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Research article

Frequency-dependent damping in the linear wave equation

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Abstract: We propose a model for frequency-dependent damping in the linear wave equation. After proving well-posedness of the problem, we study qualitative properties of the energy. In the one-dimensional case, we provide an explicit analysis for special choices of the damping operator. Finally, we show, in special cases, that solutions split into a dissipative and a conservative part.

Keywords: wave equation; frequency-dependent damping; well-posedness; energy decay
Mathematics Subject Classification: 35L05, 35L20, 35B40

1. Introduction

Sliding friction phenomena are nowadays recognized as the simultaneous occurrence of different physical mechanisms that escape the attempt to reduce them to a simple relation, like a friction law. In fact, entire areas of scientific research, such as Tribology, are specialized in the study of these problems [1–3].

From a mathematical (PDEs) perspective, a reasonable goal relies on detecting one or more key features that can enter into a rational model whose analysis reveals significant and new aspects of the phenomenon under examination. It was observed by various authors that in sliding friction, the occurrence of local instabilities emerging at different scales plays a decisive role in this physical manifestation [2]. Following this suggestion, we address here the simplest problem related to a linear wave equation in which a damping effect can occur only at certain spatial scales. This toy model is thought to investigate the effects of multiscale character in the wave propagation and the corresponding long-time behavior.

The reference model we have in mind is a variant of the linearly damped wave equation:

$$\begin{cases} \partial_{tt}u(t,x) - \Delta u(t,x) + \partial_t u(t,x) = 0\,, & (t,x) \in (0,+\infty)\times\Omega\,, \\ u(t,x) = 0\,, & (t,x) \in (0,+\infty)\times\partial\Omega\,, \\ u(0,x) = u_0(x)\,, \quad \partial_t u(0,x) = v_0(x)\,, & x \in \Omega\,. \end{cases} \tag{1.1}$$

It is well-known that the solution to this equation converges to zero (in H^1 norm) as $t \to +\infty$ exponentially fast [4, 5]. The damping term $\partial_t u$ in the equation is responsible for this decay. We are interested in studying a different scenario in which the damping term does not act on the whole function u but on suitable parts of it. In view of the above physical considerations, we could investigate the case in which damping regards a suitable frequency band of the solution. This is implemented by applying a linear operator P to the damping term in the equation, which then reads

$$\begin{cases} \partial_{tt}u(t,x) - \Delta u(t,x) + P[\partial_t u(t,\cdot)](x) = 0\,, & (t,x) \in (0,+\infty)\times\Omega\,, \\ u(t,x) = 0\,, & (t,x) \in (0,+\infty)\times\partial\Omega\,, \\ u(0,x) = u_0(x)\,, \quad \partial_t u(0,x) = v_0(x)\,, & x \in \Omega\,. \end{cases} \tag{1.2}$$

We illustrate the simplest model we have in mind, leaving more general assumptions on P to the content of this work. With the aim of capturing different behaviors at different spatial scales, one can consider the special case $\Omega = \mathbb{R}^d$, $d \geq 1$, and the operator P given by

$$\widehat{P[v]}(t,\xi) = \chi_{\{|\xi|>1\}}(\xi)\widehat{v}(t,\xi)\,, \quad \xi \in \mathbb{R}^d\,,$$

where $\widehat{v}(t,\xi)$ denotes the Fourier transform with respect to the spatial variable x and $\chi_{\{|\xi|>1\}}(\xi)$ is the characteristic function of the complement of the unit ball. In this special case, due to the linearity of the equation, the different behavior of the solution at different scales becomes evident by inspecting it in the frequency variable. Indeed, in such a case, the problem is split into

$$\begin{cases} \partial_{tt}\widehat{u}(t,\xi) + |\xi|^2\widehat{u}(t,\xi) + \partial_t\widehat{u}(t,\xi) = 0\,, & t \in (0,+\infty)\,,\ |\xi| > 1\,, \quad \text{(damped regime)} \\ \partial_{tt}\widehat{u}(t,\xi) + |\xi|^2\widehat{u}(t,\xi) = 0\,, & t \in (0,+\infty)\,,\ |\xi| \leq 1\,, \quad \text{(undamped regime)} \\ + \text{ initial conditions.} \end{cases} \tag{1.3}$$

These different regimes are reflected in the long-time behavior of the solution. Relying on the structure in Eq (1.3) exhibited in the case $\Omega = \mathbb{R}^d$, one expects in the case $\Omega \subset \mathbb{R}^d$ bounded that for high frequencies (*i.e.*, small spatial scales), the solution behaves like the one to Eq (1.1) and converges exponentially fast to zero (dissipative regime). In contrast, for low frequencies (*i.e.*, large spatial scales) the solution behaves like the one to the undamped wave equation and exhibits no decay (conservative regime). It is worth mentioning the recent results in [6] and the references therein for a detailed analysis of the decay in the case $\Omega = \mathbb{R}^d$ when χ is localized at high frequencies.

We describe the results obtained in this paper concerning the problem (1.2). In Section 2, under suitable assumptions on the operator P, we prove well-posedness of the problem (1.2) and we study qualitative decay properties of the energy. Section 3 is devoted to an explicit analysis for special choices of the operator P in the one-dimensional case. Finally, in Section 4, we show that, in special cases, solutions to problem (1.2) split into a dissipative and a conservative part.

Let us emphasize that the linear framework addressed in the present work constitutes the simplest setting in which we aim to investigate the role of the spatial scales in dissipative phenomena. We are aware that a correct scenario to grasp the very nature of this physics should include nonlinear terms. More complex behaviors may occur when damping is coupled with adhesion [7–9], peeling [10–14], and in more general nonlinear settings [15, 16]. We plan to address these issues in future works.

2. Description of the model and well-posedness

In this section we introduce the model, we prove well-posedness of the problem (1.2), and we study some qualitative properties of the energy.

2.1. The model

Let $d \geq 1$ and assume that

(Ω1) $\Omega \subset \mathbb{R}^d$ is a bounded, connected, open set with boundary of class C^2.*

We study the initial boundary value problem

$$\begin{cases} \partial_{tt}u(t,x) - \Delta u(t,x) + P[\partial_t u(t,\cdot)](x) = 0\,, & (t,x) \in (0,+\infty)\times\Omega\,, \\ u(t,x) = 0\,, & (t,x) \in (0,+\infty)\times\partial\Omega\,, \\ u(0,x) = u_0(x)\,, \quad \partial_t u(0,x) = v_0(x)\,, & x \in \Omega\,, \end{cases} \tag{2.1}$$

where $P\colon L^2(\Omega) \to L^2(\Omega)$ satisfies the following properties:

(P1) P is a bounded linear operator;

(P2) P is monotone, *i.e.*, $\langle P[u], u\rangle_{L^2(\Omega)} \geq 0$ for all $u \in L^2(\Omega)$.

2.2. The operator P

Some general comments on the properties (P1) and (P2) of the operator P are in order. For more explicit examples of operators P satisfying the properties (P1) and (P2), we refer to Section 3.

Remark 2.1. If $P = \mathrm{Id}_{L^2(\Omega)}$, we recover the classical wave equation with damping, *i.e.*, Eq (2.1) reduces to

$$\begin{cases} \partial_{tt}u(t,x) - \Delta u(t,x) + \partial_t u(t,x) = 0\,, & (t,x) \in (0,+\infty)\times\Omega\,, \\ u(t,x) = 0\,, & (t,x) \in (0,+\infty)\times\partial\Omega\,, \\ u(0,x) = u_0(x)\,, \quad \partial_t u(0,x) = v_0(x)\,, & x \in \Omega\,. \end{cases} \tag{2.2}$$

To explain in which sense the term $P[\partial_t u(t,\cdot)]$ may model a frequency-dependent damping, we point out the following example.

Example 2.2. If $P\colon L^2(\Omega) \to L^2(\Omega)$ is an orthogonal projection, then the properties (P1) and (P2) are satisfied. Indeed, orthogonal projections in a Hilbert space are characterized as linear self-adjoint idempotent operators. Continuity follows from the fact that

$$\|P[v]\|^2_{L^2(\Omega)} = \langle P[v], P[v]\rangle_{L^2(\Omega)} = \langle P^2[v], v\rangle_{L^2(\Omega)} = \langle P[v], v\rangle_{L^2(\Omega)} \leq \|P[v]\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}\,.$$

*These assumptions are enough to apply Poincaré's inequality in $H^1_0(\Omega)$ and the elliptic regularity theory up to the boundary.

From the inequality above, we also obtain that P is monotone, since

$$\langle P[v], v\rangle_{L^2(\Omega)} = \|P[v]\|^2_{L^2(\Omega)} \geq 0\,.$$

If an orthogonal projection is interpreted as a filter that selects some of the frequencies in the wave $u(t,\cdot)$, then the term $P[\partial_t u(t,\cdot)]$ can be seen as a damping term acting on those selected frequencies. For more details on this interpretation, see Section 3.

