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Research article

Numerical approximation of fourth-order fractional diffusion-wave systems using finite difference and discontinuous Galerkin method

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Abstract: This study develops a finite difference/local discontinuous Galerkin (LDG) framework for solving a fourth-order fractional diffusion-wave equation. The temporal fractional operator is approximated through a finite difference approach, achieving a truncation accuracy of $O((\Delta t)^{3-\alpha})$, where Δt denotes the time increment and α represents the fractional order. For spatial discretization, LDG technique is employed, which leads to a fully implicit discrete formulation of the considered model. By applying mathematical induction, we establish the unconditional stability and convergence of the proposed algorithm. A series of computational experiments is presented to verify the theoretical error bounds.

Keywords: fourth-order fractional diffusion-wave system; time fractional derivative; stability; convergence
Mathematics subject classification: 65M12; 65M06; 35S10

1. Introduction

Fractional calculus has attracted considerable attention over the past decades due to its novel and diverse applications in finance, engineering, biology, chemistry, and related fields [1–3]. Time-fractional diffusion equations (T-FDEs) constitute a class of linear integro-differential equations [4–6]. Although some analytical solutions have been obtained, explicit solutions for most fractional partial differential equations (FPDEs) are highly complex or even intractable, which limits their practical applicability in scientific and engineering problems. Therefore, the development of efficient and straightforward numerical schemes for T-FDEs is of significant importance. Existing numerical approaches for FPDEs include finite difference methods [7–10], finite element methods [11–13], spectral methods [14–16], discontinuous Galerkin methods [17–20], as well as homotopy perturbation and variational techniques [21–24].

In this work, we focus on the fourth-order fractional diffusion-wave equation

$$
\begin{aligned}
&\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \mu^2 \frac{\partial^4 u(x,t)}{\partial x^4} = f(x,t), \quad x \in (a,b),\ t \in (0,T], \\
&u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in [a,b],
\end{aligned} \tag{1.1}
$$

where $1 < \alpha < 2$ denotes the fractional order in time, μ is a constant coefficient, and f, u_0, u_1 are given functions. In this study, the solution is assumed to be either periodic or compactly supported. The Caputo fractional derivative is employed, defined by

$$
\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{\partial^2 u(x,s)}{\partial s^2} \frac{ds}{(t-s)^{\alpha-1}}, \quad t > 0,\ 1 < \alpha < 2, \tag{1.2}
$$

where $\Gamma(\cdot)$ represents the Gamma function.

In certain applications, a fourth-order spatial derivative is essential, such as in wave propagation along beams and in modeling the formation of grooves on flat surfaces. Hu and Zhang proposed a Crank–Nicolson scheme [25] and an implicit compact scheme [26] for the diffusion-wave equation. Jafari et al. [27] addressed a fourth-order fractional diffusion-wave equation in a bounded domain using the decomposition method. Agarwal [28] presented a general solution for fractional diffusion equations involving a fourth-order spatial derivative in both bounded and unbounded domains. Qiu et al. [29] investigated a meshfree method based on spatio-temporal homogenization functions for one-dimensional fourth-order fractional diffusion-wave equations. Liu et al. [30] developed a mixed finite element method for a time-fractional fourth-order partial differential equation and rigorously studied its stability and convergence properties. Wang et al. [31] proposed a mixed spectral element method combined with second-order time-stepping schemes to solve a two-dimensional nonlinear fourth-order fractional diffusion equation and analyzed its accuracy and efficiency. Bai et al. [32] developed a multicontinuum modeling approach for time-fractional diffusion-wave equations in heterogeneous media and investigated the accuracy and applicability of the proposed method. Cao and Xu [33] proposed an adaptive-coefficient finite difference frequency-domain method for the time-fractional diffusive-viscous wave equation in geophysical applications and analyzed its numerical accuracy and computational efficiency. Qing and Li [34] proposed a meshless generalized finite difference method for solving the time-fractional diffusion-wave equation and investigated its stability and numerical accuracy.

The remainder of this paper is organized as follows: Section 2 introduces essential notations and theoretical preliminaries. In Section 3, we develop a finite difference/LDG method for the fractional diffusion-wave system and establish its stability and error estimates via mathematical induction. Numerical experiments are reported in Section 4, and conclusions are drawn in the final section.

2. Notations and auxiliary results

Consider a mesh covering the computational domain

$$
a = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = b,
$$

where the cells are defined by $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ for $j = 1, \ldots, N$, and the corresponding cell sizes are $\Delta x_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$. Denote

$$h = \max_{1\le j\le N} \Delta x_j.$$

Let $u^+_{j+\frac{1}{2}}$ and $u^-_{j+\frac{1}{2}}$ represent the values of u at $x_{j+\frac{1}{2}}$ approached from the right cell I_{j+1} and the left cell I_j, respectively.

The piecewise-polynomial space of degree at most k is defined as

$$V_h^k = \{v : v|_{I_j} \in P^k(I_j),\ x \in I_j,\ j = 1, \ldots, N\}.$$

For error analysis, we introduce two types of projections in the one-dimensional spatial domain $\Omega = [a, b]$. The standard L^2 projection $\mathcal{P}$ satisfies, for each j,

$$\int_{I_j} (\mathcal{P}\omega(x) - \omega(x))\, v(x)\, dx = 0, \quad \forall v \in P^k(I_j), \tag{2.1}$$

and the special projections $\mathcal{P}^\pm$ are defined by, for each j,

$$\begin{aligned}
&\int_{I_j} (\mathcal{P}^+\omega(x) - \omega(x))v(x)\, dx = 0, \quad \forall v \in P^{k-1}(I_j), \quad \mathcal{P}^+\omega(x^+_{j-\frac{1}{2}}) = \omega(x_{j-\frac{1}{2}}),\\
&\int_{I_j} (\mathcal{P}^-\omega(x) - \omega(x))v(x)\, dx = 0, \quad \forall v \in P^{k-1}(I_j), \quad \mathcal{P}^-\omega(x^-_{j+\frac{1}{2}}) = \omega(x_{j+\frac{1}{2}}).
\end{aligned} \tag{2.2}$$

The following approximation property holds for the projections $\mathcal{P}$ and $\mathcal{P}^\pm$ [35–37]:

$$\|\omega^e\| + h\|\omega^e\|_\infty + h^{1/2}\|\omega^e\|_{\tau_h} \le Ch^{k+1}, \tag{2.3}$$

where $\omega^e = \mathcal{P}\omega - \omega$ or $\omega^e = \mathcal{P}^\pm\omega - \omega$.

Throughout this paper, C denotes a generic positive constant whose value may vary in different occurrences. The scalar inner product on $L^2(D)$ is denoted by $(\cdot,\cdot)_D$ with the associated norm $\|\cdot\|_D$. When $D = \Omega$, the subscript is omitted.

3. Numerical scheme

In this section, we present the numerical discretization for Eq (1.1) and discuss the stability and convergence properties of the proposed scheme.

3.1. Discretization of the time-fractional derivative

Let the time interval $[0, T]$ be uniformly partitioned with step size $\Delta t = T/M$, $M \in \mathbb{N}$, and define the discrete time points $t_n = n\Delta t$, $n = 0, 1, \ldots, M$.

