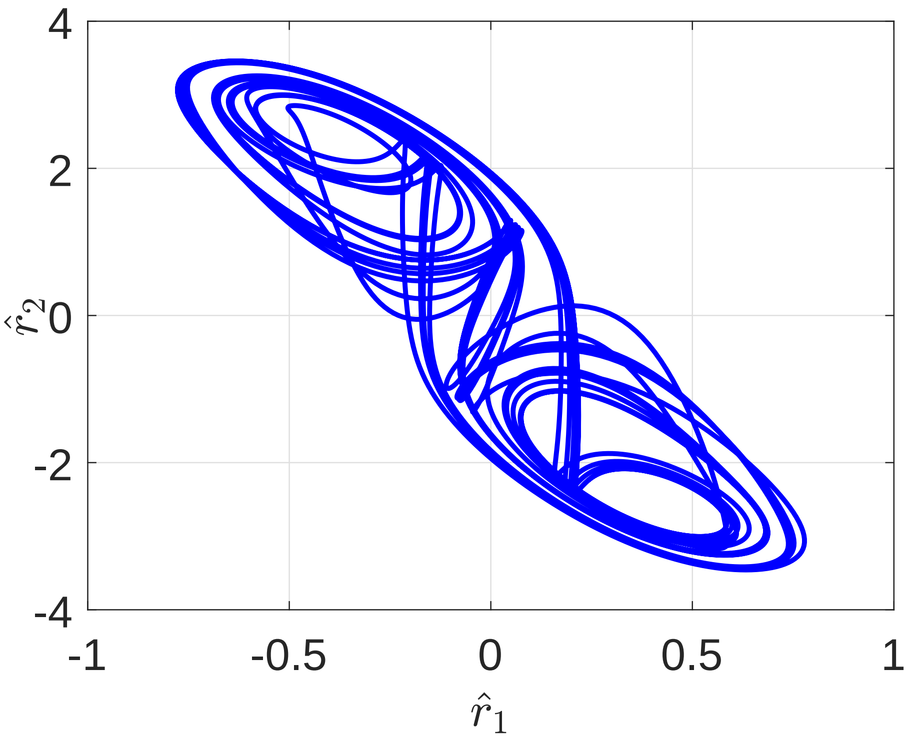

Figure 3. Phase-plane trajectory of the unforced response HN with the initial condition $[0.9, 0.7]^T$.

Set $k = 10$, $\Delta = 0$. Table 1 presents admissible upper bound of λ_2 obtained by Theorem 1 and Theorem 1 in [13] with $\xi(t) = e(t), C = I$, and $\triangle K = 0$. In this case, γ is fixed with a value of 0.1. It is shown from Table 1 that admissible upper bounds of λ_2 based on Theorem 1 in the present paper are larger than those on the basis of the method in [13]. Moreover, when λ_1 is fixed with a value of 0.4, Table 2 presents admissible minimum $\mathcal{H}_\infty$ disturbance attenuation bound γ. It is obvious from the horizontal direction of Table 2 that the admissible minimum $\mathcal{H}_\infty$ disturbance attenuation bound increases gradually as λ_2 increases. In the vertical direction, the values of γ based on Theorem 1 are consistently smaller than those on the basis of the method in [13], which suggests that the present condition allows for a stronger anti-interference ability.

Table 1. Admissible upper bound of λ_2.

	$\lambda_1 = 0.4$	$\lambda_1 = 0.5$	$\lambda_1 = 0.6$	$\lambda_1 = 0.7$
Theorem 1 [13]	1.22	1.32	1.42	1.52
Theorem 1	1.64	1.74	1.84	1.94

Table 2. Admissible minimum $\mathcal{H}_\infty$ disturbance attenuation bound γ.

	$\lambda_2 = 0.8$	$\lambda_2 = 1.0$	$\lambda_2 = 1.2$	$\lambda_2 = 1.4$
Theorem 1 [13]	0.072	0.080	0.097	0.140
Theorem 1	0.070	0.073	0.078	0.086

Let $\lambda_1 = 0.5$, $\lambda_2 = 1.1$, $\beta = 0.01$, $\beta_0 = 0.7$, and $\gamma = 0.1$. Then, with Algorithm 1, the minimal (optimal) control gain k^* is found to be 8.3722, and the corresponding R_1, R_2 are calculated as

$$R_1 = \begin{bmatrix} 0.0236 & 0.0007 \\ 0.0007 & 0.0335 \end{bmatrix}, R_2 = \begin{bmatrix} 0.0039 & -0.0027 \\ -0.0027 & 0.0089 \end{bmatrix}.$$

Hence, the $\mathcal{H}_\infty$ synchronization of the drive-response delayed HNs can be guaranteed based on $k^* = 8.3722$ determined by Theorem 2.

