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Research article

Optimal control of queuing systems governed by integro-differential equations

Nurehemaiti Yiming[1,2,*]

[1] College of Mathematics and System Science, Xinjiang University, Urumqi 830017, China
[2] Xinjiang Key Laboratory of Applied Mathematics, Xinjiang University, Urumqi 830017, China

* **Correspondence:** Email: nyiming@xju.edu.cn.

Abstract: In this research, we advanced the optimal control theory for queuing systems that are characterized by integro-differential equations. Our primary goal was to identify an optimal service rate that minimizes a performance criterion, which is a composite of the system state at the final time and the cost associated with the optimal service rate. The optimal service rate was defined by an optimality system, and this formulation essentially translated the problem into a bilinear control problem within a nonreflexive Banach space, utilizing L^1-optimization techniques. We provided a rigorous proof of the existence of an optimal controller and offered a detailed characterization of the optimal control. Additionally, a comparison was made with traditional steady-state results to highlight the differences and improvements. Finally, numerical analysis was conducted on the theoretical results.

Keywords: queueing systems; integro-differential equations; optimal control; L^1−optimisation

1. Introduction

Queuing systems are extensively employed across sectors, including the service industry, transportation, manufacturing, entertainment and leisure, online services, distribution services, and many others (see, e.g., [1, 2]). However, these systems may encounter numerous challenges such as long queues, inconsistent service quality, suboptimal queuing strategies, lack of real-time information, customer behavior issues, system malfunctions, seasonal demand fluctuations, and service capacity constraints, among others [3]. To address these challenges effectively, it is essential to delve into the optimal design and optimal control of queuing systems.

Optimal control theory focuses on identifying control laws for a given system to achieve optimality criteria, and it has extensive applications in science and engineering. In the context of queuing systems, optimal design and optimal control are two distinct yet complementary approaches. Optimal design of queuing models, also known as economic models or static models, involves setting the parameters of a queuing system before it is put into operation. This process aims to establish the most efficient and cost-effective configuration based on predefined performance metrics. On the other hand, optimal control of queuing models, referred to as rate-control models or dynamic models, treats the parameters as control variables. These variables can be adjusted dynamically in response to changes in the system's state. This flexibility enables the system to adapt to varying conditions in real-time, thereby optimizing performance and resource utilization throughout its operation.

Scholars have extensively explored the optimal design and optimal control problems of queuing models. For instance, Miller employed a dynamic programming approach to derive the explicit form of the optimal control in scenarios where the cost function is a combination of average queue length, number of lost jobs, and service resources [4]. Yiannis et al. investigated the optimal strategy for a queuing system with exponentially distributed time periods, focusing on an M/M/1 queue. The authors derived the optimal decision-making strategies for arriving customers on whether to join the queue and for waiting customers on whether to remain in the queue [5]. Shekhar et al. examined a cost optimization problem for a finite-buffer M/M/1/N queuing model with an emergency vacation policy for the server. The authors applied the bat algorithm to determine the optimal steady-state performance indicators at the minimum cost [6]. Wang et al. analyzed the strategic behavior of customers and social optimization in an M/M/1 constant retrial queue with an N-policy. They conducted a sensitivity analysis of equilibrium/optimal rates and corresponding social welfare with respect to steady-state indicators [7]. Xu et al. [8] discussed how to optimize the management and control of queuing systems with complex features, such as two-stage heterogeneous services, retries, conflicts, and delayed vacations. For a comprehensive overview of the optimal design and optimal control problems in queuing models, the monograph by Stidham provides a detailed reference [9]. These aforementioned queuing models are typically formulated using a set of high-dimensional ordinary differential equations (ODEs).

The M/G/1 queuing models are constructed by defining state variables that depend on service time and waiting time. As a result, these models are typically formulated using partial differential equations (PDEs) rather than ordinary differential equations (ODEs). A review of the history of queuing theory reveals that a significant number of M/G/1 queuing systems have been described by PDEs, as documented in monographs such as [10, 11]. Many scholars have investigated the steady-state indices, dynamic indices, and steady-state optimal problems of M/G/1 queuing systems described by PDEs (see for example [12–16]). However, research on the dynamic optimal control of queuing models from the perspective of PDE control remains limited. The optimal control theory for systems governed by PDEs has been a central research topic in the field of distributed parameter systems control since the 1960s. From a practical standpoint, optimal control problems involving state and control constraints are significant and natural, as highlighted in studies such as [17–19]. For a comprehensive overview of the optimal control of PDEs, the monograph by Troltzsch [20] provides an in-depth reference.

In this paper, we address real-time service-rate control in M/G/1 queues with optional service, modeled by hyperbolic integro-differential equations. Unlike static designs that fix rates beforehand, we treat the service rate as a dynamic control variable in L^1-based nonreflexive Banach space and derive its optimal, implementable feedback law. In this context, the service rate functions as the control input of the system. Fortunately, in the realm of optimal control, several researchers have made significant contributions to the study of age-structured population models [21, 22], pest-pathogen systems [23], and reliable systems [24, 25], all within L^1-based nonreflexive Banach spaces. Our primary objective of this article is to design the optimal service rate for a queuing system described by hyperbolic PDEs by leveraging the methods and insights from [21–23, 25]. This approach aims to minimize unnecessary service waste in dynamic queuing systems. The result may transform traditional static personnel configuration or speed decisions into dynamic knobs that respond to observed workload, eliminating the need for large safety margins and enabling cost-effective, delay sensitive operations in call centers, hospital testing laboratories, and on-demand production lines. To the best of our knowledge, this represents the first application of this method to the design of optimal service rates in queuing systems.

The remainder of this paper is structured as follows. In Section 2, we provide a rigorous mathematical proof for the existence of an optimal solution. In Section 3, we characterize the optimal control. To illustrate the relationship between the optimal service rate and the system state, in section 4, we present a simplified example. Specifically, we consider a system with no additional optional service and a constant regular service rate, demonstrating that our results encompass the static optimal problem. In section 6, we conduct some numerical analysis on the above results.

2. Mathematical model and optimal design

In this paper, we consider the optimal control of the M/G/1 queuing system with additional optional service and no waiting capacity [13, 26]. The mathematical model of this system can be described by the following integro-differential equations

$$\frac{dp_0(t)}{dt} = -\lambda p_0(t) + \int_0^\infty r\mu_1(x)p_1(x,t)dx + \int_0^\infty \mu_2(x)p_2(x,t)dx, \quad t \in (0,\infty), \tag{2.1a}$$

$$\frac{\partial p_i(x,t)}{\partial x} + \frac{\partial p_i(x,t)}{\partial t} = -\mu_i(x)p_i(x,t), \quad i = 1,2, \quad (x,t) \in (0,\infty)\times(0,\infty), \tag{2.1b}$$

$$p_1(0,t) = \lambda p_0(t), \quad p_2(0,t) = \int_0^\infty (1-r)\mu_1(x)p_1(x,t)dx, \quad t \in (0,\infty), \tag{2.1c}$$

$$p_0(0) = 1, \quad p_i(x,0) = 0,\ i = 1,2, \quad x \in (0,\infty). \tag{2.1d}$$

Here, $p_0(t)$ represent the probability that the server is idle at time t, while $p_i(x,t)$ $(i = 1,2)$ denote the probability that, at time t, the server is providing the i-th service with an elapsed service time falling within the interval $(x, x + dx)$. Additionally, λ signifies the arrival rate of customers. The probability that customers opt for the second service after receiving the regular service is denoted by $1 - r$, while r represents the probability that customers depart from the system immediately after receiving the regular service. Last, $\mu_i(x)\mathrm{d}x$ represents the first-order probability that the i-th service will be completed within the time interval $(x, x + \mathrm{d}x)$, given that it has not been completed up to time x. This quantity satisfies the conditions $\mu_i(x) \geq 0$ and $\int_0^\infty \mu_i(x)dx = \infty$.

In [26], the mathematical model (2.1a)–(2.1d) were first established using the supplementary variable technique, and optimization is not addressed. Subsequently, the author derived the time-dependent solution of the system (2.1a)–(2.1d) in terms of the Laplace transform and obtained the expression for the steady-state solution under the steady-state assumption. The well-posedness and asymptotic behavior of the system (2.1a)–(2.1d) have been further explored in [12, 13, 27], utilizing C_0-semigroup theory and spectral analysis, but never consider $\mu_i(\cdot)$ a control variable.

