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Research article

Regularized homogenization on irregularly perforated domains

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Abstract: We studied stochastic homogenization of a quasi-linear parabolic partial differential equation (PDE) with nonlinear microscopic Robin conditions on a perforated domain. The focus of our work lies in the underlying geometry that does not allow standard stochastic homogenization techniques to be applied directly. Instead, we introduced a concept of regularized homogenization: We proved homogenization on a regularized but still random geometry and demonstrated afterwards that the form of the homogenized equation was independent from the regularization, though the explicit values of the coefficients depended on the regularization. Then, we passed to the regularization limit to obtain the anticipated limit equation where the coefficients were finally independent from the intermediate regularizations. We provided evidence that the regularized homogenization and the classical stochastic homogenization coincided on geometries that indeed allowed stochastic homogenization. Furthermore, we showed that Boolean models of Poisson point processes were covered by our approach.

Keywords: compensated compactness; Robin boundary condition; continuum percolation; Poisson point process

1. Introduction

Soon after the groundbreaking introduction of stochastic homogenization by Papanicolaou and Varadhan [22] and Kozlov [12], research developed a natural interest in the homogenization of randomly perforated domains. A good summary of the existing methods up to 1994 can be found in [13]. By the same time, Zhikov [28] provided a homogenization result for linear parabolic equations on stationary randomly perforated domains. Bourgeat et al. [27] introduced a concept of two-scale convergence in the mean for stochastic problems, where the two-scale limit is performed simultaneously over all realizations. This method was the first to apply the two-scale convergence idea to a stochastic setting, but it was not capable to deal with nonlinearities that require strong convergence of solutions (see also [10]). Hence, we cannot benefit of this approach in our work. It then became silent for a decade. In [30], Zhikov and Piatnitsky reopened the case by introducing stochastic two-scale convergence as a generalization of [20, 1, 29] to the stochastic setting, particularly to random measures that comprise random perforations and random lower-dimensional structures in a natural way. The method was generalized to various applications in discrete and continuous homogenization [16, 3, 4] and recently also to an unfolding method [19, 10].

Concerning the homogenization on randomly perforated domains, there seems to be few results in the literature, with [7, 5, 23] being the closest related work from the PDE point of view. We emphasize that there is a further discipline in stochastic homogenization, studying critical regimes of scaling for holes in a perforated domain of the Stokes equation; see [6] and references therein.

In this work, we focus on the geometric aspects in the homogenization of quasi-linear parabolic equations and go beyond any recent assumptions on the random geometry. Given $\varepsilon > 0$, we consider a bounded domain $Q \subset \mathbb{R}^d$ perforated by a random set G^ε and write $Q^\varepsilon := Q \backslash G^\varepsilon$. Typically, $G^\varepsilon \approx \varepsilon G$ where G is a stationary random set and G^ε is additionally regularized close to ∂Q [7, 5, 23]. Also, we assume that Q^ε is connected, for simplicity of calculations and presentation, and our geometric model will be regularized in such a way; see also Remark 5.

We then study the following PDE on Q^ε for the time interval $I = [0, T]$:

$$\begin{aligned}
\partial_t u^\varepsilon - \nabla \cdot (A(u^\varepsilon)\nabla u^\varepsilon) &= f && \text{in } I \times Q^\varepsilon \\
A(u^\varepsilon)\nabla u^\varepsilon \cdot \nu &= 0 && \text{on } I \times \partial Q \\
A(u^\varepsilon)\nabla u^\varepsilon \cdot \nu &= \varepsilon h(u^\varepsilon) && \text{on } I \times \partial Q^\varepsilon \backslash \partial Q \\
u^\varepsilon(0, x) &= u_0(x) && \text{in } Q^\varepsilon
\end{aligned} \tag{1.1}$$

with ν being the outer normal vector.

In case of a fully linear PDE, i.e., $h(\cdot) = const$ and $A(\cdot) = const$, this problem was homogenized already in the aforementioned [28] and later reconsidered in [30]. In this linear case, one benefits from the regularity of the limit solution and the weak convergence of the ε-solutions that is given a priori.

However, the nonlinear case is more difficult. Weak convergence of solutions is no longer sufficient. Thus, one needs to establish strong convergence of the u^ε. As we will discuss below, a lack of a uniformly continuous family of extension operators or, more generally, a degeneration of the homogenized matrix will cause the whole argumentation to break down. Hence, typical assumptions in the literature, such as minimal smoothness (see Definition 23) of G and uniform boundedness of the hole sizes, ensure the existence of uniformly bounded extension operators $\mathcal{U}_{\varepsilon,\bullet} : W^{1,2}(Q^\varepsilon) \to W^{1,2}(Q)$ [7]. This in turn implies weak compactness of $\mathcal{U}_{\varepsilon,\bullet} u^\varepsilon$ in $W^{1,2}(Q)$, a property of uttermost importance to pass to the homogenization limit in the nonlinear terms. Other approaches are conceivable, e.g., exploiting the Frechet–Riesz–Kolmogorov compactness theorem, but in application the prerequisites are hard to prove.

If all limit passages go through, the homogenized limit as $\varepsilon \to 0$ reads for some positive definite matrix $\mathcal{A}_{(G)}$ as

$$\begin{aligned}
C_{1,(G)}\partial_t u - \operatorname{div}(A(u)\mathcal{A}_{(G)}\nabla u) - C_{2,(G)}h(u) &= C_{1,(G)}f && \text{in } I \times Q \\
A(u)\mathcal{A}_{(G)}\nabla u \cdot \nu &= 0 && \text{on } I \times \partial Q \\
u(0, x) &= C_{1,(G)}u_0(x) && \text{in } Q,
\end{aligned} \tag{1.2}$$

which represents the macroscopic behavior of our object. We note at this point that positivity of $\mathcal{A}_{(G)}$ is, in general, nontrivial but can be shown for minimally smooth domain examples (see Sections 8 and 9).