Define $v(x,t) = \frac{\partial u(x,t)}{\partial t}$. Using the standard three-point backward difference formula, we have

$$v(x, t_i) = \frac{\partial u(x, t_i)}{\partial t} = \frac{3u(x, t_i) - 4u(x, t_{i-1}) + u(x, t_{i-2})}{2\Delta t} + r_1^n,$$

where the truncation error satisfies $|r_1^n| \le C(\Delta t)^2$.

Then, the Caputo fractional derivative at t_n can be approximated as

$$
\begin{aligned}
\frac{\partial^\alpha u(x,t_n)}{\partial t^\alpha} &= \frac{1}{\Gamma(2-\alpha)}\int_0^{t_n}\frac{\partial v(x,s)}{\partial s}\frac{ds}{(t_n-s)^{\alpha-1}}\\
&= \frac{1}{\Gamma(2-\alpha)}\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\frac{v(x,t_{i+1})-v(x,t_i)}{\Delta t}\frac{ds}{(t_n-s)^{\alpha-1}}+r_2^n\\
&= \frac{(\Delta t)^{1-\alpha}}{\Gamma(3-\alpha)}\left[v(x,t_n)+\sum_{i=1}^{n-1}(b_{n-i}-b_{n-i-1})v(x,t_i)-b_{n-1}v(x,t_0)\right]+r_2^n\\
&= \frac{(\Delta t)^{1-\alpha}}{\Gamma(3-\alpha)}\left[\frac{3u(x,t_n)-4u(x,t_{n-1})+u(x,t_{n-2})}{2\Delta t}\right.\\
&\quad+\sum_{i=1}^{n-1}(b_{n-i}-b_{n-i-1})\frac{3u(x,t_i)-4u(x,t_{i-1})+u(x,t_{i-2})}{2\Delta t}\\
&\quad\left.-\,b_{n-1}v(x,t_0)\right]+r_3^n,
\end{aligned}
\tag{3.1}
$$

where

$$b_0=1,\quad b_i=(i+1)^{2-\alpha}-i^{2-\alpha},\quad i=1,2,\ldots,$$

and for $i=1$, we set $u(x,-1)=u(x,0)-\Delta t\,u_1(x)+O((\Delta t)^2)$ by the Taylor expansion.

Following [15], the truncation error satisfies $|r_2^n|\le C(\Delta t)^{3-\alpha}$, which implies $|r_3^n|\le C(\Delta t)^{3-\alpha}$.

The coefficients b_i satisfy

$$b_i>0,\quad i=1,2,\ldots,n,\qquad 1=b_0>b_1>b_2>\cdots>b_n,\quad b_n\to 0 \text{ as } n\to\infty. \tag{3.2}$$

Substituting Eq (3.1) into Eq (1.1), we get

$$
\begin{aligned}
3u(x,t_n)+\beta\mu^2\frac{\partial^4 u(x,t_n)}{\partial x^4} &= \sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})(3u(x,t_i)-4u(x,t_{i-1})+u(x,t_{i-2}))\\
&\quad+2\Delta t\,b_{n-1}v(x,t_0)+\beta f(x,t_n)+4u(x,t_{n-1})-u(x,t_{n-2})+\beta r_3^n,
\end{aligned}
$$

where $\beta=2(\Delta t)^\alpha\Gamma(3-\alpha)$.

Let u^k denote the numerical approximation of $u(x,t_k)$ and $f^n=f(x,t_n)$. Then the time-discrete version of problem (1.1) can be written as

$$
\begin{aligned}
3u^n-\beta\frac{\partial^2 u^n}{\partial x^2} &= \sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})\left(3u^i-4u^{i-1}+u^{i-2}\right)\\
&\quad+2\Delta t\,b_{n-1}v^0+\beta f^n+4u^{n-1}-u^{n-2},
\end{aligned}
\tag{3.3}
$$

where the initial value u^{-1} is defined by

$$u^{-1}=u^0-\Delta t\,u_1(x),$$

consistent with the second-order backward difference approximation.

3.2. Fully discrete schemes

We first rewrite Eq (1.1) as a first-order system

$$p = u_x, \quad q = p_x, \quad s = q_x, \quad \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \mu^2 s_x = f. \tag{3.4}$$

Let $u_h^n, p_h^n, q_h^n, s_h^n \in V_h^k$ denote the approximations of $u(\cdot, t_n), p(\cdot, t_n), q(\cdot, t_n), s(\cdot, t_n)$, respectively, and define $f^n(x) = f(x, t_n)$. The fully discrete LDG scheme is defined as follows: Find $u_h^n, p_h^n, q_h^n, s_h^n \in V_h^k$ such that for all test functions $v, w, \rho, \xi \in V_h^k$,

$$\begin{aligned}
&3\int_\Omega u_h^n v\,dx - \beta\mu^2\left(\int_\Omega s_h^n v_x\,dx - \sum_{j=1}^{N}\left((\widehat{s_h^n}v^-)_{j+\frac{1}{2}} - (\widehat{s_h^n}v^+)_{j-\frac{1}{2}}\right)\right)\\
&\qquad = \sum_{i=1}^{n-1}(b_{n-i-1} - b_{n-i})\int_\Omega (3u_h^i - 4u_h^{i-1} + u_h^{i-2})v\,dx + 2\Delta t\, b_{n-1}\int_\Omega v_h^0 v\,dx\\
&\qquad\quad + 4\int_\Omega u_h^{n-1} v\,dx - \int_\Omega u_h^{n-2} v\,dx + \beta\int_\Omega f^n v\,dx,\\
&\int_\Omega s_h^n w\,dx + \int_\Omega q_h^n w_x\,dx - \sum_{j=1}^{N}\left((\widehat{q_h^n}w^-)_{j+\frac{1}{2}} - (\widehat{q_h^n}w^+)_{j-\frac{1}{2}}\right) = 0,\\
&\int_\Omega q_h^n \rho\,dx + \int_\Omega p_h^n \rho_x\,dx - \sum_{j=1}^{N}\left((\widehat{p_h^n}\rho^-)_{j+\frac{1}{2}} - (\widehat{p_h^n}\rho^+)_{j-\frac{1}{2}}\right) = 0,\\
&\int_\Omega p_h^n \xi\,dx + \int_\Omega u_h^n \xi_x\,dx - \sum_{j=1}^{N}\left((\widehat{u_h^n}\xi^-)_{j+\frac{1}{2}} - (\widehat{u_h^n}\xi^+)_{j-\frac{1}{2}}\right) = 0.
\end{aligned} \tag{3.5}$$

The initial approximations u_h^{-1}, u_h^0, v_h^0 are taken as the L^2 projections of $u(\cdot,-1), u(\cdot,0), u_1(\cdot,0)$, respectively,

$$\begin{aligned}
\int_\Omega u_h^{-1}\phi\,dx &= \int_\Omega \mathcal{P}u(x,-1)\phi\,dx = \int_\Omega u(x,-1)\phi\,dx,\\
\int_\Omega u_h^0\phi\,dx &= \int_\Omega \mathcal{P}u(x,0)\phi\,dx = \int_\Omega u_0(x)\phi\,dx,\\
\int_\Omega v_h^0\phi\,dx &= \int_\Omega \mathcal{P}u_1(x,0)\phi\,dx = \int_\Omega u_1(x)\phi\,dx, \quad \forall\phi \in V_h^k.
\end{aligned}$$

The "hat" terms in Eq (3.5) arising from integration by parts represent the numerical fluxes. These are single-valued functions defined on cell interfaces and should be chosen to ensure stability. A simple choice is given by

$$\begin{aligned}
\widehat{u_h^n} &= (u_h^n)^-, \quad \widetilde{s_h^n} = (s_h^n)^+, \quad \widehat{s_h^n} = \widetilde{s_h^n} - \tau_1[u_h],\\
\widehat{q_h^n} &= (q_h^n)^-, \quad \widetilde{p_h^n} = (p_h^n)^+, \quad \widehat{p_h^n} = \widetilde{p_h^n} - \tau_2[q_h].
\end{aligned} \tag{3.6}$$

We note that the choice of fluxes (3.6) is not unique. The key principle is to select $\widehat{u_h^n}$ and $\widetilde{s_h^n}, \widetilde{p_h^n}$ from opposite sides, and $\widehat{q_h^n}$ and $\widetilde{s_h^n}, \widetilde{p_h^n}$ from opposite sides, to guarantee stability [38, 39].