In this paper, we are interested in minimizing the costs associated with time-dependent states $p_0(t)$, $p_i(\cdot,t)$ $(i = 1,2)$ and service rates $\mu_i(\cdot)$ of dynamic system described by Eq (2.1a)–(2.1d) on the domain $(0, L) \times (0, T)$, where L and T are positive real numbers:

$$\frac{dp_0(t)}{dt} = -\lambda p_0(t) + \int_0^L r\mu_1(x)p_1(x,t)dx + \int_0^L \mu_2(x)p_2(x,t)dx, \quad t \in (0,T), \tag{2.2a}$$

$$\frac{\partial p_i(x,t)}{\partial x} + \frac{\partial p_i(x,t)}{\partial t} = -\mu_i(x)p_i(x,t), \quad i = 1,2, \quad (x,t) \in (0,L) \times (0,T), \tag{2.2b}$$

$$p_1(0,t) = \lambda p_0(t), \quad p_2(0,t) = \int_0^L (1-r)\mu_1(x)p_1(x,t)dx, \quad t \in (0,T), \tag{2.2c}$$

$$p_0(0) = 1, \quad p_i(x,0) = 0, \ i = 1,2, \quad x \in (0,L). \tag{2.2d}$$

We take the service rate $\mu(\cdot) := (\mu_1(\cdot), \mu_2(\cdot))$, as the control, consider the set of admissible control as

$$U_{ad} = \{\mu \in (L^\infty(0,L))^2 \mid 0 \le \mu_i(x) \le \sup_{x \in (0,L)} \mu_i(x) := \bar{\mu}_i, \ i = 1,2\}, \tag{2.3}$$

and the corresponding solution, $p(\cdot,\cdot,\mu) = p(\mu)$, of system (2.2a)–(2.2d) as the state variable, where $p = (p_0, p_1, p_2)$. Moreover, our performance cost functional is a combination of the final time state and the cost of the applying the control. Our goal is to seek an $\mu^* \in U_{ad}$, such that

$$J(\mu^*) = \min\{J(\mu) \mid \mu \in U_{ad}\}, \tag{2.4}$$

where J is the cost functional as

$$\begin{aligned} J(\mu) &= \alpha_0 p_0(T) + \sum_{i=1}^2 \int_0^L [\alpha_i p_i(x,T) + \beta_i \mu_i(x) + \gamma_i \mu_i^2(x)]dx \\ &:= \int_0^L [\alpha p(x,T) + \beta\mu(x) + \gamma\mu^2(x)]dx \end{aligned} \tag{2.5}$$

subject to the govering system (2.2a)–(2.2d), here $\alpha = (\alpha_0 L^{-1}, \alpha_1, \alpha_2), \beta = (\beta_1, \beta_2), \gamma = (\gamma_1, \gamma_2)$, and parameters α_i $(i = 0,1,2)$ and β_i, γ_i $(i = 1,2)$ are nonnegative constants and stand for the states and control weights, respectively. The choice of the cost functional $J(\mu)$ in Eq (2.5) is motivated by the need to balance operational efficiency and economic expenditure in service systems. The terminal-state terms weighted by α_0, α_i penalize any residual workload (idle probability or unfinished services) at the end of the planning horizon T, reflecting customer-dissatisfaction or late-delivery costs. The terms $\beta_i\mu_i(r) + \gamma_i\mu_i^2(r)$ quantify the instantaneous cost of deploying the service rates $\mu_i(r)$. The linear part represents direct operating expenses (energy, staff wages), while the quadratic part discourages overly aggressive service acceleration, which would incur additional wear, overtime premiums, or equipment degradation. Minimizing $J(\mu)$ therefore yields a control strategy that simultaneously reduces unfinished work and avoids prohibitively expensive service intensification, guaranteeing a physically realizable and economically viable operation. The quadratic term penalization in Eq (2.5) generally enhances regularity for Eq (2.4).

3. Existence of optimal control

Establishing optimal control existence for problem (2.4) requires analyzing solution regularity for Eq (2.2a)–(2.2d).

Lemma 3.1. *If $p(\cdot,\cdot) = (p_0(\cdot), p_1(\cdot,\cdot), p_2(\cdot,\cdot))$ is the solution to Eq (2.2a)–(2.2d), then $p(\cdot,\cdot)$ satisfies*

$$p(x,t) = \begin{cases} \begin{pmatrix} e^{-\lambda t} + \int_0^t e^{\lambda(\tau-t)} \int_0^\tau [r\mu_1(x)p_1(x,t) + \mu_2(x)p_2(x,t)]dxd\tau \\ \lambda p_0(t-x)e^{-\int_0^x \mu_1(\tau)d\tau} \\ (1-r)\int_0^L p_1(x,t-x)\mu_1(x)dxe^{-\int_0^x \mu_2(\tau)d\tau} \end{pmatrix}, & x < t, \\ \begin{pmatrix} e^{-\lambda t} + \int_0^t e^{\lambda(\tau-t)} \int_0^\tau [r\mu_1(x)p_1(x,t) + \mu_2(x)p_2(x,t)]dxd\tau \\ 0 \\ 0 \end{pmatrix}, & x \geq t. \end{cases} \tag{3.1}$$

Moreover, $p(x,t) > 0$, for any $x, t \geq 0$,

$$p_0(t) + \int_0^L p_1(x,t)dx + \int_0^L p_2(x,t)dx = 1, \tag{3.2}$$

and

$$p_0 \in W^{1,\infty}(0,T),\ \ p_i \in L^\infty(0,T;W^{1,1}(0,L)) \cap W^{1,\infty}(0,T;L^1(0,L)), \quad i = 1,2. \tag{3.3}$$

Proof. **Step 1: Solution $p(\cdot,\cdot)$ satisfies Eq (3.1) and is positive.** Let $\xi = x - t$ and $Q_i(t) = p_i(\xi + t, t)$. Then, by Eq (2.1b), we have

$$\frac{dQ_i(t)}{dt} = \frac{\partial p_i(x,t)}{\partial x} + \frac{\partial p_i(x,t)}{\partial t} = -\mu_i(\xi + t)Q_i(t),\ \ i = 1,2. \tag{3.4}$$

For $\xi < 0$, i.e., $x < t$, by integrating Eq (3.4) from $-\xi$ to t and using $Q_i(-\xi) = p_i(0, t-x)$ to obtain

$$p_i(x,t) = Q_i(t) = Q_i(-\xi)e^{-\int_{-\xi}^t \mu_i(\xi+\tau)d\tau} = p_i(0,t-x)e^{-\int_0^x \mu_i(\tau)d\tau}, \quad i = 1,2. \tag{3.5}$$

For $\xi \geq 0$, i.e., $x \geq t$, integrating Eq (3.4) from 0 to t and using $Q_i(0) = p_i(\xi, 0) = 0$, we have

$$p_i(x,t) = Q_i(t) = Q_i(0)e^{-\int_0^t \mu_i(\xi+\tau)d\tau} = 0, \quad i = 1,2, \tag{3.6}$$

where $p_i(\xi, 0) = 0$ due to the initial condition (2.1d). Since Eq (3.6) and

$$\int_0^L \mu_i(x)p_i(x,t)dx = \int_0^t \mu_i(x)p_i(x,t)dx$$

in Eq (2.1a), we obtain

$$p_0(t) = e^{-\lambda t} + \int_0^t e^{\lambda(\tau-t)} \int_0^\tau [r\mu_1(x)p_1(x,t) + \mu_2(x)p_2(x,t)]dxd\tau. \tag{3.7}$$

Hence, by using boundary conditions (2.1c) and Eqs (3.5)–(3.7), we see that $p(\cdot,\cdot)$ is the solution of Eq (3.1). The positivity of $p(\cdot,\cdot)$ is due to the positivity of the corresponding semigroup of system (2.1a)–(2.1d) by Theorem 3 of [27].

Step 2: System (2.1a)–(2.1d) is conservative. Integrating Eq (2.1b) from 0 to L and using boundary condition (2.1c), we have

$$\begin{aligned}\frac{d}{dt}\int_0^L p_1(x,t)dx &= \lambda p_0(t) - \int_0^L \mu_1(x)p_1(x,t)dx,\\ \frac{d}{dt}\int_0^L p_2(x,t)dx &= (1-r)\int_0^L \mu_1(x)p_1(x,t)dx - \int_0^L \mu_1(x)p_1(x,t)dx.\end{aligned} \tag{3.8}$$

Hence, Eqs (2.1a) and (3.8) give

$$\frac{d}{dt}p_0(t) + \frac{d}{dt}\int_0^L p_1(x,t)dx + \frac{d}{dt}\int_0^L p_2(x,t)dx = 0, \text{ for all } t \in (0,T), \tag{3.9}$$

which implies

$$p_0(t) + \int_0^L p_1(x,t)dx + \int_0^L p_2(x,t)dx = 1. \tag{3.10}$$

Therefore, system (2.1a)–(2.1d) are conservative.

Step 3: We prove that Eq (3.3) holds. By Eq (3.10), it is easy to see that

$$\sup_{t\in[0,T]} p_0(t) \le 1, \quad \sup_{t\in[0,T]}\int_0^L p_i(x,t)dx \le 1, \ i = 1,2. \tag{3.11}$$

With Eqs (2.1a) and (3.11), the Hölder inequality, and $\mu_i(\cdot) \in L^\infty(0,L)$, $\mu_i(\cdot) \le \bar{\mu}_i$ $(i = 1,2)$, we have

$$\begin{aligned}\sup_{t\in[0,T]}\left|\frac{dp_0}{dt}\right| &\le \lambda \sup_{t\in[0,T]} p_0(t) + r\sup_{t\in[0,T]}\int_0^L \mu_1(x)p_1(x,t)dx + \sup_{t\in[0,T]}\int_0^L \mu_2(x)p_2(x,t)dx\\ &\le \lambda + r\bar{\mu}_1 + \bar{\mu}_2.\end{aligned} \tag{3.12}$$

Thus, Eqs (3.11) and (3.12) imply that $p_0(\cdot) \in W^{1,\infty}(0,T)$. Moreover, for $x < t$, using Eqs (2.1b), (2.1c), and (3.5), we derive

$$\begin{aligned}\frac{\partial p_1(x,t)}{\partial t} &= \lambda\frac{dp_0(t-x)}{dt}e^{-\int_0^x \mu_1(\tau)d\tau},\\ \frac{\partial p_1(x,t)}{\partial x} &= -\lambda\frac{dp_0(t-x)}{dt}e^{-\int_0^x \mu_1(\tau)d\tau} - \lambda\mu_1 p_0(t-x)e^{-\int_0^x \mu_1(\tau)d\tau}.\end{aligned} \tag{3.13}$$