Remark 2.3. Since P is acting only on the spatial variable, it commutes with the differentiation with respect to time. We explain this fact in a more rigorous way. Given $\varphi \in C^\infty_c(\mathbb{R}\times\Omega)$, we have that $t \mapsto P[\varphi]$ belongs to $C^\infty(\mathbb{R}; L^2(\Omega))$ and

$$\partial_t P[\varphi(t,\cdot)](x) = P[\partial_t \varphi(t,\cdot)](x)\,, \quad \text{for all } t\in\mathbb{R}\,,\ x\in\Omega\,.$$

This follows from the fact that, by linearity of P, for all $t \in \mathbb{R}$, $h \neq 0$, $x \in \Omega$, we have that

$$\frac{P[\varphi(t+h,\cdot)](x) - P[\varphi(t,\cdot)](x)}{h} = P\Big[\frac{\varphi(t+h,\cdot)-\varphi(t,\cdot)}{h}\Big](x)\,,$$

Exploiting boundedness of P and passing to the limit as $h \to 0$, we deduce that $t \in \mathbb{R} \mapsto P[\varphi(t,\cdot)] \in L^2(\Omega)$ is differentiable and we obtain the claimed identity (2.3). We iterate the argument for higher-order derivatives to conclude that $t \mapsto P[\varphi(t,\cdot)]$ belongs to $C^\infty(\mathbb{R}; L^2(\Omega))$.

2.3. Well-posedness

We recast Eq (2.1) as an ODE in a Hilbert space. By writing Eq (2.1) as the first-order system

$$\begin{cases} \partial_t u(t,x) - v(t,x) = 0\,, & (t,x)\in(0,+\infty)\times\Omega\,,\\ \partial_t v(t,x) - \Delta u(t,x) + P[v(t,\cdot)](x) = 0\,, & (t,x)\in(0,+\infty)\times\Omega\,,\\ u(t,x) = 0\,, & (t,x)\in(0,+\infty)\times\partial\Omega\,,\\ u(0,x) = u_0(x)\,, \quad v(0,x) = v_0(x)\,, & x\in\Omega\,, \end{cases}$$

then Eq (2.1) can be interpreted as the Cauchy problem for curves $U\colon [0,+\infty) \to H$

$$\begin{cases} \dfrac{\mathrm{d}U}{\mathrm{d}t}(t) + \mathscr{A}U(t) = 0\,, \quad t\in(0,+\infty)\,,\\ U(0) = U_0\,, \end{cases} \tag{2.3}$$

where

- $U(t) = (u(t), v(t))$ and $U_0 = (u_0, v_0)$;
- $H = H^1_0(\Omega)\times L^2(\Omega)$ is the Hilbert space endowed with the scalar product (we are exploiting Poincaré 's inequality)

$$\langle (u,v),(w,z)\rangle_H = \langle \nabla u, \nabla w\rangle_{L^2(\Omega)} + \langle v, z\rangle_{L^2(\Omega)}\,;$$

- the linear operator $\mathscr{A}\colon \mathrm{Dom}(\mathscr{A}) \to H$ is defined by

$$\mathscr{A}(u,v) = (-v, -\Delta u + P[v])\,, \quad (u,v)\in\mathrm{Dom}(\mathscr{A})\,;$$

- the domain $\mathrm{Dom}(\mathscr{A})$ is given by

$$\mathrm{Dom}(\mathscr{A}) = \{(u, v) \in H \,:\, u \in H^2(\Omega) \cap H_0^1(\Omega)\,,\ v \in H_0^1(\Omega)\}\,.$$

We start with the key result that guarantees the well-posedness of the problem, relying on the theory of maximal monotone operators [17].

Lemma 2.4. *Assume (Ω1) and (P1)–(P2). Then the operator* $\mathscr{A}\colon \mathrm{Dom}(\mathscr{A}) \to H$ *is maximal monotone.*

Proof. We have to show the following properties:

(i) $\mathscr{A}$ is monotone, *i.e.*, for all $(u, v), (w, z) \in \mathrm{Dom}(\mathscr{A})$, we have

$$\langle \mathscr{A}(u, v) - \mathscr{A}(w, z), (u - w, v - z)\rangle_H \geq 0\,;$$

(ii) $\mathscr{A}$ is maximal monotone, which is equivalent to saying that $\mathscr{A} + \lambda \mathrm{Id}_H$ is surjective for some (and hence all) $\lambda > 0$.

Let us prove *(i)*. By linearity of $\mathscr{A}$, it is enough to show that $\mathscr{A}$ satisfies

$$\langle \mathscr{A}(u, v), (u, v)\rangle_H \geq 0\,, \quad (u, v) \in \mathrm{Dom}(\mathscr{A})\,.$$

Let $(u, v) \in \mathrm{Dom}(\mathscr{A})$. Integrating by parts and by $(P2)$, we have that

$$\begin{aligned}\langle \mathscr{A}(u, v), (u, v)\rangle_H &= \langle (-v, -\Delta u + P[v] + u), (u, v)\rangle_H \\ &= -\langle \nabla v, \nabla u\rangle_{L^2(\Omega)} + \langle -\Delta u, v\rangle_{L^2(\Omega)} + \langle P[v], v\rangle_{L^2(\Omega)} \\ &= -\langle \nabla v, \nabla u\rangle_{L^2(\Omega)} + \langle \nabla u, \nabla v\rangle_{L^2(\Omega)} + \langle P[v], v\rangle_{L^2(\Omega)} = \langle P[v], v\rangle_{L^2(\Omega)} \geq 0\,.\end{aligned}$$

Let us prove *(ii)*. We show that $\mathscr{A} + \mathrm{Id}_H$ is surjective. Let $(f, g) \in H$. We show that it is possible to find $(u, v) \in \mathrm{Dom}(\mathscr{A})$ such that

$$\begin{cases} -v + u = f\,, \\ -\Delta u + P[v] + v = g\,. \end{cases}$$

Indeed, the first equation gives $v = -f + u$. We substitute in the second equation to obtain that

$$-\Delta u + P[-f + u] + u - f = g \implies -\Delta u + P[u] + u = f + P[f] + g\,.$$

The problem then reduces to finding $u \in H^2(\Omega)$ such that

$$\begin{cases} -\Delta u + P[u] + u = h\,, & \text{in } \Omega\,, \\ u = 0\,, & \text{on } \partial\Omega\,, \end{cases} \tag{2.4}$$

where $h = f + P[f] + g \in L^2(\Omega)$. This problem has a unique solution $u \in H_0^1(\Omega)$ by the Lax-Milgram theorem. Indeed, the bilinear form

$$b(u, u') = \langle \nabla u, \nabla u'\rangle_{L^2(\Omega)} + \langle u, u'\rangle_{L^2(\Omega)} + \langle P[u], u'\rangle_{L^2(\Omega)}\,, \quad u, u' \in H_0^1(\Omega)\,,$$

satisfies

$$|b(u,u')| \le C\|u\|_{H_0^1(\Omega)}\|u'\|_{H_0^1(\Omega)}, \quad u,u' \in H_0^1(\Omega),$$

by the continuity of P and Poincaré's inequality. Moreover, it is coercive, since

$$b(u,u) = \|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} + \langle P[u],u\rangle_{L^2(\Omega)} \ge \|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} \ge c\|u\|^2_{H_0^1(\Omega)},$$

by ($P2$) and Poincaré's inequality.

Then we read Eq (2.4) as

$$\begin{cases} -\Delta u = h - P[u] - u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

with the right-hand side $h - P[u] - u \in L^2(\Omega)$. By the elliptic regularity theory, we conclude that $u \in H^2(\Omega) \cap H_0^1(\Omega)$. This concludes the proof.

Thanks to the maximal monotonicity of $\mathscr{A}$, we can apply the theory of maximal monotone operators to guarantee the well-posedness of the Cauchy problem (2.3). We have the following classical result.

Theorem 2.5. *Assume (Ω1) and (P1) and (P2). Then the operator $\mathscr{A}$ generates a continuous semigroup $\{S(t)\}_{t\ge0}$ on H, i.e., for all $t \ge 0$ the operator $S(t)\colon H \to H$ is linear with $\|S(t)\| \le 1$, $S(0) = \mathrm{Id}_H$, $S(t+s) = S(t)S(s)$ for all $t,s \ge 0$, and $\lim_{t\to0}\|S(t)U_0 - U_0\|_H = 0$ for all $U_0 \in H$. Moreover, if $U_0 \in \mathrm{Dom}(\mathscr{A})$, then $U(t) = S(t)U_0$ belongs to $C^1([0,+\infty);H) \cap C([0,+\infty);\mathrm{Dom}(\mathscr{A}))$ and is the unique solution to Eq* (2.3).

Proof. The result follows from the theory of maximal monotone operators; see, *e.g.*, [17, Theorem I.2.2.1 & Remark I.2.2.3].

2.4. Energy-dissipation balance

In this subsection we show that the energy of the solution to Eq (2.1) is non-increasing in time, and we provide an explicit expression for the energy dissipated by the damping term. Given $(u,v) \in H_0^1(\Omega) \times L^2(\Omega)$, we define the energy functional by

$$E(u,v) = \frac{1}{2}\int_\Omega |v(x)|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla u(x)|^2\,\mathrm{d}x.$$

Theorem 2.6. *Assume (Ω1) and (P1) and (P2). Let $U_0 = (u_0,v_0) \in H_0^1(\Omega) \times L^2(\Omega)$. Let $U(t) = (u(t),v(t))$ be the unique solution to Eq* (2.3) *with initial datum U_0 provided by Theorem 2.5. Then the following energy-dissipation balance holds:*

$$E(u(t),v(t)) + \int_0^t \langle P[v(s)],v(s)\rangle_{L^2(\Omega)}\,\mathrm{d}s = E(u_0,v_0), \quad t \in [0,+\infty). \tag{2.5}$$

In particular, the energy of the solution is non-increasing in time, i.e.,

$$E(u(s),v(s)) \ge E(u(t),v(t)), \quad 0 \le s \le t.$$

Proof. Let us start by proving Eq (2.5) in the case $U_0 = (u_0, v_0) \in \mathrm{Dom}(\mathscr{A})$. In this case, by Theorem 2.5, the solution $U(t) = (u(t), v(t))$ belongs to $C^1([0,+\infty); H) \cap C([0,+\infty); \mathrm{Dom}(\mathscr{A}))$. It follows that $t \mapsto E(u(t), v(t))$ is differentiable. Differentiating the energy functional, substituting the equation in (2.1), and integrating by parts, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(u(t),v(t)) = \frac{\mathrm{d}}{\mathrm{d}t}E(u(t),\partial_t u(t)) &= \langle \partial_t u(t), \partial_{tt} u(t)\rangle_{L^2(\Omega)} + \langle \nabla u(t), \nabla \partial_t u(t)\rangle_{L^2(\Omega)} \\ &= \langle \partial_t u(t), \Delta u(t) - P[\partial_t u(t)]\rangle_{L^2(\Omega)} - \langle \Delta u(t), \partial_t u(t)\rangle_{L^2(\Omega)} \\ &= -\langle P[\partial_t u(t)], \partial_t u(t)\rangle_{L^2(\Omega)}\,.\end{aligned}$$

Integrating in time, we obtain Eq (2.5).