3.3. Stability and convergence

For simplicity of notation, and without loss of generality, we consider the case $f = 0$ in the following analysis.

Theorem 3.1. *Under periodic or compactly supported boundary conditions, the fully discrete LDG scheme (3.5) is unconditionally stable. Specifically, there exists a positive constant C depending on u, T, α, such that*

$$\|u_h^n\| \le C(\|u_h^0\| + \Delta t\|u_1(x)\|), \quad n = 1, 2, \ldots, M. \tag{3.7}$$

Proof. We prove the theorem by mathematical induction.

When $n = 1$, the scheme (3.5) reduces to

$$3\int_\Omega u_h^1 v\,dx - \beta\mu^2\Big(\int_\Omega s_h^1 v_x\,dx - \sum_{j=1}^{N}\big((\widehat{s_h^1}v^-)_{j+\frac{1}{2}} - (\widehat{s_h^1}v^+)_{j-\frac{1}{2}}\big)\Big)$$
$$= 2\Delta t b_0\int_\Omega v_h^0 v\,dx + 4\int_\Omega u_h^0 v\,dx - \int_\Omega u_h^{-1} v\,dx,$$

and

$$\begin{aligned}
&\int_\Omega s_h^1 w\,dx + \int_\Omega q_h^1 w_x\,dx - \sum_{j=1}^{N}\big((\widehat{q_h^1}w^-)_{j+\frac{1}{2}} + (\widehat{q_h^1}w^+)_{j-\frac{1}{2}}\big) = 0,\\
&\int_\Omega q_h^1 \rho\,dx + \int_\Omega p_h^1 \rho_x\,dx - \sum_{j=1}^{N}\big((\widehat{p_h^1}\rho^-)_{j+\frac{1}{2}} + (\widehat{p_h^1}\rho^+)_{j-\frac{1}{2}}\big) = 0,\\
&\int_\Omega p_h^1 \xi\,dx + \int_\Omega u_h^1 \xi_x\,dx - \sum_{j=1}^{N}\big((\widehat{u_h^1}\xi^-)_{j+\frac{1}{2}} + (\widehat{u_h^1}\xi^+)_{j-\frac{1}{2}}\big) = 0.
\end{aligned} \tag{3.8}$$

Taking the test functions

$$v = u_h^1, \quad \xi = -\beta\mu^2 s_h^1, \quad w = \beta\mu^2 p_h^1, \quad \rho = \beta\mu^2 q_h^1,$$

and using the fluxes (3.6), we obtain

$$\begin{aligned}
&3\|u_h^1\|^2 + \beta\mu^2\|q_h^1\|^2 + \beta\mu^2\sum_{j=1}^{N}(\tau_1[u_h^1]^2 + \tau_2[q_h^1]^2)_{j-\frac{1}{2}}\\
&\quad + \beta\mu^2\sum_{j=1}^{N}\big(\Psi(p_h^1, u_h^1)_{j+\frac{1}{2}} - \Psi(p_h^1, u_h^1)_{j-\frac{1}{2}} + \Theta(p_h^1, u_h^1)_{j-\frac{1}{2}}\big)\\
&\quad = 2\Delta t b_0\int_\Omega v_h^0 u_h^1\,dx + 4\int_\Omega u_h^0 u_h^1\,dx - \int_\Omega u_h^{-1} u_h^1\,dx\\
&\quad \le \frac{1}{8}(2\Delta t\|v_h^0\| + 4\|u_h^0\| + \|u_h^{-1}\|)^2 + 2\|u_h^1\|^2.
\end{aligned} \tag{3.9}$$

Here, $\Psi(\cdot,\cdot)$ and $\Theta(\cdot,\cdot)$ are defined as in the original statement, and a straightforward calculation shows that $\Theta(p_h^1, u_h^1)_{j-\frac{1}{2}} = 0$ for the chosen fluxes.

From Eq (3.9), we get

$$\|u_h^1\| \le 2\Delta t\|v_h^0\| + 4\|u_h^0\| + \|u_h^{-1}\|. \tag{3.10}$$

By the definition of u_h^{-1}, we have

$$\|u_h^{-1}\| \le C(\|u_h^0\| + \Delta t\|u_1(x)\|),$$

and similarly

$$\|v_h^0\| \le \|u_1(x)\|.$$

Combining these estimates with Eq (3.10) yields

$$\|u_h^1\| \le C(\|u_h^0\| + \Delta t\|u_1(x)\|).$$

Assume that

$$\|u_h^m\| \le C(\|u_h^0\| + \Delta t\|u_1(x)\|), \quad m = 1, 2, \dots, P. \tag{3.11}$$

We need to show that the inequality also holds for $m = P + 1$. Taking

$$v = u_h^{P+1}, \quad \xi = -\beta\mu^2 s_h^{P+1}, \quad w = \beta\mu^2 p_h^{P+1}, \quad \rho = \beta\mu^2 q_h^{P+1}$$

in Eq (3.5) and applying the same arguments as above, we obtain

$$\|u_h^{P+1}\| \le C(\|u_h^0\| + \Delta t\|u_1(x)\|).$$

This completes the proof of unconditional stability.

Theorem 3.2. *Let $u(x, t_n)$ be the exact solution of problem (1.1), sufficiently smooth such that $u \in H^{m+1}$ with $0 \le m \le k + 1$. Let u_h^n denote the numerical solution obtained from the fully discrete LDG scheme (3.5). Then the following error estimates hold:*

- *For $1 < \alpha < 2$,*

$$\|u(x, t_n) - u_h^n\| \le \frac{CT^\alpha}{2-\alpha}\Big((\Delta t)^{-\alpha}h^{k+1} + (\Delta t)^{3-\alpha} + (\Delta t)^{-\alpha/2}h^{k+\frac{1}{2}} + h^{k+1}\Big),$$

- *As $\alpha \to 2$,*

$$\|u(x, t_n) - u_h^n\| \le CT\Big((\Delta t)^{-2}h^{k+1} + \Delta t + (\Delta t)^{-1}h^{k+\frac{1}{2}} + h^{k+1}\Big),$$

where C is a constant depending on u, T, and α.