This, together with inequality (3.12) and $\int_0^L \mu_1(x)e^{-\int_0^x \mu_1(\tau)d\tau} \le \int_0^\infty \mu_1(x)e^{-\int_0^x \mu_1(\tau)d\tau} = 1$, it is easy to calculate that

$$\begin{aligned}&\sup_{t\in[0,T]}\int_0^L\left|\frac{\partial p_1}{\partial x}\right|dx\\ &\le \lambda\sup_{t\in[0,T]}\int_0^L\left|\frac{dp_0(t-x)}{dt}\right|e^{-\int_0^x \mu_1(\tau)d\tau}dx + \sup_{t\in[0,T]}\int_0^L \lambda\mu_1(x)p_0(t-x)e^{-\int_0^x \mu_1(\tau)d\tau}dx\\ &\le \lambda\sup_{t\in[0,T]}\left|\frac{dp_0}{dt}\right|\int_0^L e^{-\int_0^x \mu_1(\tau)d\tau}dx + \lambda\sup_{t\in[0,T]}p_0(t)\int_0^L \mu_1(x)e^{-\int_0^x \mu_1(\tau)d\tau}dx\\ &\le \lambda L(\lambda + r\bar{\mu}_1 + \bar{\mu}_2) + \lambda.\end{aligned} \tag{3.14}$$

Then, combining inequalities (3.11) with Eq (3.14), we derive that $p_1(\cdot,\cdot) \in L^\infty(0,T;W^{1,1}(0,L))$. In addition, using Eqs (2.1b), (3.11), (3.14) and the Hölder inequality, we obtain

$$\begin{aligned}\sup_{t\in[0,T]}\int_0^L\left|\frac{\partial p_1}{\partial t}\right|dx &\le \sup_{t\in[0,T]}\int_0^L\left|\frac{\partial p_1}{\partial x}\right|dx+\sup_{t\in[0,T]}\int_0^L\mu_1(x)p_1(x,t)dx\\ &\le \lambda L(\lambda+r\bar{\mu}_1+\bar{\mu}_2)+\lambda+\bar{\mu}_1.\end{aligned}\tag{3.15}$$

This, together with inequality (3.11), we see that $p_1(\cdot,\cdot)\in W^{1,\infty}(0,T;L^1(0,L))$. For $x<t$, using Eqs (2.1b), (2.1c) and (3.5), we obtain

$$\begin{aligned}\frac{\partial p_2(x,t)}{\partial t} &= (1-r)\int_0^L\mu_1(x)\frac{\partial p_1(x,t-x)}{\partial t}dxe^{-\int_0^x\mu_2(\tau)d\tau},\\ \frac{\partial p_2(x,t)}{\partial x} &= -(1-r)\int_0^L\mu_1(x)\frac{\partial p_1(x,t-x)}{\partial t}dxe^{-\int_0^x\mu_2(\tau)d\tau}\\ &\quad -(1-r)\mu_2(x)\int_0^L\mu_1(x)p_1(x,t-x)dxe^{-\int_0^x\mu_2(\tau)d\tau}.\end{aligned}\tag{3.16}$$

Thus, combining Eq (3.16) with inequalities (3.11), (3.15) and using the Hölder inequality and inequality

$$\int_0^L\mu_2(x)e^{-\int_0^x\mu_2(\tau)d\tau}\le\int_0^\infty\mu_2(x)e^{-\int_0^x\mu_2(\tau)d\tau}=1,$$

we obtain

$$\begin{aligned}\sup_{t\in[0,T]}\int_0^L\left|\frac{\partial p_2}{\partial x}\right|dx &\le (1-r)\sup_{t\in[0,T]}\int_0^L\mu_1(x)\left|\frac{\partial p_1(x,t-x)}{\partial t}\right|dx\int_0^L e^{-\int_0^x\mu_2(\tau)d\tau}dx\\ &\quad+(1-r)\sup_{t\in[0,T]}\int_0^L\mu_1(x)p_1(x,t-x)dx\int_0^L\mu_2(x)e^{-\int_0^x\mu_2(\tau)d\tau}dx\\ &\le (1-r)\bar{\mu}_1L\sup_{t\in[0,T]}\int_0^L\left|\frac{\partial p_1}{\partial t}\right|dx+(1-r)\bar{\mu}_1\sup_{t\in[0,T]}\int_0^L p_1(x,t)dx\\ &\le (1-r)\bar{\mu}_1\{L[\lambda L(\lambda+r\bar{\mu}_1+\bar{\mu}_2)+\lambda+\bar{\mu}_1]+1\}.\end{aligned}\tag{3.17}$$

Therefore, using inequalities (3.11) and (3.17), we obtain that $p_2(\cdot,\cdot)\in L^\infty(0,T;W^{1,1}(0,L))$. In addition, using Eqs (2.1b), (3.11), and (3.17) and the Hölder inequality, we have

$$\begin{aligned}\sup_{t\in[0,T]}\int_0^L\left|\frac{\partial p_2}{\partial t}\right|dx &\le \sup_{t\in[0,T]}\int_0^L\left|\frac{\partial p_2}{\partial x}\right|dx+\sup_{t\in[0,T]}\int_0^L\mu_2(x)p_2(x,t)dx\\ &\le (1-r)\bar{\mu}_1\{L[\lambda L(\lambda+r\bar{\mu}_1+\bar{\mu}_2)+\lambda+\bar{\mu}_1]+1\}+\bar{\mu}_2.\end{aligned}\tag{3.18}$$

That is, $p_2(\cdot,\cdot)\in W^{1,\infty}(0,T;L^1(0,L))$. Therefore, Eq (3.3) holds.

Lemma 3.2. *Let* $F: U_{ad}\to L^\infty(0,T)\times(L^\infty(0,T;L^1(0,L)))^2 := X_1$ *by* $(F(t)\mu)(x)=p(x,t)$ *for* $(x,t)\in[0,L)\times[0,T)$. *Then, the embedding* $Range(F)\hookrightarrow C[0,T]\times(C([0,T];L^1(0,L)))^2$ *is compact.*

Proof. Using lemma 3.1, we obtain that the range of F, which satisfies

$$R(F) \subset \left\{ p(\cdot,\cdot) \in X_1 \;\middle|\; \begin{array}{l} p(\cdot,\cdot) = (p_0(\cdot), p_1(\cdot,\cdot), p_2(\cdot,\cdot)), p_0(\cdot) \in W^{1,\infty}(0,T), \\ p_i(\cdot,\cdot) \in L^\infty(0,T; W^{1,1}(0,L)), \frac{\partial p_i(\cdot,t)}{\partial t} \in L^\infty(0,T; L^1(0,L)),\; i = 1,2 \end{array} \right\}.$$

By the Sobolev imbedding theorem [28, p. 85], it is easy to see that

$$W^{1,\infty}(0,T) \hookrightarrow C[0,T] \text{ and } W^{1,1}(0,L) \hookrightarrow L^1(0,L)$$

are compact for $0 < L, T < \infty$. Moreover, by Aubin-Lions-Simon lemma [29, p. 102] the embedding

$$L^\infty(0,T; W^{1,1}[0,L)) \cap W^{1,\infty}(0,T; L^1[0,L)) \hookrightarrow C(0,T; L^1(0,L))$$

is compact. This instantaneously produces the required outcome.

Theorem 3.1. *There exists an optimal pair $(p^*, \mu^*) \in C[0,T] \times (C(0,T; L^1(0,L)))^2 \times U_{ad}$ such that p^* is the solution of system (2.1a)–(2.1d) and μ^* is an optimal control that minimizes the objective functional $J(\mu)$ over U_{ad}.*

Proof. Using Lemma 3.1 and $\mu \in U_{ad}$, we have

$$\begin{aligned} 0 \le J(\mu) &= \alpha_0 p_0(T) + \sum_{i=1}^{2} \int_0^L [\alpha_i p_i(x,T) + \beta_i \mu_i(x) + \gamma_i \mu_i^2(x)] dx \\ &\le \alpha_0 + \sum_{i=1}^{2} [\alpha_i + (\beta_i \bar{\mu}_i + \gamma_i \bar{\mu}_i^2) L]. \end{aligned} \tag{3.19}$$

The set $\{J(\mu) \mid \mu \in U_{ad}\}$ is non-empty and bounded from below. Let the minimizing sequence $\{\mu_n(\cdot)\} = \{(\mu_{1,n}(\cdot), \mu_{2,n}(\cdot))\}_{n\ge 1}$ be such that

$$J(\mu^*) = \lim_{n\to\infty} J(\mu_n) = \inf_{\mu \in U_{ad}} J(\mu).$$

Since $0 \le \mu_{i,n}(\cdot) \le \bar{\mu}_i, i = 1,2; n \ge 1$, $\{\mu_n\}$ is uniformly bounded in $(L^\infty(0,L))^2$ and $(L^2(0,L))^2$. Thus, there exists a convergence subsequence, denoted by $\{\mu_n\}(\cdot)$, such that

$$\begin{aligned} \mu_n(\cdot) = (\mu_{1,n}(\cdot), \mu_{2,n}(\cdot)) \rightharpoonup \mu^* = (\mu_1^*, \mu_2^*) \quad &\text{weakly star in} \quad (L^\infty(0,L))^2, \\ \mu_n(\cdot) = (\mu_{1,n}(\cdot), \mu_{2,n}(\cdot)) \rightharpoonup \mu^* = (\mu_1^*, \mu_2^*) \quad &\text{weakly in} \quad (L^2(0,L))^2. \end{aligned} \tag{3.20}$$

Denote by $\{p_n(\cdot,\cdot)\} = \{(p_{0,n}(\cdot), p_{1,n}(\cdot,\cdot), p_{2,n}(\cdot,\cdot))\}$ the state sequence corresponding to $\{\mu_n(\cdot)\}$ with identical initial data

$$p_n(\cdot,0) = (p_{0,n}(0), p_{1,n}(\cdot,0), p_{2,n}(\cdot,0)) = (1,0,0).$$

Then, by Lemma 3.2, choose a subsequence denoted $\{p_n\}$, there exists p^*, such that

$$p_n = (p_{0,n}, p_{1,n}, p_{2,n}) \to p^* = (p_0^*, p_1^*, p_2^*) \quad \text{strongly in} \quad C[0,T] \times (C(0,T; L^1(0,L)))^2. \tag{3.21}$$

