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Research article

Ground states for the NLS on non-compact graphs with an attractive potential

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Abstract: We consider the subcritical nonlinear Schrödinger equation on non-compact quantum graphs with an attractive potential supported in the compact core, and investigate the existence and the nonexistence of ground states, defined as minimizers of the energy at fixed L^2-norm, or mass. We finally reach the following picture: for small and large mass there are ground states. Moreover, according to the metric features of the compact core of the graph and to the strength of the potential, there may be an interval of intermediate masses for which there are no ground states. The study was inspired by the research on quantum waveguides, in which the curvature of a thin tube induces an effective attractive potential.

Keywords: nonlinear Schrödinger equation; quantum graphs; ground states; concentration compactness techniques

1. Introduction

The nonlinear Schrödinger (NLS) equation is currently used to model the dynamics of several physical systems. In particular, it provides an effective description of the evolution of the wave function of a Bose-Einstein condensate (BEC), namely an ultracold system made of many identical bosons in a peculiar phase [1–3]. When extended to graphs, the NLS equation can describe evolutionary phenomena in networks, with applications to various contexts in science and engineering [4–8]. In particular, it describes the behaviour of particles traveling through branched structures of quantum wires in the presence of nonlinearities.

Here we investigate the existence of ground states for the energy functional

$$E(u,\mathcal{G}) = \frac{1}{2}\int_{\mathcal{G}} |u'(x)|^2 dx - \frac{1}{p}\int_{\mathcal{G}} |u(x)|^p dx - \frac{1}{2}\int_{\mathcal{K}} w(x)|u(x)|^2 dx, \qquad 2 < p < 6, \tag{1.1}$$

on a metric graph $\mathcal{G}$ with at least an unbounded edge, i.e., a halfline, under the mass constraint

$$\|u\|^2_{L^2(\mathcal{G})} = \mu,$$

which means that we seek solutions in the mass constrained space

$$H^1_\mu(\mathcal{G}) := \{u \in H^1(\mathcal{G}),\ \|u\|^2_{L^2(\mathcal{G})} = \mu\}. \tag{1.2}$$

In Eq (1.1), the symbol $\mathcal{K}$ denotes a compact subset of $\mathcal{G}$ that supports the action of the continuous potential $w \geq 0$. Notice that with such a choice on the sign of w, the contribution to Eq (1.1) of the potential term is nonpositive and, since it is supported on a compact set, it goes to zero at infinity and so, roughly speaking, can be considered as attractive. For simplicity, we identify $\mathcal{K}$ with the compact core of $\mathcal{G}$, namely the subgraph made of all bounded edges of $\mathcal{G}$. The meaning of the integrals and of the norms in Eqs (1.1) and (1.2) are made precise in Section 2.

The energy functional (1.1) is obtained from the standard NLS energy

$$E_{\mathrm{NLS}}(u,\mathcal{G}) = \frac{1}{2}\int_{\mathcal{G}} |u'|^2\, dx - \frac{1}{p}\int_{\mathcal{G}} |u|^p\, dx, \tag{1.3}$$

by adding a term that describes the effect of an external potential w. Throughout the paper the potential is chosen as nonnegative, continuous, and, as already mentioned, supported on the compact core $\mathcal{K}$ of the graph $\mathcal{G}$. The limitation on p means that we restrict our analysis to the L^2-subcritical case. It is well-known that for $p > 6$ the constrained functional $E(\cdot,\mathcal{G})$ is not lower bounded, while the case $p = 6$ is more delicate and deserves to be treated separately [9]. Several investigations on the existence of the ground states of (1.3) on graphs were conducted in recent years, starting from the analysis on star graphs [10, 11] and later extended to general graphs [12–15]. Moreover, research has focused on the search for stationary states [16, 17] and other special solutions [18–21]. The starting point of the analysis of the NLS on metric graphs is the existence of ground states at every mass for the functional $E_{\mathrm{NLS}}(\cdot,\mathbb{R})$, given by the soliton

$$\phi_\mu(x) = \mu^\alpha C_p \operatorname{sech}^{\frac{\alpha}{\beta}}(c_p \mu^\beta x), \tag{1.4}$$

where c_p and C_p are positive constants and the powers α and β are

$$\alpha = \frac{2}{6-p}, \qquad \beta = \frac{p-2}{6-p}. \tag{1.5}$$

Up to translations and multiplication by a constant phase, the function ϕ_μ is the unique Ground State at mass μ of E_{NLS} on the line [22, 23].

Let us recall an existence criterion singled out in Cor. 3.4 of [12] for a noncompact graph:

Existence criterion. *Fix $2 < p < 6$ and let $\mathcal{G}$ be a graph with n infinite edges, $n \geq 1$. If there exists $v \in H^1_\mu(\mathcal{G})$ such that*

$$E_{\mathrm{NLS}}(v,\mathcal{G}) \leq E_{\mathrm{NLS}}(\phi_\mu,\mathbb{R}), \tag{1.6}$$

then there exists a Ground State for E_{NLS} on $\mathcal{G}$ at mass μ.

The criterion provides a conceptually simple method to prove the existence of a Ground State for graphs with an infinite edge: it is sufficient to exhibit a function whose energy lies below the energy of the soliton. However, in practice the use of such a criterion can be quite cumbersome as the construction of such a function may be non-trivial.

Furthermore, the converse of the existence criterion holds too, i.e., if a Ground State v at mass μ exists, then it satisfies Eq (1.6). The reason is that the soliton Eq (1.4) can be approximated arbitrarily well by functions supported on the infinite edge, so the infimum of the energy at a given mass, and therefore energy of a possible ground state, cannot exceed the quantity $E_{\text{NLS}}(\mathbb{R})$ (Theorem 2.2 in [12]). This remains true for the energy functional (1.1), where the presence of an attractive potential further lowers the infimum of the constrained energy.

A preliminary result of the present paper consists in the extension of the existence criterion to the case with the potential w (Lemma 3.1). We exploit then such generalization to prove two results of existence and one of nonexistence of ground states.

The first is Theorem 4.1, that establishes that if one concentrates a soliton in a region where the potential gives a negative contribution, then the existence criterion is fulfilled and a Ground State exists. Of course, this can be done if the mass is fixed as large enough, in order to allow to confine the soliton in such a region.

On the other side of the range of the mass, the content of Theorem 4.2, namely the existence of ground states for small mass, is not new, as it was already proved in [24] for every attractive potential. Such ground states were proven to arise as a nonlinear bifurcation from linear ground states. Here we give a different proof based on the extension of the existence criterion to the functional $E(u, \mathcal{G})$ (Lemma 3.1).

The last achievement, proven in Theorem 5.1, is a nonexistence result that holds for a class of graphs and some interval of masses, under the hypothesis of a weak potential. To our knowledge this is the first nonexistence theorem for an attractive potential on graphs, except for the case of a Dirac's delta potential, and generalizes a nonexistence result in [12] that holds in the absence of potentials. Owing to Theorems 4.1 and 4.2 the picture we get is the following: in the presence of a negative potential there are two mass thresholds $\mu_\star \leq \mu^\star$. Existence is guaranteed below $\mu_\star$ and above $\mu^\star$. Between the two, there may be an interval of masses in which ground states do not exist. The construction of explicit examples in which the nonexistence occurs is not straightforward, and we exhibit one such example in Section 5, inspired by the analogous result proven in [13]. A more detailed investigation on nonexistence is in order, requiring a more thorough analytical and numerical effort.

Besides the case of an external potential, our analysis applies to the case of a potential induced by the curvature of the graph [25]. More specifically, we recall that for a waveguide modeled as a curved tube Ω_Γ around a curve Γ, one can express the Laplacian with Dirichlet boundary conditions by using a transformation from the ordinary cartesian coordinates to a system centred in a straightened tube Ω_0. This procedure results in an effective operator with a potential that depends on the signed curvature $\gamma(x)$, where x is the longitudinal coordinate that coincides with the arclength, i.e.,

$$V(x,\xi) := -\frac{\gamma^2(x)}{4(1+\xi\gamma(x))^2} + \frac{\xi\gamma''(x)}{2(1+\xi\gamma(x))^3} + \frac{5}{4}\frac{\xi^2\gamma'(x)}{(1+\xi\gamma(x))^4},$$

where ξ is the transverse coordinate. In the thin-waveguide limit $\xi \to 0$ one has $V(x) \approx -\frac{\gamma^2(x)}{4}$, and the transversal contribution remains present in the effective potential for finite ξ. When we try to rigorously apply this limit to branched networks, the main source of issues is the behaviour of eigenfunctions around the vertices (see Sec. 8.3–8.5 in [26]). Here we circumvent this complication by avoiding vertices, and get inspired by this limit to study the effects of adding a purely attractive potential to the edges of a generic graph $\mathcal{G}$.