To characterize the $\mathcal{H}_\infty$ disturbance attenuation performance, define

$$W(t) \triangleq \sqrt{\int_0^t e^T(s)We(s)ds \Big/ \int_0^\infty \omega^T(s)\omega(s)ds}.$$

The curve of $W(t)$ under the zero initial condition is shown in Figure 4. It is clear that $W(\infty) = 0.0309 < \gamma = 0.1$, which means the HDAL of the SES is guaranteed. The corresponding state trajectories of the SES with $\omega(t) \equiv 0$ are displayed in Figure 5, demonstrating the successful achievement of asymptotic stability in the SES.

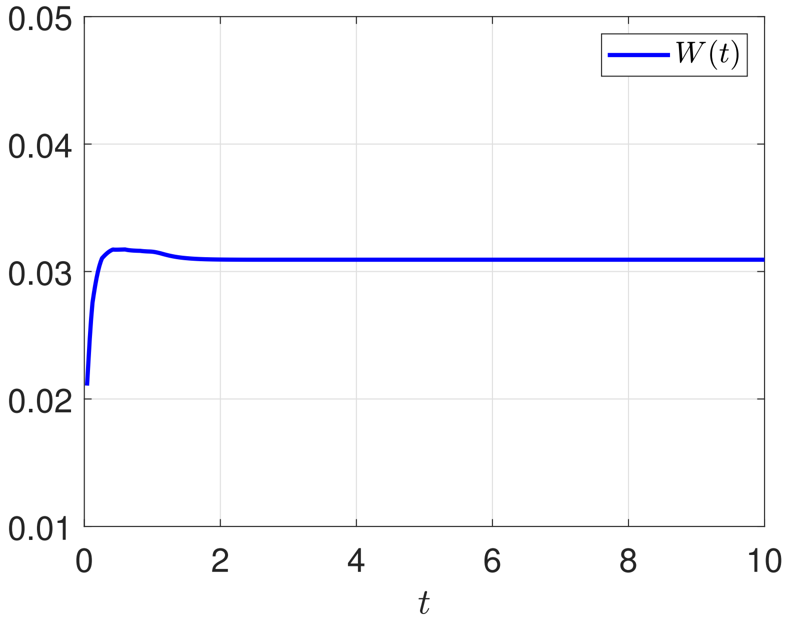

Figure 4. State trajectory of $W(t)$ under the zero initial condition.

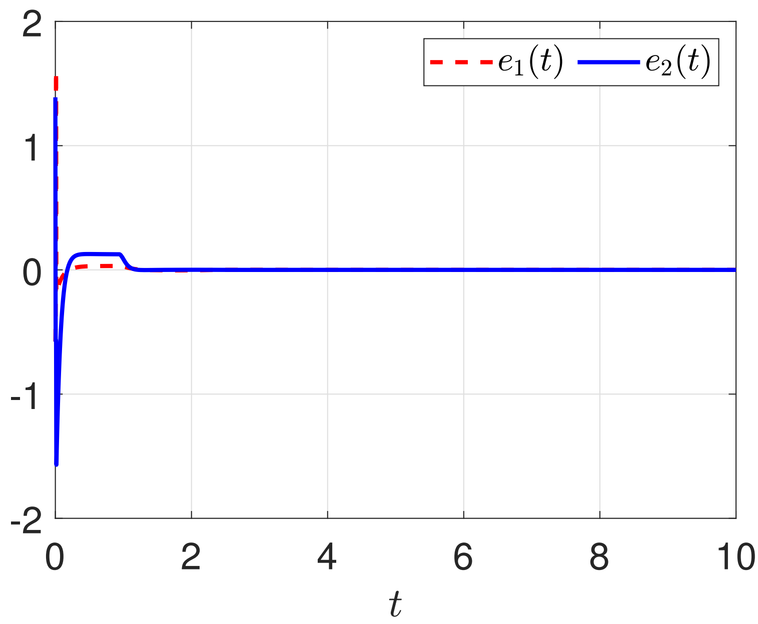

Figure 5. State trajectories of the SES with $\omega(t) \equiv 0$.

5. Conclusions

In this paper, QC is adopted to address the $\mathcal{H}_\infty$ synchronization problem of drive-response delayed HNs. Theorem 1 establishes a delay-dependent sufficient condition for achieving $\mathcal{H}\infty$ synchronization within the framework of QC, based on a linear combination of Lyapunov functional candidates

$V(t)$ together with two efficient matrix inequalities. Thanks to the introduction of some decoupling techniques, Theorem 2 presents an alternative synchronization criterion characterized by reduced nonlinearities. Furthermore, with fixed bounds on the time delay and the HDAL, Algorithm 1 is proposed to determine the minimum required gain k^* for the quantized controller. A numerical example is provided to demonstrate the applicability and superiority of the present theoretical results.

The quantizer adopted in the present work is static with fixed parameters. One limitation is that the quantization range cannot be adjusted dynamically. Extending the proposed framework to dynamic QC that incorporate a time-varying scaling factor to avoid signal saturation is a natural direction for future research. Other potential extensions include combining the proposed approach with event-triggered mechanisms and applying it to more general classes of delayed RNNs.

Use of AI tools declaration

The author declares they have not used Artificial Intelligence (AI) tools in the creation of this paper.

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Conflict of interest

The author declares no conflicts of interest in this paper.

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