Unfortunately, canonical perforation models are neither minimally smooth nor is the size of their holes uniformly bounded. Our toy model of choice will be the Boolean model $\Xi\mathbb{X}_{\text{poi}} := \overline{\bigcup_{x\in\mathbb{X}_{\text{poi}}} \mathbb{B}_r(x)}$ (see Definition 1) driven by a Poisson point process $\mathbb{X}_{\text{poi}}$. It clearly reveals the following general issues for the homogenization analysis:

i) ${\Xi\mathbb{X}_{\text{poi}}}^{\complement} = \mathbb{R}^d \backslash \Xi\mathbb{X}_{\text{poi}}$ is not connected due to areas that are encircled.

ii) Two distinct balls can lie arbitrarily close to each other or – in case they intersect – have arbitrary small overlap. This implies that

- the connected components in $\Xi\mathbb{X}_{\text{poi}}$ develop arbitrarily large local Lipschitz constants: Two balls of equal radius intersecting at an angle α have the Lipschitz constant $\tan((\pi - \alpha)/2)$ at the points of intersection, and
- there is no $\delta > 0$ such that for every $p \in \partial{\Xi\mathbb{X}_{\text{poi}}}^{\complement}$ the surface $\mathbb{B}_\delta(p) \cap \partial{\Xi\mathbb{X}_{\text{poi}}}^{\complement}$ is a graph of a function: If $x, y \in \mathbb{X}_{\text{poi}}$ with $|x - y| = 2r + \eta$ and $|p - x| = r$, $|p - y| = r + \eta$, $\mathbb{B}_\delta(p) \cap \partial\Xi\mathbb{X}_{\text{poi}}$ can be a graph only if $\delta < \eta$.

The first issue can be fixed by considering a "filled-up model" $\boxminus\mathbb{X}_{\text{poi}}$ in Definition 1. Furthermore, we will see that – under mild regularity conditions – the procedure of filling up a perforation does not change the effective conductivity $\mathcal{A}_{(G)}$ at all (Theorem 58). Unfortunately, the second issue poses an actual problem. In a recent work [9], one of the authors has shown that in some cases an extension operator $\mathcal{U}_{\varepsilon,\bullet}\colon W^{1,p}(Q^\varepsilon) \to W^{1,q}(Q)$, $1 \le q < p$, can be constructed for some geometries including the Boolean model (strictly speaking, this was shown for an extension from the balls to the complement in the percolation case). However, [9] also suggests that the Boolean model for the Poisson point process requires $p > 2$ in order for $\mathcal{U}_{\varepsilon,\bullet}$ to be properly defined for some $q \ge 1$.

Due to these severe analytical difficulties, we need other approaches to the problem. We call our approach 'regularized homogenization' and it consists of an approximation of the random geometry G by G_n, performing homogenization, and afterward letting $n \to \infty$. In our particular example, it consists of the following steps:

i) Given a general stationary ergodic (admissible) random point process $\mathbb{X}$, we construct a regularization $\mathbb{X}^{(n)} := \mathtt{F}_n\mathbb{X}$ (see Definition 3) such that the set $\boxminus\mathbb{X}^{(n)}$ is uniformly minimally smooth for given $n \in \mathbb{N}$.

ii) Given $n \in \mathbb{N}$, we perform homogenization for the smoothed geometry $\boxminus\mathbb{X}^{(n)}$ instead of $\boxminus\mathbb{X}$ (see Lemma 7).

iii) We pass to the limit $n \to \infty$ along a subsequence to obtain the anticipated homogenized limit problem (see Theorem 56), where the coefficients are independent from the intermediate regularizations (Corollary 39). This happens under the assumption that $\boxminus\mathbb{X}^{\complement}$ is statistically connected (see Definition 15).

iv) We show that the Poisson point process in the subcritical regime is a valid example for our general homogenization result (see Section 9).

We are thus in a position to prove an indirect homogenization result. This seems to us an appropriate intermediate step on the way to a full homogenization result, which may be achieved in the future using further developed homogenization techniques based on a better understanding of the interaction of geometry and homogenization. Let us note that we focus on fixed radii $r > 0$ for the sake of presentation. In fact, random radii pose no issue for the procedure as briefly mentioned in Remark 18 as long as the remaining conditions are satisfied.

This paper is structured as follows:

- In Section 3, we introduce the core objects and state the main result. This includes the thinned point processes $\mathbb{X}^{(n)}$ and its filled-up Boolean model $\boxminus\mathbb{X}^{(n)}$.
- In Section 4, we prove relevant properties of the thinning map and the thinned point processes, most importantly, minimal smoothness of $\boxminus\mathbb{X}^{(n)}$ (Theorem 25) and $\boxminus\mathbb{X}^{(n)} \to \boxminus\mathbb{X}$ in a certain sense (Lemma 29).
- Section 5 deals with the cell solutions and the definition of the effective conductivity $\mathcal{A}$.
- The homogenization theory for minimally smooth holes is sketched in Section 6 on the basis of stochastic two-scale convergence. Due to the considerations in Section 5, the underlying probability space is a compact separable metric space.
- In Section 7, we show that the homogenized solutions to Eq (1.2) for $G = \boxminus\mathbb{X}^{(n)}$ converge and that their limit is a solution to the anticipated limit problem for $G = \boxminus\mathbb{X}$.
- Section 8 establishes a criterion for statistical connectedness (nondegeneracy of the effective conductivity $\mathcal{A}$) using percolation channels. We follow the ideas in [13, Chapter 9] where a discrete model was considered.
- In Section 9, we show that the Poisson point process $\mathbb{X}_{\text{poi}}$ is indeed admissible, which follows from readily available percolation results. Showing statistical connectedness of $\boxminus\mathbb{X}_{\text{poi}}^{\complement}$ is much harder. We do so using the criterion established in Section 8 and a version of [11, Theorem 11.1]. As the original [11, Theorem 11.1] is a statement about percolation channels on the $\mathbb{Z}^2$-lattice, we need to adjust both the statement and the proof to our setting.