The paper is organized as follows: In Section 2, we recall some well-known estimates and give some preliminary results. The goal of Section 3 is to prove Lemma 3.1, namely the existence criterion for the functional $E(u, \mathcal{G})$. In Section 4, we exploit Lemma 3.1 and prove the existence of ground states for large (Theorem 4.1) and small (Theorem 4.2) masses. In Section 5, we prove a nonexistence theorem (Theorem 5.1) that identifiess a class of graphs for which, for some values of the mass, ground states do not exist despite the presence of the potential.

2. General framework

A metric graph $\mathcal{G}$ embedded in an Euclidean space of dimension d is a metric space defined by the pair $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} \subset \mathbb{R}^d$ is a set of points called vertices, and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is a set whose elements are called edges, that are interpreted as connections between couples of vertices. The metric structure is defined by establishing that every edge $e \in \mathcal{E}$ is homeomorphic to a real interval $I_e = [0, \ell_e]$. If $\ell_e = \infty$, then the edge e is a halfline and $\mathcal{G}$ is noncompact. We restrict to the case of a finite number of edges and vertices (that excludes in particular the case of periodic graphs) so in this context noncompact means that at least one edge is a halfline. We denote by $\mathcal{H}_i$ the i.th halfline of $\mathcal{G}$. We shall not distinguish in notation between a point in $\mathcal{G}$ and its coordinate x in the corresponding interval I_e. Moreover, we assume that $\mathcal{G}$ is connected.

A function $u : \mathcal{G} \to \mathbb{C}$ is a family of functions $u_e : I_e \to \mathbb{C}$, one for every edge, namely $u \equiv \{u_e\}_{e\in\mathcal{E}}$. Differential operators are naturally defined edge by edge, namely $u' = \{u'_e\}_{e\in\mathcal{E}}$, and integrals on $\mathcal{G}$ are defined by

$$\int_{\mathcal{G}} u(x)\,dx = \sum_{e\in\mathcal{E}} \int_0^{\ell_e} u(x_e)\,dx_e.$$

Furthermore, L^p-spaces are defined in the natural way, i.e.,

$$L^p(\mathcal{G}) = \{u : \mathcal{G} \to \mathbb{C},\ \|u_e\|_{L^p(I_e)} < \infty\ \forall e \in \mathcal{E}\}, \quad \|u\|^p_{L^p(\mathcal{G})} = \sum_{e\in\mathcal{E}} \|u_e\|^p_{L^p(I_e)},$$

and analogously for the Sobolev spaces, in particular

$$H^1(\mathcal{G}) = \{u : \mathcal{G} \to \mathbb{C},\ \|u_e\|_{H^1(I_e)} < \infty\ \forall e \in \mathcal{E}\}, \quad \|u\|^2_{H^1(\mathcal{G})} = \sum_{e\in\mathcal{E}} \|u_e\|^2_{H^1(I_e)}.$$

The space $H^1_\mu(\mathcal{G})$, already mentioned in Sec. 1, in which we seek the minimizers of the functional $E(\cdot, \mathcal{G})$, is then defined as

$$H^1_\mu(\mathcal{G}) = \{u,\ u \in H^1_\mu(\mathcal{G}),\ \|u\|^2_{L^2(\mathcal{G})} = \mu\}.$$

For a more comprehensive introduction to variational calculus on metric graphs, see e.g., [12].

Typically one defines the compact core of a graph as the complement of the halflines. In principle, this notion does not coincide with that of the support of a given compactly supported potential. However, one can consider a larger compact set by taking the union $\mathcal{K}'$ of the compact core of the graph with the support of the potential. The complement of $\mathcal{K}'$ contains then n halflines $\mathcal{H}_i'$, which in general are subsets of the original n halflines $\mathcal{H}_i$. The complement of the union of the halflines $\mathcal{H}_i'$ contains both the compact core of the graph $\mathcal{G}$ and the support of w, and can be used in the results of the paper in the place of $\mathcal{K}$. For the sake of simplicity, in the following we refer to $\mathcal{K}$, the compact core of the graph, as the support of the potential. Of course, all results remain valid if the compact support of w does not coincide with $\mathcal{K}$.

In summary, the hypotheses on w are the following:

w is continuos, nonnegative, and supported on $\mathcal{K}$, the compact core of $\mathcal{G}$.

For notational purposes we introduce the symbols

$$T(u) = \int_{\mathcal{G}} |u'|^2\,dx, \qquad V(u) = \int_{\mathcal{G}} |u|^p\,dx, \qquad W(u) = \int_{\mathcal{K}} w(x)|u|^2\,dx,$$

so that

$$E(u,\mathcal{G}) = \frac{1}{2}T(u) - \frac{1}{p}V(u) - \frac{1}{2}W(u).$$

Furthermore, we denote

$$\mathcal{I}_{\mathcal{G}}(\mu) := \inf_{u\in H^1_\mu(\mathcal{G})} E(u,\mathcal{G})$$

$$\mathcal{I}_{\mathrm{NLS},\mathcal{G}}(\mu) := \inf_{u\in H^1_\mu(\mathcal{G})} E_{\mathrm{NLS}}(u,\mathcal{G}).$$

Remark 2.1. As mentioned in Section 1, the unique positive solution to the problem of the mass-constrained minimization of the functional (1.3) in the case $\mathcal{G} = \mathbb{R}$ is given, up to translations and multiplications by a constant phase, by soliton ϕ_μ defined in Eq (1.4), so

$$\mathcal{I}_{\mathrm{NLS},\mathbb{R}}(\mu) = E_{\mathrm{NLS}}(\phi_\mu,\mathbb{R}) = -\theta_p\mu^{2\beta+1} < 0,$$

where $\theta_p := -E_{\mathrm{NLS}}(\phi_1,\mathbb{R}) > 0$.

Remark 2.2. As mentioned in Section 1 and proved in [12], if $\mathcal{G}$ is a noncompact graph then

$$\mathcal{I}_{\mathrm{NLS},\mathcal{G}}(\mu) \le -\theta_p\mu^{2\beta+1}.$$

Since we are considering a nonnegative w, we get

$$\mathcal{I}_{\mathcal{G}}(\mu) \le \mathcal{I}_{\mathrm{NLS},\mathcal{G}}(\mu) \le -\theta_p\mu^{2\beta+1}. \tag{2.1}$$

Occasionally we use the shorthand notation

$$\|w\|_\infty := \max_{x\in\mathcal{K}} |w(x)|.$$

2.1. General Properties

In this section, we recall some well-known properties that are widely used in the context of the nonlinear Schrödinger equation on noncompact graphs.

Proposition 2.3 (Gagliardo-Nirenberg inequalities)**.** *There exists $M_p > 0$ such that*

$$\|u\|^p_{L^p(\mathcal{G})} \le M_p \|u\|^{\frac{p}{2}+1}_{L^2(\mathcal{G})} \|u'\|^{\frac{p}{2}-1}_{L^2(\mathcal{G})}. \tag{2.2}$$

Moreover

$$\|u\|^2_{L^\infty(\mathcal{G})} \le 2\|u\|_{L^2(\mathcal{G})}\|u'\|_{L^2(\mathcal{G})}. \tag{2.3}$$

for every $u \in H^1(\mathcal{G})$ and every metric graph $\mathcal{G}$ with n infinite edges, $n \ge 1$.

We refer for proofs to Proposition 4.1 in [27] and Proposition 2.1 in [13].

Proposition 2.4. *Let $\mathcal{K}$ be a compact metric graph with total length $|\mathcal{K}|$, $1 \le r \le p \le \infty$ and $s = 1/r - 1/p$. Then for every $u \in H^1(\mathcal{K})$*

$$\|u\|_{L^r(\mathcal{K})} \le \|u\|_{L^p(\mathcal{K})}|\mathcal{K}|^s.$$

Proof. This is a direct consequence of the Hölder inequality

$$\int_{\mathcal{K}} |fg| dx \le (\int_{\mathcal{K}} |f|^p dx)^{\frac{1}{p}} (\int_{\mathcal{K}} |g|^q dx)^{\frac{1}{q}},$$

where $p, q > 0$ and $1/p + 1/q = 1$, if we consider $f = |u|^r$ and $g \equiv 1$ on $\mathcal{K}$.

Since we need lower and upper bounds for the terms appearing in the energy (1.3), we prove the following result.