Proof. It is straightforward to verify that the exact solution of partial differential equation

(PDE) (1.1) satisfies

$$\begin{aligned}
3\int_\Omega u(x,t_n)v\,dx - \beta\mu^2\Big(\int_\Omega s(x,t_n)v_x\,dx - \sum_{j=1}^{N}\big((s(x,t_n)v^-)_{j+\frac12} - (s(x,t_n)v^+)_{j-\frac12}\big)\Big)&\\
= \sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})\int_\Omega (3u(x,t_i)-4u(x,t_{i-1})+u(x,t_{i-2}))v\,dx&\\
+ 2\Delta t\, b_{n-1}\int_\Omega v(x,t_0)v\,dx + 4\int_\Omega u(x,t_{n-1})v\,dx&\\
- \int_\Omega u(x,t_{n-2})v\,dx + \beta\int_\Omega f(x,t_n)v\,dx + \beta\int_\Omega r_3^n v\,dx,&\\
\int_\Omega s(x,t_n)w\,dx + \int_\Omega q(x,t_n)w_x\,dx - \sum_{j=1}^{N}\big((q(x,t_n)w^-)_{j+\frac12} - (q(x,t_n)w^+)_{j-\frac12}\big) = 0,&\\
\int_\Omega q(x,t_n)\rho\,dx + \int_\Omega p(x,t_n)\rho_x\,dx - \sum_{j=1}^{N}\big((p(x,t_n)\rho^-)_{j+\frac12} - (p(x,t_n)\rho^+)_{j-\frac12}\big) = 0,&\\
\int_\Omega p(x,t_n)\xi\,dx + \int_\Omega u(x,t_n)\xi_x\,dx - \sum_{j=1}^{N}\big((u(x,t_n)\xi^-)_{j+\frac12} - (u(x,t_n)\xi^+)_{j-\frac12}\big) = 0,&
\end{aligned} \tag{3.12}$$

for all $v, w, \rho, \xi \in H^1(I_j)$, $j = 1, \ldots, N$.

Denote the error terms as

$$\begin{aligned}
e_u^n &= u(x,t_n) - u_h^n = \mathcal{P}^- e_u^n - (\mathcal{P}^- u(x,t_n) - u(x,t_n)),\\
e_s^n &= s(x,t_n) - s_h^n = \mathcal{P}^+ e_s^n - (\mathcal{P}^+ s(x,t_n) - s(x,t_n)),\\
e_q^n &= q(x,t_n) - q_h^n = \mathcal{P}^- e_q^n - (\mathcal{P}^- q(x,t_n) - q(x,t_n)),\\
e_p^n &= p(x,t_n) - p_h^n = \mathcal{P}^+ e_p^n - (\mathcal{P}^+ p(x,t_n) - p(x,t_n)).
\end{aligned} \tag{3.13}$$

Subtracting the fully discrete LDG scheme (3.5) from Eq (3.12) and applying the numerical fluxes (3.6), we obtain the following error equation:

$$\begin{aligned}
3\int_\Omega e_u^n v\,dx - \beta\mu^2\Big(\int_\Omega e_s^n v_x\,dx - \sum_{j=1}^{N}\big((e_s^n)^+ v^-_{j+\frac12} - (e_s^n)^+ v^+_{j-\frac12}\big)\Big)&\\
- \sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})\int_\Omega (3e_u^i - 4e_u^{i-1} + e_u^{i-2})v\,dx&\\
- 4\int_\Omega e_u^{n-1}v\,dx + \int_\Omega e_u^{n-2}v\,dx + \beta\int_\Omega r_3^n v\,dx&\\
+ \int_\Omega e_s^n w\,dx + \int_\Omega e_q^n w_x\,dx - \sum_{j=1}^{N}\big((e_q^n)^- w^-_{j+\frac12} - (e_q^n)^- w^+_{j-\frac12}\big)&\\
+ \int_\Omega e_q^n \rho\,dx + \int_\Omega e_p^n \rho_x\,dx - \sum_{j=1}^{N}\big((e_p^n)^+ \rho^-_{j+\frac12} - (e_p^n)^+ \rho^+_{j-\frac12}\big)&
\end{aligned} \tag{3.14}$$

$$+\int_\Omega e_p^n\xi\,dx+\int_\Omega e_u^n\xi_x\,dx-\sum_{j=1}^{N}((e_u^n)^-\xi^-_{j+\frac{1}{2}}-(e_u^n)^-\xi^+_{j-\frac{1}{2}})$$
$$+\beta\tau_1\sum_{j=1}^{N}[e_u^n][v]_{j-\frac{1}{2}}+\tau_2\sum_{j=1}^{N}[e_q^n][\rho]_{j-\frac{1}{2}}=0.$$

Using the notation (3.13), the error equation (3.3) can be rewritten as

$$\begin{aligned}
&3\int_\Omega \mathcal{P}^-e_u^n v dx-\beta\mu^2(\int_\Omega \mathcal{P}^+e_s^n v_x dx-\sum_{j=1}^{N}(((\mathcal{P}^+e_s^n)^+v^-)_{j+\frac{1}{2}}\\
&-((\mathcal{P}^+e_s^n)^+v^+)_{j-\frac{1}{2}}))+\int_\Omega \mathcal{P}^+e_s^n w dx+\int_\Omega \mathcal{P}^-e_q^n w_x dx\\
&-\sum_{j=1}^{N}(((\mathcal{P}^-e_q^n)^-w^-)_{j+\frac{1}{2}}-((\mathcal{P}^-e_q^n)^-w^+)_{j-\frac{1}{2}})\\
&+\int_\Omega \mathcal{P}^-e_q^n\rho dx+\int_\Omega \mathcal{P}^+e_p^n\rho_x dx-\sum_{j=1}^{N}(((\mathcal{P}^+e_p^n)^+\rho^-)_{j+\frac{1}{2}}\\
&-((\mathcal{P}^+e_p^n)^+\rho^+)_{j-\frac{1}{2}})+\int_\Omega \mathcal{P}^+e_p^n\xi dx+\int_\Omega \mathcal{P}^-e_u^n\xi_x dx\\
&-\sum_{j=1}^{N}(((\mathcal{P}^-e_u^n)^-\xi^-)_{j+\frac{1}{2}}-((\mathcal{P}^-e_u^n)^-\xi^+)_{j-\frac{1}{2}})\\
&+\beta\tau_1\sum_{j=1}^{N}[\mathcal{P}^-e_u^n][v]_{j-\frac{1}{2}}+\tau_2\sum_{j=1}^{N}[\mathcal{P}^-e_q^n][\rho]_{j-\frac{1}{2}}\\
&=\sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})\int_\Omega(3\mathcal{P}^-e_u^i-4\mathcal{P}^-e_u^{i-1}+\mathcal{P}^-e_u^{i-2})v dx\\
&+4\int_\Omega \mathcal{P}^-e_u^{n-1}v dx-\int_\Omega \mathcal{P}^-e_u^{n-2}v dx-\beta\int_\Omega r_3^n v dx\\
&+3\int_\Omega(\mathcal{P}^-u(x,t_n)-u(x,t_n))v dx-\beta\mu^2(\int_\Omega(\mathcal{P}^+s(x,t_n)-s(x,t_n))v_x dx\\
&-\sum_{j=1}^{N}(((\mathcal{P}^+s(x,t_n)-s(x,t_n))^+v^-)_{j+\frac{1}{2}}-((\mathcal{P}^+s(x,t_n)-s(x,t_n))^+v^+)_{j-\frac{1}{2}}))\\
&-\sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})\int_\Omega(3(\mathcal{P}^-u(x,t_i)-u(x,t_i))-4(\mathcal{P}^-u(x,t_{i-1})-u(x,t_{i-1}))\\
&+(\mathcal{P}^-u(x,t_{i-2})-u(x,t_{i-2})))v dx-4\int_\Omega(\mathcal{P}^-u(x,t_{n-1})-u(x,t_{n-1}))v dx\\
&+\int_\Omega(\mathcal{P}^-u(x,t_{n-2})-u(x,t_{n-2}))v dx\\
&+\int_\Omega(\mathcal{P}^+s(x,t_n)-s(x,t_n))w dx+\int_\Omega(\mathcal{P}^-q(x,t_n)-q(x,t_n))w_x dx
\end{aligned} \tag{3.15}$$

