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Research article

Convergence analysis of a second-order finite difference scheme on Shishkin-type meshes for a singularly perturbed Volterra integro-differential equation

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Abstract: In this paper, a novel second-order numerical method on a Shishkin mesh is constructed to solve a singularly perturbed Volterra integro-differential equation. The proposed numerical scheme employs a second-order backward differentiation formula (BDF2) for discretizing the first-order derivative term, while utilizing the trapezoidal rule to approximate the integral term. Specifically, at the grid transition point, a first-order finite difference approximation is implemented to handle the first-order derivative computation. Subsequently, comprehensive truncation error estimations and rigorous convergence analyses are systematically conducted. Finally, two numerical examples are performed to verify the theoretical findings.

Keywords: Volterra integro-differential equation; singularly perturbed; Shishkin mesh; BDF2

1. Introduction

In this paper, our primary focus is developing a second-order accurate numerical method for solving the following singularly perturbed Volterra integro-differential equation (SPVIDE):

$$\begin{cases} \varepsilon u'(x) + a(x)u(x) + \displaystyle\int_0^x K(x,s)u(s)\mathrm{d}s = f(x), \ x \in \Omega := (0,1], \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where $0 < \varepsilon \ll 1$ and u_0 is a given constant. The functions $a(x)$, $K(x,s)$ and $f(x)$ are assumed to be sufficiently smooth. Moreover, there exists a positive constant α, such that $a(x) \geq \alpha > 0$, $x \in [0,1]$. Under these conditions, the problem (1.1) has a unique solution $u(x)$, and its derivatives have the following bounds (see [1]):

$$|u^k(x)| \leq C(1+\varepsilon^{-k}e^{-\alpha x/\varepsilon}), \quad x \in \Omega, \ 0 \leq k \leq 3,$$

where C is a positive constant. Obviously, as $\varepsilon \to 0$, the solution $u(x)$ of (1.1) typically exhibits an initial layer at $x = 0$.

It is widely recognized that SPVIDEs arise in diffusion-dissipation processes, filament stretching problems, epidemic dynamics, synchronous control systems, and so forth [2–5]. The primary characteristic and challenge of such problems is that the first-order derivative term contains a perturbation parameter ε, requiring numerical methods to be ε-uniformly convergent. To the best of our knowledge, two distinct categories of numerical solution methods have been developed to solve SPVIDEs. The first is the layer-adapted meshes approach (Shishkin meshes and Bakhvalov meshes, see, e.g., [1, 6, 7]), which is specifically designed to handle the boundary layer or the initial layer effectively by using the prior information of the exact solution; The second is the adaptive grid method [8–10], which is derived through the application of a posteriori error estimation techniques.

It is crucial to highlight that the numerical methods proposed in this body of literature demonstrate merely first-order accuracy with parameter-uniform convergence, revealing a fundamental limitation in their approximation capabilities. For this reason, Yapman and Amiraliyev [11, 12] developed second-order exponentially fitted schemes on Shishkin-type meshes to solve SPVIDEs and singularly perturbed Volterra delay-integro-differential equations (SPVDIDEs), respectively. Based on the Richardson extrapolation technique, the authors in [13, 14] presented a second-order layer-adapted meshes approach and a second-order adaptive grid method, respectively. Moreover, the Crank-Nicolson scheme was constructed on Shishkin-type meshes and Bakhvalov-type meshes to solve singularly perturbed problems [15], while the ε-uniform second-order convergences were shown. In the very recent past, a discontinuous Galerkin method was proposed on Bakhvalov-type meshes for SPVIDEs [16], and the $(k+1)$-st order of accuracy has been reached, where k is the degree of the piecewise polynomial space.

Recently, building upon the foundational framework of the second-order variable-step-size backward differentiation formula (BDF2) [17], Liao et al. [18, 19] proposed a second-order finite difference method (non-hybrid schemes) on Shishkin-type meshes and Bakhvalov-type meshes for SPVIDEs, and utilized the discretization BDF2 convolution kernels theory (see [17]) to derive the corresponding stability results. Besides, the mesh ratios near the transition points of Shishkin grids and Bakhvalov grids is related to $\frac{1}{\varepsilon}$, which introduces certain difficulties in the stability analysis of the proposed numerical methods in [18, 19]. Here, to overcome the limitations of the mesh ratio near transition points, this paper proposes a modified BDF2-based numerical framework on Shishkin-type grids for solving Eq (1.1). Crucially, we introduce a stability analysis methodology that circumvents the conventional reliance on discrete convolution kernel theory for BDF2 schemes as described in [17].

The outline of this paper is organized as follows: The discretization scheme on a Shishkin mesh is constructed in Section 2. Section 3 introduces the local truncation error estimates. In Section 4, the convergence result of our proposed scheme is proved through rigorous theoretical analysis. Some numerical results are presented in Section 5. Finally, some concluding discussions are provided in Section 6.