2. Notation

General notation

- $\mathcal{M}(\mathbb{R}^d)$: Space of Radon measures on $\mathbb{R}^d$ equipped with the vague topology
- $\mathcal{S}(\mathbb{R}^d) \subset \mathcal{M}(\mathbb{R}^d)$: Space of boundedly finite point clouds/point measures in $\mathbb{R}^d$
- $A^{\complement}$: Complement of a set A
- $\mathcal{B}(X)$: Borel-σ-algebra of the topological space X
- $\mathcal{L}^d$: d-dimensional Lebesgue-measure
- $\mathcal{H}^d$: d-dimensional Hausdorff-measure
- $\mathcal{H}^d_{\llcorner A}$: Restriction of $\mathcal{H}^d$ to A, i.e. $\mathcal{H}^d_{\llcorner A}(B) := \mathcal{H}^d(B \cap A)$
- $o := 0_{\mathbb{R}^d} \in \mathbb{R}^d$: Origin in $\mathbb{R}^d$
- $\mathbb{1}_A$: Indicator/characteristic function of a set A

Specific notation introduced later

- $\mathbb{B}_r(A)$: Open r-neighborhood around A. (Definition 1)
- $\Xi\mathbb{x}$ and $\boxminus\mathbb{x}$: Boolean model of $\mathbb{x}$ and its filled version (Definition 1)
- $C_{\mathbb{x}}(x)$: Cluster of x in $\mathbb{x} \in \mathcal{S}(\mathbb{R}^d)$ (Definition 3)
- $\mathbb{x}^{(n)}$ for $\mathbb{x} \in \mathcal{S}(\mathbb{R}^d)$: $\mathbb{x}^{(n)} = \mathtt{F}_n\mathbb{x}$ with thinning map $\mathtt{F}_n$ (Definition 3)
- $Q^\varepsilon_{\mathbb{x}}$ and $J^\varepsilon(Q, \mathbb{x})$: Perforated domain and index set generating perforations (Definition 4)
- $\tau_x\colon \mathcal{M}(\mathbb{R}^d) \to \mathcal{M}(\mathbb{R}^d)$: Shift-operator in $\mathcal{M}(\mathbb{R}^d)$ (Definition 8)
- $\lambda(\mu)$: Intensity of random measure μ (Definition 8)
- $\mu_{\mathbb{x}}$: $\mathcal{H}^{d-1}$ restricted to $\partial \boxminus \mathbb{x}$ (Definition 30)
- $\mathcal{A}$ and $\alpha_{\mathcal{A}}$: Effective conductivity and smallest eigenvalue of $\mathcal{A}$ (Definition 37)
- $\mathcal{U}$ and $\mathcal{T}$: Extension and trace operators (Theorem 43 and Theorem 51)
- μ^ε: Scaled measure (Assumption 46)

3. Setting and main result

3.1. Generating minimally smooth perforations

We start by introducing some concepts from the theory of point processes. We will not formulate the concepts in full generality but only as general as needed for our purpose. Let $d \geq 2$ and let $\mathcal{S}(\mathbb{R}^d)$ be the set of boundedly finite point clouds in $\mathbb{R}^d$ (i.e., point clouds without accumulation points) and $\mathcal{M}(\mathbb{R}^d)$ the space of Radon measures with the vague topology, that is, the smallest topology on $\mathcal{M}(\mathbb{R}^d)$ such that

$$\mu \mapsto \int_{\mathbb{R}^d} f \, \mathrm{d}\mu$$

is continuous for every $f \in C^\infty_c(\mathbb{R}^d)$. Every $\mathbb{x} \in \mathcal{S}(\mathbb{R}^d)$ can be identified with a Borel measure through the correspondence

$$\mathbb{x}(A) = \sum_{x \in \mathbb{x}} \delta_x(A).$$

Hence, we identify $\mathcal{S}(\mathbb{R}^d) \subset \mathcal{M}(\mathbb{R}^d)$.

Our perforation model of interest is the Boolean model driven by a point cloud $\mathbb{x} \in \mathcal{S}(\mathbb{R}^d)$. While it is a natural way to generate perforations, we require its complement to be connected for suitable $\mathbb{x}$. Hence, the perforation $\Xi\mathbb{x}$ needs to be filled up in order to remove all finite-sized connected components from its complement. These can be easily identified as they do not admit a path of infinite diameter.

Definition 1 (Boolean model Ξ of a point cloud and filled-up model $\boxminus$ (see Figure 1))**.** Let $\mathbb{x} \in \mathcal{S}(\mathbb{R}^d)$. The *Boolean model* of $\mathbb{x}$ for a radius $r > 0$ is

$$\Xi\mathbb{x} := \overline{\bigcup_{x \in \mathbb{x}} \mathbb{B}_r(x)} = \overline{\mathbb{B}_r(\mathbb{x})},$$

where $\mathbb{B}_r(x)$ is the open ball of radius r around x and $\mathbb{B}_r(A) := \bigcup_{x \in A} \mathbb{B}_r(x)$. We define the *filled-up Boolean model* $\boxminus\mathbb{x}$ of $\mathbb{x}$ for radius r through its complement, i.e.,

$$\boxminus\mathbb{x}^\complement := \{x \in \mathbb{R}^d \,|\, \exists \gamma\colon [0, \infty) \to \Xi\mathbb{x}^\complement \text{ continuous and } \gamma(0) = x, \limsup_{t \to \infty} |\gamma(t)| = \infty\}.$$

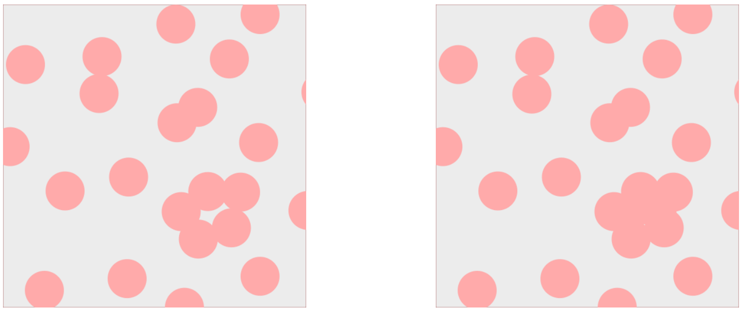

Figure 1. Initial Boolean model $\Xi\mathbb{x}$ vs filled-up Boolean model $\boxminus\mathbb{x}$.