$$
\begin{aligned}
&- \sum_{j=1}^{N} (((\mathcal{P}^- q(x,t_n) - q(x,t_n))^- w^-)_{j+\frac{1}{2}} - ((\mathcal{P}^- q(x,t_n) - q(x,t_n))^- w^+)_{j-\frac{1}{2}}) \\
&+ \int_\Omega (\mathcal{P}^- q(x,t_n) - q(x,t_n)) \rho dx + \int_\Omega (\mathcal{P}^+ p(x,t_n) - p(x,t_n)) \rho_x dx \\
&- \sum_{j=1}^{N} (((\mathcal{P}^+ p(x,t_n) - p(x,t_n))^+ \rho^-)_{j+\frac{1}{2}} - ((\mathcal{P}^+ p(x,t_n) - p(x,t_n))^+ \rho^+)_{j-\frac{1}{2}}) \\
&+ \int_\Omega (\mathcal{P}^+ p(x,t_n) - p(x,t_n)) \xi dx + \int_\Omega (\mathcal{P}^- u(x,t_n) - u(x,t_n)) \xi_x dx \\
&- \sum_{j=1}^{N} (((\mathcal{P}^- u(x,t_n) - u(x,t_n))^- \xi^-)_{j+\frac{1}{2}} - ((\mathcal{P}^- u(x,t_n) - u(x,t_n))^- \xi^+)_{j-\frac{1}{2}}).
\end{aligned}
$$

Noting that for any $a, b \in \mathbb{R}$,

$$
ab \le \varepsilon a^2 + \frac{1}{4\varepsilon} b^2,
$$

choosing the test functions

$$
v = \mathcal{P}^- e_u^n, \quad \xi = -\beta\mu^2 \mathcal{P}^+ e_s^n, \quad w = \beta\mu^2 \mathcal{P}^+ e_p^n, \quad \rho = \beta\mu^2 \mathcal{P}^- e_q^n,
$$

in Eq (3.3), and using the properties (2.1) and (2.2), we obtain

$$
\begin{aligned}
&3 \int_\Omega (\mathcal{P}^- e_u^n)^2 dx + \beta\mu^2 \int_\Omega (\mathcal{P}^- e_q^n)^2 dx + \beta\mu^2 \tau_1 \sum_{j=1}^{N} [\mathcal{P}^- e_u^n]^2_{j-\frac{1}{2}} + \beta\tau_2 \sum_{j=1}^{N} [\mathcal{P}^- e_q^n]^2_{j-\frac{1}{2}} \\
&= \sum_{i=1}^{n-1} (b_{n-i-1} - b_{n-i}) \int_\Omega (3\mathcal{P}^- e_u^i - 4\mathcal{P}^- e_u^{i-1} + \mathcal{P}^- e_u^{i-2}) \mathcal{P}^- e_u^n dx \\
&+ 4 \int_\Omega \mathcal{P}^- e_u^{n-1} \mathcal{P}^- e_u^n dx - \int_\Omega \mathcal{P}^- e_u^{n-2} \mathcal{P}^- e_u^n dx \\
&- \sum_{i=1}^{n-1} (b_{n-i-1} - b_{n-i}) \int_\Omega (3(\mathcal{P}^- u(x,t_i) - u(x,t_i)) - 4(\mathcal{P}^- u(x,t_{i-1}) \\
&- u(x,t_{i-1})) + (\mathcal{P}^- u(x,t_{i-2}) - u(x,t_{i-2}))) \mathcal{P}^- e_u^n dx \\
&- 4 \int_\Omega (\mathcal{P}^- u(x,t_{n-1}) - u(x,t_{n-1})) \mathcal{P}^- e_u^n dx - \beta \int_\Omega r_3^n \mathcal{P}^- e_u^n dx \\
&+ \int_\Omega (\mathcal{P}^- u(x,t_{n-2}) - u(x,t_{n-2})) \mathcal{P}^- e_u^n dx \\
&+ 3 \int_\Omega (\mathcal{P}^- u(x,t_n) - u(x,t_n)) \mathcal{P}^- e_u^n dx + \beta\mu^2 \int_\Omega (\mathcal{P}^- q(x,t_n) - q(x,t_n)) \mathcal{P}^- e_q^n dx \\
&+ \beta\mu^2 \sum_{j=1}^{N} ((\mathcal{P} s(x,t_n) - s(x,t_n))^+ [\mathcal{P}^- e_u^n])_{j-\frac{1}{2}} + \beta\mu^2 \sum_{j=1}^{N} ((\mathcal{P} p(x,t_n) - p(x,t_n))^+ [\mathcal{P}^- e_q^n])]_{j-\frac{1}{2}}
\end{aligned} \tag{3.16}
$$

$$
\begin{aligned}
&\le \frac{1}{8}\Big(\sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})(3\|\mathcal{P}^- e_u^i\| + 4\|\mathcal{P}^- e_u^{i-1}\| + \|\mathcal{P}^- e_u^{i-2}\|) + 4\|\mathcal{P}^- e_u^{n-1}\| \\
&+ \|\mathcal{P}^- e_u^{n-2}\| + \beta\|r_3^n\| + \sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})(3\|\mathcal{P}^- u(x,t_i) - u(x,t_i)\| \\
&+ 4\|\mathcal{P}^- u(x,t_{i-1}) - u(x,t_{i-1})\| + \|\mathcal{P}^- u(x,t_{i-2}) - u(x,t_{i-2})\|) \\
&+ 4\|\mathcal{P}^- u(x,t_{n-1}) - u(x,t_{n-1})\| + \|\mathcal{P}^- u(x,t_{n-2}) - u(x,t_{n-2})\| \\
&+ 3\|\mathcal{P}^- u(x,t_n) - u(x,t_n)\|\Big)^2 + 2\|\mathcal{P}^- e_u^n\|^2 \\
&+ \frac{\beta\mu^2}{2}\|\mathcal{P}^- q(x,t_n) - q(x,t_n)\|^2 + \frac{\beta\mu^2}{2}\|\mathcal{P}^- e_q^n\|^2 \\
&+ \frac{\beta\mu^2}{4\varepsilon}\sum_{j=1}^{N}((\mathcal{P}s(x,t_n) - s(x,t_n))^+)^2_{j-\frac{1}{2}} + \beta\mu^2\varepsilon\sum_{j=1}^{N}[\mathcal{P}^- e_u^n]^2_{j-\frac{1}{2}} \\
&+ \frac{\beta\mu^2}{4\varepsilon}\sum_{j=1}^{N}((\mathcal{P}p(x,t_n) - p(x,t_n))^+)^2_{j-\frac{1}{2}} + \beta\mu^2\varepsilon\sum_{j=1}^{N}[\mathcal{P}^- e_q^n]^2_{j-\frac{1}{2}}.
\end{aligned}
$$