Notation. In this paper, C denotes a positive constant independent of ε and the mesh parameter N, which may take different values in different places. For our analysis we shall assume that $\varepsilon \le CN^{-1}$.

2. The discretization scheme

Let N be an even and positive integer and $\sigma = \min\left\{\frac{1}{2}, \frac{2\varepsilon}{\alpha}\ln N\right\}$ be a grid transition point. Then the Shishkin-type meshes can be constructed as follows:

$$\bar{\Omega}^N := \{0 = x_0 < x_1 < \cdots < x_N = 1\},$$

where the point x_i is defined by

$$x_i = \begin{cases} \frac{2\sigma i}{N}, & 0 \le i \le N/2, \\ \sigma + \frac{2(1-\sigma)}{N}(i - N/2), & N/2+1 \le i \le N. \end{cases}$$

To simplify the notation, we set $g_i = g(x_i)$ for any function g and set $K_{i,k} = K(x_i, x_k)$ for a bivariate function K. In addition, for $i = 1, \cdots, N$, let $h_i := x_i - x_{i-1}$ be the step sizes, which are given by

$$h_i = \begin{cases} h = \frac{4}{\alpha}\varepsilon N^{-1}\ln N, & 0 \le i \le N/2, \\ H = 2(1-\sigma)N^{-1}, & N/2+1 \le i \le N. \end{cases}$$

For a given mesh function $v = \{v_i\}_{i=0}^N$, define the following backward differentiation operators:

$$L_1^N v_i := \begin{cases} \dfrac{v_i - v_{i-1}}{h_i}, & i = 1,\, N/2+1, \\ \dfrac{3v_i - 4v_{i-1} + v_{i-2}}{2h_i}, & else. \end{cases}$$

Then we obtain the following discretization scheme of problem (1.1):

$$\begin{cases} \varepsilon L_1^N u_i^N + a_i u_i^N + L_2^N u_i^N = f_i, & 1 \le i \le N, \\ u_0^N = u_0, \end{cases} \tag{2.1}$$

where u^N is the approximation solution of $u(x)$ on $\bar{\Omega}^N$ and L_2^N is the trapezoidal formula, presented as

$$L_2^N u_i^N := \sum_{k=1}^{i} \frac{h_k}{2}\left(K_{i,k-1}u_{k-1}^N + K_{ik}u_k^N\right), \quad 1 \le i \le N.$$

3. Truncation error

For $i = 1, \cdots, N$, let R_i be the local truncation error of scheme (2.1) at point x_i. Then

$$R_i = \varepsilon\left(L_1^N u_i - u_i'\right) + L_2^N u_i - \int_0^{x_i} K(x_i, s)u(s)\mathrm{d}s := R_{i,1} + R_{i,2}, \tag{3.1}$$

where

$$\begin{aligned} R_{i,1} &= \varepsilon\left(L_1^N u_i - u_i'\right), \\ R_{i,2} &= L_2^N u_i - \int_0^{x_i} K(x_i, s)u(s)\mathrm{d}s. \end{aligned}$$

Lemma 3.1. *For $i = 1, \cdots, N$, one has*

$$|R_1| \le CN^{-1} \ln N, \tag{3.2}$$
$$|R_i| \le C(e^{-\alpha x_i/\varepsilon} + \varepsilon)N^{-2} \ln^2 N, \quad 2 \le i \le N/2, \tag{3.3}$$
$$|R_i| \le CN^{-2}, \quad N/2+1 \le i \le N. \tag{3.4}$$

Proof. In order to prove this lemma, we need to give the upper bounds of $\left|R_{i,1}\right|$ and $\left|R_{i,2}\right|$, respectively. Now, we will present a detailed proof process as follows:

At first, by using the Taylor series expansion, one has

$$\left|R_{1,1}\right| \le \frac{\varepsilon}{h} \int_0^{x_1} \left|u''(t)\right| t \mathrm{d}t \le \varepsilon \int_0^{x_1} (1 + \varepsilon^{-2} e^{-\alpha t/\varepsilon}) \mathrm{d}t \le CN^{-1} \ln N$$

and

$$\begin{aligned}
\left|R_{N/2+1,1}\right| &\le \frac{\varepsilon}{H} \int_{x_{N/2}}^{x_{N/2+1}} \left|u''(t)\right| (t - x_{N/2}) \mathrm{d}t \\
&\le \int_{x_{N/2}}^{x_{N/2+1}} \varepsilon (1 + \varepsilon^{-2} e^{-\alpha t/\varepsilon}) \mathrm{d}t \\
&\le \varepsilon H + e^{-\alpha\sigma/\varepsilon} \left(1 - e^{-\alpha H/\varepsilon}\right) \\
&\le C\varepsilon N^{-1} + CN^{-2} \\
&\le CN^{-2}.
\end{aligned}$$