In the following, we verify that p^* is the solution of system (2.1a)–(2.1d) corresponding to μ^*. It suffices to show that $\sup_{t\in[0,T]}|p_{0,n}(t)-p_0^*(t)|\to 0$ and $\sup_{t\in[0,T]}\|p_{i,n}(\cdot,t)-p_i^*(\cdot,t)\|_{L^1(0,L)}\to 0$ as $n\to 0$. Using Eqs (3.1), (3.20) and (3.21), we have

$$
\begin{aligned}
&\sup_{t\in[0,T]}|p_{0,n}(t)-p_0^*(t)| \le r\sup_{t\in[0,T]}\left|\int_0^t e^{\lambda(\tau-t)}\int_0^\tau \mu_{1,n}(x)[p_{1,n}(x,\tau)-p_1^*(x,\tau)]dxd\tau\right|\\
&+\sup_{t\in[0,T]}\left|\int_0^t e^{\lambda(\tau-t)}\int_0^\tau p_1^*(x,\tau)[\mu_{1,n}(x)-\mu_1^*(x)]dxd\tau\right|\\
&+\sup_{t\in[0,T]}\left|\int_0^t e^{\lambda(\tau-t)}\int_0^\tau \mu_{2,n}(x)[p_{2,n}(x,\tau)-p_2^*(x,\tau)]dxd\tau\right|\\
&+\sup_{t\in[0,T]}\left|\int_0^t e^{\lambda(\tau-t)}\int_0^\tau p_2^*(x,\tau)[\mu_{2,n}(x)-\mu_2^*(x)]dxd\tau\right|\\
&\le r\bar{\mu}_1\sup_{t\in[0,T]}\|p_{1,n}(\cdot,t)-p_1^*(\cdot,t)\|_{L^1[0,L)}\\
&+\sup_{t\in[0,T]}\left|\int_0^L[\mu_{1,n}(x)-\mu_1^*(x)]\int_x^T e^{\lambda(\tau-t)}p_1^*(x,\tau)d\tau dx\right|\\
&+\bar{\mu}_2\sup_{t\in[0,T]}\|p_{2,n}(\cdot,t)-p_2^*(\cdot,t)\|_{L^1[0,L)}\\
&+\sup_{t\in[0,T]}\left|\int_0^L[\mu_{2,n}(x)-\mu_2^*(x)]\int_x^T e^{\lambda(\tau-t)}p_2^*(x,\tau)d\tau dx\right|\to 0\quad\text{as}\quad n\to 0,
\end{aligned}
\tag{3.22}
$$

$$
\begin{aligned}
&\sup_{t\in[0,T]}\int_0^L|p_{1,n}(x,t)-p_1^*(x,t)|dx \le \lambda\sup_{t\in[0,T]}\int_0^L|p_{0,n}(t-x)-p_0^*(t-x)|e^{-\int_0^x\mu_{1,n}(\tau)d\tau}dx\\
&+\lambda\sup_{t\in[0,T]}\int_0^L p_0^*(t-x)\left|e^{-\int_0^x\mu_{1,n}(\tau)d\tau}-e^{-\int_0^x\mu_1^*(\tau)d\tau}\right|dx\\
&\le\lambda\sup_{t\in[0,T]}|p_{0,n}(t)-p_0^*(t)|\int_0^L e^{-\int_0^x\mu_{1,n}(\tau)d\tau}dx\\
&+\lambda\sup_{t\in[0,T]}p_0^*(t)\int_0^L\left|e^{-\int_0^x\mu_{1,n}(\tau)d\tau}-e^{-\int_0^x\mu_1^*(\tau)d\tau}\right|dx\to 0\quad\text{as}\quad n\to 0,
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
&\sup_{t\in[0,T]}\int_0^L|p_{2,n}(x,t)-p_2^*(x,t)|dx\\
&\le(1-r)\sup_{t\in[0,T]}\int_0^L\mu_{1,n}(x)|p_{1,n}(x,t)-p_1^*(x,t)|dx\int_0^L e^{-\int_0^x\mu_{2,n}(\tau)d\tau}dx\\
&+(1-r)\sup_{t\in[0,T]}\int_0^L p_1^*(x,t)|\mu_{1,n}(x)-\mu_1^*(x)|dx\int_0^L e^{-\int_0^x\mu_{2,n}(\tau)d\tau}dx\\
&+(1-r)\sup_{t\in[0,T]}\int_0^L p_1^*(x,t)\mu_1^*(x)dx\left|\int_0^L e^{-\int_0^x\mu_{2,n}(\tau)d\tau}dx-\int_0^L e^{-\int_0^x\mu_2^*(\tau)d\tau}dx\right|
\end{aligned}
$$

$$\to 0 \quad \text{as} \quad n \to 0. \tag{3.24}$$

Hence, by Eqs (3.23) and (3.24), we obtain that p^* is the solution of system (2.1a)–(2.1d) corresponding to μ^*. Last, for the quadratic term of the control in the objective functional, the lower semi-continuity of the L^2 norm with respect to weak convergence of control sequence gives

$$\int_0^L (\mu_i^*(x))^2 dx \le \varliminf_{n\to\infty} \int_0^L (\mu_{i,n}(x))^2 dx, \quad i = 1, 2.$$

Therefore, by using Eq (3.20) and (3.21), we obtain

$$\begin{aligned} J(\mu^*) &= \int_0^L [\alpha p^*(x,T) + \beta \mu^*(x) + \gamma(\mu^*(x))^2] dx \\ &\le \varliminf_{n\to\infty} J(\mu_n) = \inf_{\mu \in U_{ad}} J(\mu). \end{aligned}$$

This means that μ^* is an optimal control.

4. Characterization of an optimal control

To obtain a characterization of an optimal control, we need to derive an optimality system that consists of the original state system coupled with an adjoint system. To obtain the necessary conditions for the optimality system, we will need to differentiate the objective functional $J(\mu)$ with respect to control μ.

Lemma 4.1. *The map $\mu \in U_{ad} \to p = p(\mu) \in C[0,T] \times (C(0,T; L^1(0,L)))^2$ is differentiable in following sense*

$$\begin{aligned} &\frac{p_0(\mu + \varepsilon h) - p_0(\mu)}{\varepsilon} \to z_0 \quad \textit{weakly in} \quad C[0,T], \\ &\frac{p_i(\mu + \varepsilon h) - p_i(\mu)}{\varepsilon} \to z_i \quad \textit{weakly in} \quad C(0,T; L^1(0,L)),\ i = 1, 2, \end{aligned}$$

as $\varepsilon \to 0$, for small $\varepsilon > 0$, where $\mu + \varepsilon h \in U_{ad}$ and $h = (h_1, h_2) \in (L^\infty(0,L))^2$. Moreover, $z = (z_0, z_1, z_2) = z(p, h)$ satisfies

$$\begin{cases} \dfrac{dz_0(t)}{dt} = -\lambda z_0(t) + r \displaystyle\int_0^L [\mu_1(x) z_1(x,t) + h_1(x) p_1(x,t)] dx \\ + \displaystyle\int_0^L [\mu_2(x) z_2(x,t) + h_2(x) p_2(x,t)] dx, \\ \dfrac{\partial z_i(x,t)}{\partial x} + \dfrac{\partial z_i(x,t)}{\partial t} = -\mu_i(x) z_i(x,t) - h_i(x) p_i(x,t),\ i = 1, 2, \\ z_1(0,t) = \lambda z_0(t), \quad z_2(0,t) = (1 - r) \displaystyle\int_0^L [\mu_1(x) z_1(x,t) + h_1(x) p_1(x,t)] dx, \\ z_0(0) = 0, \quad z_i(x,0) = 0,\ i = 1, 2. \end{cases} \tag{4.1}$$

Proof. We vary the control from μ to $\mu + \varepsilon h$, where $h \in (L^\infty(0, L))^2$ is an arbitrary variation. For simplicity, we will use $p_0(t, \mu) = p_0(\mu), p_i(x, t, \mu) = p_i(\mu), \mu_i(x) = \mu_i$ and $h_i(x) = h_i,\ i = 1, 2$ for $(x, t) \in (0, L) \times (0, T)$. By Eq (3.1), it is easy to see that

$$\begin{aligned} p_0(\mu + \varepsilon h) - p_0(\mu) &= \int_0^t e^{\lambda(\tau - t)} \int_0^\tau \{ r\mu_1 [p_1(\mu + \varepsilon h) - p_1(\mu)] + r\varepsilon h_1 p_1(\mu + \varepsilon h) \\ &\quad + \mu_2 [p_2(\mu + \varepsilon h) - p_2(\mu)] + \varepsilon h_2 p_2(\mu + \varepsilon h)] \} dx d\tau, \end{aligned} \tag{4.2a}$$

$$\begin{aligned} p_1(\mu + \varepsilon h) - p_1(\mu) &= \lambda [p_0(\mu + \varepsilon h, t - x) - p_0(\mu, t - x)] e^{-\int_0^x (\mu_1 + \varepsilon h_1) d\tau} \\ &\quad + \lambda p_0(\mu, t - x) \left(e^{-\int_0^x (\mu_1 + \varepsilon h_1) d\tau} - e^{-\int_0^x \mu_1 d\tau} \right), \end{aligned} \tag{4.2b}$$

$$\begin{aligned} p_2(\mu + \varepsilon h) - p_2(\mu) &= (1 - r) \int_0^L \{ \mu_1 [p_1(\mu + \varepsilon h, x, t - x) - p_1(\mu, x, t - x)] \\ &\quad + \varepsilon h_1 p_1(\mu + \varepsilon h, x, t - x) \} dx e^{-\int_0^x (\mu_2 + \varepsilon h_2) d\tau} \\ &\quad - (1 - r) \int_0^L p_1(\mu, x, t - x) \mu_1 dx \left(e^{-\int_0^x (\mu_2 + \varepsilon h_2) d\tau} - e^{-\int_0^x \mu_2 d\tau} \right), \end{aligned} \tag{4.2c}$$