Proposition 2.5. *Let $\mathcal{G}$ be a metric graph with n infinite edges, $n \ge 1$. For all $u \in H^1_\mu(\mathcal{G})$ such that*

$$E(u, \mathcal{G}) \;\le\; \frac{1}{2} \mathcal{I}_{\mathcal{G}}(\mu) \;<\; 0, \tag{2.4}$$

the following estimates hold:

$$\min\{C_1\mu^{2\beta+1}, C_2 g^{-\frac{2p}{p-2}}\mu^{3(2\beta+1)}\} \le T(u) \le C_3\mu^{2\beta+1} + C_4\|w\|_\infty \mu \tag{2.5}$$

$$\min\{C_1\mu^{2\beta+1}, C_2 g^{-\frac{p}{2}}\mu^{(2\beta+1)\frac{p}{2}}\} \le V(u) \le C_3\mu^{2\beta+1} + C_4\|w\|_\infty^{\frac{p-2}{4}}\mu^{\frac{p}{2}} \tag{2.6}$$

$$\min\{C_1\mu^{\beta+1}, C_2 g^{-\frac{p}{p-2}}\mu^{3\beta+2}\} \le \|u\|^2_{L^\infty(\mathcal{G})} \le C_3\mu^{\beta+1} + C_4\|w\|_\infty^{\frac{1}{2}}\mu \tag{2.7}$$

for some constants $C_1, C_2, C_3, C_4 > 0$, where $g := \|w\|_\infty |\mathcal{K}|^{\frac{p-2}{p}}$.

Proof. Let $u \in H^1_\mu(\mathcal{G})$ satisfy hypothesis Eq (2.4). If

$$W(u) < \frac{1}{2}\theta_p \mu^{2\beta+1}, \tag{2.8}$$

then by Eqs (2.4), (2.8), and (2.1)

$$\begin{aligned}\frac{1}{2}T(u)-\frac{1}{p}V(u) &= E(u,\mathcal{G})+\frac{1}{2}W(u) \ < \ \frac{1}{2}\mathcal{I}_{\mathcal{G}}(\mu)+\frac{1}{4}\theta_p\mu^{2\beta+1} \\ &\leq \ -\frac{1}{2}\theta_p\mu^{2\beta+1}+\frac{1}{4}\theta_p\mu^{2\beta+1} \ = \ -\frac{1}{4}\theta_p\mu^{2\beta+1}.\end{aligned} \tag{2.9}$$

So the estimates of Lemma 2.6 in [13] hold, namely

$$\begin{aligned} C_1\mu^{2\beta+1} &\leq T(u)\leq C_3\mu^{2\beta+1} \\ C_1\mu^{2\beta+1} &\leq V(u)\leq C_3\mu^{2\beta+1} \\ C_1\mu^{\beta+1} &\leq \|u\|^2_{L^\infty(\mathcal{G})}\leq C_3\mu^{\beta+1}. \end{aligned} \tag{2.10}$$

On the other hand, if

$$W(u)\geq \frac{1}{2}\theta_p\mu^{2\beta+1}, \tag{2.11}$$

then we use the inequality Eq (2.2) and obtain

$$V(u)\leq M_p\mu^{\frac{p+2}{4}}T(u)^{\frac{p-2}{4}}, \tag{2.12}$$

that, combined with (2.4) yields

$$T(u)-\frac{2M_p}{p}\mu^{\frac{p+2}{4}}T(u)^{\frac{p-2}{4}} < W(u)\leq \|w\|_\infty\mu.$$

Since $\frac{p-2}{4}<1$, using Young's inequality there exists $C_4>0$ such that

$$T(u)\leq C_4\|w\|_\infty\mu+C_4\mu^{2\beta+1},$$

thus by Eq (2.2)

$$V(u)\leq C_4\|w\|_\infty^{\frac{p-2}{4}}\mu^{\frac{p}{2}}+C_4\mu^{2\beta+1},$$

and by Eq (2.3)

$$\|u\|^2_{L^\infty(\mathcal{G})}\leq C_4\|w\|_\infty^{\frac{1}{2}}\mu+C_4\mu^{\beta+1}.$$

To obtain the lower bound of $V(u)$ we apply Proposition 2.4 with $r=2$. Denoting $g=\|w\|_\infty|\mathcal{K}|^{\frac{p-2}{p}}$ it holds that

$$W(u)\leq gV(u)^{\frac{2}{p}},$$

and from (2.11) one gets

$$C_2g^{-\frac{p}{2}}\mu^{(2\beta+1)\frac{p}{2}}\leq V(u).$$

Using (2.12) yields

$$C_2g^{-\frac{2p}{p-2}}\mu^{3(2\beta+1)}\leq T(u).$$

Finally, using the fact that $V(u)\leq\mu\|u\|_\infty^{p-2}$ we obtain

$$C_2g^{-\frac{p}{p-2}}\mu^{3\beta+2}\leq\|u\|^2_\infty.$$

Summing up, in the case Eq (2.11) we obtain the inequalities

$$\begin{aligned} C_2 g^{-\frac{2p}{p-2}}\mu^{3(2\beta+1)} &\le T(u) \le C_4\mu^{2\beta+1} + C_4\|w\|_\infty \mu \\ C_2 g^{-\frac{p}{2}}\mu^{(2\beta+1)\frac{p}{2}} &\le V(u) \le C_4\mu^{2\beta+1} + C_4\|w\|_\infty^{\frac{p-2}{4}}\mu^{\frac{p}{2}} \\ C_2 g^{-\frac{p}{p-2}}\mu^{3\beta+2} &\le \|u\|^2_{L^\infty(\mathcal{G})} \le C_4\mu^{\beta+1} + C_4\|w\|_\infty^{\frac{1}{2}}\mu. \end{aligned} \tag{2.13}$$

By Eqs (2.10) and (2.13), the proof is complete.

Remark 2.6. If u is a Ground State at mass μ, then it satisfies Eq (2.4), so the estimates in Proposition 2.5 hold. Moreover, from Eq (2.8) one sees that the effect of the potential does not appear in the estimates if $\mu^{2\beta} > 2\frac{\|w\|_\infty}{\theta_p}$. Then, the presence of the potential is effective in the small mass regime.

Remark 2.7. We shall use Proposition 2.5 in the proof of Theorem 5.1. We need, however, the inequalities Eq (2.10) only, since the hypotheses of the theorem imply the assumption Eq (2.8). Nonetheless, we chose to prove the more general inequalities Eqs (2.5)–(2.7) for the sake of completeness.

In the proof of Theorem 5.1 we shall need the notion of monotone rearrangement of a function on a metric graph. Such a notion was first introduced in [28], where the monotone rearrangement is defined for functions on metric graphs by mimicking the analogous classical notion for functions on an interval (see e.g., [29]). Moreover, in [28] the Pólya-Szegő inequality was proved too, stating that the monotone rearrangement does not increase the kinetic energy. In [12] the theory was extended to the symmetric rearrangement. Here we shall need the monotone rearrangement only, so we can directly quote the part of Proposition 3.1 in [12] which is relevant for the proof of Theorem 5.1.

Proposition 2.8 (Monotone rearrangement)**.** *Let $\mathcal{G}$ be a connected metric graph and let $u \in H^1(\mathcal{G})$ be nonnegative. Denote by u^* the monotone rearrangement of u as defined in [12, 28] on the interval $I^* = [0, |\mathcal{G}|)$, where $|\mathcal{G}|$ denotes the total length of $\mathcal{G}$, i.e., the sum, possibly infinite, of the lengths of all edges of $\mathcal{G}$. Then, $u^* \in H^1(I^*)$ and*

$$\|u^*\|_{L^r(I^*)} = \|u\|_{L^r(\mathcal{G})}, \quad r \in [1, +\infty] \tag{2.14}$$

$$\|(u^*)'\|_{L^2(I^*)} \le \|u'\|_{L^2(\mathcal{G})}. \tag{2.15}$$

Notice that for a general complex-valued function $u \in H^1(\mathcal{G})$, one has $\||u|\|_{L^r(\mathcal{G})} = \|u\|_{L^r(\mathcal{G})}$ and $\||u|'\|_{L^2(\mathcal{G})} \le \|u'\|_{L^2(\mathcal{G})}$. Thus, setting $u^* = |u|^*$ both identity (2.14) and Pólya-Szegő inequality Eq (2.15) hold.

Remark 2.9. (Potential induced by curvature) From Remark 2.3 in [13] one has that if $u \in H^1(\mathcal{G})$, then the quantities

$$\mu^{-2\beta-1}\|u'\|^2_{L^2(\mathcal{G})}, \quad \mu^{-2\beta-1}\|u\|^p_{L^p(\mathcal{G})}, \quad \mu^{-\beta-1}\|u\|^2_{L^\infty(\mathcal{G})}$$

are invariant under the following rescaling of $\mathcal{G}$ and u:

$$\mathcal{G} \mapsto t^{-\beta}\mathcal{G}, \quad u(\cdot) \mapsto t^\alpha u(t^\beta \cdot).$$

Notice that u is rescaled with the mass as solitons do. The potential term shows the same invariance if one imposes the scaling

$$w(\cdot) \to t^{2\beta} w(\cdot), \tag{2.16}$$

i.e., if it scales as the inverse of the square of a length. In fact

$$\begin{aligned}\int_{\mathcal{K}} w(x)|u(x)|^2 dx &\mapsto \int_{t^{-\beta}\mathcal{K}} t^{2\beta} w(x)|t^{\alpha} u(t^{\beta} x)|^2 t^{-\beta} d(t^{\beta} x)\\ &= t^{\beta+2\alpha} \int_{\mathcal{K}} w(x)|u(x)|^2 dx\end{aligned}$$

and since $\beta + 2\alpha = 2\beta + 1$ from (1.5), one recovers the same scaling law and thus the quantity $\mu^{-2\beta-1} E(u, \mathcal{G})$ is invariant while the mass is mapped from μ to $t\mu$.