Choosing a sufficiently small $\varepsilon \le \{\tau_1, \tau_2\}$, we obtain

$$
\begin{aligned}
&\int_\Omega (\mathcal{P}^- e_u^n)^2\,dx + \frac{\beta\mu^2}{2}\int_\Omega (\mathcal{P}^- e_q^n)^2\,dx \\
&\le \Bigg(\sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})(3\|\mathcal{P}^- e_u^i\| + 4\|\mathcal{P}^- e_u^{i-1}\| + \|\mathcal{P}^- e_u^{i-2}\|) \\
&+ 4\|\mathcal{P}^- e_u^{n-1}\| + \|\mathcal{P}^- e_u^{n-2}\| + \beta\|r_3^n\| \\
&+ \sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})\big(3\|\mathcal{P}^- u(x,t_i) - u(x,t_i)\| + 4\|\mathcal{P}^- u(x,t_{i-1}) - u(x,t_{i-1})\| \\
&\quad + \|\mathcal{P}^- u(x,t_{i-2}) - u(x,t_{i-2})\|\big) \\
&+ 4\|\mathcal{P}^- u(x,t_{n-1}) - u(x,t_{n-1})\| + \|\mathcal{P}^- u(x,t_{n-2}) - u(x,t_{n-2})\| \\
&+ 3\|\mathcal{P}^- u(x,t_n) - u(x,t_n)\|\Bigg)^2 \\
&+ \frac{\beta\mu^2}{2}\|\mathcal{P}^- q(x,t_n) - q(x,t_n)\|^2 \\
&+ \frac{\beta\mu^2}{4\varepsilon}\sum_{j=1}^{N}\big((\mathcal{P}s(x,t_n) - s(x,t_n))^+\big)^2_{j-\frac{1}{2}} \\
&+ \frac{\beta\mu^2}{4\varepsilon}\sum_{j=1}^{N}\big((\mathcal{P}p(x,t_n) - p(x,t_n))^+\big)^2_{j-\frac{1}{2}}.
\end{aligned}
\tag{3.17}
$$

Hence, we have

$$\begin{aligned}
\|\mathcal{P}^- e_u^n\| \le & \sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})(3\|\mathcal{P}^- e_u^i\| + 4\|\mathcal{P}^- e_u^{i-1}\| + \|\mathcal{P}^- e_u^{i-2}\|) \\
& + 4\|\mathcal{P}^- e_u^{n-1}\| + \|\mathcal{P}^- e_u^{n-2}\| + \beta\|r_3^n\| \\
& + \sum_{i=1}^{n-1}(b_{n-i-1}-b_{n-i})(3\|\mathcal{P}^- u(x,t_i)-u(x,t_i)\| + 4\|\mathcal{P}^- u(x,t_{i-1})-u(x,t_{i-1})\| \\
& \quad + \|\mathcal{P}^- u(x,t_{i-2})-u(x,t_{i-2})\|) \\
& + 4\|\mathcal{P}^- u(x,t_{n-1})-u(x,t_{n-1})\| + \|\mathcal{P}^- u(x,t_{n-2})-u(x,t_{n-2})\| \\
& + 3\|\mathcal{P}^- u(x,t_n)-u(x,t_n)\| \\
& + \sqrt{\frac{\beta\mu^2}{2}}\|\mathcal{P}^- q(x,t_n)-q(x,t_n)\| \\
& + \sqrt{\frac{\beta\mu^2}{4\varepsilon}\sum_{j=1}^{N}((\mathcal{P}s(x,t_n)-s(x,t_n))^+)_{j-\frac{1}{2}}^2} \\
& + \sqrt{\frac{\beta\mu^2}{4\varepsilon}\sum_{j=1}^{N}((\mathcal{P}p(x,t_n)-p(x,t_n))^+)_{j-\frac{1}{2}}^2}.
\end{aligned} \tag{3.18}$$

Now, we prove Theorem 3.2 in two steps. First, we establish the error estimate

$$\|\mathcal{P}^- e_u^n\| \le b_{n-1}^{-1} C\Big(h^{k+1} + (\Delta t)^3 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}}\Big). \tag{3.19}$$

Considering the case $n = 1$ in Eq (3.18), we have

$$\begin{aligned}
\|\mathcal{P}^- e_u^1\| \le & 4\|\mathcal{P}^- e_u^0\| + \|\mathcal{P}^- e_u^{-1}\| + \beta\|r_3^1\| \\
& + 4\|\mathcal{P}^- u(x,t_0)-u(x,t_0)\| + \|\mathcal{P}^- u(x,t_{-1})-u(x,t_{-1})\| \\
& + 3\|\mathcal{P}^- u(x,t_1)-u(x,t_1)\| \\
& + \sqrt{\frac{\beta\mu^2}{2}}\|\mathcal{P}^- q(x,t_1)-q(x,t_1)\| \\
& + \sqrt{\frac{\beta\mu^2}{4\varepsilon}\sum_{j=1}^{N}((\mathcal{P}s(x,t_1)-s(x,t_1))^+)_{j-\frac{1}{2}}^2} \\
& + \sqrt{\frac{\beta\mu^2}{4\varepsilon}\sum_{j=1}^{N}((\mathcal{P}p(x,t_1)-p(x,t_1))^+)_{j-\frac{1}{2}}^2}.
\end{aligned} \tag{3.20}$$

Noting that

$$\mathcal{P}^- e_u^0 = 0, \quad \|r_3^1\| \le C(\Delta t)^{3-\alpha}$$

and applying the projection property (2.3), we immediately obtain

$$\|\mathcal{P}^- e_u^1\| \le C\Big(h^{k+1} + (\Delta t)^3 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}}\Big). \tag{3.21}$$

Next, we assume the following inequality holds for some $K \geq 1$,

$$\|\mathcal{P}^- e_u^m\| \leq b_{m-1}^{-1} C\Big(h^{k+1} + (\Delta t)^3 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}}\Big), \quad m = 1, 2, \ldots, K. \tag{3.22}$$

When $n = K + 1$ in Eq (3.18), we obtain

$$\begin{aligned}
\|\mathcal{P}^- e_u^{K+1}\| \leq & \sum_{i=1}^{K} (b_{K-i} - b_{K+1-i})(3\|\mathcal{P}^- e_u^i\| + 4\|\mathcal{P}^- e_u^{i-1}\| + \|\mathcal{P}^- e_u^{i-2}\|) \\
& + 4\|\mathcal{P}^- e_u^K\| + \|\mathcal{P}^- e_u^{K-1}\| + \beta\|r_3^{K+1}\| \\
& + \sum_{i=1}^{K} (b_{K-i} - b_{K+1-i})(3\|\mathcal{P}^- u(x, t_i) - u(x, t_i)\| + 4\|\mathcal{P}^- u(x, t_{i-1}) - u(x, t_{i-1})\| \\
& \quad + \|\mathcal{P}^- u(x, t_{i-2}) - u(x, t_{i-2})\|) + 4\|\mathcal{P}^- u(x, t_K) - u(x, t_K)\| \\
& + \|\mathcal{P}^- u(x, t_{K-1}) - u(x, t_{K-1})\| + 3\|\mathcal{P}^- u(x, t_{K+1}) - u(x, t_{K+1})\| \\
& + \sqrt{\frac{\beta\mu^2}{2}} \|\mathcal{P}^- q(x, t_{K+1}) - q(x, t_{K+1})\| \\
& + \sqrt{\frac{\beta\mu^2}{4\varepsilon} \sum_{j=1}^{N} ((\mathcal{P} s(x, t_{K+1}) - s(x, t_{K+1}))^+)^2_{j-\frac{1}{2}}} \\
& + \sqrt{\frac{\beta\mu^2}{4\varepsilon} \sum_{j=1}^{N} ((\mathcal{P} p(x, t_{K+1}) - p(x, t_{K+1}))^+)^2_{j-\frac{1}{2}}}.
\end{aligned}$$