Then, from the convergence of $p_i(\mu + \varepsilon h)$ to $p_i(\mu)$ as $\varepsilon \to 0$ and differentiability of p_i, we can pass the limit in the right hand side of Eq (4.2a)–(4.2c) and get the weak convergence of the $\frac{p_i(\mu + \varepsilon h) - p_i(\mu)}{\varepsilon}$ to the desired sensitivities z_i $(i = 0, 1, 2)$ as $\varepsilon \to 0$, which satisfies system (4.1). In fact, by Eq (4.2a)–(4.2c), the boundary condition (2.1c) and the initial condition (2.1d), we directly calculate that

$$\begin{aligned} \frac{dz_0(t)}{dt} &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \frac{dp_0(\mu + \varepsilon h) - dp_0(\mu)}{dt} \\ &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[-\lambda p_0(\mu + \varepsilon h) + r \int_0^L (\mu_1 + \varepsilon h_1) p_1(\mu + \varepsilon h) dx \right. \\ &\quad \left. + \int_0^L (\mu_2 + \varepsilon h_2) p_2(\mu + \varepsilon h) dx + \lambda p_0(\mu) - r \int_0^L \mu_1 p_1(\mu) dx - \int_0^L \mu_2 p_2(\mu) dx \right] \\ &= -\lambda_0 \lim_{\varepsilon \to 0} \frac{p_0(\mu + \varepsilon h) - p_0(\mu)}{\varepsilon} + r \int_0^L \mu_1 \lim_{\varepsilon \to 0} \frac{p_1(\mu + \varepsilon h) - p_1(\mu)}{\varepsilon} dx \\ &\quad + \int_0^L \mu_2 \lim_{\varepsilon \to 0} \frac{p_2(\mu + \varepsilon h) - p_2(\mu)}{\varepsilon} dx + \int_0^L \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \varepsilon h_1 p_1(\mu + \varepsilon h) dx \\ &\quad + \int_0^L \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \varepsilon h_2 p_2(\mu + \varepsilon h) dx \\ &= -\lambda z_0(t) + r \int_0^L [\mu_1(x) z_1(x, t) + h_1(x) p_1(x, t)] dx \\ &\quad + \int_0^L [\mu_2(x) z_2(x, t) + h_2(x) p_2(x, t)] dx, \quad t \in (0, T), \end{aligned} \tag{4.3a}$$

$$
\begin{aligned}
\frac{\partial z_i(x,t)}{\partial x} + \frac{\partial z_i(x,t)}{\partial t} &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\frac{\partial p_i(\mu+\varepsilon h)-\partial p_i(\mu)}{\partial x} + \frac{\partial p_i(\mu+\varepsilon h)-\partial p_i(\mu)}{\partial t}\right]\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\frac{\partial p_i(\mu+\varepsilon h)}{\partial x} + \frac{\partial p_i(\mu+\varepsilon h)}{\partial t} - \left(\frac{\partial p_i(\mu)}{\partial x} + \frac{\partial p_i(\mu)}{\partial t}\right)\right]\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[-(\mu_i+\varepsilon h_i)p_i(\mu+\varepsilon h) + \mu_i p_i(\mu)\\
&= -\mu_i \lim_{\varepsilon\to 0}\frac{p_i(\mu+\varepsilon h)-p_i(\mu)}{\varepsilon} - \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\varepsilon h_i p_i(\mu+\varepsilon h)\\
&= -\mu_i(x)z_i(x,t) - h_i(x)p_i(x,t),\ i=1,2, \quad (x,t)\in(0,L)\times(0,T),
\end{aligned}
\tag{4.3b}
$$

$$
z_1(0,t) = \lim_{\varepsilon\to 0}\frac{p_1(\mu+\varepsilon h,0,t)-p_1(\mu,0,t)}{\varepsilon} = \lim_{\varepsilon\to 0}\frac{\lambda p_0(\mu+\varepsilon h)-\lambda p_0(\mu)}{\varepsilon} = \lambda z_0(t), \tag{4.3c}
$$

$$
\begin{aligned}
z_2(0,t) &= \lim_{\varepsilon\to 0}\frac{p_2(\mu+\varepsilon h,0,t)-p_2(\mu,0,t)}{\varepsilon}\\
&= \lim_{\varepsilon\to 0}\frac{(1-r)\int_0^L(\mu_1+\varepsilon h_1)p_1(\mu+\varepsilon h)dx - (1-r)\int_0^L \mu_1 p_1(\mu)dx}{\varepsilon}\\
&= (1-r)\int_0^L \mu_1 \lim_{\varepsilon\to 0}\frac{p_1(\mu+\varepsilon h)-p_1(\mu)}{\varepsilon}dx + (1-r)\int_0^L h_1 \lim_{\varepsilon\to 0} p_1(\mu+\varepsilon h)dx\\
&= (1-r)\int_0^L [\mu_1(x)z_1(x,t) + h_1(x)p_1(x,t)]dx, \quad t\in[0,T],
\end{aligned}
\tag{4.3d}
$$

$$
z_0(0) = \lim_{\varepsilon\to 0}\frac{p_0(\mu+\varepsilon h,0)-p_0(\mu,0)}{\varepsilon} = \lim_{\varepsilon\to 0}\frac{0-0}{\varepsilon} = 0, \tag{4.3e}
$$

$$
z_i(x,0) = \lim_{\varepsilon\to 0}\frac{p_i(\mu+\varepsilon h,x,0)-p_0(\mu,x,0)}{\varepsilon} = \lim_{\varepsilon\to 0}\frac{0-0}{\varepsilon} = 0,\ i=1,2, \quad x\in[0,L]. \tag{4.3f}
$$

Hence, by combining Eq (4.3a)–(4.3f), we obtain that $z = z(p,\mu)$ satisfies system (4.1).

To proceed with the optimal control characterization, we need to introduce the Lagrangian functional (see [20, p. 85]) for the three adjoint variables q_0, q_1, q_2 corresponding to p_0, p_1, p_2. The Lagrangian functional is given by:

$$
\begin{aligned}
&\mathcal{L}(p,\mu,q)\\
&:= J(\mu) + \int_0^T q_0(t)\left[\frac{dp_0(t)}{dt} + \lambda p_0(t) - \int_0^L r\mu_1(x)p_1(x,t)\,dx - \int_0^L \mu_2(x)p_2(x,t)\,dx\right]dt\\
&+ \sum_{m=1}^{2}\int_0^L\int_0^T q_m(x,t)\left[\frac{\partial p_m(x,t)}{\partial t} + \frac{\partial p_m(x,t)}{\partial x} + \mu_m(x)p_m(x,t)\right]dt\,dx\\
&+ \int_0^T \nu_1(t)\,[p_1(0,t) - \lambda p_0(t)]\,dt + \int_0^T \nu_2(t)\left[p_2(0,t) - \int_0^L (1-r)\mu_1(x)p_1(x,t)\,dx\right]dt\\
&+ \zeta_0\,[p_0(0)-1] + \int_0^L \zeta_1(x)p_1(x,0)dx + \int_0^L \zeta_2(x)p_2(x,0)dx,
\end{aligned}
$$

where $q_i, \zeta_i, i = 1,2,3$ and $\nu_m, m = 1,2$ are Lagrange multipliers.

We are ready to characterize the optimal control using the above Lagrangian functional and deriving the optimality system by differentiating $J(\mu)$ with respect to μ at an optimal control.

Theorem 4.1. *Given an optimal control* $\mu^* = (\mu_1^*, \mu_2^*) \in U_{ad}$ *and the corresponding solution* $p(\cdot,\cdot)$ *to Eq (2.1a)–(2.1d), then there exists adjoint variables* $q(\cdot,\cdot) = (q_0(\cdot), q_1(\cdot,\cdot), q_2(\cdot,\cdot))$*, satisfying this system*

$$\begin{cases} \dfrac{dq_0}{dt} = \lambda[q_0(t) - q_1(0,t)], \quad t \in (0,T), \\ \dfrac{\partial q_1}{\partial x} + \dfrac{\partial q_1}{\partial t} = \mu_1(x)[q_1(x,t) - rq_0(t) - (1-r)q_2(0,t)], \ (x,t) \in (0,L)\times(0,T), \\ \dfrac{\partial q_2}{\partial x} + \dfrac{\partial q_2}{\partial t} = \mu_2(x)[q_2(x,t) - q_0(t)], \quad (x,t) \in (0,L)\times(0,T), \\ q_0(T) = \alpha_0, \ q_i(x,T) = \alpha_i, \ i = 1,2, \quad x \in [0,L] \end{cases} \tag{4.4}$$

Furthermore, $\mu^* = (\mu_1^*, \mu_2^*)$ *can be explicitly characterized as*

$$\begin{aligned} \mu_1^*(x) &= \max\left\{0, \min\left\{\frac{1}{2\gamma_1}\int_0^T p_1(x,t)[q_1(x,t) - rq_0(t)]dt - \frac{\beta_1}{2\gamma_1}, \bar{\mu}_1\right\}\right\} \quad \textit{and} \\ \mu_2^*(x) &= \max\left\{0, \min\left\{\frac{1}{2\gamma_2}\int_0^T p_2(x,t)[q_2(x,t) - q_0(t)]dt - \frac{\beta_2}{2\gamma_2}, \bar{\mu}_2\right\}\right\}. \end{aligned} \tag{4.5}$$

In particular, when there is no additional optional service, the optimal control becomes

$$\mu_1^*(x) = \max\left\{0, \min\left\{\frac{1}{2\gamma_1}\int_0^T p_1(x,t)[q_1(x,t) - q_0(t)]dt - \frac{\beta_1}{2\gamma_1}, \bar{\mu}_1\right\}\right\}.$$