Assuming that $U_0 = (u_0, v_0) \in H^1_0(\Omega) \times L^2(\Omega)$, we approximate it by a sequence $U^n_0 = (u^n_0, v^n_0) \in \mathrm{Dom}(\mathscr{A})$ such that $U^n_0 \to U_0$ in H as $n \to +\infty$. By the continuity of the semigroup obtained in Theorem 2.5, we have that $U^n(t) = (u^n(t), v^n(t)) = S(t)U^n_0 \to U(t) = (u(t), v(t)) = S(t)U_0$ in H as $n \to +\infty$, for all $t \geq 0$. The energy-dissipation balance Eq (2.5) holds for U^n_0 and $U^n(t)$, *i.e.*,

$$E(u^n(t), v^n(t)) + \int_0^t \langle P[v^n(s)], v^n(s)\rangle_{L^2(\Omega)}\,\mathrm{d}s = E(u^n_0, v^n_0)\,, \quad t \in [0,+\infty)\,.$$

By the continuity of the energy functional with respect to the norm of H, we pass to the limit as $n \to +\infty$ in the above equation to obtain Eq (2.5) for U_0 and $U(t)$.

Finally, to show that the energy is non-increasing in time, we observe that

$$E(u(t), v(t)) + \int_s^t \langle P[v(\tau)], v(\tau)\rangle_{L^2(\Omega)}\,\mathrm{d}\tau = E(u(s), v(s))\,, \quad 0 \leq s \leq t\,,$$

and, by the monotonicity of P, we have that $\langle P[v(\tau)], v(\tau)\rangle_{L^2(\Omega)} \geq 0$ for all $\tau \geq 0$. This concludes the proof.

Remark 2.7. If $P = \mathrm{Id}_{L^2(\Omega)}$, Eq (2.1) reduces to the damped wave Eq (2.2); see Remark 2.1. A classical result states that the energy of the solution to the damped wave equation decays exponentially fast to zero. We recall the proof of this result, which relies on the perturbed energy functional

$$E_\lambda(u, v) = \frac{1}{2}\int_\Omega |v(x)|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla u(x)|^2\,\mathrm{d}x + \lambda\int_\Omega u(x)v(x)\,\mathrm{d}x\,,$$

where $\lambda > 0$ is a parameter to be chosen. Assuming that the initial datum $U_0 = (u_0, v_0)$ belongs to $\mathrm{Dom}(\mathscr{A})$, we have that the solution $U(t) = (u(t), v(t))$ provided by Theorem 2.5 belongs to $C^1([0,+\infty); H) \cap C([0,+\infty); \mathrm{Dom}(\mathscr{A}))$. We can differentiate the perturbed energy functional and substitute the equation to obtain that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_\lambda(u(t), v(t)) &= -\|v(t)\|^2_{L^2(\Omega)} + \lambda\|v(t)\|^2_{L^2(\Omega)} + \lambda\langle u(t), \Delta u(t)\rangle_{L^2(\Omega)} - \lambda\langle u(t), v(t)\rangle_{L^2(\Omega)} \\ &= -(1-\lambda)\|v(t)\|^2_{L^2(\Omega)} - \lambda\|\nabla u(t)\|^2_{L^2(\Omega)} - \lambda\langle u(t), v(t)\rangle_{L^2(\Omega)}\,.\end{aligned}$$

Choosing $0 < \lambda < \frac{1}{2}$, we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}E_\lambda(u(t), v(t)) \leq -C_\lambda E_\lambda(u(t), v(t))\,,$$

for a suitable $C_\lambda > 0$, which implies that

$$E_\lambda(u(t), v(t)) \leq E_\lambda(u_0, v_0)e^{-\gamma t},$$

with $\gamma > 0$ depending on λ. Then we estimate, by Poincaré's inequality, that

$$\left|\langle u(t), v(t)\rangle_{L^2(\Omega)}\right| \leq \frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|v(t)\|^2_{L^2(\Omega)} \leq C\|\nabla u(t)\|_{L^2(\Omega)} + \frac{1}{2}\|v(t)\|^2_{L^2(\Omega)},$$

where $C > 0$ depends on the Poincaré constant of Ω. By choosing λ suitably small, we obtain the estimate

$$\frac{1}{C}E_\lambda(u(t), v(t)) \leq E(u(t), v(t)) \leq CE_\lambda(u(t), v(t)),$$

which implies that

$$E(u(t), v(t)) \leq CE(u_0, v_0)e^{-\gamma t}.$$

The estimate above is extended to the case $U_0 = (u_0, v_0) \in H^1_0(\Omega) \times L^2(\Omega)$ by approximation. This energy decay implies that the solution to the damped wave equation converges exponentially fast to zero in the energy space.

Note that the constant γ appearing in the exponential decay estimate depends on the domain Ω (through the Poincaré constant) and does not depend on the initial datum U_0.

3. Explicit solution in the one-dimensional case for filtered damping

In this section we provide an explicit solution to the problem (2.1) in the one-dimensional case, *i.e.*, $\Omega = (0, L)$, assuming a specific structure for the operator P. This will allow us to understand the asymptotic behavior of the solution in some specific cases.

3.1. Formulation of the problem in one dimension

We aim to solve the problem

$$\begin{cases} \partial_{tt}u - \partial_{xx}u + P[\partial_t u(t,\cdot)](x) = 0, & (t, x) \in (0, +\infty) \times (0, L), \\ u(t, 0) = u(t, L) = 0, & t \in [0, +\infty), \\ u(0, x) = u_0(x), \quad u_t(0, x) = v_0(x) & x \in (0, L), \end{cases} \tag{3.1}$$

where $u_0 \in H^1_0(0, L)$ and $v_0 \in L^2(0, L)$.

3.2. Fourier series notation

To solve the problem explicitly, we will regard solutions as L-periodic functions and use Fourier series.[†] We write

$$u_0(x) = \sum_{k\in\mathbb{Z}} \widehat{u}_k(0)e^{2\pi ikx/L}, \quad v_0(x) = \sum_{k\in\mathbb{Z}} \widehat{v}_k(0)e^{2\pi ikx/L}, \quad u(t, x) = \sum_{k\in\mathbb{Z}} \widehat{u}_k(t)e^{2\pi ikx/L},$$

[†]Because of the Dirichlet boundary conditions, one could use a different orthonormal system, *e.g.*, writing $u_0(x) = \sum_{n\in\mathbb{N}} a_n \sin\left(\frac{n\pi x}{L}\right)$. We prefer to use Fourier series to make the computations easier to follow.

where the Fourier coefficients are given by

$$\widehat{u}_k(t) = \frac{1}{L}\int_0^L u(t,x)e^{-2\pi ikx/L}\,\mathrm{d}x\,, \quad k \in \mathbb{Z}\,,$$

and analogously for u_0 and v_0. Since u_0 and v_0 are real-valued, we have that

$$\widehat{u}_{-k}(0) = \overline{\widehat{u}_k(0)}\,, \quad \widehat{v}_{-k}(0) = \overline{\widehat{v}_k(0)}\,.$$

3.3. Assumption on the operator P

We analyze a specific form for the damping term $P[\partial_t u(t,\cdot)]$, assuming that P acts as a frequency filter. More precisely, we assume that

($P3$) there exists a bounded sequence $(\widehat{\phi}_k)_{k\in\mathbb{Z}}$ with $\widehat{\phi}_k \geq 0$ for all $k \in \mathbb{Z}$ such that

$$(\widehat{P[v]})_k = \widehat{\phi}_k \widehat{v}_k\,.$$

Remark 3.1. If P satisfies ($P3$), by Parseval's identity, we have that

$$\|P[v]\|^2_{L^2(0,L)} = \sum_{k\in\mathbb{Z}} |\widehat{\phi}_k \widehat{v}_k|^2 \leq \sup_{k\in\mathbb{Z}} |\widehat{\phi}_k|^2 \|v\|^2_{L^2(0,L)}\,,$$

whence ($P1$) holds. Moreover,

$$\langle P[v], v\rangle_{L^2(0,L)} = \sum_{k\in\mathbb{Z}} \widehat{\phi}_k \widehat{v}_k^2 \geq 0\,,$$

i.e., ($P2$) holds.