Similarly, for $2 \le i \le N/2$, it is easy to get

$$\begin{aligned}
\left|R_{i,1}\right| &\le C\frac{\varepsilon}{h} \left[\int_{x_{i-1}}^{x_i} \left|u'''(t)\right| (t - x_{i-1})^2 \mathrm{d}t + \int_{x_{i-2}}^{x_i} \left|u'''(t)\right| (t - x_{i-2})^2 \mathrm{d}t\right] \\
&\le C\varepsilon h \int_{x_{i-2}}^{x_i} (1 + \varepsilon^{-3} e^{-\alpha t/\varepsilon}) \mathrm{d}t \\
&\le CN^{-1} \ln N \int_{x_{i-2}}^{x_i} \varepsilon^2 (1 + \varepsilon^{-3} e^{-\alpha t/\varepsilon}) \mathrm{d}t \\
&\le CN^{-1} \ln N \left[\varepsilon^2 h + e^{-\alpha x_i/\varepsilon} \left(e^{\alpha \cdot 2h/\varepsilon} - 1\right)\right] \\
&\le CN^{-1} \ln N (\varepsilon^3 N^{-1} \ln N + e^{-\alpha x_i/\varepsilon} N^{-1} \ln N) \\
&\le Ce^{-\alpha x_i/\varepsilon} N^{-2} \ln^2 N,
\end{aligned}$$

where we have used the simple fact that $\varepsilon^3 \le CN^{-3} \le e^{-\alpha x_i/\varepsilon}$, $1 \le i \le N/2$.

Meanwhile, when $N/2+2 \le i \le N$, we have

$$\begin{aligned}
\left|R_{i,1}\right| &\le C\frac{\varepsilon}{H}\left[\int_{x_{i-1}}^{x_i}\left|u'''(t)\right|(t-x_{i-1})^2\mathrm{d}t+\int_{x_{i-2}}^{x_i}\left|u'''(t)\right|(t-x_{i-2})^2\mathrm{d}t\right]\\
&\le C\int_{x_{i-2}}^{x_i}\left(\varepsilon+\varepsilon^{-2}e^{-\alpha t/\varepsilon}\right)(t-x_{i-2})\mathrm{d}t\\
&\le C\left[\int_{x_{i-2}}^{x_i}\left(\sqrt{\varepsilon}+\varepsilon^{-1}e^{-\alpha t/2\varepsilon}\right)\mathrm{d}t\right]^2\\
&\le C\left(\sqrt{\varepsilon}N^{-1}+e^{-\alpha\sigma/2\varepsilon}\right)^2\\
&\le CN^{-2},
\end{aligned}$$

where we have used the fact that

$$\int_a^b \phi(x)(x-a)^{k-1}\mathrm{d}x \le \frac{1}{k}\left[\int_a^b \phi(x)^{\frac{1}{k}}\mathrm{d}x\right]^k$$

in which $k \in \mathbb{N}^*$ and ϕ is a positive decreasing function on $[a,b]$.

The next thing to do in the proof is to derive the bounds for $\left|R_{i,2}\right|$, $i=1,\cdots,N$.

By using the Taylor series expansion, one has

$$\begin{aligned}
R_{i,2} = \sum_{k=1}^{i}\int_{x_{k-1}}^{x_k}\Bigg\{&\frac{x_k-s}{h_k}\int_{x_{k-1}}^{x_k}\left[K(x_i,t)u(t)\right]''(t-x_{k-1})\mathrm{d}t\\
&-\int_s^{x_k}\left[K(x_i,t)u(t)\right]''(t-s)\mathrm{d}t\Bigg\}\mathrm{d}s.
\end{aligned}$$

So that for $1 \le i \le N/2$, there holds

$$\begin{aligned}
\left|R_{i,2}\right| &\le C\sum_{k=1}^{i}\int_{x_{k-1}}^{x_k}\mathrm{d}s\int_{x_{k-1}}^{x_k}\left(\left|u(t)\right|+\left|u'(t)\right|+\left|u''(t)\right|\right)(t-x_{k-1})\mathrm{d}t\\
&\le Ch^2\sum_{k=1}^{i}\int_{x_{k-1}}^{x_k}(1+\varepsilon^{-2}e^{-\alpha t/\varepsilon})\mathrm{d}t\\
&\le C\varepsilon N^{-2}\ln^2 N\int_0^{\sigma}\varepsilon(1+\varepsilon^{-2}e^{-\alpha t/\varepsilon})\mathrm{d}t\\
&\le CN^{-3}\ln^2 N\left(\varepsilon/2+1-e^{-\alpha\sigma/\varepsilon}\right)\\
&\le CN^{-3}\ln^2 N.
\end{aligned}$$