Of course, for a generic potential the scaling assumption (2.16) is meaningless. Yet, there is one special case in which it is the correct scaling law, i.e., the already mentioned case of the potential $-\gamma^2(x)/4$ induced by the presence of a curvature in the edges. Indeed the curvature γ scales as the inverse of a length, namely $\gamma \mapsto t^{\beta}\gamma$, which is consistent with the fact that, by definition, the curvature is the inverse of the radius of the osculating circle.

3. Existence criterion for $E(\cdot, \mathcal{G})$

Here, we extend the existence criterion given in Sec. 1 to the functional (1.3) by proving the following lemma.

Lemma 3.1 (Existence criterion for $E(\cdot, \mathcal{G})$)**.** *Fix $2 < p < 6$ and let $\mathcal{G}$ be a metric graph with n infinite edges, $n \geq 1$. If there exists $v \in H^1_{\mu}(\mathcal{G})$ such that*

$$E(v, \mathcal{G}) \leq E_{\mathrm{NLS}}(\phi_{\mu}, \mathbb{R}), \tag{3.1}$$

then $E(\cdot, \mathcal{G})$ admits a Ground State at mass μ.

In order to prove Lemma 3.1, we follow the line of Section 3 in [13]. Thus, as a first step we show that the strict concavity of $\mathcal{I}_{\mathcal{G}}$ as a function of μ is preserved in the presence of the potential term.

Proposition 3.2. *$\mathcal{I}_{\mathcal{G}}$ is strictly concave and strictly subadditive as a function of the mass μ.*

Proof. Consider a sequence $\{v_n\} \subset H^1_{\mu}(\mathcal{G})$ that minimizes $E(\cdot, \mathcal{G})$ at mass μ. Since $\mathcal{I}_{\mathcal{G}}(\mu) < 0$, eventually $E(v_n, \mathcal{G}) \leq \frac{1}{2}\mathcal{I}_{\mathcal{G}}(\mu)$. Let us restrict to the elements of the sequence that satisfy such inequality.

By Proposition 2.5, one has $V(v_n) \geq \min(C_1\mu^{2\beta+1}, C_2 g^{-\frac{p}{2}} \mu^{(2\beta+1)\frac{p}{2}})$. Let us now introduce the sequence $\{u_n\} \subset H^1_1(\mathcal{G})$, defined as $u_n = \frac{v_n}{\sqrt{\mu}}$. Since $V(u_n) = \mu^{-\frac{p}{2}} V(v_n)$, one has

$$\mu^{\frac{p}{2}} V(u_n) = V(v_n) \geq \min(C_1\mu^{2\beta+1}, C_2 g^{-\frac{p}{2}} \mu^{(2\beta+1)\frac{p}{2}}),$$

that implies that every function u_n belongs to the set

$$U := \left\{ u \in H^1(\mathcal{G}),\ \int_{\mathcal{G}} |u|^2 dx = 1,\ \mu^{\frac{p}{2}} V(u) \geq \min(C_1\mu^{2\beta+1},\ C_2 g^{-\frac{p}{2}} \mu^{(2\beta+1)\frac{p}{2}}) \right\},$$

with C_1, C_2 and g as in Proposition 2.5. Then we consider the family of functions f_u defined by

$$f_u(\mu) := E(\sqrt{\mu}u, \mathcal{G}) = \frac{\mu}{2}\int_{\mathcal{G}} |u'|^2 dx - \frac{\mu^{\frac{p}{2}}}{p} V(u) - \frac{\mu}{2}\int_{\mathcal{K}} w(x)|u|^2 dx, \quad u \in U,$$

so that the following identities hold:

$$\mathcal{I}_{\mathcal{G}}(\mu) = \lim_{n\to\infty} E(v_n, \mathcal{G}) = \lim_{n\to\infty} f_{u_n}(\mu), \tag{3.2}$$

thus

$$\mathcal{I}_{\mathcal{G}}(\mu) = \inf_{u\in U} f_u(\mu).$$

Now, since $u \in U$ implies $V(u) > 0$,

$$f_u''(\mu) = -\frac{p-2}{4}\mu^{\frac{p}{2}-2}V(u) < 0,$$

which implies that $\mathcal{I}_{\mathcal{G}}$ is concave. Furthermore, since f_u is uniformly strictly concave, $\mathcal{I}_{\mathcal{G}}$ is strictly concave too on every interval $[a, b] \subset (0, \infty)$ and also strictly subadditive since $\mathcal{I}_{\mathcal{G}}(0) = 0$.

Now we analyze the behaviour of the minimizing sequences.

Proposition 3.3. *Any minimizing sequence $\{u_n\} \subset H^1_\mu(\mathcal{G})$ for the functional $E(\cdot, \mathcal{G})$ defined in* (1.1) *with $2 < p < 6$, is weakly compact in $H^1(\mathcal{G})$.*

Proof. Using Eq (2.2), the fact that $2 < p < 6$ and a straightforward estimate for the potential term, we get the lower bound

$$E(u, \mathcal{G}) \geq \frac{1}{2}\|u_n'\|^2_{L^2(\mathcal{G})} - \frac{M_p}{p}\mu^{\frac{p}{4}+\frac{1}{2}}\|u_n'\|^{\frac{p}{2}-1}_{L^2(\mathcal{G})} - \frac{\mu}{2}\|w\|_\infty,$$

so that $\|u_n'\|_{L^2(\mathcal{G})}$ is bounded, otherwise the sequence $E(u_n, \mathcal{G})$ would diverge as $p < 6$ and the sequence u_n could not be minimizing. Since $\|u_n\|_{L^2(\mathcal{G})} = \sqrt{\mu}$, the sequence is bounded in $H^1(\mathcal{G})$ and then weakly compact by the Banach-Alaoglu Theorem.

In the next result we characterise the behaviour of weakly convergent minimizing sequences, to which we reduce owing to Proposition 3.3.

Proposition 3.4. *If $\{u_n\}$ is a minimizing sequence for the functional $E(\cdot, \mathcal{G})$ at mass μ, and $u_n \rightharpoonup u$ in $H^1(\mathcal{G})$, then one of the two following cases occurs:*
(i) $u_n \to 0$ in $L^\infty_{loc}(\mathcal{G})$ and $u \equiv 0$;
(ii) $u \in H^1_\mu(\mathcal{G})$ is a minimizer and $u_n \to u$ strongly in $H^1(\mathcal{G}) \cap L^p(\mathcal{G})$.

Proof. The sequence u_n converges weakly to u in $L^2(\mathcal{G})$ too, so let

$$m = \mu - \|u\|^2_{L^2(\mathcal{G})} \in [0, \mu]$$

be the loss of mass in the limit. If $m = \mu$, then case (i) occurs because $u_n \to 0$ strongly in $L^\infty_{loc}(\mathcal{G})$ and so $u \equiv 0$.

Furthermore, following [13], we can state that the case $0 < m < \mu$ never occurs. Indeed, first we note that from the fact that $u_n \rightharpoonup u$ in $L^2(\mathcal{G})$ one has

$$\|u_n - u\|^2_{L^2(\mathcal{G})} = \|u_n\|^2_{L^2(\mathcal{G})} + \|u\|^2_{L^2(\mathcal{G})} - 2\Re(u_n, u) \to m.$$

Moreover, using Brezis-Lieb's lemma [30] one gets

$$\frac{1}{p}\int_{\mathcal{G}}|u_n|^p dx-\frac{1}{p}\int_{\mathcal{G}}|u_n-u|^p dx-\frac{1}{p}\int_{\mathcal{G}}|u|^p dx=o(1),$$

$$\frac{1}{2}\int_{\mathcal{G}}|u_n'|^2 dx-\frac{1}{2}\int_{\mathcal{G}}|u_n'-u'|^2 dx-\frac{1}{2}\int_{\mathcal{G}}|u'|^2 dx=o(1),$$

$$\frac{1}{2}\int_{\mathcal{K}}w(x)|u_n|^2 dx-\frac{1}{2}\int_{\mathcal{K}}w(x)|u_n-u|^2 dx-\frac{1}{2}\int_{\mathcal{K}}w(x)|u|^2 dx=o(1).$$

Therefore

$$E(u_n,\mathcal{G})=E(u_n-u,\mathcal{G})+E(u,\mathcal{G})+o(1)$$

as $n\to\infty$. Now, since by concavity $\mathcal{I}_{\mathcal{G}}$ is continuous, one obtains

$$\mathcal{I}_{\mathcal{G}}(\mu)\ \geq\ \mathcal{I}_{\mathcal{G}}(m)+E(u,\mathcal{G})\ \geq\ \mathcal{I}_{\mathcal{G}}(m)+\mathcal{I}_{\mathcal{G}}(\|u\|^2_{L^2(\mathcal{G})}).$$

From Proposition 3.2, the function $\mathcal{I}_{\mathcal{G}}$ is strictly subadditive, therefore if $0<m<\mu$ then

$$\mathcal{I}_{\mathcal{G}}(\mu)<\mathcal{I}_{\mathcal{G}}(\mu-\|u\|^2_{L^2(\mathcal{G})})+\mathcal{I}_{\mathcal{G}}(\|u\|^2_{L^2(\mathcal{G})}).$$

which is a contradiction.