Remark 2. We observe that

$$\Xi(\mathbb{x}+x)=\Xi(\mathbb{x})+x \qquad \text{and} \qquad \boxminus(\mathbb{x}+x)=\boxminus\mathbb{x}+x.$$

As discussed in the introduction, we need to "smoothen" the geometry in order to be able to apply standard homogenization methods. Given a Lipschitz domain $P\subset\mathbb{R}^d$, we define for $p\in\partial P$

$$\delta(p):=\frac{1}{2}\sup_{\delta'>0}\left\{\partial P \text{ is Lipschitz-graph in } \mathbb{B}_{\delta'}(p)\right\},$$

and because $\delta\colon\partial P\to\mathbb{R}_{\geq 0}$ is continuous [9], we can define for bounded P

$$\delta(P):=\min_{p\in\partial P}\delta(p).$$

Definition 3 (Thinning maps $\mathtt{F}_n$ (see Figure 2))**.** Let $\mathbb{x}\in\mathcal{S}(\mathbb{R}^d)$ be a point cloud. We denote the *cluster* of x in $\mathbb{x}$ by

$$C_{\mathbb{x}}(x):=\{y\in\mathbb{x}\,|\,\exists\text{path from } x \text{ to } y \text{ inside } \Xi\mathbb{x}\}.$$

We set

$$\mathtt{F}_{1,n}\mathbb{x}:=\left\{x\in\mathbb{x}\,|\,\forall y\in\mathbb{x}\colon d(x,y)\notin(0,\tfrac{1}{n})\cup(2r-\tfrac{1}{n},\,2r+\tfrac{1}{n})\right\}, \tag{3.1}$$

$$\mathtt{F}_{2,n}\mathbb{x}:=\left\{x\in\mathbb{x}\,|\,\#C_{\mathbb{x}}(x)\leq n,\ \delta(\mathbb{B}_r(C_{\mathbb{x}}(x)))\geq\tfrac{1}{n}\right\}, \tag{3.2}$$

and define the *thinning map* $\mathtt{F}_n$

$$\mathtt{F}_n\colon\mathcal{S}(\mathbb{R}^d)\to\mathcal{S}(\mathbb{R}^d), \qquad \mathbb{x}^{(n)}:=\mathtt{F}_n\mathbb{x}:=(\mathtt{F}_{2,n}\circ\mathtt{F}_{1,n})\mathbb{x}.$$

$\mathtt{F}_n$ can be understood as a generalization of the classical Matérn construction [15, 26]. For an arbitrary $\mathbb{x}\in\mathcal{S}(\mathbb{R}^d)$, we see that $(\boxminus\mathbb{x}^{(n)})^{\complement}$ is always minimally smooth (see Definition 23). Furthermore, if $\mathbb{X}$ is a stationary point process (as defined later), then the same holds for $\mathbb{X}^{(n)}=\mathtt{F}_n\mathbb{X}$. We note that $\mathtt{F}_n$ is in general not monotone in n, i.e., $\mathtt{F}_m\mathbb{x}\not\subset\mathtt{F}_n\mathbb{x}$ for $m\leq n$.

Given a scale $\varepsilon>0$, we define the perforation domain Q^ε such that the perforations have some minimal distance from the boundary ∂Q:

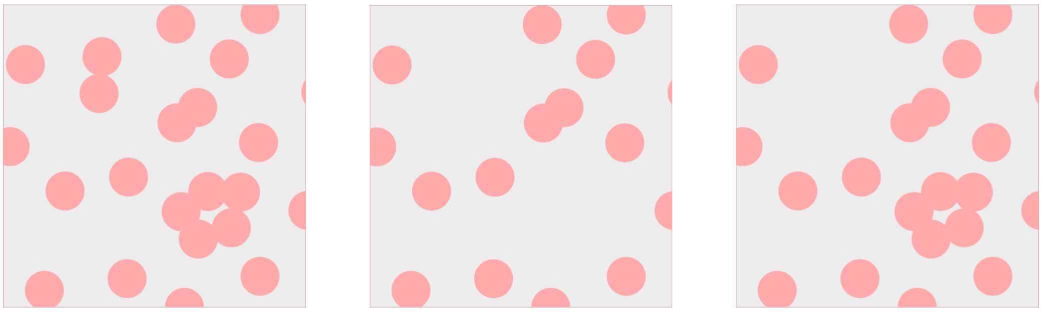

Figure 2. Thinning of point clouds under $\mathtt{F}_n$ pictured via the Boolean model Ξ. From left to right are $\mathbb{x}$, $\mathbb{x}^{(2)}$, and $\mathbb{x}^{(5)}$.

Definition 4 (Perforation of domain Q^ε)**.** Let $\mathbb{x} \in \mathcal{S}(\mathbb{R}^d)$. We set

$$J^\varepsilon(\mathbb{x}, Q) := \left\{x \in \mathbb{x} \,\middle|\, \operatorname{dist}\!\left(\varepsilon C_{\mathbb{x}}(x),\, Q^{\complement}\right) > 2\varepsilon r\right\}, \qquad G^\varepsilon_{\mathbb{x}} := \varepsilon \boxminus (J^\varepsilon(\mathbb{x}, Q))$$

as well as the *perforated domain*

$$Q^\varepsilon_{\mathbb{x}} := Q \setminus G^\varepsilon_{\mathbb{x}}.$$

One quickly verifies that $Q^\varepsilon_{\mathbb{x}^{(n)}}$ is minimally smooth (Definition 23); see Theorem 25.

Remark 5. By construction, our perforation model $G^\varepsilon_{\mathbb{x}} := \varepsilon \boxminus (J^\varepsilon(\mathbb{x}, Q))$ ensures connectedness of its complement $\mathbb{R}^d \setminus G^\varepsilon_{\mathbb{x}}$, resp., of $Q^\varepsilon_{\mathbb{x}}$. We prefer this approach of "filling the holes of the holes" because it allows for an easy use of extension operators in a way that we can safely apply compact embeddings. Otherwise, once $\mathbb{R}^d \setminus G^\varepsilon_{\mathbb{x}}$ becomes disconnected, we would have to rely on additional regularity assumptions on the initial data in those holes, which is then carried on by the evolution equation and would lead to additional tedious calculations. Otherwise, we would have to introduce a second PDE on the holes with suitable coupling boundary conditions. While we do not claim that a treatment of such a situation is impossible, we claim that the additional effort would not be justified by the additional expected insight.

3.2. Homogenization for minimally smooth perforations

We make the following parameter assumptions on our partial differential Eq (1.1).

Assumption 6 (Parameters of PDE)**.** *Let $I = [0, T] \subset \mathbb{R}$ and $Q \subset \mathbb{R}^d$ be a bounded, connected open domain. We assume that*

- $u_0 \in W^{1,2}(Q)$
- $f \in L^2(I; L^2(Q))$
- $h\colon \mathbb{R} \to \mathbb{R}$ *is Lipschitz continuous with Lipschitz constant* L_h
- $A\colon \mathbb{R} \to \mathbb{R}$ *is continuous with* $0 < \inf(A)$ *and* $\sup(A) < \infty$.