Similarly, for $N/2+1 \le i \le N$, it is easy to show that

$$\begin{aligned}\left|R_{i,2}\right| &\le CN^{-3}\ln^2 N + CN^{-1}\sum_{k=N/2+1}^{i}\int_{x_{k-1}}^{x_k}(1+\varepsilon^{-2}e^{-\alpha t/\varepsilon})(t-x_{k-1})\mathrm{d}t\\ &\le CN^{-3}\ln^2 N + CN^{-1}\sum_{k=N/2+1}^{i}\left[\int_{x_{k-1}}^{x_k}\left(1+\varepsilon^{-1}e^{-\alpha t/2\varepsilon}\right)\mathrm{d}t\right]^2\\ &\le CN^{-3}\ln^2 N + CN^{-1}\sum_{k=N/2+1}^{i}\left(N^{-1}+e^{-\alpha\sigma/2\varepsilon}\right)^2\\ &\le CN^{-2}.\end{aligned}$$

Combining Eq (3.1), we complete the proof of this lemma.

4. Convergence analysis

Before conducting convergence analysis for our proposed numerical scheme, we present the following important identity.

Lemma 4.1. (Telescope formula for BDF2 [20]) *For arbitrary* (v^n, v^{n-1}, v^{n-2}), *it holds that*

$$\begin{aligned}\left(\frac{3}{2}v^n - 2v^{n-1}+\frac{1}{2}v^{n-2}\right)v^n = \frac{1}{4}\Big(&|v^n|^2-\left|v^{n-1}\right|^2+\left|2v^n-v^{n-1}\right|^2\\ &-\left|2v^{n-1}-v^{n-2}\right|^2+\left|v^n-2v^{n-1}+v^{n-2}\right|^2\Big).\end{aligned}$$

Let $e_i = u_i - u_i^N$. Then we obtain the following error equation:

$$\varepsilon L_1^N e_i + a_i e_i + L_2^N e_i = R_i, \quad 1\le i\le N, \;\; e_0 = 0. \tag{4.1}$$

Theorem 4.1. *For* $i = 1,\cdots,N$, *one has*

$$|e_i| \le CN^{-2}\ln^2 N.$$

Proof. For $i=1$, it follows from Eq (4.1) that

$$\varepsilon\frac{e_1-e_0}{h} + a_1 e_1 + \frac{h}{2}(K_{10}e_0 + K_{11}e_1) = R_1.$$

By using $h = \frac{4}{\alpha}\varepsilon N^{-1}\ln N$, $e_0 = 0$ and Eq (3.2), it is easy to obtain

$$|e_1| \le CN^{-1}\ln N\,|R_1| \le CN^{-2}\ln^2 N. \tag{4.2}$$

For $2\le i\le N/2$, by multiplying both sides of Eq (4.1) by e_i, yields

$$\begin{aligned}\varepsilon\frac{3e_i-4e_{i-1}+e_{i-2}}{2h}e_i &= -a_i\left|e_i\right|^2 - e_i\sum_{k=1}^{i}\frac{h}{2}\left(K_{i,k-1}e_{k-1}+K_{ik}e_k\right)+e_iR_i\\ &\le -a_i\left|e_i\right|^2 + C\varepsilon N^{-1}\ln N\left|e_i\right|\sum_{k=1}^{i}\left|e_k\right| + \left|e_i\right|\left|R_i\right|.\end{aligned}\tag{4.3}$$

Then, applying Lemma 4.1 to the left-hand side of Eq (4.3) and Cauchy inequality to the right-hand side, one has

$$
\begin{aligned}
&\frac{\varepsilon}{4h}\left(|e_i|^2-|e_{i-1}|^2+|2e_i-e_{i-1}|^2-|2e_{i-1}-e_{i-2}|^2\right)\\
&\le \varepsilon\frac{3e_i-4e_{i-1}+e_{i-2}}{2h}e_i\\
&\le -a_i|e_i|^2+C\varepsilon N^{-1}\ln N|e_i|\sum_{k=1}^{i}|e_k|+|e_i||R_i|\\
&\le -a_i|e_i|^2+C\varepsilon N^{-1}\ln N\sum_{k=1}^{i}\left(|e_i|^2+|e_k|^2\right)+\alpha|e_i|^2+\frac{1}{4\alpha}|R_i|^2\\
&\le C\varepsilon\ln N|e_i|^2+C\varepsilon N^{-1}\ln N\sum_{k=1}^{i}|e_k|^2+C|R_i|^2.
\end{aligned}
$$