Example 3.2. If P is obtained via a convolution with an L^1 function, *i.e.*,

$$P[v](x) = \phi * v(x) = \frac{1}{L}\int_0^L \phi(x-y)v(y)\,\mathrm{d}y\,,$$

with $\phi \in L^1(0,L)$ extended periodically, then ($P3$) is satisfied. Indeed, we have that

$$(\widehat{P[v]})_k = (\widehat{\phi * v})_k = \widehat{\phi}_k \widehat{v}_k\,.$$

3.4. Solving the PDE with Fourier series

Taking the Fourier series in the equation in (3.1), we obtain that the Fourier coefficients $\widehat{u}_k(t)$ satisfy the following system of ordinary differential equations:

$$\partial_{tt}\widehat{u}_k(t) + (\tfrac{2\pi k}{L})^2 \widehat{u}_k(t) + \widehat{\phi}_k \partial_t \widehat{u}_k(t) = 0\,, \quad k \in \mathbb{Z}\,. \tag{3.2}$$

This can be solved explicitly. The characteristic equation is

$$\lambda^2 + \widehat{\phi}_k \lambda + (\tfrac{2\pi k}{L})^2 = 0\,,$$

and its solutions are, possibly counted with multiplicity,

$$\lambda_k^{\pm} = \frac{-\widehat{\phi}_k \pm \sqrt{\widehat{\phi}_k^2 - \left(\frac{4\pi k}{L}\right)^2}}{2}. \tag{3.3}$$

Case 1: $\widehat{\phi}_k^2 - \left(\frac{4\pi k}{L}\right)^2 > 0$ (Overdamping). The roots obtained in Eq (3.3) are real and distinct. The general solution of Eq (3.2) is

$$\widehat{u}_k(t) = A_k e^{t\lambda_k^+} + B_k e^{t\lambda_k^-}, \tag{3.4}$$

where A_k and B_k are complex constants that depend on the initial data. To find them, we impose

$$\begin{cases} A_k + B_k = \widehat{u}_k(0), \\ A_k\lambda_k^+ + B_k\lambda_k^- = \widehat{v}_k(0). \end{cases}$$

Solving this system, we obtain

$$A_k = -\frac{\widehat{u}_k(0)\lambda_k^- - \widehat{v}_k(0)}{\lambda_k^+ - \lambda_k^-}, \quad B_k = \frac{\widehat{u}_k(0)\lambda_k^+ - \widehat{v}_k(0)}{\lambda_k^+ - \lambda_k^-}.$$

Substituting in Eq (3.4), we obtain

$$\begin{aligned} \widehat{u}_k(t) &= -\frac{\widehat{u}_k(0)\lambda_k^- - \widehat{v}_k(0)}{\lambda_k^+ - \lambda_k^-} e^{t\lambda_k^+} + \frac{\widehat{u}_k(0)\lambda_k^+ - \widehat{v}_k(0)}{\lambda_k^+ - \lambda_k^-} e^{t\lambda_k^-} \\ &= \widehat{u}_k(0)\frac{-\lambda_k^- e^{t\lambda_k^+} + \lambda_k^+ e^{t\lambda_k^-}}{\lambda_k^+ - \lambda_k^-} + \widehat{v}_k(0)\frac{e^{t\lambda_k^+} - e^{t\lambda_k^-}}{\lambda_k^+ - \lambda_k^-} \\ &= \widehat{u}_k(0)\frac{(\frac{1}{2}\widehat{\phi}_k + \frac{1}{2}\sqrt{\widehat{\phi}_k^2 - (\frac{4\pi k}{L})^2})e^{t\lambda_k^+} + (-\frac{1}{2}\widehat{\phi}_k + \frac{1}{2}\sqrt{\widehat{\phi}_k^2 - (\frac{4\pi k}{L})^2})e^{t\lambda_k^-}}{\sqrt{\widehat{\phi}_k^2 - (\frac{4\pi k}{L})^2}} \\ &\quad + \widehat{v}_k(0)\frac{e^{t\lambda_k^+} - e^{t\lambda_k^-}}{\sqrt{\widehat{\phi}_k^2 - (\frac{4\pi k}{L})^2}} \\ &= \widehat{u}_k(0)\frac{e^{t\lambda_k^+} + e^{t\lambda_k^-}}{2} + \left(\widehat{\phi}_k\widehat{u}_k(0) + 2\widehat{v}_k(0)\right)\frac{1}{\sqrt{\widehat{\phi}_k^2 - (\frac{4\pi k}{L})^2}}\frac{e^{t\lambda_k^+} - e^{t\lambda_k^-}}{2}. \end{aligned}$$

Writing the explicit expression of $\lambda_k^{\pm}$, we obtain

$$\begin{aligned} \widehat{u}_k(t) = e^{-t\frac{\widehat{\phi}_k}{2}}\Big[&\cosh\left(\tfrac{t}{2}\sqrt{\widehat{\phi}_k^2 - (\tfrac{4\pi k}{L})^2}\right)\widehat{u}_k(0) \\ &+ \frac{1}{\frac{1}{2}\sqrt{\widehat{\phi}_k^2 - (\frac{4\pi k}{L})^2}}\sinh\left(\tfrac{t}{2}\sqrt{\widehat{\phi}_k^2 - (\tfrac{4\pi k}{L})^2}\right)\left(\tfrac{\widehat{\phi}_k}{2}\widehat{u}_k(0) + \widehat{v}_k(0)\right)\Big]. \end{aligned}$$

Case 2: $\widehat{\phi}_k^2 - \left(\frac{4\pi k}{L}\right)^2 < 0$ (Damped oscillations). The roots obtained in Eq (3.3) are complex. The algebra to obtain the solution is the same as in the previous case. We obtain

$$\begin{aligned} \widehat{u}_k(t) = e^{-t\frac{\widehat{\phi}_k}{2}}\Big[&\cos\left(\tfrac{t}{2}\sqrt{(\tfrac{4\pi k}{L})^2 - \widehat{\phi}_k^2}\right)\widehat{u}_k(0) \\ &+ \frac{1}{\frac{1}{2}\sqrt{(\frac{4\pi k}{L})^2 - \widehat{\phi}_k^2}}\sin\left(\tfrac{t}{2}\sqrt{(\tfrac{4\pi k}{L})^2 - \widehat{\phi}_k^2}\right)\left(\tfrac{\widehat{\phi}_k}{2}\widehat{u}_k(0) + \widehat{v}_k(0)\right)\Big]. \end{aligned}$$

Case 3: $\widehat{\phi}_k^2 - \left(\frac{4\pi k}{L}\right)^2 = 0$. The roots obtained in Eq (3.3) are real and coincide. This means that the general solution of Eq (3.2) is

$$\widehat{u}_k(t) = A_k e^{-t\frac{\widehat{\phi}_k}{2}} + B_k t e^{-t\frac{\widehat{\phi}_k}{2}} . \tag{3.5}$$

To find the constants A_k and B_k, we impose

$$\begin{cases} A_k = \widehat{u}_k(0) , \\ A_k\left(- \frac{\widehat{\phi}_k}{2}\right) + B_k = \widehat{v}_k(0) . \end{cases}$$

Solving this system, we obtain

$$A_k = \widehat{u}_k(0) , \quad B_k = \frac{\widehat{\phi}_k}{2}\widehat{u}_k(0) + \widehat{v}_k(0) .$$

Substituting in Eq (3.5), we obtain

$$\widehat{u}_k(t) = e^{-t\frac{\widehat{\phi}_k}{2}}\left[\widehat{u}_k(0) + \left(\frac{\widehat{\phi}_k}{2}\widehat{u}_k(0) + \widehat{v}_k(0)\right)t\right] .$$

3.5. Result in one dimension

In the previous subsection we have obtained the explicit solution to the problem (3.1) in the one-dimensional case. The result is summarized in the following theorem.

Theorem 3.3. *Assume that $\Omega = (0, L)$ and that the operator P satisfies (P3). Let $u_0 \in H_0^1(0, L)$ and $v_0 \in L^2(0, L)$. Let $u(t, x)$ be the unique solution to Eq* (3.1) *with initial data u_0 and v_0. Then the solution $u(t, x)$ is given by*

$$u(t, x) = \sum_{k\in\mathbb{Z}} \widehat{u}_k(t) e^{2\pi i k x/L} ,$$

where the Fourier coefficients $\widehat{u}_k(t)$ are given by the following expressions:

- *If $\widehat{\phi}_k^2 - \left(\frac{4\pi k}{L}\right)^2 > 0$, then*

$$\widehat{u}_k(t) = e^{-t\frac{\widehat{\phi}_k}{2}}\Big[\cosh\left(\tfrac{t}{2}\sqrt{\widehat{\phi}_k^2 - (\tfrac{4\pi k}{L})^2}\right)\widehat{u}_k(0) + \frac{1}{\frac{1}{2}\sqrt{\widehat{\phi}_k^2 - (\frac{4\pi k}{L})^2}} \sinh\left(\tfrac{t}{2}\sqrt{\widehat{\phi}_k^2 - (\tfrac{4\pi k}{L})^2}\right)\left(\tfrac{\widehat{\phi}_k}{2}\widehat{u}_k(0) + \widehat{v}_k(0)\right)\Big] .$$

- *If $\widehat{\phi}_k^2 - \left(\frac{4\pi k}{L}\right)^2 < 0$, then*

$$\widehat{u}_k(t) = e^{-t\frac{\widehat{\phi}_k}{2}}\Big[\cos\left(\tfrac{t}{2}\sqrt{(\tfrac{4\pi k}{L})^2 - \widehat{\phi}_k^2}\right)\widehat{u}_k(0) + \frac{1}{\frac{1}{2}\sqrt{(\frac{4\pi k}{L})^2 - \widehat{\phi}_k^2}} \sin\left(\tfrac{t}{2}\sqrt{(\tfrac{4\pi k}{L})^2 - \widehat{\phi}_k^2}\right)\left(\tfrac{\widehat{\phi}_k}{2}\widehat{u}_k(0) + \widehat{v}_k(0)\right)\Big] . \tag{3.6}$$

- *If $\widehat{\phi}_k^2 - \left(\frac{4\pi k}{L}\right)^2 = 0$, then*

$$\widehat{u}_k(t) = e^{-t\frac{\widehat{\phi}_k}{2}}\left[\widehat{u}_k(0) + \left(\frac{\widehat{\phi}_k}{2}\widehat{u}_k(0) + \widehat{v}_k(0)\right)t\right] . \tag{3.7}$$

3.6. Examples

We provide some examples of the explicit solution in the one-dimensional case. In the next two examples, we show that the solution converges exponentially fast to the solution projected on the null space of the frequency filter P.