Proof. Consider $\mu \in U_{ad}$ and $h \in (L^\infty(0,L))^2$, such that $\mu + \varepsilon h \in U_{ad}$ for small $\varepsilon > 0$. Then, the derivative of $J(\mu)$ with respect to in the direction h satisfies

$$\begin{aligned} 0 \ &\le J'(\mu)h = \lim_{\varepsilon\to 0}\frac{J(\mu+\varepsilon h) - J(\mu)}{\varepsilon} \\ &= \alpha\int_0^L \lim_{\varepsilon\to 0}\frac{p(\mu+\varepsilon h, x, T) - p(\mu, x, T)}{\varepsilon}dx + \beta\int_0^L \lim_{\varepsilon\to 0}\frac{\mu(x)+\varepsilon h(x) - \mu(x)}{\varepsilon}dx \\ &+\gamma\int_0^L \lim_{\varepsilon\to 0}\frac{[\mu(x)+\varepsilon h(x)]^2 - \mu^2(x)}{\varepsilon}dx \\ &= \alpha\int_0^L z(x,T)dx + \beta\int_0^L h(x)dx + \gamma\int_0^L \mu(x)h(x)dx \\ &= \alpha_0 z_0(T) + \sum_{i=1}^2 \alpha_i\int_0^L z_i(x,T)dx + \sum_{i=1}^2\int_0^L [\beta_i + 2\gamma_i\mu_i(x)]h_i(x)dx. \end{aligned} \tag{4.6}$$

First, we focus just on the terms $\alpha_0 z_0(T) + \sum_{i=1}^2 \alpha_i \int_0^L z_i(x,T)dx$ of Eq (4.6). Substituting from the adjoint system (4.4), integrating by parts and using system (4.1), we obtain

$$\alpha_0 z_0(T) + \sum_{i=1}^2 \alpha_i\int_0^L z_i(x,T)dx = z_0(T)q_0(T) + \sum_{i=1}^2\int_0^L z_i(x,T)q_i(x,T)dx$$

$$
\begin{aligned}
&= \int_0^T \left(\frac{dz_0}{dt}q_0 + z_0\frac{dq_0}{dt}\right)dt + \sum_{i=1}^{2}\int_0^L\int_0^T\left(\frac{\partial z_i}{\partial t}q_i + z_i\frac{\partial q_i}{\partial t}\right)dtdx \\
&= \int_0^T \left[\left(-\lambda z_0 + r\int_0^L(\mu_1 z_1 + h_1 p_1)dx + \int_0^L(\mu_2 z_2 + h_2 p_2)dx\right)q_0 + z_0\frac{dq_0}{dt}\right]dt \\
&+ \sum_{i=1}^{2}\int_0^L\int_0^T\left[\left(-\frac{\partial z_i}{\partial x} - \mu_i z_i - h_i p_i\right)q_i + z_i\frac{\partial q_i}{\partial t}\right]dtdx \\
&= \int_0^T z_0(t)\left(\frac{dq_0}{dt} - \lambda q_0(t) + \lambda q_1(0,t)\right)dt \\
&+ \int_0^L\int_0^T z_1(x,t)\left[\frac{\partial q_1}{\partial x} + \frac{\partial q_1}{\partial t} - \mu_1(x)q_1(x,t) + rq_0(t) + (1-r)q_1(0,t)\right]dtdx \\
&+ \int_0^L\int_0^T z_2(x,t)\left[\frac{\partial q_2}{\partial x} + \frac{\partial q_2}{\partial t} - \mu_2(x)q_2(x,t) + \mu_2(x)q_0(t)\right]dtdx \\
&+ \int_0^L\int_0^T \{h_1(x)p_1(x,t)[rq_0(t) - q_1(x,t)] + h_2(x)p_2(x,t)[q_0(t) - q_2(x,t)]\}dtdx \\
&= \int_0^L\int_0^T \{h_1(x)p_1(x,t)[rq_0(t) - q_1(x,t)]dtdx + h_2(x)p_2(x,t)[q_0(t) - q_2(x,t)]\}dxdt. \qquad (4.7)
\end{aligned}
$$

Then, using the last expression $\sum_{i=1}^{2}\int_0^L[\beta_i + 2\gamma_i\mu_i(x)]h_i(x)dx$ of Eq (4.6), we obtain

$$
\begin{aligned}
0 \le J'(\mu)h &= \int_0^L h_1(x)\left[\int_0^T p_1(x,t)[rq_0(t) - q_1(x,t)]dt + \beta_1 + 2\gamma_1\mu_1(x)\right]dx \\
&+ \int_0^L h_2(x)\left[\int_0^T p_2(x,t)[q_0(t) - q_2(x,t)]dt + \beta_2 + 2\gamma_2\mu_2(x)\right]dx.
\end{aligned} \qquad (4.8)
$$

Hence, by the arbitrary variation of $h = (h_1, h_2) \in (L^\infty(0, L))^2$, and bounds on the control set U_{ad}, we obtain

$$
\mu_1^*(x) = \max\left\{0, \min\left\{\frac{1}{2\gamma_1}\int_0^T p_1(x,t)[q_1(x,t) - rq_0(t)]dt - \frac{\beta_1}{2\gamma_1}, \bar{\mu}_1\right\}\right\} \quad \text{and}
$$
$$
\mu_2^*(x) = \max\left\{0, \min\left\{\frac{1}{2\gamma_2}\int_0^T p_2(x,t)[q_2(x,t) - q_0(t)]dt - \frac{\beta_2}{2\gamma_2}, \bar{\mu}_2\right\}\right\}.
$$

In particular, if there is no additional optional service, i.e., $r = 1$, then the system (2.1a)–(2.1d) become

$$
\begin{cases}
\dfrac{dp_0(t)}{dt} = -\lambda p_0(t) + \displaystyle\int_0^L \mu_1(x)p_1(x,t)dx, \ t \in (0,T), \\
\dfrac{\partial p_1(x,t)}{\partial x} + \dfrac{\partial p_1(x,t)}{\partial t} = -\mu_1(x)p_1(x,t), \ (x,t) \in (0,L)\times(0,T), \\
p_1(0,t) = \lambda p_0(t), \ p_0(0) = 1, \ p_1(x,0) = 0, \ x \in [0,L], t \in [0,T].
\end{cases} \qquad (4.9)
$$

By using the similar approach as Eqs (4.6)–(4.8), we have

$$\mu_1^*(x) = \max\left\{0, \min\left\{\frac{1}{2\gamma_1}\int_0^T p_1(x,t)[q_1(x,t) - q_0(t)]dt - \frac{\beta_1}{2\gamma_1}, \bar{\mu}_1\right\}\right\}.$$

5. Optimal service rate of $M/M/1/1$ queuing system

To illustrate the state-dependent relationship of optimal service rate, we examine system (2.2a)–(2.2d) with constant regular service and no optional service. That is, we consider the situation $r = 1$ and $\mu_1(\cdot) = \mu_1 := \mu$ as a constant. Let $p_1(\cdot) = \int_0^L p_1(x,\cdot)dx$. Then, system (2.2a)–(2.2d) becomes the $M/M/1/1$ queuing system given by [11, p. 121], which can be rewritten as the following differential difference equations:

$$\begin{cases} p_0'(t) = -\lambda p_0(t) + \mu p_1(t), \quad t \in (0,T), \\ p_1'(t) = \lambda p_0(t) - \mu p_1(t), \quad t \in (0,T), \\ p_0(0) = 1, \ p_1(0) = 0. \end{cases} \tag{5.1}$$

The cost functional J of system (5.1) can be written as

$$J(\mu) = \alpha_0 p_0(T) + \alpha_1 p_1(T) + \beta_1\mu + \gamma_1\mu^2.$$

Given an optimal control μ^* and the corresponding solution $p(\cdot) = (p_0(\cdot), p_1(\cdot))$ to system (5.1), there exists adjoint variables $q(\cdot) = (q_0(\cdot), q_1(\cdot))$, satisfying the following system

$$\begin{cases} q_0'(t) = \lambda[q_0(t) - q_1(t)], \quad t \in (0,T), \\ q_1'(t) = -\mu[q_0(t) - q_1(t)], \quad t \in (0,T), \\ q_i(T) = \alpha_i, \quad i = 1, 2. \end{cases} \tag{5.2}$$

By using the similar approach as Theorem 4.1, the optimal control is obtained as follows:

$$\mu^* = \max\left\{0, \min\left\{\frac{1}{2\gamma_1}\int_0^T p_1(t)(q_1(t) - q_0(t))dt - \frac{\beta_1}{2\gamma_1}, \bar{\mu}\right\}\right\}.$$

Moreover, solve system (5.1) and using $p_1(\cdot) = 1 - p_0(\cdot)$, we obtain

$$p_1(t) = \frac{\lambda}{\lambda+\mu}\left(1 - e^{-(\lambda+\mu)t}\right), \quad t \in (0,T). \tag{5.3}$$

In addition, by system (5.2), it is easy to calculate that

$$\begin{aligned} q_0(t) &= \frac{\lambda(\alpha_0 - \alpha_1)}{\lambda+\mu}e^{(\lambda+\mu)(T-t)} + \frac{\mu\alpha_0 + \lambda\alpha_1}{\lambda+\mu}, \\ q_1(t) &= \frac{\mu(\alpha_1 - \alpha_0)}{\lambda+\mu}e^{(\lambda+\mu)(T-t)} + \frac{\mu\alpha_0 + \lambda\alpha_1}{\lambda+\mu}. \end{aligned} \tag{5.4}$$

Hence, using Eqs (5.3) and (5.4), the arbitrary of μ, and $0 \le \mu \le \bar{\mu}$, it is easy to calculate that the optimality condition becomes

$$\mu^* = \max\left\{0, \min\left\{\frac{\alpha_1 - \alpha_0}{2\gamma_1(\lambda+\mu)^2}\left[\frac{\lambda}{2}\left(e^{(\lambda+\mu)T} - e^{-(\lambda+\mu)T}\right) + \mu\left(e^{(\lambda+\mu)T} - 1\right)\right] - \frac{\beta_1}{2\gamma_1}, \bar{\mu}\right\}\right\}.$$