Moreover, multiply both sides by $4h/\varepsilon$ and sum up from 2 to i to obtain

$$
|e_i|^2-|e_1|^2+|2e_i-e_{i-1}|^2-|2e_1-e_0|^2\le C\varepsilon N^{-1}\ln^2N\sum_{k=1}^{i}|e_k|^2+CN^{-1}\ln N\sum_{k=2}^{i}|R_i|^2,
$$

which implies

$$
\begin{aligned}
|e_i|^2&\le C|e_1|^2+C\varepsilon N^{-1}\ln^2N\sum_{k=1}^{i}|e_k|^2+CN^{-1}\ln N\sum_{k=2}^{i}|R_i|^2\\
&\le CN^{-4}\ln^4N+C\varepsilon N^{-1}\ln^2N\sum_{k=1}^{i}|e_k|^2+CN^{-5}\ln^5N\sum_{k=2}^{i}\left(e^{-\alpha x_i/\varepsilon}+\varepsilon\right)^2\\
&\le CN^{-4}\ln^4N+C\varepsilon N^{-1}\ln^2N\sum_{k=1}^{i}|e_k|^2+CN^{-5}\ln^5N\sum_{k=2}^{i}e^{-2\alpha x_i/\varepsilon}\\
&\le CN^{-4}\ln^4N+C\varepsilon N^{-1}\ln^2N\sum_{k=1}^{i}|e_k|^2.
\end{aligned}
\tag{4.4}
$$

Then, it follows from Eqs (4.2) and (4.4) and the discrete Grönwall inequality [21, Lemma 3.2] that

$$
|e_i|\le CN^{-2}\ln^2N,\quad 1\le i\le N/2.
$$

For $i=N/2+1$, by using Eq (4.1) and the simple calculation, it holds that

$$
\begin{aligned}
|e_i|&\le\left|\frac{\varepsilon}{H}+a_i+\frac{H}{2}K_{ii}\right|^{-1}\left(\frac{\varepsilon}{H}|e_{i-1}|+C\varepsilon N^{-1}\ln N\sum_{k=1}^{i-1}|e_k|+CN^{-1}|e_{i-1}|+|R_i|\right)\\
&\le CN^{-2}\ln^2N+C\varepsilon N^{-1}\ln N\cdot N^{-1}\ln^2N+CN^{-3}\ln^2N+CN^{-2}\\
&\le CN^{-2}\ln^2N.
\end{aligned}
$$

Similarly, for $N/2+2\le i\le N$, we have

$$
\left(\frac{3\varepsilon}{2H}+a_i\right)|e_i|\le 2\frac{\varepsilon}{H}|e_{i-1}|+\frac{\varepsilon}{2H}|e_{i-2}|+CN^{-1}\sum_{k=N/2+1}^{i}|e_k|+CN^{-2}.\tag{4.5}
$$

Furthermore, for a sufficiently small value of ε satisfying $\varepsilon \le CN^{-1}$, yields,

$$\frac{3\varepsilon}{2H} + a_i \ge \frac{3\varepsilon}{2H} + \alpha \ge 3\frac{\varepsilon}{H}. \tag{4.6}$$

By applying inequality Eq (4.6) to Eq (4.5), it is straightforward to demonstrate that

$$|e_i| \le \frac{2}{3}|e_{i-1}| + \frac{1}{6}|e_{i-2}| + CN^{-1}\sum_{k=N/2+1}^{i}|e_k| + CN^{-2}.$$

By simple calculation, the above inequality is equivalent to

$$|e_i| + p\,|e_{i-1}| - q\left(|e_{i-1}| + p\,|e_{i-2}|\right) \le CN^{-1}\sum_{k=N/2+1}^{i}|e_k| + CN^{-2}, \tag{4.7}$$

with $p = \frac{\sqrt{10}-2}{6}$, $q = \frac{\sqrt{10}+2}{6}$.

Furthermore, for $N/2+2 \le j \le i$, from Eq (4.7), yields,

$$\begin{aligned} &|e_j| + p|e_{j-1}| - q(|e_{j-1}| + p|e_{j-2}|) \\ &\le CN^{-1}\sum_{k=N/2+1}^{j}|e_k| + CN^{-2} \\ &\le CN^{-1}\sum_{k=N/2+1}^{i}|e_k| + CN^{-2}. \end{aligned} \tag{4.8}$$

Multiplying both sides of Eq (4.8) by q^{i-j}, summing over j from $N/2+2$ to i and noting that $\sum_{n=0}^{\infty} q^n \le C$, we obtain

$$\begin{aligned} |e_i| &\le C(|e_{N/2}| + |e_{N/2+1}|) + CN^{-1}\sum_{k=N/2+1}^{i}|e_k| + CN^{-2} \\ &\le CN^{-2}\ln^2 N + CN^{-1}\sum_{k=N/2+1}^{i}|e_k|. \end{aligned}$$

Then by using the discrete Grönwall inequality, one has

$$|e_i| \le CN^{-2}\ln^2 N, \quad N/2+1 \le i \le N.$$

This completes the proof.

5. Numerical results and discussion

In this section, we present two numerical examples to test the accuracy and efficiency of our proposed numerical method.