Example 3.4. Let $(\widehat{\phi}_k)_{k\in\mathbb{Z}}$ be such that

$$\widehat{\phi}_k = \begin{cases} 1\,, & \text{if } |k| \geq k_0\,, \\ 0\,, & \text{if } |k| < k_0\,, \end{cases}$$

for some $k_0 \in \mathbb{N}$. This means that the operator $P\colon L^2(0,L) \to L^2(0,L)$ is an orthogonal projection on Fourier modes with high frequencies. Indeed, it is idempotent, *i.e.*, $P^2 = P$, and self-adjoint.

Assume that $1 - 4\left(\frac{2\pi k}{L}\right)^2 < 0$ for $|k| \geq k_0$. In this case, the solution is given by Eq (3.6). In particular,

$$\widehat{u}_k(t) = \cos\left(t\tfrac{2\pi|k|}{L}\right)\widehat{u}_k(0) + \frac{1}{\frac{2\pi|k|}{L}}\sin\left(t\tfrac{2\pi|k|}{L}\right)\widehat{v}_k(0)\,,$$

for $|k| < k_0$, and

$$\widehat{u}_k(t) = e^{-\frac{t}{2}}\Big[\cos\Big(\tfrac{t}{2}\sqrt{(\tfrac{4\pi k}{L})^2 - 1}\Big)\widehat{u}_k(0) + \frac{2}{\sqrt{(\frac{4\pi k}{L})^2 - 1}}\sin\Big(\tfrac{t}{2}\sqrt{(\tfrac{4\pi k}{L})^2 - 1}\Big)\Big(\tfrac{1}{2}\widehat{u}_k(0) + \widehat{v}_k(0)\Big)\Big]\,, \tag{3.8}$$

for $|k| \geq k_0$.

From the explicit solutions we can deduce the asymptotic behavior of the solution. First of all, we observe that the null space of the linear operator $P\colon L^2(0,L) \to L^2(0,L)$ is given by

$$N(P) = \Big\{v = \sum_{|k|<k_0} \widehat{v}_k e^{2\pi i k x/L} \in L^2(0,L)\Big\}\,.$$

We define $Q = \mathrm{Id} - P\colon L^2(0,L) \to L^2(0,L)$, which is the orthogonal projection on the null space of P, *i.e.*, the space of Fourier modes with low frequencies. We observe that

$$Q\widehat{[u(t,\cdot)]}_k = \begin{cases} \cos\left(t\tfrac{2\pi|k|}{L}\right)\widehat{u}_k(0) + \frac{1}{\frac{2\pi|k|}{L}}\sin\left(t\tfrac{2\pi|k|}{L}\right)\widehat{v}_k(0)\,, & \text{if } |k| < k_0\,, \\ 0\,, & \text{if } |k| \geq k_0\,. \end{cases}$$

This implies that $\|u(t,\cdot) - Q[u(t,\cdot)]\|^2_{H^1(0,L)}$ can be estimated, using Parseval's identity, simply in terms of the Fourier coefficients in Eq (3.8), giving

$$\|u(t,\cdot) - Q[u(t,\cdot)]\|_{H^1(0,L)} \leq Ce^{-\frac{t}{2}}\Big(\|u_0\|_{H^1(0,L)} + \|v_0\|_{L^2(0,L)}\Big)\,.$$

Moreover, by linearity, $Q[u(t,\cdot)]$ is the solution to the problem (3.1) (which becomes an undamped wave equation, since $PQ = QP = 0$) with initial data $Q[u_0]$ and $Q[v_0]$, *i.e.*, the initial data projected on the null space of P. We have shown that the solution $u(t,x)$ to the problem (3.1) converges exponentially fast to the solution to the problem with initial data projected on the null space of P. See also Figure 1 for a numerical simulation.

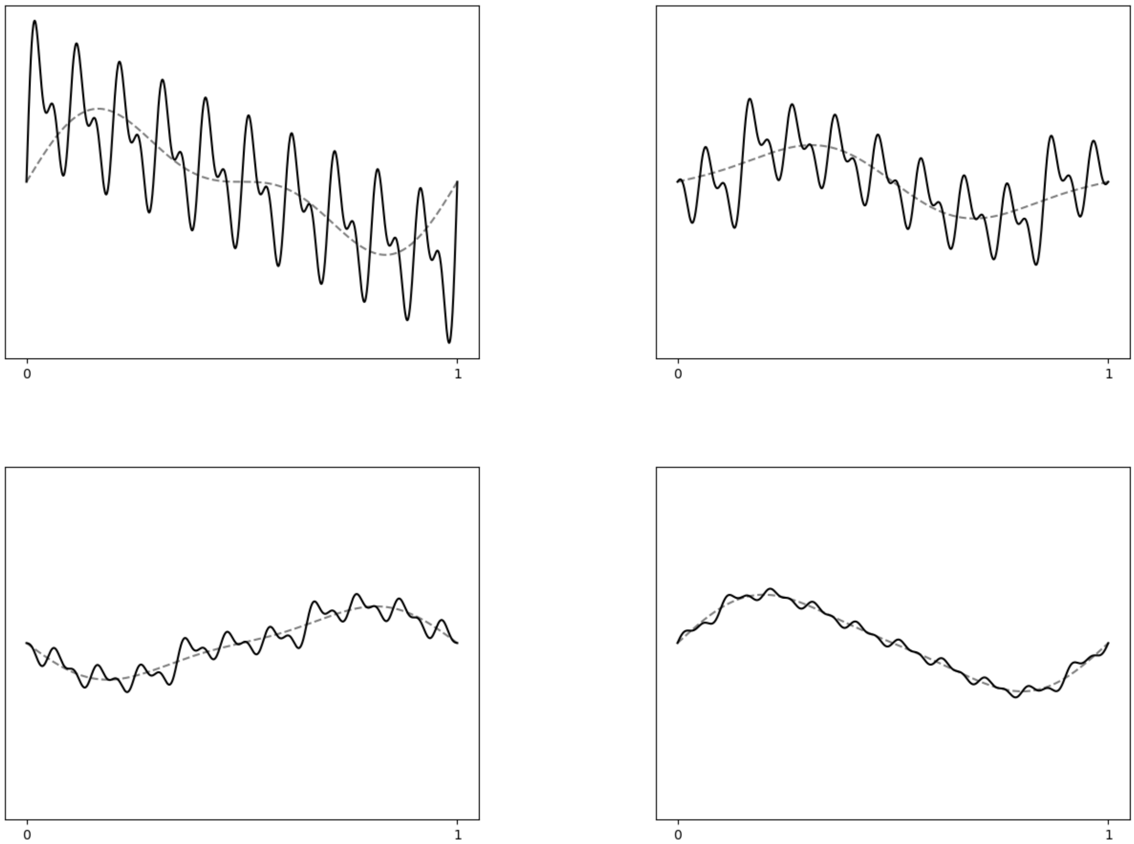

Figure 1. Numerical simulation of the solution to the problem (3.1) with $L = 1$, u_0 built on modes with frequencies ranging from 0 to 20, $v_0 = 0$, and ϕ such that $\widehat{\phi}_k = 1$ for $|k| < 3$ and $\widehat{\phi}_k = 0$ for $|k| \geq 3$. The figure shows on top left the initial condition u_0 in solid black and its projection $Q[u_0]$ on the null space of P in dashed black. The other frames show the solution $u(t, \cdot)$ at different times. The solution converges exponentially fast to the solution built only on the modes with low frequencies.

Example 3.5. Let $L = 1$ for simplicity. Assume that

$$\widehat{\phi}_k = \begin{cases} 0\,, & \text{if } |k| \neq 1\,, \\ 4\pi\,, & \text{if } |k| = 1\,. \end{cases}$$

Consider the initial data

$$u_0 = 0\,, \quad v_0 = 2\cos(2\pi x) = e^{2\pi i x} + e^{-2\pi i x}\,.$$

Then

$$\widehat{v}_k(0) = \begin{cases} 0\,, & \text{if } |k| \neq 1\,, \\ 1\,, & \text{if } |k| = 1\,. \end{cases}$$

By Eq (3.7), the solution $u(t,x)$ satisfies

$$\widehat{u}_k(t) = \begin{cases} 0\,, & \text{if } |k| \neq 1\,, \\ te^{-\frac{t}{2}}\,, & \text{if } |k| = 1\,. \end{cases}$$

We observe that the null space of the operator P is given by

$$N(P) = \Big\{ v = \sum_{|k|\neq 1} \widehat{v}_k e^{2\pi i k x} \in L^2(0,1) \Big\}\,.$$

The orthogonal projection $Q = \mathrm{Id} - P$ on the null space of P is simply the projection on the modes with $|k| \neq 1$. This implies that $u(t,\cdot) - Q[u(t,\cdot)]$ is given by the modes with $|k| = 1$. It follows that

$$\|u(t,\cdot) - Q[u(t,\cdot)]\|^2_{H^1(0,1)} = (1 + (2\pi)^2)|\widehat{u}_1(t)|^2 + (1 + (-2\pi)^2)|\widehat{u}_{-1}(t)|^2 = 2(1 + (2\pi)^2)t^2 e^{-t}\,.$$

Taking the square root, we obtain that the rate of convergence is not $e^{-t/2}$ as in Example 3.4, but it is still exponential $e^{-\gamma t}$ for $\gamma \in (0, \frac{1}{2})$.