If $m=0$ then $u_n\to u$ strongly in $L^2(\mathcal{G})$. Now, since $p>2$ and $\|u_n\|_{L^\infty}\leq C$, from

$$\|u_n-u\|^p_{L^p(\mathcal{G})}\leq\|u_n-u\|^{p-2}_{L^\infty(\mathcal{G})}\|u_n-u\|^2_{L^2(\mathcal{G})}\leq C\|u_n-u\|^2_{L^2(\mathcal{G})}\to 0, \tag{3.3}$$

one gets $u_n\to u$ strongly in $L^p(\mathcal{G})$ too.

Furthermore

$$\begin{aligned}\int_{\mathcal{K}}w(x)\left||u_n(x)|^2-|u(x)|^2\right|dx\ &\leq C\int_{\mathcal{K}}w(x)|u_n(x)-u(x)|^2 dx\\ &\leq C\|w\|_\infty\|u_n-u\|^2_{L^2(\mathcal{G})}\to 0\end{aligned} \tag{3.4}$$

so $W(u_n)$ converges to $W(u)$.

Then, from Eqs (3.3), (3.4), and the fact that u_n is a minimizing sequence,

$$\begin{aligned}\|u_n'\|^2_{L^2(\mathcal{G})}\ &=E(u_n,\mathcal{G})+\frac{1}{p}\|u_n\|^p_{L^p(\mathcal{G})}+\frac{1}{2}\int_{\mathcal{K}}w(x)|u_n(x)|^2 dx\\ &\to\mathcal{I}(\mu)++\frac{1}{p}\|u\|^p_{L^p(\mathcal{G})}+\frac{1}{2}\int_{\mathcal{K}}w(x)|u(x)|^2 dx\\ &\leq E(u,\mathcal{G})+\frac{1}{p}\|u\|^p_{L^p(\mathcal{G})}+\frac{1}{2}\int_{\mathcal{K}}w(x)|u(x)|^2 dx\ =\ \|u'\|^2_{L^2(\mathcal{G})}.\end{aligned} \tag{3.5}$$

where the last passage exploits $u\in H^1_\mu(\mathcal{G})$. It follows

$$\lim\|u_n'\|^2_{L^2(\mathcal{G})}\ \leq\ \|u'\|^2_{L^2(\mathcal{G})}. \tag{3.6}$$

On the other hand, since u' is the weak limit in $L^2(\mathcal{G})$ of u_n', it must be

$$\liminf\|u_n'\|^2_{L^2(\mathcal{G})}\ \geq\ \|u'\|^2_{L^2(\mathcal{G})}. \tag{3.7}$$

From Eqs (3.6) and (3.7), one concludes $\lim\|u_n'\|^2_{L^2(\mathcal{G})}\ =\ \|u'\|^2_{L^2(\mathcal{G})}$, so u_n converges to u strongly in $H^1(\mathcal{G})$, and therefore $E(u,\mathcal{G})=\lim E(u_n,\mathcal{G})$ and u is a Ground State for $E(\cdot,\mathcal{G})$ at mass μ. The proof is complete.

We are ready to prove Lemma 3.1.

Proof of Lemma 3.1. Given a minimizing sequence $\{u_n\} \subset H^1_\mu(\mathcal{G})$ we need to exclude case (i) of Proposition 3.4. Assume (i). Then $u_n \to 0$ in $L^\infty_{loc}(\mathcal{G})$, that implies $u_n \to 0$ in $L^\infty(\mathcal{K})$. Since the potential is supported on $\mathcal{K}$, then $E(u_n, \mathcal{G}) = E_{\mathrm{NLS}}(u_n, \mathcal{G}) + o(1)$ and one can repeat the argument in Theorem 3.3 of [13]. Such argument shows that such a minimizing sequence leads to an energy level not lower than the threshold $-\theta_p \mu^{2\beta+1}$. Therefore, if there exists $v \in H^1_\mu(\mathcal{G})$ such that the condition (3.1) is satisfied, then case (i) of Proposition 3.4 is ruled out, so case (ii) is verified. Thus we can assume the existence of a minimizing sequence $\{u_n\}$ that strongly converges to u in $H^1 \cap L^p(\mathcal{G})$ and u is a Ground State.

4. Existence

It is well known that a graph $\mathcal{G}$ does not support ground states for $E_{\mathrm{NLS}}(\cdot, \mathcal{G})$ at any mass, provided that it satisfies the so-called Assumption H (see [12]), i.e., if every point of $\mathcal{G}$ belongs to a trail that contains two halflines. This is the case, for instance, of the graph in Figure (1), called the 2-bridge.

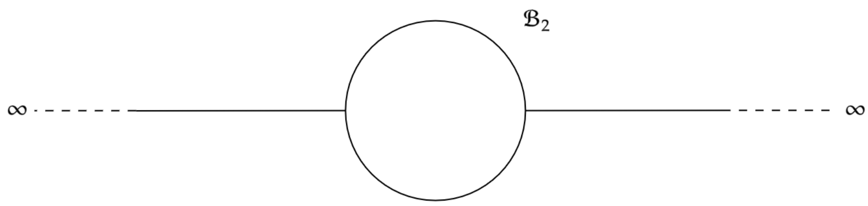

Figure 1. The 2-bridge graph satisfies the assumption (H) that prevents the existence of ground states for E_{NLS} at every mass.

On the other hand, the presence of an attractive potential can make it energetically convenient to concentrate the mass in the support of the potential and therefore, at least for some values of the mass, ground states can exist. In the present section we show that this is indeed the case if the mass is large or small enough. We achieve the results by directly applying Lemma 3.1, i.e., exhibiting a function of mass μ whose energy is below the threshold $-\theta_p \mu^{2\beta+1}$.

4.1. Existence for large mass

Here we give the first result on existence of ground states.

Theorem 4.1. *Let $\mathcal{G}$ be a graph with $n \geq 1$ halflines and let $w \geq 0$ be a continuous, non identically vanishing function supported on the compact core $\mathcal{K}$ of $\mathcal{G}$.*

If μ is large enough, then there exists a Ground State for $E(\cdot, \mathcal{G})$ at mass μ.

Proof. Consider a point $\bar{x}$ in $\mathcal{K}$ such that $\bar{x}$ is not a vertex of $\mathcal{G}$ and $w(\bar{x}) > 0$. Such a point exists, since otherwise by continuity w would be zero in all vertices too, and then identically zero. With a slight abuse of notation, we denote by $\bar{x}$ the coordinate of the point $\bar{x}$ as an element of the interval $I_{\bar{e}}$ that represents the edge $\bar{e}$ in which $\bar{x}$ lies. By continuity, there is $\ell > 0$ such that the interval

$[\bar{x} - \ell/2, \bar{x} + \ell/2]$ belongs to $\bar{e}$ and $w > 0$ in $[\bar{x} - \ell/2, \bar{x} + \ell/2]$. We denote $\kappa := \min_{[\bar{x}-\ell/2, \bar{x}+\ell/2]} w(x)$, so that $\kappa > 0$.

Consider now the family of functions $v_\mu \in H^1(\mathbb{R})$ defined in the following way:

$$v_\mu(x) = (\phi_\mu(x - \bar{x}) - \phi_\mu(\ell/2))\chi_{[\bar{x}-\ell/2, \bar{x}+\ell/2]}(x),$$

where χ_A denotes the characteristic function of the real subset A.

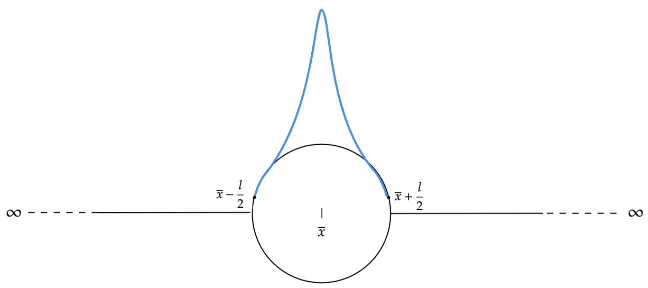

Figure 2. A pictorial representation of the function $\widetilde{f_\mu}$ on the 2-bridge graph.