Static optimal problem: In the following, we consider the optimal control problem in the steady-state for system (5.1) by [9, p. 4]. Since the well-posedness and asymptotic behavior results in [12, 27], the steady-state assumption $p_i = \lim_{t\to\infty} p_i(t)$ $(i = 0, 1)$ given by [26] holds true. Our static optimization problem would be to choose $\mu_1 \in [0, \bar{\mu}_1]$ to minimize

$$J(\mu_1) = \alpha_1 p_1 + \beta_1 \mu_1. \tag{5.5}$$

From Eq (5.3), together with the fact $p_0 + p_1 = 1$, we have

$$p_1 = \frac{\lambda}{\lambda + \mu_1}. \tag{5.6}$$

Then, using Eq (5.6) and taking the first and second order derivatives of $J(\mu_1)$, we obtain

$$J(\mu_1) = \frac{\alpha_1\lambda}{\lambda+\mu_1} + \beta_1\mu_1, \quad J'(\mu_1) = \frac{-\lambda\alpha_1}{(\lambda+\mu_1)^2} + \beta_1, \quad J''(\mu_1) = \frac{2\lambda\alpha_1}{(\lambda+\mu_1)^3}. \tag{5.7}$$

Notice that $J''(\mu_1) > 0$ for $\lambda, \mu_1, \alpha_1 > 0$, so that $J(\mu_1)$ is convex (see, e.g., [9, p. 5]). Moreover, $J(\mu_1) \to \infty$ as $\mu_1 \to \infty$. Hence, we can solve this problem by differentiating $J(\mu_1)$ and setting $J'(\mu_1) = 0$. This yields the following expression for the unique optimal value of the service rate (see e.g., [9, p. 5]):

$$\mu_1^* = \sqrt{\frac{\lambda\alpha_1}{\beta_1}} - \lambda.$$

The optimal value of the objective function (5.5) is thus given by

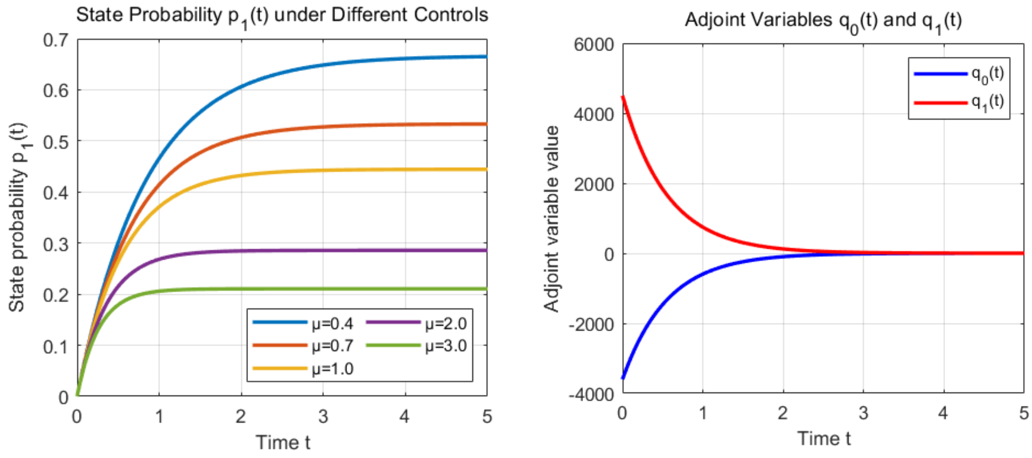
$$J(\mu_1^*) = 2\sqrt{\lambda\beta_1\alpha_1} - \lambda\beta_1.$$

This is similar to the result in [30].

6. Numerical analysis results description

In this section, we conduct numerical analysis on the results in Section 5 and obtain the following results using Matlab. The following figures illustrate that the results obtained in this paper are correct.

Figure 1(a) shows the time evolution of the state probability $p_1(t)$ under different control intensities μ. In this figure, we take $\lambda = 0.8, T = 5$, and all curves start from $p_1(t) = 0$ and exhibit exponential growth over time, gradually approaching their respective steady-state values $\frac{\lambda}{\lambda+\mu}$. When the μ value is small (e.g., $\mu = 0.1$), the system converges slowly but reaches a higher steady-state probability. In contrast, when μ is large (e.g., $\mu = 3$), the convergence rate increases significantly, but the steady-state probability decreases accordingly. This visually demonstrates the critical trade-off between the system's convergence speed and its final state distribution, which is fundamental for understanding system dynamics and designing control strategies.



(a) (b)

Figure 1. The time evolution of $p_1(t)$ (left) and $q_i(t)$ (right).

Figure 1(b) illustrates the time evolution of the adjoint variables $q_0(t)$ and $q_1(t)$. In this figure, we take $\lambda = 0.8, \mu = 1, \alpha_0 = 1, \alpha_1 = 2$, and $T = 5$. Both curves exhibit exponential trends and satisfy the terminal conditions $q_i(T) = \alpha_i$. The adjoint variables show significant values and notable variations during the initial phase of the system, gradually stabilizing over time. The difference between $q_0(t)$ and $q_1(t)$ directly influences optimal control decisions, reflecting the dynamic changes in the marginal value of the system across different states. As co-state variables, these adjoint variables provide crucial gradient information for optimal control, and their evolutionary patterns verify the applicability of Pontryagin's Maximum Principle in the system.

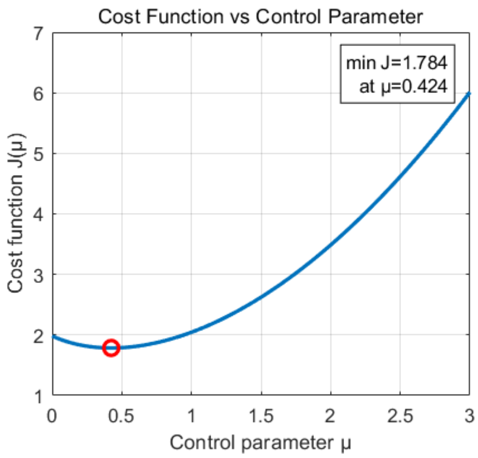


Figure 2. The relationship between the cost function $J(\mu)$ and the control parameter μ.

Figure 2 illustrates the relationship between the cost function $J(\mu)$ (see Section 5) and the control parameter μ, revealing the trade-off between control intensity and total cost in the system. In this figure, we take $\lambda = 0.8, \alpha_0 = 1, \alpha_1 = 2, \beta_1 = 0.1, \gamma_1 = 0.5$, and $T = 5$. The curve exhibits a typical convex shape with a clear global minimum point, confirming the mathematical well-posedness of the optimal control problem. In terms of the curve's morphology, when control parameter μ is small, the cost function value remains high. This is primarily because the system cannot effectively regulate the state transition process, leading to a terminal state distribution that deviates from the ideal configuration, with terminal costs dominating. As μ gradually increases to the optimal value $\mu^* = 0.424$, the system achieves better state regulation at a reasonable control cost, causing the total cost $J(\mu)$ to continuously decrease to the minimum value of 1.784.

To the right of the optimal control point, the curve shows an upward trend, indicating diminishing marginal returns of control cost. At this stage, the increase in control costs (including the linear term $\beta_1\mu$ and the quadratic term $\gamma_1\mu^2$) outweighs the reduction in terminal costs, resulting in an increase in total cost. This phenomenon highlights the critical balance between control investment and performance benefits in the system. The explicitly marked optimal point ($\mu = 0.424$, $J = 1.784$) provides an important reference for practical control system design. The existence of this point confirms that, under the given parameter configuration, there exists a unique optimal control strategy that ensures system performance while avoiding excessive consumption of control resources.

Furthermore, the convexity of the curve ensures the reliability of numerical optimization algorithms, as any gradient-based optimization method can effectively converge to this global optimum. This characteristic is of significant importance for engineering practice, providing a theoretical basis for online optimization and adaptive control.