The exponential rate of convergence is not the general behavior of the solution. It is strongly related to the structure of the frequency filter P, as we show in the next example, where the rate of convergence is subexponential.

Example 3.6. In this example we show that the universal exponential decay is not always present. We provide an example of P such that the following statement is not true: There exist $\gamma > 0$ and $M > 0$ such that for all initial data $u_0 \in H^1_0(0,1)$ and $v_0 \in L^2(0,1)$, the solution $u(t,\cdot)$ with initial data u_0 and v_0 satisfies

$$\limsup_{t\to+\infty} \frac{\|u(t,\cdot)\|_{H^1(0,1)}}{e^{-\gamma t}(\|u_0\|_{H^1(0,1)} + \|v_0\|_{L^2(0,1)})} \leq M\,. \tag{3.9}$$

Let $L = 1$ for simplicity. Let us fix $(\widehat{\phi}_k)_{k\in\mathbb{Z}}$ that satisfies the following properties:

- $\widehat{\phi}_k > 0$ for all $k \in \mathbb{Z}$;
- $\widehat{\phi}_k^2 - (4\pi k)^2 < 0$ for all $k \in \mathbb{Z}$, $k \neq 0$;
- $\widehat{\phi}_k = \widehat{\phi}_{-k}$ for all $k \in \mathbb{Z}$;
- $\liminf_{|k|\to+\infty} \widehat{\phi}_k = 0$.

The first condition is required just to simplify the example. Indeed, it implies that the null space of the operator P is given by $N(P) = \{0\}$. Hence, in this example we do not need to consider the orthogonal projection Q, and we simply have to analyze the convergence of the solutions to the zero function.

Our claim is the following.

Claim: Let $\gamma > 0$. Let $M > 0$. There exist initial data $u_0 \in H^1_0(0,1)$ and $v_0 \in L^2(0,1)$ and a sequence of times $t_n \to +\infty$ such that the solution $u(t_n,\cdot)$ with initial data u_0 and v_0 satisfies[‡]

$$\|u(t_n,\cdot)\|_{H^1(0,1)} > Me^{-\gamma t_n}(\|u_0\|_{H^1(0,1)} + \|v_0\|_{L^2(0,1)}) \quad \text{for all } n \in \mathbb{N}\,.$$

We divide the proof of the claim into several steps.

[‡]This is precisely the negation of Eq (3.9).

Step 1: Fixing a suitable frequency k_0. Let us fix $\gamma > 0$ and $M > 0$. We let $k_0 \in \mathbb{N} \setminus \{0\}$ (depending on γ) be such that

$$\widehat{\phi}_{k_0} < \frac{\gamma}{2} \quad \text{and} \quad 2(1 + (2\pi k_0)^2) > \frac{\gamma^2}{2}\,. \tag{3.10}$$

Step 2: Constructing the initial datum. In this step we define the initial data u_0 and v_0. We let

$$u_0(x) = 2\sin(2\pi k_0 x)\,.$$

Note that $u_0 \in H^1_0(0,1)$ and

$$\widehat{u}_k(0) = \begin{cases} -i\,, & \text{if } k = k_0\,, \\ i\,, & \text{if } k = -k_0\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Moreover, we let $v_0 = -\frac{1}{2}P[u_0] \in L^2(0,1)$, so that

$$\widehat{v}_k(0) = -\frac{\widehat{\phi}_k}{2}\widehat{u}_k(0) \implies \frac{\widehat{\phi}_k}{2}\widehat{u}_k(0) + \widehat{v}_k(0) = 0\,, \quad \text{for all } k \in \mathbb{Z}\,.$$

Note that

$$\|u_0\|^2_{H^1(0,1)} = \sum_{k\in\mathbb{Z}} (1 + (2\pi k)^2)|\widehat{u}_k(0)|^2 = 2(1 + (2\pi k_0)^2)\,,$$

and, by Eq (3.10),

$$\|v_0\|^2_{L^2(0,1)} = \sum_{k\in\mathbb{Z}} |\widehat{v}_k(0)|^2 = |\widehat{v}_{k_0}(0)|^2 + |\widehat{v}_{-k_0}(0)|^2 = \frac{\widehat{\phi}^2_{k_0}}{2} < \frac{\gamma^2}{2} < 2(1 + (2\pi k_0)^2) = \|u_0\|^2_{H^1(0,1)}\,. \tag{3.11}$$

Step 4: Computing the solution. By Eq (3.6), the solution $u(t,x)$ to the problem (3.1) with initial data u_0 and v_0 has Fourier coefficients given by

$$\begin{aligned} \widehat{u}_k(t) &= e^{-t\frac{\widehat{\phi}_k}{2}} \cos\left(\tfrac{t}{2}\sqrt{(4\pi k)^2 - \widehat{\phi}^2_k}\right)\widehat{u}_k(0) \\ &= \begin{cases} e^{-t\frac{\widehat{\phi}_k}{2}} \cos\left(\tfrac{t}{2}\sqrt{(4\pi k)^2 - \widehat{\phi}^2_k}\right), & \text{if } k = k_0\,, \\ -e^{-t\frac{\widehat{\phi}_k}{2}} \cos\left(\tfrac{t}{2}\sqrt{(4\pi k)^2 - \widehat{\phi}^2_k}\right), & \text{if } k = -k_0\,, \\ 0\,, & \text{otherwise}\,. \end{cases} \end{aligned}$$

Step 4: Estimating the H^1 norm of the solution. By Eqs (3.10) and (3.11), we get

$$\begin{aligned} \|u(t,\cdot)\|^2_{H^1} &= \sum_{k\in\mathbb{Z}} (1 + (2\pi k)^2)|\widehat{u}_k(t)|^2 = e^{-t\widehat{\phi}_{k_0}} \cos^2\left(\tfrac{t}{2}\sqrt{(4\pi k_0)^2 - \widehat{\phi}^2_{k_0}}\right) 2(1 + (2\pi k_0)^2) \\ &\geq e^{-\gamma t/2} \cos^2\left(\tfrac{t}{2}\sqrt{(4\pi k_0)^2 - \widehat{\phi}^2_{k_0}}\right) 2(1 + (2\pi k_0)^2) \\ &\geq e^{\gamma t/2} e^{-\gamma t} \cos^2\left(\tfrac{t}{2}\sqrt{(4\pi k_0)^2 - \widehat{\phi}^2_{k_0}}\right) \|u_0\|^2_{H^1(0,1)} \\ &\geq e^{\gamma t/2} e^{-\gamma t} \cos^2\left(\tfrac{t}{2}\sqrt{(4\pi k_0)^2 - \widehat{\phi}^2_{k_0}}\right) \frac{1}{2}(\|u_0\|^2_{H^1(0,1)} + \|v_0\|^2_{L^2(0,1)})\,. \end{aligned} \tag{3.12}$$

Step 5: Choice of sequence of times. First of all, we construct a sequence of times $t_n \to +\infty$ such that

$$\cos^2\left(\tfrac{t_n}{2}\sqrt{(4\pi k_0)^2 - \widehat{\phi}^2_{k_0}}\right) \geq \tfrac{1}{2}\,, \quad \text{for all } n \in \mathbb{N}\,. \tag{3.13}$$

For, it is enough to choose $t_n = \frac{2\pi n}{\sqrt{(4\pi k_0)^2 - \widehat{\phi}^2_{k_0}}}$. By choosing $n \geq n_0$ with n_0 large enough, we can additionally ensure that

$$\frac{1}{4}e^{\gamma t_n/2} > M\,, \quad \text{for all } n \geq n_0\,. \tag{3.14}$$

Putting Eqs (3.13) and (3.14) in Eq (3.12), we obtain that

$$\begin{aligned}\|u(t_n,\cdot)\|^2_{H^1} &\geq \frac{1}{2}e^{\gamma t_n/2}e^{-\gamma t_n}\cos^2\left(\tfrac{t_n}{2}\sqrt{(4\pi k_0)^2 - \widehat{\phi}^2_{k_0}}\right)(\|u_0\|^2_{H^1(0,1)} + \|v_0\|^2_{L^2(0,1)}) \\ &\geq Me^{-\gamma t_n}(\|u_0\|^2_{H^1(0,1)} + \|v_0\|^2_{L^2(0,1)})\,.\end{aligned}$$

This concludes the proof of the claim.

4. Exponential decay for projected solutions

In this section we show that, under suitable assumptions on P, the solution is split into two components: One that decays exponentially fast (the projected solution) and one that solves the undamped wave equation (the orthogonal component).

The precise assumptions on the bounded linear operator $P\colon L^2(\Omega) \to L^2(\Omega)$ are the following:

(A1) P commutes pointwise[§] with the Dirichlet Laplacian, *i.e.*, for every $\varphi \in H^2(\Omega) \cap H^1_0(\Omega)$ we have that $P[\varphi] \in H^2(\Omega) \cap H^1_0(\Omega)$ and $P[\Delta\varphi] = \Delta P[\varphi]$.