One has

$$\begin{aligned}\int_{\mathbb{R}} |v_\mu|^2 dx &= \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} |\phi_\mu - \phi_\mu(\frac{\ell}{2})|^2 dx = 2\int_0^{\frac{\ell}{2}} |\phi_\mu - \phi_\mu(\frac{\ell}{2})|^2 dx \\ &= 2\int_0^{\frac{\ell}{2}} \phi_\mu(x)^2 dx + 2\int_0^{\frac{\ell}{2}} \phi_\mu(\frac{\ell}{2})^2 dx - 4\int_0^{\frac{\ell}{2}} \phi_\mu(x)\phi_\mu(\frac{\ell}{2}) dx \\ &= \mu - 2\int_{\frac{\ell}{2}}^{\infty} \phi_\mu(x)^2 dx + \ell\phi_\mu(\frac{\ell}{2})^2 - 4\phi_\mu(\frac{\ell}{2})\int_0^{\frac{\ell}{2}} \phi_\mu(x) dx.\end{aligned} \tag{4.1}$$

From the explicit form of the soliton (1.4), since $\ell > 0$ one immediately has $\phi_\mu(\ell/2) \le C\mu^\alpha e^{-c\mu^\beta}$. Moreover, since $2\alpha - \beta = 1$,

$$\int_{\frac{\ell}{2}}^{\infty} \phi_\mu(x)^2 dx \le \phi_\mu(\frac{\ell}{2}) \int_0^{\infty} \phi_\mu(x) dx \le C\mu e^{-c\mu^\beta}, \tag{4.2}$$

where C and c are positive constants independent of μ. Then, recalling that v_μ is a cut and lowered version of ϕ_μ, and that in (4.1) both $\phi_\mu(\ell/2)$ and $\phi_\mu^2(\ell/2)$ appear, we conclude

$$0 \le \mu - \|v_\mu\|^2_{L^2(\mathbb{R})} \le C(\mu^{2\alpha} + \mu)e^{-c\mu^\beta}. \tag{4.3}$$

Furthermore,

$$\|v_\mu - \phi_\mu(\cdot - \bar{x})\|^2_{L^2(\mathbb{R})} = 2\int_0^{\ell/2} \phi_\mu^2(\ell/2) dx + 2\int_{\ell/2}^{\infty} \phi_\mu^2(x)\, dx \le C(\mu^{2\alpha} + \mu)e^{-c\mu^\beta}, \tag{4.4}$$

and

$$\|v'_\mu - \phi'_\mu(\cdot - \bar{x})\|^2_{L^2(\mathbb{R})} = 2\int_{\ell/2}^{\infty} (\phi'_\mu)^2(x)\, dx \le C\mu^{2\beta} \int_{\ell/2}^{\infty} \phi_\mu^2(x)\, dx \le C\mu^{2\beta+1} e^{-c\mu^\beta} \tag{4.5}$$

where the last estimate is obtained from inequality

$$|\phi'_\mu(x)| \;=\; C\mu^{\alpha+\beta}\frac{|\sinh(\mu^\beta c_p x)|}{\cosh^{\frac{\alpha}{\beta}+1}(\mu^\beta c_p x)} \leq C\mu^\beta \phi_\mu(x),$$

and by Eq (4.2).

Now we define the function $f_\mu = \frac{\sqrt{\mu}}{\|v_\mu\|_{L^2(\mathbb{R})}} v_\mu$ and compute

$$\begin{aligned} \|f_\mu - v_\mu\|_{L^2(\mathbb{R})} &= \sqrt{\mu} - \|v_\mu\|_{L^2(\mathbb{R})} \;=\; \frac{\mu - \|v_\mu\|^2_{L^2(\mathbb{R})}}{\sqrt{\mu} + \|v_\mu\|_{L^2(\mathbb{R})}} \;\leq\; C(\sqrt{\mu} + \mu^{2\alpha-\frac{1}{2}})e^{-c\mu^\beta} \\ \|f'_\mu - v'_\mu\|_{L^2(\mathbb{R})} &\leq (\sqrt{\mu} - \|v_\mu\|_{L^2(\mathbb{R})})\frac{\|v'_\mu\|_{L^2(\mathbb{R})}}{\|v_\mu\|_{L^2(\mathbb{R})}} \;\leq\; C(\sqrt{\mu} + \mu^{2\alpha-\frac{1}{2}})e^{-c\mu^\beta}\frac{\|v'_\mu\|_{L^2(\mathbb{R})}}{\|v_\mu\|_{L^2(\mathbb{R})}}. \end{aligned} \tag{4.6}$$

From Eq (4.2)

$$\|v_\mu\|_{L^2(\mathbb{R})} \geq \sqrt{\mu - C(\mu + \mu^{2\alpha})e^{-c\mu^\beta}},$$

while, since $\|\phi'_\mu\|_{L^2(\mathbb{R})} \leq C\mu^{\beta+\frac{1}{2}}$, from Eq (4.5) one gets

$$\|v'_\mu\|_{L^2(\mathbb{R})} \;\leq\; \|\phi'_\mu\|_{L^2(\mathbb{R})} + C\mu^{\beta+\frac{1}{2}}e^{-c\mu^\beta} \;\leq\; C\mu^{\beta+\frac{1}{2}}(1 + e^{-c\mu^\beta}).$$

Thus, for the second inequality in Eq (4.6) one gets

$$\|f'_\mu - v'_\mu\|_{L^2(\mathbb{R})} \leq\; C(\sqrt{\mu} + \mu^{2\alpha-\frac{1}{2}})e^{-c\mu^\beta}\frac{\mu^{\beta+\frac{1}{2}}(1 + e^{-c\mu^\beta})}{\sqrt{\mu - C(\mu + \mu^{2\alpha})e^{-c\mu^\beta}}}, \tag{4.7}$$

where we used Eqs (4.2)–(4.5). From Eqs (4.6) and (4.7) one then concludes that both $\|f_\mu - v_\mu\|_{L^2(\mathbb{R})}$ and $\|f'_\mu - v'_\mu\|$ vanish as μ goes to infinity.

Furthermore, from Eqs (2.2) and (2.3)

$$\|f_\mu - v_\mu\|^p_{L^p(\mathbb{R})} \leq C\|f'_\mu - v'_\mu\|^{\frac{p}{2}-1}_{L^2(\mathbb{R})}\|f_\mu - v_\mu\|^{\frac{p}{2}+1}_{L^2(\mathbb{R})} \to 0, \quad \mu \to \infty. \tag{4.8}$$

Let us introduce the function $\widetilde{f_\mu}$, defined as f_μ on the edge $\bar{e}$ and zero on all other edges. Obviously, $\widetilde{f_\mu}$ belongs to $H^1(\mathcal{G})$ and its $L^2(\mathcal{G})$, $L^p(\mathcal{G})$, and $H^1(\mathcal{G})$ norms are the same as the corresponding ones of f_μ as a function on $\mathbb{R}$. Now, from Eqs (4.6) and (4.5) one has

$$\begin{aligned} \Big|\|\widetilde{f'_\mu}\|_{L^2(\mathcal{G})} - \|\phi'_\mu\|_{L^2(\mathbb{R})}\Big| &= \Big|\|f'_\mu\|_{L^2(\mathbb{R})} - \|\phi'_\mu\|_{L^2(\mathbb{R})}\Big| \leq \|f'_\mu - \phi'_\mu\|_{L^2(\mathbb{R})} \\ &\leq \|f'_\mu - v'_\mu\|_{L^2(\mathbb{R})} + \|v'_\mu - \phi'_\mu\|_{L^2(\mathbb{R})} \to 0, \quad \mu \to \infty, \end{aligned} \tag{4.9}$$

and analogously from Eq (4.8)

$$\|\widetilde{f_\mu}\|^p_{L^p(\mathcal{G})} - \|\phi_\mu\|^p_{L^p(\mathbb{R})} \to 0, \quad \mu \to \infty. \tag{4.10}$$

By Eqs (4.9) and (4.10)

$$\begin{aligned} E(\widetilde{f_\mu}, \mathcal{G}) - E_{\mathrm{NLS}}(\phi_\mu, \mathbb{R}) &= E_{\mathrm{NLS}}(\widetilde{f_\mu}, \mathcal{G}) - E_{\mathrm{NLS}}(\phi_\mu, \mathbb{R}) - \frac{1}{2}\int_{\mathcal{G}} w(x)|\widetilde{f_\mu}(x)|^2 dx \\ &\leq -\frac{\kappa}{2}\int_{-\ell/2}^{-\ell/2} \phi^2_\mu(x)\, dx + o(1) \\ &\leq -\frac{\kappa}{2}\mu + o(1), \quad \mu \to \infty, \end{aligned}$$

so $E(\widetilde{f_\mu}, \mathcal{G}) < E_{\mathrm{NLS}}(\phi_\mu, \mathbb{R})$ for μ large enough and by Lemma 3.1 the proof is complete.