Generalized time derivatives will always be considered under the evolution triple $W^{1,2}(Q) \hookrightarrow L^2(Q) \hookrightarrow W^{1,2}(Q)^*$ *or* $W^{1,2}(Q^\varepsilon_{\mathbb{x}^{(n)}}) \hookrightarrow L^2(Q^\varepsilon_{\mathbb{x}^{(n)}}) \hookrightarrow W^{1,2}(Q^\varepsilon_{\mathbb{x}^{(n)}})^*$ *in the case of a perforated domain* $Q^\varepsilon_{\mathbb{x}^{(n)}}$.

Lemma 7 (Solution to PDE for minimally smooth holes)**.** *Let* $\mathbb{x} \in \mathcal{S}(\mathbb{R}^d)$ *and* $n \in \mathbb{N}$. *Under Assumption 6, we have on* $Q^{\varepsilon}_{\mathbb{x}^{(n)}}$*: There exists a weak solution* $u^{\varepsilon} \in L^2(I; W^{1,2}(Q^{\varepsilon}_{\mathbb{x}^{(n)}}))$ *with generalized time derivative* $\partial_t u^{\varepsilon} \in L^2(I; W^{1,2}(Q^{\varepsilon}_{\mathbb{x}^{(n)}})^*)$ *to Eq (1.1) on* $Q^{\varepsilon}_{\mathbb{x}^{(n)}}$ *instead of* Q^{ε}.

This u^{ε} *satisfies for some* $C > 0$ *independent of* ε *depending only on* Q*,*n*,* f *and* u_0 *but not on* ε

$$\operatorname*{ess\,sup}_{t\in I} \|u^{\varepsilon}(t)\|_{L^2(Q^{\varepsilon}_{\mathbb{x}^{(n)}})} + \|u^{\varepsilon}\|_{L^2(I;W^{1,2}(Q^{\varepsilon}_{\mathbb{x}^{(n)}}))} \le C$$
$$\|\partial_t u^{\varepsilon}\|_{L^2(I;W^{1,2}(Q^{\varepsilon}_{\mathbb{x}^{(n)}})^*)} \le C.$$

The proof is given in Section 6 (Theorem 53).

The next step is passing to the limit $\varepsilon \to 0$. We do so in the case of $\mathbb{x}$ being a realization of a stationary ergodic point process $\mathbb{X}$ as defined below:

Definition 8 (Random measure and shift-operator τ_x)**.** A *random measure* $\mu_\bullet$ is a random variable with values in $\mathcal{M}(\mathbb{R}^d)$. It induces a probability distribution $\mathbb{P}$ on $\mathcal{M}(\mathbb{R}^d)$. Given the continuous map

$$\tau_x \colon \mathcal{M}(\mathbb{R}^d) \to \mathcal{M}(\mathbb{R}^d), \qquad \tau_x \xi(A) := \xi(A + x), \tag{3.3}$$

a random measure is *stationary* if $\mathbb{P}(F) = \mathbb{P}(\tau_x F)$ for every $F \in \mathcal{B}(\mathcal{M}(\mathbb{R}^d))$ and every $x \in \mathbb{R}^d$. In line with the above setting, a random *point process* $\mathbb{X}$ is a random measure with $\mathbb{P}(\mathcal{S}(\mathbb{R}^d)) = 1$, and one quickly verifies that $\mathbb{X}$ is stationary if for every $N \in \mathbb{N}$, $x \in \mathbb{R}^d$, and bounded open $A \subset \mathbb{R}^d$, it holds

$$\mathbb{P}\Big(\mathbb{x} \in \mathcal{S}(\mathbb{R}^d) \colon \mathbb{x}(A) = N\Big) = \mathbb{P}\Big(\mathbb{x} \in \mathcal{S}(\mathbb{R}^d) \colon \mathbb{x}(A + x) = N\Big).$$

We call a stationary random measure $\mu_\bullet$ *ergodic* if the σ-algebra of τ-invariant sets is trivial under its distribution $\mathbb{P}$.

Remark 9 (*Compatibility of thinning with shifts*). The thinning map $\mathtt{F}_n$ is compatible with the shift τ_x, i.e., on $\mathcal{S}(\mathbb{R}^d)$

$$\mathtt{F}_n \circ \tau_x = \tau_x \circ \mathtt{F}_n.$$

Lemma 10 (Homogenized PDE for minimally smooth domains)**.** *Let* $\mathbb{X}$ *be a stationary ergodic point process and* $n \in \mathbb{N}$ *be fixed. For almost every realization* $\mathbb{x}$ *of* $\mathbb{X}$*, we have under Assumption 6:*

For $\varepsilon > 0$*, let* $u^{\varepsilon} \in L^2(I; W^{1,2}(Q^{\varepsilon}_{\mathbb{x}^{(n)}}))$ *be a solution to Eq (1.1) on* $Q^{\varepsilon}_{\mathbb{x}^{(n)}}$ *instead of* Q^{ε}. *Given any sequence* $\varepsilon \to 0$*, we find a subsequence (still denoted as* $\varepsilon \to 0$*) and corresponding* $\tilde{u}^{\varepsilon} \in L^2(I; W^{1,2}(Q))$ *with* $\tilde{u}^{\varepsilon}|_{Q^{\varepsilon}_{\mathbb{x}^{(n)}}} = u^{\varepsilon}$*, such that* $\tilde{u}^{\varepsilon} \to u_n$ *strongly in* $L^2(I; L^2(Q))$ *for some* $u_n \in L^2(I; W^{1,2}(Q))$ *with generalized time derivative* $\partial_t u_n \in L^2(I; W^{1,2}(Q)^*)$*. This* u_n *is a weak solution to*

$$\begin{aligned} C_{1,\mathbb{P}^{(n)}} \partial_t u_n - \nabla \cdot (A(u_n) \mathcal{A}^{(n)} \nabla u_n) - C_{2,\mathbb{P}^{(n)}} h(u_n) &= C_{1,\mathbb{P}^{(n)}} f && \text{in } I \times Q \\ A(u_n) \mathcal{A}^{(n)} \nabla u_n \cdot \nu &= 0 && \text{on } I \times \partial Q \\ u_n(0, x) &= C_{1,\mathbb{P}^{(n)}} u_0(x) && \text{in } Q, \end{aligned} \tag{3.4}$$

with constants $C_{i,\mathbb{P}^{(n)}} > 0$ *depending on the distribution* $\mathbb{P}^{(n)}$ *of* $\mathbb{X}^{(n)}$ *and* $\mathcal{A}^{(n)}$ *being a symmetric positive semi-definite matrix – the so-called effective conductivity based on the event that the origin is not covered by* $\boxminus\mathbb{X}^{(n)}$ *(see Definition 37).*

Proof. This is shown in Section 6 (Theorem 55) using two-scale convergence.