(A2) $P\colon L^2(\Omega) \to L^2(\Omega)$ is an orthogonal projection, *i.e.*, $P^2 = P$ and P is self-adjoint;

Example 4.1. An operator P of the form of Example 3.4 satisfies (A1) and (A2).

Remark 4.2. Assume that P satisfies (A1) and (A2). Then it also satisfies ($P1$) and ($P2$); see Example 2.2.

Remark 4.3. Assume that P satisfies (A1) and (A2). Let us show that P preserves $H^1_0(\Omega)$, *i.e.*, $P(H^1_0(\Omega)) \subset H^1_0(\Omega)$. Let $u \in H^1_0(\Omega)$ and let us show that $P[u] \in H^1_0(\Omega)$. Let us fix an approximating sequence $u_j \in H^2(\Omega) \cap H^1_0(\Omega)$ such that $u_j \to u$ in $H^1_0(\Omega)$. By (A1), we have that $P[u_j] \in H^2(\Omega) \cap H^1_0(\Omega)$. Moreover, $P[u_j] \to P[u]$ in $L^2(\Omega)$, as P is bounded. Let us estimate $\sup_j \|\nabla P[u_j]\|^2_{L^2(\Omega)}$. We integrate by parts and we exploit (A1) and (A2) to obtain that

$$\begin{aligned}\|\nabla P[u_j]\|^2_{L^2(\Omega)} &= -\langle P[u_j](x), \Delta P[u_j]\rangle_{L^2(\Omega)} = -\langle P[u_j], P[\Delta u_j]\rangle_{L^2(\Omega)} = -\langle P^2[u_j], \Delta u_j\rangle_{L^2(\Omega)} \\ &= -\langle P[u_j], \Delta u_j\rangle_{L^2(\Omega)} = \langle \nabla P[u_j], \nabla u_j\rangle_{L^2(\Omega)} \leq \|\nabla P[u_j]\|_{L^2(\Omega)}\|\nabla u_j\|_{L^2(\Omega)}\,,\end{aligned}$$

from which we deduce that

$$\sup_j \|\nabla P[u_j]\|_{L^2(\Omega)} \leq \sup_j \|\nabla u_j\|_{L^2(\Omega)} < +\infty\,.$$

It follows that $P[u_j] \rightharpoonup P[u]$ weakly in $H^1_0(\Omega)$, proving the claim.

[§]We use this nomenclature to distinguish the assumption from strong commutation. See, *e.g.*, [18, VIII.5]

To the aim of the splitting result, we start with a preliminary result. The proof is classical, as it relies on the linearity of the equation. We provide a proof for the sake of completeness.

Theorem 4.4. *Assume that P satisfies (A1) and (A2). Let $u_0 \in H_0^1(\Omega)$ and $v_0 \in L^2(\Omega)$. Let $(u(t), v(t))$ be the unique solution to Eq* (2.1) *with initial datum (u_0, v_0) provided by Theorem 2.5. Then $(P[u(t)], P[v(t)])$ is the unique solution (in the sense of Theorem 2.5) to the problem:*

$$\begin{cases} \partial_{tt} w - \Delta w + \partial_t w = 0\,, & (t,x) \in (0,+\infty) \times \Omega\,, \\ w = 0\,, & (t,x) \in (0,+\infty) \times \partial\Omega\,, \\ w(0,x) = P[u_0](x)\,, \quad \partial_t w(0,x) = P[v_0](x)\,, & x \in \Omega\,. \end{cases} \tag{4.1}$$

Moreover, there exist constants $M > 0$ and $\gamma > 0$ such that

$$\|P[u(t)]\|^2_{H^1(\Omega)} + \|P[v(t)]\|^2_{L^2(\Omega)} \leq M e^{-\gamma t}\,, \quad \text{for all } t \geq 0\,, \tag{4.2}$$

where $M > 0$ depends on the energy of the initial data (u_0, v_0) and $\gamma > 0$ depends on the domain Ω.

Proof. We split the proof into several steps.

Step 1: First of all, we show that $t \mapsto u(t,\cdot) \in H_0^1(\Omega)$ is also a distributional solution to Eq (2.1), *i.e.*, we have that

$$\begin{aligned} &\int_0^{+\infty} \int_\Omega u(t,x)\Big(\partial_{tt}\varphi(t,x) - \Delta\varphi(t,x) - \partial_t P[\varphi(t,\cdot)](x)\Big)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = -\int_\Omega u_0(x)\partial_t\varphi(0,x)\,\mathrm{d}x + \int_\Omega \Big(u_0(x)P[\varphi(0,\cdot)](x) + v_0(x)\varphi(0,x)\Big)\,\mathrm{d}x\,, \end{aligned} \tag{4.3}$$

for all $\varphi \in C_c^\infty(\mathbb{R} \times \Omega)$.

To see this, let us first work in the case $U_0 \in \mathrm{Dom}(\mathscr{A})$, using the notation of Section 2. Then $U(t) = (u(t), v(t)) = S(t)U_0$ belongs to $C^1([0,+\infty); H) \cap C([0,+\infty); \mathrm{Dom}(\mathscr{A}))$ and is the unique solution to Eq (2.3). We fix a curve $t \mapsto \Phi(t) \in H$ given by $\Phi(t) = (0, \varphi(t,\cdot))$ with $\varphi \in C_c^\infty(\mathbb{R} \times \Omega)$. We have that

$$\begin{aligned} -\langle U_0, \Phi(0)\rangle_H &= \int_0^{+\infty} \frac{\mathrm{d}}{\mathrm{d}t}\langle U(t), \Phi(t)\rangle_H\,\mathrm{d}t \\ &= -\int_0^{+\infty} \langle \mathscr{A} U(t), \Phi(t)\rangle_H\,\mathrm{d}t + \int_0^{+\infty} \Big\langle U(t), \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t)\Big\rangle_H\,\mathrm{d}t\,, \end{aligned}$$

which reads, using the self-adjointness of P,

$$\begin{aligned} -\langle v_0, \varphi(0)\rangle_{L^2(\Omega)} &= \int_0^{+\infty} \langle \Delta u(t) - P[v(t)], \varphi(t)\rangle_{L^2(\Omega)}\,\mathrm{d}t - \int_0^{+\infty} \langle v(t), \partial_t\varphi(t)\rangle_{L^2(\Omega)}\,\mathrm{d}t \\ &= \int_0^{+\infty} \Big(\langle u(t), \Delta\varphi(t)\rangle_{L^2(\Omega)} - \langle v(t), P[\varphi(t)]\rangle_{L^2(\Omega)}\Big)\,\mathrm{d}t \\ &\quad + \int_0^{+\infty} \langle v(t), \partial_t\varphi(t)\rangle_{L^2(\Omega)}\,\mathrm{d}t \\ &= \int_0^{+\infty} \Big(\langle u(t), \Delta\varphi(t)\rangle_{L^2(\Omega)} - \Big\langle \frac{\mathrm{d}}{\mathrm{d}t}u(t), P[\varphi(t)]\Big\rangle_{L^2(\Omega)}\Big)\,\mathrm{d}t \\ &\quad + \int_0^{+\infty} \Big\langle \frac{\mathrm{d}}{\mathrm{d}t}u(t), \partial_t\varphi(t)\Big\rangle_{L^2(\Omega)}\,\mathrm{d}t\,. \end{aligned}$$

Then we substitute in the previous equation the following two identities:

$$
\begin{aligned}
-\langle u_0, \partial_t\varphi(0)\rangle_{L^2(\Omega)} &= \int_0^{+\infty} \frac{\mathrm{d}}{\mathrm{d}t}\langle u(t), \partial_t\varphi(t)\rangle_{L^2(\Omega)}\,\mathrm{d}t \\
&= \int_0^{+\infty} \Big\langle \frac{\mathrm{d}}{\mathrm{d}t}u(t), \partial_t\varphi(t)\Big\rangle_{L^2(\Omega)}\,\mathrm{d}t + \int_0^{+\infty} \langle u(t), \partial_{tt}\varphi(t)\rangle_{L^2(\Omega)}\,\mathrm{d}t\,,
\end{aligned}
$$

and, using Remark 2.3,

$$
\begin{aligned}
-\langle u_0, P[\varphi(0)]\rangle_{L^2(\Omega)} &= \int_0^{+\infty} \frac{\mathrm{d}}{\mathrm{d}t}\langle u(t), P[\varphi(t)]\rangle_{L^2(\Omega)}\,\mathrm{d}t \\
&= \int_0^{+\infty} \Big\langle \frac{\mathrm{d}}{\mathrm{d}t}u(t), P[\varphi(t)]\Big\rangle_{L^2(\Omega)}\,\mathrm{d}t + \int_0^{+\infty} \Big\langle u(t), \frac{\mathrm{d}}{\mathrm{d}t}P[\varphi(t)]\Big\rangle_{L^2(\Omega)}\,\mathrm{d}t \\
&= \int_0^{+\infty} \Big\langle \frac{\mathrm{d}}{\mathrm{d}t}u(t), P[\varphi(t)]\Big\rangle_{L^2(\Omega)}\,\mathrm{d}t + \int_0^{+\infty} \langle u(t), P[\partial_t\varphi(t)]\rangle_{L^2(\Omega)}\,\mathrm{d}t\,,
\end{aligned}
$$

to obtain that

$$
\begin{aligned}
&\langle u_0, \partial_t\varphi(0)\rangle_{L^2(\Omega)} - \langle u_0, P[\varphi(0)]\rangle_{L^2(\Omega)} - \langle v_0, \varphi(0)\rangle_{L^2(\Omega)} \\
&\quad= \int_0^{+\infty} \Big(\langle u(t), \Delta\varphi(t)\rangle_{L^2(\Omega)} + \langle u(t), P[\partial_t\varphi(t)]\rangle_{L^2(\Omega)}\Big)\,\mathrm{d}t - \int_0^{+\infty} \langle u(t), \partial_{tt}\varphi(t)\rangle_{L^2(\Omega)}\,\mathrm{d}t\,,
\end{aligned}
$$

which is precisely Eq (4.3).