Noting that

$$b_{i-1}^{-1} < b_i^{-1}, \quad i \geq 1$$

and applying the induction hypothesis (3.22), we immediately obtain

$$\|\mathcal{P}^- e_u^{K+1}\| \leq b_K^{-1} C\Big(h^{k+1} + (\Delta t)^3 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}}\Big).$$

Hence, by the principle of mathematical induction, the inequality (3.19) holds for all $n \geq 1$.
By some analysis, we know that

$$n^{-\alpha} b_{n-1}^{-1} \to \frac{1}{2-\alpha} \quad \text{as } n \text{ increases.}$$

Thus, from Eq (3.19), we have

$$\begin{aligned}
\|\mathcal{P}^- e_u^n\| &\leq b_{n-1}^{-1} C\Big(h^{k+1} + (\Delta t)^3 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}}\Big) \\
&\leq n^{\alpha} n^{-\alpha} b_{n-1}^{-1} C\Big(h^{k+1} + (\Delta t)^3 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}}\Big) \\
&\leq \frac{CT^{\alpha}}{2-\alpha}\Big((\Delta t)^{-\alpha} h^{k+1} + (\Delta t)^{3-\alpha} + (\Delta t)^{-\frac{\alpha}{2}} h^{k+\frac{1}{2}}\Big),
\end{aligned}$$

where $T = n\Delta t$.

Notice that this estimate becomes meaningless as $\alpha \to 2$ since $\frac{1}{2-\alpha} \to \infty$. Therefore, we need to reconsider the estimate for α near 2.

Suppose instead that

$$\|\mathcal{P}^- e_u^n\| \le nC\Big(h^{k+1} + (\Delta t)^3 + (\Delta t)^{\frac{\alpha}{2}} h^{k+\frac{1}{2}}\Big). \tag{3.23}$$

By using similar techniques as in the proof of Eq (3.19), we can easily verify Eq (3.23) (the details are omitted for brevity). In particular, when $\alpha \to 2$, this gives

$$\|\mathcal{P}^- e_u^n\| \le TC\Big((\Delta t)^{-2} h^{k+1} + \Delta t + (\Delta t)^{-1} h^{k+\frac{1}{2}}\Big).$$

Finally, Theorem 3.2 follows by applying the triangle inequality and the interpolation property (2.3).

Remark. In Theorem 3.2, we observe that the error arising from the spatial approximation is influenced by the inverse of the time step. Similar phenomena have also been reported in [15, 40]. Nevertheless, for sufficiently smooth solutions, one can generally choose a spatial step h much smaller than the time step Δt, so that this influence does not significantly degrade the overall accuracy [15].

4. Numerical examples

Example 4.1. Consider the following fractional problem:

$$\begin{aligned} &\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \frac{\partial^4 u(x,t)}{\partial x^4} = \left(\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 16\pi^4 t^2\right)\sin(2\pi x), \\ &u(x,0) = 0, \quad \frac{\partial u(x,0)}{\partial t} = 0, \quad x \in [0,1]. \end{aligned} \tag{4.1}$$

The exact solution is given by

$$u(x,t) = t^2 \sin(2\pi x).$$

Table 1. Spatial accuracy test using piecewise P^k polynomials. $\alpha = 1.2, \Delta t = \frac{1}{1000}, T = 1$.

	N	L^2-error	order	L^∞-error	order
	10	0.268369171454230	-	0.425516775770436	-
P^0	20	0.143456516001119	0.90	0.233192588549476	0.87
	40	7.418297884848368E-002	0.95	0.121337346301527	0.94
	80	3.772652263985216E-002	0.98	6.183068333664676E-002	0.97
	10	1.598834727872643E-002	-	5.686767451595021E-002	-
P^1	20	4.097515165530630E-003	1.96	1.528254715234789E-002	1.90
	40	1.041526895366093E-003	1.98	3.971581531355906E-003	1.94
	80	2.628167145762841E-004	1.99	1.010930910805063E-003	1.97
	10	8.202233250136834E-004	-	3.688835453254780E-003	-
P^2	20	1.046335883710975E-004	2.97	4.924187272465639E-004	2.90
	40	1.322262615436073E-005	2.98	6.319441916376639E-005	2.96
	80	1.664864081868804E-006	2.99	7.992896861264143E-006	2.98

Choosing $\tau_1 = \tau_2 = 0.1$ and $\Delta t = 1/1000$, we consider spatial step sizes $h = 1/10, 1/20, 1/40, 1/80$. The corresponding errors measured in the L^2-norm and L^∞-norm, along with the discrete spatial convergence orders for the scheme with $\alpha = 1.2, 1.4, 1.6, 1.8$, are reported in Tables 1–4. From these results, it is evident that the numerical errors consistently attain the expected $(k + 1)$-th order of accuracy for piecewise P^k polynomials. These findings confirm not only the theoretical convergence properties of the proposed scheme but also their high efficiency and reliability in practical computations. Overall, the numerical experiments demonstrate the robustness and excellent performance of the method across different fractional orders.

To further support the unconditional convergence observed in our numerical results, one may refer to the analysis in Li and Dong [41], where unconditional error estimates for an element-free Galerkin method for the nonlinear Schrödinger equation are rigorously established. This reference provides a useful benchmark for understanding unconditional stability and convergence in fractional or nonlinear PDEs.

Table 2. Spatial accuracy test using piecewise P^k polynomials. $\alpha = 1.4, \Delta t = \frac{1}{1000}, T = 1$.

	N	L^2-error	order	L^∞-error	order
	10	0.267648256386237	-	0.425890675278256	-
P^0	20	0.143450994210472	0.90	0.233186418176963	0.87
	40	7.417976532107379E-002	0.95	0.121333514437931	0.94
	80	3.772479055786165E-002	0.98	6.182855829164957E-002	0.97
	10	1.598832244742653E-002	-	5.686789803899872E-002	-
P^1	20	4.097513364765737E-003	1.96	1.528257735763527E-002	1.90
	40	1.041526776316258E-003	1.98	3.971585348418571E-003	1.94
	80	2.628167072630438E-004	1.99	1.010931387972480E-003	1.97
	10	8.202233189327410E-004	-	3.688835494435471E-003	-
P^2	20	1.046335884353141E-004	2.97	4.924187268000053E-004	2.90
	40	1.322262608804521E-005	2.98	6.319441722133412E-005	2.96
	80	1.664863445486945E-006	2.99	7.992959526761934E-006	2.98

Table 3. Spatial accuracy test using piecewise P^k polynomials. $\alpha = 1.6, \Delta t = \frac{1}{1000}, T = 1$.