Example 1. The test problem follows [1] by taking Eq (1.1) with

$$a(x) = x+1, \quad K(x,s) = x+s,$$
$$f(x) = (\varepsilon - 2x)\cos x + (x+2)\sin x + (x - 2\varepsilon x - \varepsilon^2)e^{-x/\varepsilon} + (1+\varepsilon)x + \varepsilon^2,$$

and the initial condition is $u(0) = 1$. The exact solution of this problem is

$$u(x) = \sin x + e^{-x/\varepsilon}.$$

Let $E^N := \max_{1\le i\le N} |e_i|$ denote the maximum nodal errors. The corresponding convergence rates are then determined through the formula $r = \log_2\left(\frac{E^N}{E^{2N}}\right)$.

Example 2. The test problem follows [11] by taking Eq (1.1) with

$$a(x) = 2, \quad K(x,s) = -(x-s)e^{1-xs}, \quad f(x) = e^x - x,$$

and the initial condition is $u(0) = 1$. Since the exact solution of the test problem is not available, we present the numerical solutions in Figure 1 and use the double mesh principle to estimate the errors and convergence rates. Because mesh points for N and $2N$ do not match, we measure the maximum nodal errors by

$$E^N := \max_{1\le i\le N} \left|u_i^N - \widetilde{u}_i^{2N}\right|,$$

where $\widetilde{u}^{2N}$ represents the Hermite interpolation of the approximate solution u^{2N} and $\widetilde{u}_i^{2N} := \widetilde{u}^{2N}(x_i)$, $x_i \in \bar{\Omega}^N$. Similarly, corresponding convergence rates are calculated by $r = \log_2\left(\frac{E^N}{E^{2N}}\right)$.

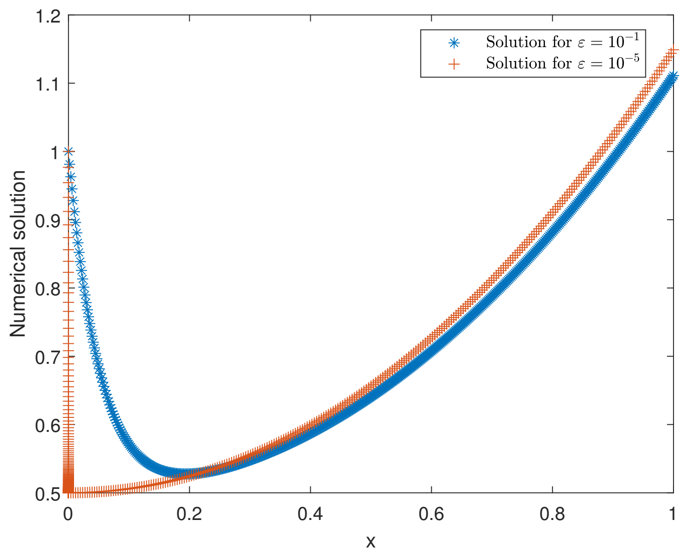

Figure 1. Nunerical solutions for $N = 512$ with $\varepsilon = 10^{-1}$ and 10^{-5}.

For different values of ε and N, Tables 1 and 2 show the maximum nodal error E^N and its convergence rates r in the above two test problems. We can see that the convergence rates r reach ε-uniform almost second-order, which is consistent with the theoretical result in Theorem 4.1.

Table 1. Results for Example 1: The maximum errors E^N and convergence rates r.

N	$\varepsilon=10^{-1}$ E^N	r	$\varepsilon=10^{-2}$ E^N	r	$\varepsilon=10^{-3}$ E^N	r	$\varepsilon=10^{-4}$ E^N	r
2^7	3.3097e-03	1.8050	1.0344e-02	1.3753	1.0349e-02	1.3757	1.0349e-02	1.3758
2^8	9.4718e-04	1.8907	3.9873e-03	1.4879	3.9879e-03	1.4880	3.9879e-03	1.4880
2^9	2.5543e-04	1.9277	1.4216e-03	1.5758	1.4217e-03	1.5759	1.4217e-03	1.5759
2^{10}	6.7138e-05	1.9611	4.7689e-04	1.6493	4.7691e-04	1.6494	4.7691e-04	1.6494
2^{11}	1.7243e-05	1.9767	1.5203e-04	1.6981	1.5203e-04	1.6981	1.5203e-04	1.6981
2^{12}	4.3808e-06	1.9869	4.6852e-05	1.7380	4.6853e-05	1.7380	4.6853e-05	1.7380
2^{13}	1.1052e-06	-	1.4046e-05	-	1.4046e-05	-	1.4046e-05	-
N	$\varepsilon=10^{-5}$ E^N	r	$\varepsilon=10^{-6}$ E^N	r	$\varepsilon=10^{-7}$ E^N	r	$\varepsilon=10^{-8}$ E^N	r
2^7	1.0349e-02	1.3758	1.0349e-02	1.3758	1.0349e-02	1.3758	1.0349e-02	1.3758
2^8	3.9880e-03	1.4880	3.9880e-03	1.4880	3.9880e-03	1.4880	3.9880e-03	1.4880
2^9	1.4217e-03	1.5759	1.4217e-03	1.5759	1.4217e-03	1.5759	1.4217e-03	1.5759
2^{10}	4.7691e-04	1.6494	4.7691e-04	1.6494	4.7691e-04	1.6494	4.7691e-04	1.6494
2^{11}	1.5203e-04	1.6981	1.5203e-04	1.6981	1.5203e-04	1.6981	1.5203e-04	1.6981
2^{12}	4.6853e-05	1.7380	4.6853e-05	1.7380	4.6853e-05	1.7380	4.6853e-05	1.7380
2^{13}	1.4046e-05	-	1.4046e-05	-	1.4046e-05	-	1.4046e-05	-