If $U_0 \in H$ (not necessarily in $\mathrm{Dom}(\mathscr{A})$), then Eq (4.3) is obtained by approximating U_0 in the H-norm with a sequence in $\mathrm{Dom}(\mathscr{A})$ and then passing to the limit.

Step 2: In the condition (4.3), it is enough to test the equation with $\varphi(t,x) = \zeta(t)\psi(x)$, where $\zeta \in C_c^\infty(\mathbb{R})$ and $\psi \in C_c^\infty(\Omega)$. Hence, it reads

$$
\begin{aligned}
&\int_0^{+\infty}\int_\Omega u(t,x)\Big(\partial_{tt}\zeta(t)\psi(x) - \zeta(t)\Delta\psi(x) - \partial_t\zeta(t)P[\psi](x)\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad= -\int_\Omega u_0(x)\partial_t\zeta(0)\psi(x)\,\mathrm{d}x + \int_\Omega \Big(u_0(x)\zeta(0)P[\psi](x) + v_0(x)\zeta(0)\psi(x)\Big)\,\mathrm{d}x\,,
\end{aligned}
\tag{4.4}
$$

for all $\zeta \in C_c^\infty(\mathbb{R})$ and $\psi \in C_c^\infty(\Omega)$.

Step 3: Note that, by an approximation argument, Eq (4.4) can be tested with $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$.

Step 4: Given $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$, by (A1) we have that $P[\psi] \in H^2(\Omega) \cap H_0^1(\Omega)$. Hence, we can use $P[\psi]$ in Eq (4.4) instead of ψ to obtain that

$$
\begin{aligned}
&\int_0^{+\infty}\int_\Omega u(t,x)\Big(\partial_{tt}\zeta(t)P[\psi](x) - \zeta(t)\Delta P[\psi](x) - \partial_t\zeta(t)P[P[\psi]](x)\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad= -\int_\Omega u_0(x)\partial_t\zeta(0)P[\psi](x)\,\mathrm{d}x + \int_\Omega \Big(u_0(x)\zeta(0)P[P[\psi]](x) + v_0(x)\zeta(0)P[\psi](x)\Big)\,\mathrm{d}x\,.
\end{aligned}
$$

Using the properties $P[\Delta\psi] = \Delta P[\psi]$ from (A1) and $P^2 = P$ from (A2), we obtain that

$$
\begin{aligned}
&\int_0^{+\infty}\int_\Omega u(t,x)\Big(\partial_{tt}\zeta(t)P[\psi](x) - \zeta(t)P[\Delta\psi](x) - \partial_t\zeta(t)P[\psi](x)\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad= -\int_\Omega u_0(x)\partial_t\zeta(0)P[\psi](x)\,\mathrm{d}x + \int_\Omega (u_0(x) + v_0(x))\zeta(0)P[\psi](x)\,\mathrm{d}x\,.
\end{aligned}
$$

Finally, since P is self-adjoint, we have that

$$\int_0^{+\infty}\int_\Omega P[u(t,\cdot)](x)\Big(\partial_{tt}\zeta(t)\psi(x) - \zeta(t)\Delta\psi(x) - \partial_t\zeta(t)\psi(x)\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$= -\int_\Omega P[u_0](x)\partial_t\zeta(0)\psi(x)\,\mathrm{d}x + \int_\Omega \Big(P[u_0](x) + P[v_0](x)\Big)\zeta(0)\psi(x)\,\mathrm{d}x\,.$$

Step 6: Reasoning as for Eq (4.4), we conclude that

$$\int_0^{+\infty}\int_\Omega P[u(t,\cdot)](x)\Big(\partial_{tt}\varphi(t,x) - \Delta\varphi(t,x) - \partial_t\varphi(t,x)\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$= -\int_\Omega P[u_0](x)\partial_t\varphi(0,x)\,\mathrm{d}x + \int_\Omega \Big(P[u_0](x) + P[v_0](x)\Big)\varphi(0,x)\,\mathrm{d}x\,.$$

for all $\varphi \in C_c^\infty(\mathbb{R}\times\Omega)$, which we recognize as the definition of distributional solution to Eq (4.1).

Step 7: The distributional solution $w \in L^2((0,+\infty)\times\Omega)$ to Eq (4.1) is unique hence, *a fortiori*, it must coincide with the solution provided by Theorem 2.5. This follows from a duality argument. To be precise, let us assume that $w \in L^2((0,+\infty)\times\Omega)$ is a distributional solution to Eq (2.2) with initial data $P[u_0] = 0$ and $P[v_0] = 0$, and let us show that

$$\int_0^{+\infty}\int_\Omega w(t,x)\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t = 0\,,$$

for all $\varphi \in C_c^\infty(\mathbb{R}\times\Omega)$. Given $\varphi \in C_c^\infty(\mathbb{R}\times\Omega)$, let $T > 0$ be such that $\varphi(t,x) = 0$ for $t \geq T$. We consider a (strong) solution $\widetilde{\varphi}$ to the final-time dual problem:

$$\begin{cases} \partial_{tt}\widetilde{\varphi} - \Delta\widetilde{\varphi} - \partial_t\widetilde{\varphi} = \varphi\,, & (t,x) \in (0,+\infty)\times\Omega\,,\\ \widetilde{\varphi} = 0\,, & (t,x) \in (0,+\infty)\times\partial\Omega\,,\\ \widetilde{\varphi}(T,x) = 0\,, \quad \partial_t\widetilde{\varphi}(T,x) = 0\,, & x \in \Omega\,. \end{cases}$$

We approximate $\widetilde{\varphi}$ with a sequence $\widetilde{\varphi}_j \in C_c^\infty(\mathbb{R}\times\Omega)$ such that $\widetilde{\varphi}_j \to \widetilde{\varphi}$ in $H^2(\mathbb{R}\times\Omega)$. Since w is a distributional solution to Eq (4.1) with zero initial data, we have that

$$\int_0^{+\infty}\int_\Omega w(t,x)\Big(\partial_{tt}\widetilde{\varphi}_j(t,x) - \Delta\widetilde{\varphi}_j(t,x) - \partial_t\widetilde{\varphi}_j(t,x)\Big)\,\mathrm{d}x\,\mathrm{d}t = 0\,.$$

Passing to the limit as $j \to +\infty$, we obtain that

$$0 = \int_0^{+\infty}\int_\Omega w(t,x)\Big(\partial_{tt}\widetilde{\varphi}(t,x) - \Delta\widetilde{\varphi}(t,x) - \partial_t\widetilde{\varphi}(t,x)\Big)\,\mathrm{d}x\,\mathrm{d}t = \int_0^{+\infty}\int_\Omega w(t,x)\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t\,,$$

concluding the proof of the claim.

Step 8: The exponential decay follows from Remark 2.7.

Theorem 4.5. *Assume that P satisfies (A1)–(A2). Let $Q = \mathrm{Id} - P$. Let $u_0 \in H_0^1(\Omega)$ and $v_0 \in L^2(\Omega)$. Let $(u(t), v(t))$ be the unique solution to Eq* (2.1) *with initial datum* (u_0, v_0) *provided by Theorem 2.5. Then*

$$(u(t), v(t)) = (Q[u(t)], Q[v(t)]) + (P[u(t)], P[v(t)]) \tag{4.5}$$

where $(Q[u(t)], Q[v(t)])$ *is the unique solution to the undamped wave equation with initial datum* $(Q[u_0], Q[v_0])$*:*

$$\begin{cases} \partial_{tt} z - \Delta z = 0\,, & (t,x) \in (0,+\infty) \times \Omega\,, \\ z = 0\,, & (t,x) \in (0,+\infty) \times \partial\Omega\,, \\ z(0,x) = Q[u_0](x)\,, \quad \partial_t z(0,x) = Q[v_0](x)\,, & x \in \Omega\,, \end{cases} \tag{4.6}$$

and there exist constants $\gamma, M > 0$ *such that*

$$\|u(t) - Q[u(t)]\|_{H_0^1(\Omega)} + \|v(t) - Q[v(t)]\|_{L^2(\Omega)} \leq Me^{-\gamma t}\,, \quad \textit{for all } t \geq 0\,,$$

where M *and* γ *are as in Theorem 4.4.*

Proof. Note that Q is the orthogonal projection on the null space of P. The decomposition (4.5) follows from the fact that Q is the orthogonal projection on the orthogonal complement of the range of P. Moreover, Q satisfies (A1) and (A2) as well.

As in the proof of Proposition 4.4, one proves that $(Q[u(t)], Q[v(t)])$ is a distributional solution to the undamped wave equation (4.6). Finally, the exponential decay follows from Eq (4.2).

5. Conclusions

In this paper we have investigated a specific dissipation mechanism occurring in the damped linear wave equation, when a frequency selection operator affects the damping term. Well-posedness of the initial boundary value problem and qualitative decay properties of the energy are investigated. Owing to the linear structure of the problem, we have explicitly analyzed different cases pertaining the frequency selection. Eventually we have proved that in special cases solutions of the evolution problem split into a dissipative and a conservative part.

Author contributions

All authors contributed equally to the study and the writing of the manuscript.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that they have no conflict of interest.

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