A more general result, which immediately is implied by the proof of the previous lemma is the following Lemma 11. There are many random geometries that do not have an extension operator $W^{1,2}(Q^\varepsilon) \to W^{1,2}(Q)$ but still an extension operator $W^{1,2}(Q^\varepsilon) \to W^{1,q}(Q)$ for some $1 \leq q < 2$, see [9]. For such geometries, the desired homogenization result could be established without the bypass that we will take below. However, results in [9] will provide sufficient conditions for homogenization, but they do not provide necessary conditions.

Lemma 11. *Assume that Q^ε is such that there exists $q \in (1,2]$ such that for every family $v^\varepsilon \in W^{1,2}(Q^\varepsilon)$ with* $\sup_\varepsilon(\|\nabla v^\varepsilon\|_{L^2(Q^\varepsilon)} + \|v^\varepsilon\|_{L^2(Q^\varepsilon)}) < +\infty$*, there exists a $V^\varepsilon \in L^q(Q)$, such that $v^\varepsilon = \mathbb{1}_{Q^\varepsilon} V^\varepsilon$ and such that V^ε is pre-compact in $L^q(Q)$. Then, every convergent subsequence of the solutions u^ε of Eq (1.1) converges to a weak solution of Eq (3.5) given below.*

Remark 12 (*Estimate*). Observe that the situation in Lemma 7 provides us with an estimate $\sup_\varepsilon(\|\nabla u^\varepsilon\|_{L^2((0,T]\times Q^\varepsilon)} + \|u^\varepsilon\|_{L^2((0,T]\times Q^\varepsilon)}) < +\infty$ on the solutions u^ε, and the compact operator is a combination of the extension operator for minimally smooth domains and the standard Sobolev embedding.

Remark 13 (*Compactness*). It has to be noted that the claim of compactness is somewhat natural and for most nonlinearities, even necessary. Indeed, unless the nonlinearities are monotone operators, we are not aware of an existence theory for equations of Type (1.1), which would not rely on compactness of a sequence of approximate solutions in some L^p-space. However, if existence theory is already heavily relying on this compactness assumption, we have no hope that homogenization could cope with less.

3.3. Regularized homogenization for irregular perforations

When it comes to the final homogenization result, we will need the following assumptions on the point process $\mathbb{X}$.

Definition 14 (Admissible point process)**.** We call a point cloud $\mathbb{x} \in \mathcal{S}(\mathbb{R}^d)$ *admissible* the following holds (with $r > 0$ from Definition 1):

i) Equidistance Property: $\forall x, y \in \mathbb{x}\colon\ |x - y| \neq 2r$.
ii) Finite Clusters: For every $x \in \mathbb{x}$, we have $\#C_{\mathbb{x}}(x) < \infty$.

A stationary ergodic boundedly finite point process $\mathbb{X}$ is called *admissible* if its realizations are almost surely admissible.

Definition 15 (Statistical connectedness)**.** The random set $⊟\mathbb{X}^{\complement}$ is *statistically connected* if the effective conductivity $\mathcal{A}$ (Definition 37) based on the event that the origin is covered by $⊟\mathbb{X}^{\complement}$ is strictly positive definite.

In our setting of Boolean models of admissible point processes, the procedure of filling up a perforation does not change its effective conductivity $\mathcal{A}$ (Theorem 58). In particular, $⊟\mathbb{X}^{\complement}$ is statistically connected if, and only if, the same holds for $\Xi\mathbb{X}^{\complement}$.

Remark 16 (*On statistical connectedness*). Let us note that the existence of an infinite connected component in $⊟\mathbb{X}^{\complement}$ is a necessary requirement for $\mathcal{A} > 0$, as can be seen later in Section 8. There, we also give a criterion for statistical connectedness: the existence of sufficiently many so-called percolation channels. The procedure is based on [13, Lemma 9.7], which we adjust to the continuum setting.

We may now state the main theorem of this work.

Theorem 17 (Homogenized limit for admissible point processes)**.** *Let $\mathbb{X}$ be an admissible point process and $\boxminus\mathbb{X}^{\complement}$ statistically connected. Under Assumption 6, we have for almost every realization $\mathbbm{x}$ of $\mathbb{X}$: For every $n \in \mathbb{N}$, let u_n be a homogenized limit in Lemma 10. For every subsequence of $(u_n)_{n\in\mathbb{N}}$, we can extract a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ such that there exists a $u \in L^2(I; W^{1,2}(Q))$ with generalized time-derivative $\partial_t u \in L^2(I; W^{1,2}(Q)^*)$*

$$u_n \xrightarrow[n\to\infty]{L^2(I;W^{1,2}(Q))} u \qquad \text{and} \qquad \partial_t u_n \xrightarrow[n\to\infty]{L^2(I;W^{1,2}(Q)^*)} \partial_t u,$$

and u is a weak solution to

$$\begin{aligned} C_{1,\mathbb{P}}\partial_t u - \nabla\cdot(A(u)\mathcal{A}\nabla u) - C_{2,\mathbb{P}}h(u) &= C_{1,\mathbb{P}}f && \text{in } I\times Q \\ A(u)\mathcal{A}\nabla u\cdot\nu &= 0 && \text{on } I\times\partial Q \\ u(0,x) &= C_{1,\mathbb{P}}u_0(x) && \text{in } Q, \end{aligned} \tag{3.5}$$

with constants $C_{i,\mathbb{P}} > 0$ only depending on the distribution $\mathbb{P}$ of $\mathbb{X}$ and $\mathcal{A}$ being a symmetric positive definite matrix – the so-called effective conductivity $\mathcal{A}$ based on the event that the origin is not covered by $\boxminus\mathbb{X}$ (Definition 37). The limit u may depend on the chosen subsequence if the solution to Eq (3.5) is not unique. Otherwise, the whole sequence $(u_n)_{n\in\mathbb{N}}$ converges to u.

Proof. This is proved in Theorem 56.

Regarding the derivation of Eq (3.5), we only need local convergence of $\boxminus\mathbbm{x}^{(n)} \to \boxminus\mathbbm{x}$ (Lemma 29). Hence, Eq (3.5) is independent of the thinning procedure as long as local convergence is satisfied.