4.2. Existence for small mass

Here we give the second result on existence of ground states.

Theorem 4.2. *Let $\mathcal{G}$ be a graph with $n \geq 1$ infinite edges and let $w \geq 0$ be a continuous, non identically vanishing function supported on the compact core $\mathcal{K}$ of $\mathcal{G}$.*

If μ is small enough, then there exists a Ground State for $E(\cdot, \mathcal{G})$ at mass μ.

Proof. Let $\mu > 0$. We define the function u_μ as follows:

$$u_\mu(x) = \begin{cases} \phi_m & \text{if } \ x \in \mathcal{H}_i, \ i = 1, \dots, n \\ \phi_m(0) & \text{if } \ x \in \mathcal{K}, \end{cases}$$

where $\mathcal{H}_i$ represents the halfline associated with the index i. The parameter m is uniquely determined by imposing $\|u_\mu\|^2_{L^2(\mathbb{R})} = \mu$, namely by the identity

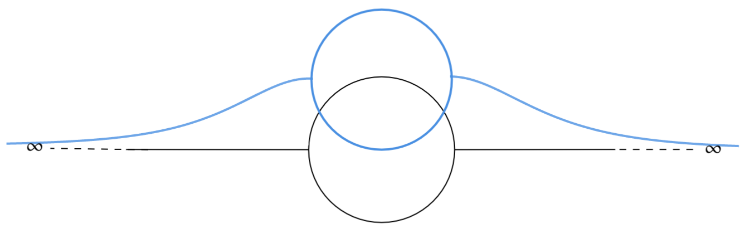

Figure 3. A representation of the function u_μ on the two-bridge graph.

$$\mu = \int_{\mathcal{G}} |u_\mu|^2 dx = \frac{n}{2} m + |\phi_m(0)|^2 |\mathcal{K}| = \frac{n}{2} m + C_p^2 |\mathcal{K}| m^{2\alpha},$$

where $|\mathcal{K}|$ is the total length of $\mathcal{K}$, while the energy of u_μ reads

$$E(u_\mu, \mathcal{G}) = -\frac{n}{2}\theta_p m^{2\beta+1} - \frac{1}{p} C_p^p m^{p\alpha} |\mathcal{K}| - C_p^2 m^{2\alpha} \frac{|\mathcal{K}|}{2} \int_{\mathcal{K}} w(x) dx.$$

In order to be less energetic than the soliton on the line with the same mass, u_μ must satisfy the condition

$$-\frac{n}{2}\theta_p m^{2\beta+1} - \frac{1}{p} C_p^p |\mathcal{K}| m^{p\alpha} - C_p^2 \frac{|\mathcal{K}|}{2} m^{2\alpha} \int_{\mathcal{K}} w(x) dx < -\theta_p \left(\frac{n}{2} m + C_p^2 |\mathcal{K}| m^{2\alpha}\right)^{2\beta+1}. \tag{4.11}$$

Since 2α is the smallest exponent in inequality Eq (4.11), it turns out that the inequality is satisfied if μ is small enough, thus by Lemma 3.1 a Ground State exists and the theorem is proved.

5. Nonexistence

Here we extend the nonexistence result given in Theorem 5.1 of [13] to the case of the presence of a weak, attractive, compactly supported potential and give sufficient conditions for the nonexistence of ground states in some intervals of the values of the mass.

Theorem 5.1. *Let $\mathcal{G}$ be a graph with $n \geq 1$ infinite edges and let $w \geq 0$ be a continuous, non identically vanishing function supported on the compact core $\mathcal{K}$ of $\mathcal{G}$. Furthermore denote by $|\mathcal{K}|$ the total length of $\mathcal{K}$ and by* $\mathrm{diam}(\mathcal{K})$ *its diameter, i.e., the maximal distance between any pair of points of $\mathcal{K}$.*

Then there exists a number $\epsilon > 0$, that depends on p only, such that, if

$$\max\left(\mu^{\beta}\mathrm{diam}(\mathcal{K}), \frac{1}{\mu^{\beta}|\mathcal{K}|}, \frac{\|w\|_\infty}{\mu^{2\beta}}\right) < \epsilon, \tag{5.1}$$

are satisfied, then the functional $E(\cdot,\mathcal{G})$ defined in (1.1) *has no Ground State at mass μ.*

Proof. We proceed by contradiction, thus we consider $\mu > 0$ that satisfies the condition Eq (5.1) and suppose that there exists a Ground State u at mass μ for $E(\cdot,\mathcal{G})$. Due to the invariance of $E(\cdot,\mathcal{G})$ under multiplication by a phase, we can assume without loss of generality that u is real and nonnegative (see e.g., Section 1 in [12]).

First we assume that $\mathcal{G}$ contains only one halfline, i.e., $n = 1$. Taking $\epsilon < \theta_p/2$, condition Eq (5.1) guarantees that inequality Eq (2.8) holds, so that estimates Eq (2.10) are valid. It is therefore possible to follow the proof of Theorem 5.1 in [13] replacing $E(\phi_\mu,\mathbb{R})$ by $E_{\mathrm{NLS}}(\phi_\mu,\mathbb{R})$ up to the last inequality, which has to be rephrased including the contribution of the potential, i.e.,

$$\begin{aligned}
&E(u,\mathcal{G}) - E_{\mathrm{NLS}}(\phi_\mu,\mathbb{R})\\
\geq&\left(C_1 - C_2\epsilon^{\frac{p-2}{2}}\right)\mu^{2\beta}\int_{\mathcal{K}}|u|^2dx + \frac{1}{2}\int_{\mathcal{K}}|u'|^2dx - C_3\mu^{\beta}\|u\|^2_{L^\infty(\mathcal{K})} - \frac{1}{2}\int_{\mathcal{K}}w(x)|u|^2dx\\
\geq&\left(C_1 - C_2\epsilon^{\frac{p-2}{2}}\right)\mu^{2\beta}\int_{\mathcal{K}}|u|^2dx + \frac{1}{2}\int_{\mathcal{K}}|u'|^2dx - C_3\mu^{\beta}\|u\|^2_{L^\infty(\mathcal{K})} - \frac{1}{2}\|w\|_\infty\int_{\mathcal{K}}|u|^2dx\\
\geq&\left(C_1 - C_2\epsilon^{\frac{p-2}{2}} - \frac{\epsilon}{2}\right)\mu^{2\beta}\int_{\mathcal{K}}|u|^2dx + \frac{1}{2}\int_{\mathcal{K}}|u'|^2dx - C_3\mu^{\beta}\|u\|^2_{L^\infty(\mathcal{K})}.
\end{aligned}$$

Using Formula (40) in [13] to estimate $\|u\|_{L^\infty(\mathcal{K})}$ one finds

$$\begin{aligned}
&E(u,\mathcal{G}) - E_{\mathrm{NLS}}(\phi_\mu,\mathbb{R})\\
\geq&\left(C_1 - C_2\epsilon^{\frac{p-2}{2}} - \frac{\epsilon}{2} - 2C_3\epsilon\right)\mu^{2\beta}\int_{\mathcal{K}}|u|^2dx + \left(\frac{1}{2} - 2C_3\epsilon\right)\int_{\mathcal{K}}|u'|^2dx\\
>&\,0,
\end{aligned}$$

uniformly in μ and provided that ϵ is small enough. By Remark 2.2 this contradicts the fact that u is a Ground State of $E(\cdot,\mathcal{G})$, and the proof is complete for the case $n = 1$.

Suppose now $n > 1$. Let $\mathcal{H}_{\tilde{\imath}}$ be the halfline in which u attains $\max_{\mathcal{G}\setminus\mathcal{K}} u$, and let us call $\widetilde{x}$ the corresponding maximum point on $\mathcal{H}_{\tilde{\imath}}$. Moreover, let us define the halfline $\widetilde{\mathcal{H}}$ as the subset of $\mathcal{H}_{\tilde{\imath}}$ corresponding to the coordinate interval $[\widetilde{x}, +\infty)$. It is convenient to use on $\widetilde{\mathcal{H}}$ the coordinate system inherited by $\mathcal{H}_{\tilde{\imath}}$ ranging from $\widetilde{x}$ to $+\infty$.

For any $i \neq \tilde{\imath}$ let us set

$$\widetilde{y}_i = \min\{x \in [\widetilde{x}, +\infty),\ u_{\tilde{\imath}}(x) = u_i(0)\}, \tag{5.2}$$

which is well-defined since by definition of $\widetilde{x}$, for every $i \neq \tilde{\imath}$ one has $u_i(0) \leq u_{\tilde{\imath}}(\widetilde{x})$, thus by continuity of $u_{\tilde{\imath}}$ there exists at least a point $\widetilde{z} \in \widetilde{\mathcal{H}}$ such that $u_{\tilde{\imath}}(\widetilde{z}) = u_i(0)$. The symbol $\widetilde{y}_i$ denotes then the minimum of such $\widetilde{z}$'s.