Table 2. Results for Example 2: The maximum errors E^N and convergence rates r.

N	$\varepsilon=10^{-1}$ E^N	r	$\varepsilon=10^{-2}$ E^N	r	$\varepsilon=10^{-3}$ E^N	r	$\varepsilon=10^{-4}$ E^N	r
2^7	3.6618e-03	1.5891	3.3200e-03	1.3360	3.3199e-03	1.3360	3.3199e-03	1.3360
2^8	1.2171e-03	1.7895	1.3151e-03	1.4386	1.3151e-03	1.4386	1.3151e-03	1.4386
2^9	3.5206e-04	1.8818	4.8520e-04	1.5643	4.8519e-04	1.5643	4.8519e-04	1.5643
2^{10}	9.5534e-05	1.9283	1.6407e-04	1.6249	1.6407e-04	1.6249	1.6407e-04	1.6249
2^{11}	2.5101e-05	1.9579	5.3197e-05	1.6899	5.3197e-05	1.6899	5.3197e-05	1.6899
2^{12}	6.4613e-06	1.9769	1.6489e-05	1.7271	1.6489e-05	1.7271	1.6489e-05	1.7271
2^{13}	1.6414e-06	-	4.9806e-06	-	4.9806e-06	-	4.9806e-06	-
N	$\varepsilon=10^{-5}$ E^N	r	$\varepsilon=10^{-6}$ E^N	r	$\varepsilon=10^{-7}$ E^N	r	$\varepsilon=10^{-8}$ E^N	r
2^7	3.3199e-03	1.3360	3.3199e-03	1.3360	3.3199e-03	1.3360	3.3199e-03	1.3360
2^8	1.3151e-03	1.4386	1.3151e-03	1.4386	1.3151e-03	1.4386	1.3151e-03	1.4386
2^9	4.8519e-04	1.5643	4.8519e-04	1.5643	4.8519e-04	1.5643	4.8519e-04	1.5643
2^{10}	1.6407e-04	1.6249	1.6407e-04	1.6249	1.6407e-04	1.6249	1.6407e-04	1.6249
2^{11}	5.3197e-05	1.6899	5.3197e-05	1.6899	5.3197e-05	1.6899	5.3197e-05	1.6899
2^{12}	1.6489e-05	1.7271	1.6489e-05	1.7271	1.6489e-05	1.7271	1.6489e-05	1.7271
2^{13}	4.9806e-06	-	4.9806e-06	-	4.9806e-06	-	4.9806e-06	-

6. Conclusions

A novel second-order numerical method on a Shishkin mesh based on the BDF2 has been developed for solving a singularly perturbed Volterra integro-differential equation. Unlike the discrete approach used in [18, 19], although we have employed a first-order upwind scheme to discretize the first-order derivatives at the interface between the fine grid and the coarse grid, this does not compromise the second-order convergence result of our proposed numerical scheme.

The linchpin of this paper's success lies in the development of an innovative stability analysis methodology we have crafted. In the future, it will be significant to extend our method to nonlinear singularly perturbed Volterra integro-differential equations and singularly perturbed Fredholm integro-differential equations.

Authors contribution

Jiwen Chen and Li-Bin Liu conducted theoretical proof, performed the experiments, and wrote the manuscript. Guangqing Long and Zaitang Huang conducted feasibility analysis of research plans, and provided critical revisions to the manuscript. All authors reviewed and approved the final manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Li-Bin Liu is a guest editor for NHM and was not involved in the editorial review or the decision to publish this article. All authors declare that there are no competing interests.