Remark 18 (*Random radii*). For simplicity, we have chosen the Boolean model with fixed radius r as our underlying model. One can easily generalize the procedure to random independent radii. Given a marked point process $\mathcal{X} = \bigcup_i (x_i, r_i)$ whose marks represent the radius r_i of the ball around x_i, we need to adjust Point 1 in Definition 14 accordingly, i.e.,

$$\forall (x, r_x), (y, r_y) \in \mathcal{X}\colon\ |x-y| \neq r_x + r_y.$$

The thinning maps $\mathtt{F}_n$ (Definition 3) also need to be modified to ensure minimal smoothness (Definition 23), e.g., balls with especially large/small radii need to be removed. Eq (3.1) has the purpose to ensure that two balls in the regularized geometry either have a minimal distance $1/n$ or, if they intersect, the intersection angle is bounded away from 0. This also has to be modified depending on radii r_1 and r_2 of two intersecting balls. Local convergence is preserved if all clusters remain finite. Hence, the rest of our procedure essentially follows as is under the assumption of $\mathcal{A} > 0$.

Remark 19 (*Homogenization procedure*). For fixed $\varepsilon > 0$, solutions $u^\varepsilon = u^\varepsilon_{\mathbbm{x}}$ to Eq (1.1) exist for admissible $\mathbbm{x} \in \mathcal{S}(\mathbb{R}^d)$ as $Q^\varepsilon_{\mathbbm{x}^{(n)}} = Q^\varepsilon_{\mathbbm{x}}$ for n large enough (Lemma 29). If $\mathbbm{x}$ is a realization of some admissible point process $\mathbb{X}$, then this is still not sufficient to pass to the limit $\varepsilon \to 0$. The missing regularity of $\boxminus\mathbb{X}$ still prevents us from establishing a priori estimates. All in all, our procedure yields the following diagram:

$$
\begin{array}{ccc}
u^{\varepsilon} & \overset{?}{\underset{\varepsilon\downarrow 0}{\dashrightarrow}} & u \\
{\scriptstyle n\uparrow\infty}\Big\uparrow & & \Big\uparrow{\scriptstyle n\uparrow\infty} \\
u_n^{\varepsilon} & \underset{\varepsilon\downarrow 0}{\longrightarrow} & u_n
\end{array}
\tag{3.6}
$$

Statistical connectedness of $\boxminus\mathbb{X}^{\complement}$ is crucial to establish $W^{1,2}(Q)$-estimates for u_n. This indicates that the direct limit passing $u^\varepsilon \to u$ might only rely on the statistical connectedness property, but we cannot answer that as of now. On the other hand, if the assumptions of Lemma 11 hold, then the diagram commutes.

3.4. Example: Poisson point processes

In order to demonstrate that the class of point process satisfying our assumptions is not empty, we show in Section 9 that the Poisson point process $\mathbb{X}_{\text{poi}}$ is indeed suitable for our framework. We obtain the following.

Theorem 20 (Admissibility and statistical connectedness for $\mathbb{X}_{\text{poi}}$)**.** *In the subcritical regime (see Assumption 62), we have for the Poisson point process $\mathbb{X}_{\text{poi}}$ that*

- $\mathbb{X}_{\text{poi}}$ *is an admissible point process.*
- $\boxminus\mathbb{X}^{\complement}_{\text{poi}}$ *is statistically connected.*

While admissibility is easily proven, statistical connectedness is much harder to deal with. Most of Section 9 is dedicated to this proof. It also builds up on Section 8 in which we show that so-called percolation channels yield statistical connectedness.

3.5. Consistency of our approach

In what follows, we collect examples where stochastic homogenization is well understood and where our new ansatz of regularized stochastic homogenization is consistent in the sense that the Diagram (3.6) commutes for these examples.

3.5.1. Linear equations where $Q_n^\varepsilon \supset Q^\varepsilon$

Let G be a stationary ergodic random set and let G_n be jointly stationary sets with G where $G_n \subset G =: G_\infty$ and where $G_n \to G$ pointwise as $n \to \infty$. We then define $Q^\varepsilon := Q \setminus \varepsilon G$ as before and additionally $Q_n^\varepsilon := Q \setminus \varepsilon G_n$. Then, $Q_n^\varepsilon \supset Q^\varepsilon$ by definition, and we have the following result.

Theorem 21. *Under the above assumptions, let $f_n^\varepsilon \in L^2(Q_n^\varepsilon)$ be extended by* 0 *to* Q *and* $\lambda > 0$.

i) For each $\varepsilon > 0$, assume that $f_n^\varepsilon \rightharpoonup f_\infty^\varepsilon =: f^\varepsilon$ as $n \to \infty$. Then, the solutions u_n^ε of the problem

$$
\begin{aligned}
-\nabla\cdot\nabla u_n^\varepsilon + \lambda u_n^\varepsilon = f_n^\varepsilon &\quad \textit{in } Q_n^\varepsilon \\
u_n^\varepsilon = 0 &\quad \textit{on } \partial Q \\
\nabla u_n^\varepsilon \cdot \nu = 0 &\quad \textit{on } \partial Q_n^\varepsilon \backslash \partial Q
\end{aligned}
\tag{3.7}
$$

satisfy $u_n^\varepsilon \rightharpoonup u_\infty^\varepsilon =: u^\varepsilon$ as $n \to \infty$, where u^ε is the solution to Eq (3.7) on $Q_\infty^\varepsilon := Q^\varepsilon$.

ii) For each $n \in \mathbb{N} \cup \{+\infty\}$, there exists a positive semi-definite $\mathcal{A}^{(n)}$ such that $f_n^\varepsilon \rightharpoonup f_n$ as $\varepsilon \to 0$ implies $u_n^\varepsilon \rightharpoonup u_n$ in $L^2(Q)$, where u_n solves

$$-\nabla \cdot \mathcal{A}^{(n)} \nabla u_n + \lambda u_n = f_n \text{ in } Q.$$

iii) If $f_n \rightharpoonup f_\infty =: f$ as $n \to \infty$, then $\mathcal{A}^{(n)} \to \mathcal{A}^{(\infty)} =: A$ and $u_n \rightharpoonup u_\infty =: u$ weakly in $L^2(Q)$. Note from Statement 2 that u solves

$$-\nabla \cdot \mathcal{A} \nabla u + \lambda u = f \text{ in } Q.$$

In other words, Diagram 3.6 commutes for linear equations.