	N	L^2-error	order	L^∞-error	order
	10	0.264187268000053	-	0.422262608804521	-
P^0	20	0.149065278286237	0.90	0.231864190270472	0.87
	40	7.417919351101487E-002	0.95	0.121332832597304	0.94
	80	3.772448055981137E-002	0.98	6.182817795510087E-002	0.97
	10	1.598831804196817E-002	-	5.686793765736797E-002	-
P^1	20	4.097513048158760E-003	1.96	1.528258267024296E-002	1.90
	40	1.041526756818823E-003	1.98	3.971586011891404E-003	1.94
	80	2.628167067819486E-004	1.99	1.010931514054292E-003	1.97
	10	8.202233183462588E-004	-	3.688835501201323E-003	-
P^2	20	1.046335885436331E-004	2.97	4.924187316523287E-004	2.90
	40	1.322262585850249E-005	2.98	6.319440504411021E-005	2.96
	80	1.664861326824847E-006	2.99	7.992891888367049E-006	2.98

Table 4. Spatial accuracy test using piecewise P^k polynomials. $\alpha = 1.8, \Delta t = \frac{1}{1000}, T = 1$.

	N	L^2-error	order	L^∞-error	order
	10	0.268602489555734	-	0.420388004556405	-
P^0	20	0.140643401016234	0.90	0.233799701362584	0.87
	40	7.415946963940129E-002	0.95	0.123508583403740	0.94
	80	3.772590217169987E-002	0.98	6.182992210789384E-002	0.97
	10	1.594211241651383E-002	-	5.682364033051722E-002	-
P^1	20	4.097514540221151E-003	1.96	1.528255760663999E-002	1.90
	40	1.041526856361772E-003	1.98	3.971582800122109E-003	1.94
	80	2.628167133465506E-004	1.99	1.010930931959475E-003	1.97
	10	8.202233236779249E-004	-	3.688835463357487E-003	-
P^2	20	1.046335885513895E-004	2.97	4.924187401504371E-004	2.90
	40	1.322262577431816E-005	2.98	6.319441252891346E-005	2.96
	80	1.664860645729182E-006	2.99	7.992970942741207E-006	2.98

We further examine the temporal convergence of the proposed scheme. For $\alpha = 1.3$ and $\alpha = 1.7$, the numerical errors and convergence rates in the L^2-norm and L^∞-norm are reported in Table 5. It is evident that, when sufficiently small spatial steps are employed, the total error is dominated by the pure temporal term $(\Delta t)^{3-\alpha}$, resulting in a temporal convergence rate of $3 - \alpha$, in excellent agreement with the theoretical predictions.

Table 5. Temporal errors and convergence rates using piecewise P^2 basis functions when $N = 200, T = 1$.

	M	L^2-error	order	L^∞-error	order
	5	4.423464658635612E-003	-	2.235465465435201E-003	-
$\alpha = 1.3$	10	1.330751848856013E-003	1.67	6.976495787941314E-004	1.68
	20	4.124358284335621E-004	1.69	2.117707524889009E-004	1.72
	40	1.323327058236111E-004	1.64	6.608997492394428E-005	1.68
	5	3.435616859211621E-003	-	2.812091403979456E-003	-
$\alpha = 1.7$	10	1.444498950548263E-003	1.25	1.119056260712752E-003	1.24
	20	5.825965873323327E-004	1.31	4.902683343623456E-004	1.28
	40	2.432596859907130E-004	1.26	2.004957399603285E-004	1.29

Example 4.2. Consider the fourth-order time-fractional diffusion-wave problem with the exact solution

$$u(x, t) = t^3 x^2 (1 - x)^2.$$

The corresponding source term $f(x, t)$ is computed from the exact solution as

$$f(x, t) = \frac{6t^{3-\alpha}}{\Gamma(4 - \alpha)} + 24.$$

For numerical tests, one can take $\Delta t = 1/1000$ and spatial step sizes $h = 1/5, 1/10, 1/20, 1/40$. Errors are measured in the L^2-norm and L^∞-norm for different fractional orders. $\alpha = 1.3, 1.7$ are plotted in Figures 1 and 2 to verify the accuracy and convergence of the proposed method.

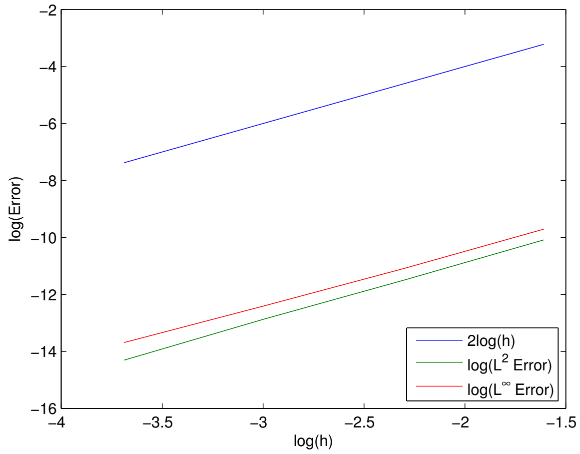

Figure 1. Spatial accuracy test using piecewise P^1 polynomials for Example 4.2 at the time $t = 1.0$ when $\Delta t = 1/1000, \alpha = 1.3$.

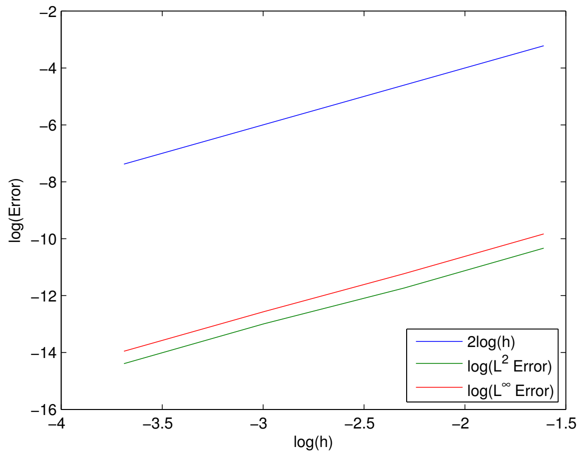

Figure 2. Spatial accuracy test using piecewise P^1 polynomials for Example 4.2 at the time $t = 1.0$ when $\Delta t = 1/1000, \alpha = 1.5$.

5. Conclusions

In this work, we study a high-order finite difference/LDG method for a fourth-order fractional diffusion-wave system. A discretization formula for approximating the fractional derivatives with $1 < \alpha < 2$ is derived. We then propose a fully discrete scheme and rigorously prove its unconditional stability and convergence. Several numerical examples are provided to demonstrate the excellent performance and accuracy of the proposed method.

For future work, it will be interesting to extend the present LDG scheme to higher-order spatial elements, such as P_3, and develop fully discrete LDG schemes for two- or three-dimensional fourth-order fractional diffusion-wave models. Thanks to the local formulation and flexibility in handling auxiliary variables, the LDG framework is well-suited for such high-dimensional extensions.

In addition, meshless methods, such as the element-free Galerkin approach [42], may provide promising alternatives for solving high-dimensional fractional partial differential equations. These directions are expected to further improve the accuracy and broaden the applicability of the proposed method.

Authors contribution

Xindong Zhang carried out the coding and contributed to writing and original draft preparation. Xiaoyan Xu and Chuan Ran contributed to the methodology development and stability analysis. Changshun Hou was responsible for the error estimation. All authors have read and agreed to the published version of the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Xindong Zhang is a guest editor for NHM and was not involved in the editorial review or the decision to publish this article. All authors declare that there are no competing interests.

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