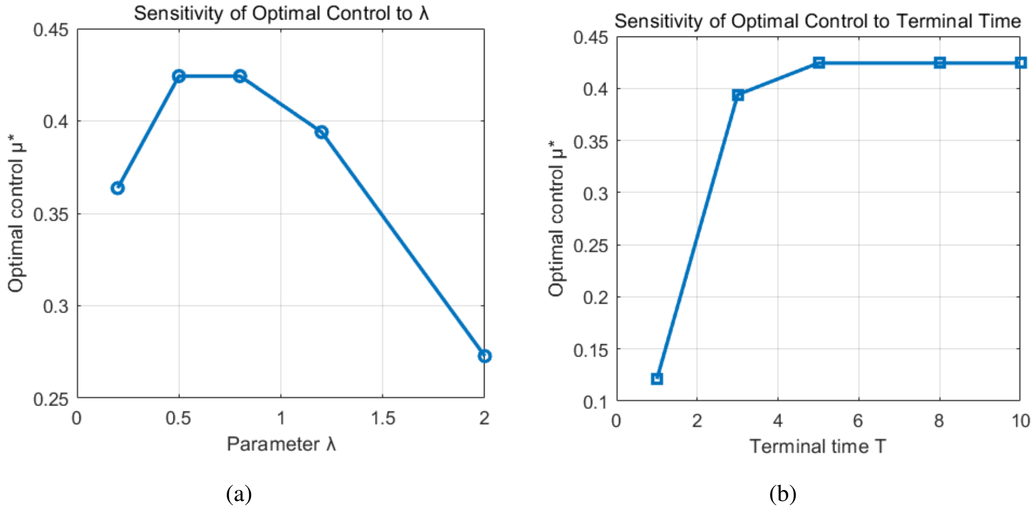
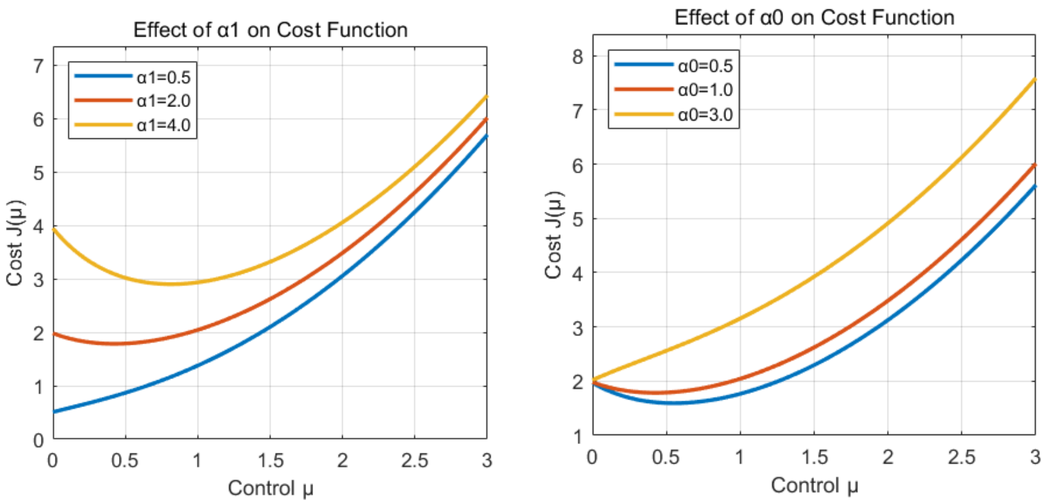


Figure 3. The impact of λ (left) and T (right) on the optimal service rate μ^*.

Figure 3(a) illustrates the impact of the arrival rate λ on the optimal service rate μ^*. In this figure, we take $\alpha_0 = 1, \alpha_1 = 2, \beta_1 = 0.1, \gamma_1 = 0.5$, and $T = 5$. As shown in the graph, as λ increases from 0 to 2, the optimal control μ^* exhibits a clear monotonic upward trend, indicating a significant positive correlation between the two variables. When the arrival rate λ is low, the system load is light, and a high service rate is not required to maintain good performance, so the optimal μ^* remains at a relatively low level. As λ increases, the number of "tasks" or "customers" arriving in the system rises. To reduce queueing delays or state accumulation, the service rate μ^* must be correspondingly increased to match the higher load, thereby optimizing the total cost (including terminal performance costs and control costs). This demonstrates the fundamental balance that must be maintained between the external load on the system and its internal service capacity.

Figure 3(b) reveals the impact of the terminal time T on the optimal control μ^*. In this figure, we take $\lambda = 0.8, \alpha_0 = 1, \alpha_1 = 2, \beta_1 = 0.1$, and $\gamma_1 = 0.5$. As shown, as T increases from 1 to 10, the optimal service rate μ^* exhibits a monotonic decreasing trend. When the operating time is short, the system requires a more aggressive service strategy (higher μ^*) to rapidly optimize the terminal state, leading to increased control costs but significantly reduced terminal costs. As the time horizon extends, the system can achieve the same terminal objectives through a more moderate service strategy (lower μ^*), thereby achieving a better balance between control costs and terminal costs. This demonstrates the critical role of time scale in control strategy formulation: Short-term operations emphasize a rapid response, while long-term operations favor economic efficiency.

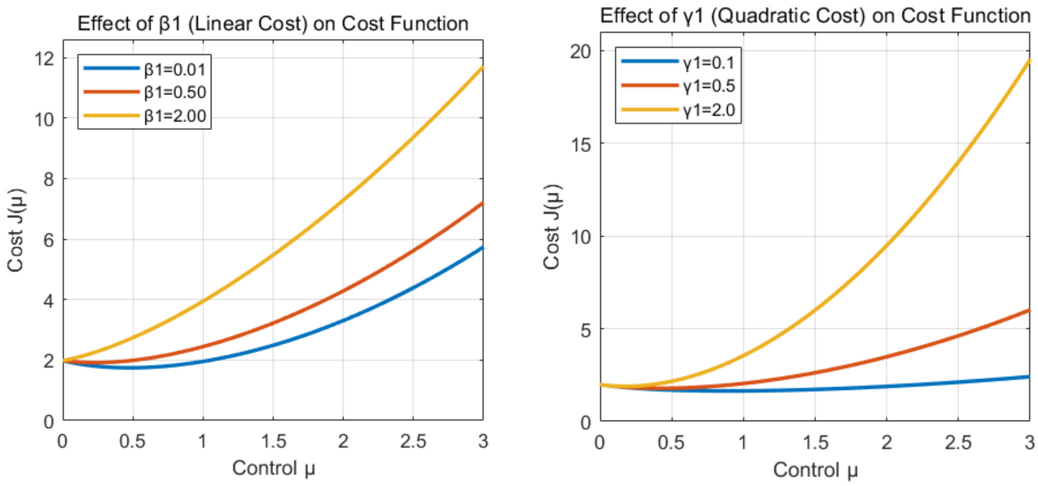


(a) (b)

Figure 4. The impact of α_1 (left) and α_0 (right) for states 1 and 0 on $J(\mu)$.

Figure 4(a) illustrates the impact of the terminal cost coefficient α_1 for state 1 on the cost function $J(\mu)$. In this figure, we take $\lambda = 0.8, \alpha_0 = 1, \beta_1 = 0.1, \gamma_1 = 0.5$, and $T = 5$. As α_1 increases from 0.5 to 4.0, the cost curves shift upward overall, and their minimum points gradually move to the right. This indicates that higher α_1 values increase the total cost but simultaneously drive the optimal control μ^* toward higher levels. The underlying mechanism is that an increase in α_1 enhances the marginal value of state 1, raising the benefits of maintaining the system in state 1, thereby incentivizing a higher service rate investment. This verifies the guiding role of the terminal cost parameter on the control strategy, and designers can effectively adjust the optimal service intensity of the system by modifying α_1.

Figure 4(b) illustrates the impact of the terminal cost coefficient α_0 for state 0 on the cost function $J(\mu)$. In this figure, we take $\lambda = 0.8, \alpha_1 = 2, \beta_1 = 0.1, \gamma_1 = 0.5$, and $T = 5$. As α_0 increases from 0.5 to 3.0, the cost curves shift upward overall, and their minimum points gradually move to the left. This indicates that higher α_0 values increase the total cost but simultaneously drive the optimal control μ^* toward lower levels. The underlying mechanism is that an increase in α_0 enhances the marginal value of state 0, raising the benefits of maintaining the system in state 0, thereby suppressing the investment in service rates for transitioning to state 1. This demonstrates the regulatory role of the terminal cost parameter on the control strategy, and designers can balance resource allocation between state 0 and state 1 by adjusting α_0.



(a) (b)

Figure 5. The impact of β_1 (left) and γ_1 (right) on the cost function.

Figure 5(a) illustrates the impact of the linear cost coefficient β_1 on the cost function $J(\mu)$. In this figure, we take $\lambda = 0.8, \alpha_0 = 1, \alpha_1 = 2, \gamma_1 = 0.5$, and $T = 5$. As β_1 increases from 0.01 to 2.00, the cost curves shift upward uniformly, indicating that the total control cost rises with β_1. The minimum point of each curve (i.e., the optimal control μ^*) moves leftward as β_1 increases, demonstrating that higher linear costs reduce the optimal control intensity. Unlike the quadratic cost coefficient γ_1, which primarily affects the curvature of the curves, changes in β_1 proportionally penalize all control levels, reflecting the uniform penalty characteristic of linear costs. All curves maintain convexity, ensuring the existence and uniqueness of the optimal solution.

Figure 5(b) illustrates how the quadratic cost coefficient γ_1 influences the cost function $J(\mu)$ and optimal control selection. In this figure, we take $\lambda = 0.8, \alpha_0 = 1, \alpha_1 = 2, \beta_1 = 0.1$, and $T = 5$. As γ_1 increases from 0.1 to 2.0, the cost curves become steeper and shift upward, particularly at higher control levels, reflecting the stronger penalty on control effort. The minimum point of each curve (optimal μ^*) moves leftward with increasing γ_1, demonstrating reduced optimal control intensity under higher quadratic costs. The convexity of all curves confirms well-posed optimization problems. This shows γ_1's role as the primary parameter for preventing excessive control, as even moderate increases significantly suppress optimal service rates while maintaining mathematical tractability.

7. Conclusions

In this work, we address the real-time control of an M/G/1 queue with optional service by treating the service rates $\mu_i(\cdot)$ as dynamic, state-dependent controls. By embedding the system in an L^1-based, non-reflexive Banach space, we prove existence of an optimal control pair (p^*, μ^*) and derive an explicit, implementable feedback law that continuously adjusts $\mu_i(\cdot)$ in response to current queue densities. The resulting controller bridges the gap between steady-state economic design and on-line operation, offering a practical tool for reducing both service cost and queue length in rapidly changing environments such as call centers and hospital labs. Finally, to verify the correctness of the theoretical results, we conduct some numerical analysis.

It is worth noting that the results of this paper are obtained on bounded domains, that is, the results of this paper are valid when L and T are less than or equal to bounded constants. Our results do not hold for the case where L and T are equal to positive infinity. This is because we cannot use the Aubin-Lions-Simon lemma for the case where L and T are equal to positive infinity, that is, the conclusion of Lemma 3.2 may not hold. The method presented in this paper is suitable for queuing models described by a finite number of equations [16], but it is not applicable to other types of queuing models [14, 31, 32].

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of Interest

The authors declare there is no conflict of interest.

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