References

1. B. C. Iragi, J. B. Munyakazi, A uniformly convergent numerical method for a singularly perturbed Volterra integro-differential equation, *Int. J. Comput. Math.*, **97** (2020), 759–771. https://doi.org/10.1080/00207160.2019.1585828
2. J. S. Angell, W. E. Olmstead, Singularly perturbed Volterra integral equations II, *SIAM J. Appl. Math.*, **47** (1987), 1150–1162. https://doi.org/10.1137/0147077
3. F. C. Hoppensteadt, An algorithm for approximate solutions to weakly filtered synchronous control systems and nonlinear renewal processes, *SIAM J. Appl. Math.*, **43** (1983), 834–843. https://doi.org/10.1137/0143054

4. A. S. Lodge, J. B. McLeod, J. A. Nohel, A nonlinear singularly perturbed Volterra integrodifferential equation occurring in polymer rheology, *Proc. R. Soc. Edinburgh Sect. A*, **80** (1978), 99–137. https://doi.org/10.1017/S0308210500010167

5. G. S. Jordan, A nonlinear singularly perturbed Volterra integrodifferential equation of nonconvolution type, *Proc. R. Soc. Edinburgh Sect. A*, **80** (1978), 235–247. https://doi.org/10.1017/S030821050001026X

6. S. Şevgin, Numerical solution of a singularly perturbed Volterra integro-differential equation, *Adv. Differ. Equ.*, **2014** (2014), 171–196. https://doi.org/10.1186/1687-1847-2014-171

7. M. Kudu, I. Amirali, G. M. Amiraliyev, A finite-difference method for a singularly perturbed delay integro-differential equation, *J. Comput. Appl. Math.*, **308** (2016), 379–390. https://doi.org/10.1016/j.cam.2016.06.018

8. J. Huang, Z. Cen, A. Xu, L. B. Liu, A posteriori error estimation for a singularly perturbed Volterra integro-differential equation, *Numer. Algorithms*, **83** (2020), 549–563. https://doi.org/10.1007/s11075-019-00693-y

9. Sumit, S. Kumar, J. Vigo-Aguiar, Analysis of a nonlinear singularly perturbed Volterra integro-differential equation, *J. Comput. Appl. Math.*, **404** (2022), 113410. https://doi.org/10.1016/j.cam.2021.113410

10. A. Jaiswal, S. Kumar, S. Kumar, A priori and a posteriori error analysis for a system of singularly perturbed Volterra integro-differential equations, *Comput. Appl. Math.*, **42** (2023), 278. https://doi.org/10.1007/s40314-023-02406-7

11. Ö. Yapman, G. M. Amiraliyev, A novel second-order fitted computational method for a singularly perturbed Volterra integro-differential equation, *Int. J. Comput. Math.*, **97** (2020), 1293–1302. https://doi.org/10.1080/00207160.2019.1614565

12. Ö. Yapman, G. M. Amiraliyev, Convergence analysis of the homogeneous second order difference method for a singularly perturbed Volterra delay-integro-differential equation, *Chaos Solitons Fractals*, **150** (2021), 111100. https://doi.org/10.1016/j.chaos.2021.111100

13. A. Panda, J. Mohapatra, I. Amirali, A second-order post-processing technique for singularly perturbed Volterra integro-differential equations, *Mediterr. J. Math.*, **18** (2021), 231. https://doi.org/10.1007/s00009-021-01873-8

14. G. Long, L. B. Liu, Z. Huang, Richardson extrapolation method on an adaptive grid for singularly perturbed Volterra integro-differential equations, *Numer. Funct. Anal. Optim.*, **42** (2021), 739–757. https://doi.org/10.1080/01630563.2021.1928698

15. S. Kumar, S. Kumar, Sumit, High-order convergent methods for singularly perturbed quasilinear problems with integral boundary conditions, *Math. Methods Appl. Sci.*, **47** (2020), 11106–11119. https://doi.org/10.1002/mma.6854

16. Y. Liao, X. Luo, Convergence of a discontinuous Galerkin method on Bakhvalov-type meshes for singularly perturbed Volterra integro-differential equations, *East Asian J. Appl. Math.*, **15** (2025), 770–786. https://doi.org/10.4208/eajam.2024-075.140824

17. H. L. Liao, Z. Zhang, Analysis of adaptive BDF2 scheme for diffusion equations, *Math. Comput.*, **90** (2021), 1207–1226. https://doi.org/10.1090/mcom/3585

18. L. B. Liu, Y. Liao, G. Long, A novel parameter-uniform numerical method for a singularly perturbed Volterra integro-differential equation, *Comput. Appl. Math.*, **42** (2023), 12. https://doi.org/10.1007/s40314-022-02142-4

19. Y. Liao, L. B. Liu, L. Ye, T. Liu, Uniform convergence analysis of the BDF2 scheme on Bakhvalov-type meshes for a singularly perturbed Volterra integro-differential equation, *Appl. Math. Lett.*, **145** (2023), 108755. https://doi.org/10.1016/j.aml.2023.108755

20. J. Liu, Simple and efficient ALE methods with provable temporal accuracy up to fifth order for the Stokes equations on time varying domains, *SIAM J. Numer. Anal.*, **51** (2023), 743–772. https://doi.org/10.1137/110825996

21. C. Li, Q. Yi, A. Chen, Finite difference methods with non-uniform meshes for nonlinear fractional differential equations, *J. Comput. Phys.*, **316** (2016), 614–631. https://doi.org/10.1016/j.jcp.2016.04.039