Now for every $i \neq \widetilde{\imath}$ we attach the origin of $\mathcal{H}_i$ to the point of coordinate $\widetilde{y}_i$ in the halfine $\widetilde{\mathcal{H}}$. We obtain in this way a graph $\widehat{\mathcal{G}}$, made of one halfline ($\widetilde{\mathcal{H}}$), to which are attached by their origins $n-1$ other halflines ($\mathcal{H}_i,\ i \neq \widetilde{\imath}$).

Let us consider the function $\widehat{u} : \widehat{\mathcal{G}} \to \mathbb{R}$, which is made of the restrictions of u to the halflines that constitute the graph $\widehat{\mathcal{G}}$. In symbols, $\widehat{u} = (\widehat{u_{\widetilde{\imath}}}, \widehat{u}_i)_{i\neq\widetilde{\imath}}$, with $\widehat{u_{\widetilde{\imath}}}(x) = u_{\widetilde{\imath}}(x)$ for every $x \in [\widetilde{x}, +\infty)$ and $\widehat{u}_i(x) = u_i(x)$ for every $x \in [0, +\infty)$. Since the restriction of $\widehat{u}$ to every halfline is in H^1, and since by definition of the points $\widetilde{y}_i$ the function $\widehat{u}$ is continuous at the vertices of $\widehat{\mathcal{G}}$, it follows that $\widehat{u} \in H^1(\widehat{\mathcal{G}})$. Moreover, $\widehat{\mathcal{G}}$ is connected and therefore one can apply Proposition 2.8 and then define the monotone rearrangement u^* of $\widehat{u}$, that is defined on $[0, +\infty)$.

Now we define the graph $\mathcal{G}'$ as the original graph $\mathcal{G}$, but with $\mathcal{H}_{\widetilde{\imath}}$ as the only halfline, i.e.,

$$\mathcal{G}' = \mathcal{G}\backslash(\cup_{i\neq\widetilde{\imath}}\mathcal{H}_i) = \mathcal{K} \cup \mathcal{H}_{\widetilde{\imath}},$$

and construct on it the function $v : \mathcal{G}' \to \mathbb{R}$ as

$$v(x) := \begin{cases} u(x), & x \in \mathcal{K} \\ u_{\widetilde{\imath}}(x), & x \in [0, \widetilde{x}] \subset \mathcal{H}_{\widetilde{\imath}} \\ u^*(x - \widetilde{x}), & x \in (\widetilde{x}, +\infty) \subset \mathcal{H}_{\widetilde{\imath}}. \end{cases}$$

From Proposition 2.8 the monotone rearrangement preserves the L^p-norms (see identity Eq (2.14)), then

$$\begin{aligned} \|v\|^r_{L^r(\mathcal{G}')} =& \|v\|^r_{L^r(\mathcal{K})} + \|v\|^r_{L^r(\mathcal{H}_{\widetilde{\imath}})} = \|v\|^r_{L^r(\mathcal{K})} + \int_0^{\widetilde{x}} |v_{\widetilde{\imath}}|^r dx + \int_{\widetilde{x}}^{+\infty} |v_{\widetilde{\imath}}|^r dx \\ =& \|u\|^r_{L^r(\mathcal{K})} + \int_0^{\widetilde{x}} |u_{\widetilde{\imath}}|^r dx + \int_{\widetilde{x}}^{+\infty} |u^*(x-\widetilde{x})|^r dx \\ =& \|u\|^r_{L^r(\mathcal{K})} + \int_0^{\widetilde{x}} |u_{\widetilde{\imath}}|^r dx + \sum_{i\neq\widetilde{\imath}} \|u\|^r_{L^r(\mathcal{H}_i)} + \int_{\widetilde{x}}^{+\infty} |u_{\widetilde{\imath}}|^r dx \\ =& \|u\|^r_{L^r(\mathcal{K})}, \end{aligned} \tag{5.3}$$

for every $r \in [1, +\infty]$. In particular, for $r = 2$ one has $\|v\|^2_{L^2(\mathcal{G}')} = \mu$.

Now, due to Eq (5.3) and to $u \equiv v$ on $\mathcal{K}$, the difference $E(u, \mathcal{G}) - E(v, \mathcal{G}')$ reduces to $T(u) - T(v)$ outside $\mathcal{K}$, i.e., on the halflines only. Thus

$$\begin{aligned} & E(u, \mathcal{G}) - E(v, \mathcal{G}') \\ &= \sum_{i\neq\widetilde{\imath}} \|u_i'\|^2_{L^2(\mathcal{H}_i)} + \int_0^{\widetilde{x}} |u_{\widetilde{\imath}}'|^2 dx + \int_{\widetilde{x}}^{+\infty} |u_{\widetilde{\imath}}'|^2 dx - \int_0^{\widetilde{x}} |v_{\widetilde{\imath}}'|^2 dx - \int_{\widetilde{x}}^{+\infty} |v_{\widetilde{\imath}}'|^2 dx \\ &= \sum_{i\neq\widetilde{\imath}} \|u_i'\|^2_{L^2(\mathcal{H}_i)} + \int_{\widetilde{x}}^{+\infty} |u_{\widetilde{\imath}}'|^2 dx - \int_{\widetilde{x}}^{+\infty} |v_{\widetilde{\imath}}'|^2 dx \\ &= \sum_{i\neq\widetilde{\imath}} \|u_i'\|^2_{L^2(\mathcal{H}_i)} + \int_{\widetilde{x}}^{+\infty} |u_{\widetilde{\imath}}'|^2 dx - \int_{\widetilde{x}}^{+\infty} |(u^*)'(x-\widetilde{x})|^2 dx \\ &\geq 0, \end{aligned} \tag{5.4}$$

where in the last passage we used inequality Eq (2.15).

Then, since u is supposed to be a Ground State for $E(\cdot, \mathcal{G})$, by Eq (5.4) it must be

$$E(v, \mathcal{G}') \leq E(u, \mathcal{G}) \leq -\theta_p \mu^{2\beta+1},$$

therefore by the existence criterion there exists a Ground State for $E(\cdot, \mathcal{G}')$ at mass μ, that contradicts the present proof in the case $n = 1$. This concludes the proof.

Remark 5.2. As an application of Theorem 5.1 we consider the graph $\mathcal{G}$ made of one halfline and a compact core $\mathcal{K}$ consisting of n edges e_i, $i = 1, \dots n$, each of length l, all attached at the origin of the halfline (see Figure 4). Thus $\operatorname{diam}(\mathcal{K}) = 2l$ and $|\mathcal{K}| = nl$.

Moreover, we take in consideration a potential w supported on the edges e_i and defined as

$$w_i(x) := \frac{\epsilon^3}{4l^{2k+2}} x^{2k}, \quad i = 1, \dots, n, \ x \in [0, l], \ k \in \mathbb{N},$$

where w_i denotes the restriction of w on the edge e_i. Obviously $w \geq 0$ and $\|w\|_\infty = \frac{\epsilon^3}{4l^2}$.

The condition (5.1) rewrites then as

$$\epsilon > \max\left(2\mu^\beta l, \frac{1}{\mu^\beta nl}, \frac{\epsilon^3}{4l^2\mu^{2\beta}}\right),$$

that, by a straightforward computation, amounts to

$$\frac{1}{nl\epsilon} < \mu^\beta < \frac{\epsilon}{2l}. \tag{5.5}$$

The inequalities Eq (5.5) can be simultaneously satisfied for n large enough. In other words, if n is large enough, then there exists an interval of masses to which Theorem 5.1 applies.

This example shows that Theorem 5.1 is not empty, in the sense that for some graphs the condition Eq (5.1) singles out a significant interval of masses.

At present, we do not have a precise estimate of the constant ϵ. It will be the subject of further investigation.

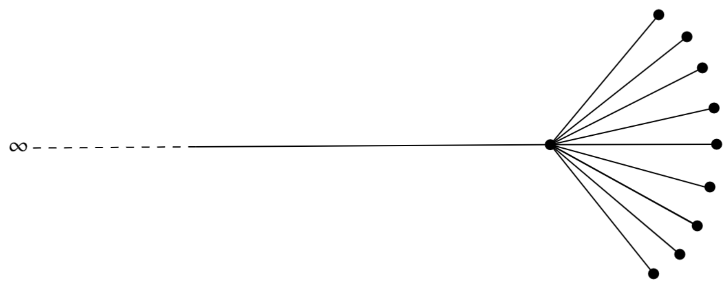

Figure 4. A n-fork graph consisting of one halfline and n edges of length l.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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