3.6. *An explicit geometry with commutative Diagram (3.6) for nonlinear problems*

We now sketch the homogenization of Eq (1.1) on a geometry that is not minimally smooth but allows for homogenization. This sample geometry was introduced in [9]: We start from a Poisson point process and erase all points that are closer to each other than a given distance threshold $s > 0$. We then construct the Delaunay triangulation and assign a pipe of random diameter $0 < D < s/2$ with a distribution $\mathbb{P}(D < x) < \exp(-1/x)$ to every edge, for all $x > 0$. Furthermore, we enrich the geometry by the balls of radius $s/4$ around the remaining points. This union of pipes and balls is stationary and ergodic and we then define Q^ε as the intersection of Q with the system of balls and pipes scaled by a factor ε. In three dimensions, the complement of Q^ε is pathwise connected and unbounded, but in two dimensions, the complement consists of bounded sets, where there is no upper bound on the diameter of these sets. Also, there is no upper bound on the local Lipschitz constant.

Writing $W^{1,2}_{(0),\partial Q}(Q^\varepsilon)$ for the functions in $W^{1,2}$ with value zero on ∂Q then, in [9, Theorem 1.15], it is shown that, for every $1 < q < 2$, there almost-surely exists an extension operator $\mathcal{U}^\varepsilon \colon W^{1,2}_{(0),\partial Q}(Q^\varepsilon) \to W^{1,q}_0(\mathbb{B}_{\varepsilon^\beta}(Q))$ that is continuous with $\beta \in (0,1)$ depending only on the random geometry, such that, for every $\varepsilon > 0$ and $u^\varepsilon \in W^{1,2}_{(0),\partial Q}(Q^\varepsilon)$, it holds that

$$\|\nabla \mathcal{U}^\varepsilon u^\varepsilon\|_{L^q(\mathbb{R}^d)} \le C \|\nabla u^\varepsilon\|_{L^2(Q^\varepsilon)} \qquad \text{and} \qquad \|\mathcal{U}^\varepsilon u^\varepsilon\|_{L^q(\mathbb{R}^d)} \le C \|u^\varepsilon\|_{L^2(Q^\varepsilon)} . \tag{3.8}$$

An inspection of the proof of [9, Theorem 1.7] reveals that it also holds for $q = p = d = 2$ therein, that is, there exists a constant $C > 0$, independent from ε, such that

$$\|\mathcal{T}^\varepsilon u^\varepsilon\|_{L^2(\partial Q^\varepsilon)} \le C \|\nabla u^\varepsilon\|_{L^2(Q^\varepsilon)} . \tag{3.9}$$

The important insight is that $\mathcal{U}^\varepsilon u^\varepsilon$ is bounded in $W^{1,q}_0(\mathbb{B}_1(Q))$ and hence, pre-compact in $L^2(\mathbb{R}^2)$, which in turn yields two-scale pre-compactness of the traces $\mathcal{T}^\varepsilon u^\varepsilon$, in a similar way as below. Thus, we can argue as in our proof of Lemma 10 to obtain the respective homogenization result. The needed two-scale convergence methods have been introduced in [8].

On the other hand, in $\mathbb{R}^2$, we can regularize the geometry by filling up holes with a diameter larger than a prescribed threshold and we can additionally prescribe a minimal thickness for the pipes. Since the generating points of the Delaunay triangulation have a minimal mutual distance, the filling of large holes provides minimal smoothness. However, Lemma 10 guarantees homogenization. Together with the proof of Theorem 17, we again find that Diagram 3.6 commutes.

3.6.1. Linear equations under norm bounds

In the previous example, we explicitly constructed the approximation in a way that $Q_n^\varepsilon \supset Q^\varepsilon$. However, this is not strictly necessary. The following result shows that the homogenization limits would also exist and look the same as above, as long as the distributions of several geometric quantities do not worsen (significantly), with the respective conditions to be taken from [9].

Theorem 22. *Let G_n be jointly stationary sets with G where $G_n \to G$ pointwise as $n \to \infty$. Assume further that there exists $C > 0$, independent of ε and n, such that Eq (3.8) holds for every ε and n. Then, the assertion in Theorem 21 still holds.*

Without going further into detail, we mention that our main results in form of Lemma 10 and Theorem 56 can be reproduced for any family of geometries satisfying Eqs (3.8) and (3.9) using Theorem 22.

3.6.2. Extrapolation to the Boolean model

It is currently not clear whether the Boolean model introduced above has a family of extension operators such as in the pipe example. Furthermore, if such a family does exist, then the diagram commutes. Hence, we propose the approximation method to derive an educated guess for a homogenized model.

4. Thinning properties, surface measure and convergence of intensities

We first establish some properties of $\mathtt{F}_n\colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$, most importantly the minimal smoothness of $\boxminus\mathbb{x}^{(n)}$.

Definition 23 (Minimal smoothness [25]). An open set $P \subset \mathbb{R}^d$ is called *minimally smooth* with constants (δ, N, M) if we may cover ∂P by a countable sequence of open sets $(U_i)_i$ such that

i) $\forall x \in \mathbb{R}^d\colon \#\{U_i \,|\, x \in U_i\} \leq N$.
ii) $\forall x \in \partial P\, \exists U_i\colon \mathbb{B}_\delta(x) \subset U_i$.
iii) For every i, $\partial P \cap U_i$ agrees (in some Cartesian system of coordinates) with the graph of a Lipschitz function whose Lipschitz semi-norm is at most M.

Lemma 24 (Uniform δ on individual clusters). *Let $\mathbb{x} \in \mathcal{S}(\mathbb{R}^d)$ be an admissible point cloud. Then, for every $x \in \mathbb{x}$*

$$\delta(\Xi(C_{\mathbb{x}}(x))) > 0.$$

Proof. Let $\mathbb{x} \in \mathcal{S}(\mathbb{R}^d)$ and assume $\delta(\Xi(C_{\mathbb{x}}(x))) = 0$ for some $x \in \mathbb{x}$. Then, there must be some $p \in \partial\Xi_{\mathbb{x}}$ with $\delta(p) = 0$. This together with bounded finiteness gives $x_p, y_p \in \mathbb{x}$ such that $p \in B_r(x_p) \cap B_r(y_p)$, in particular $|x_p - y_p| = 2r$. This contradicts the equidistance property of $\mathbb{x}$.

The thinning maps $\mathtt{F}_n$ have been constructed just to yield the following theorem.

Theorem 25 (Minimal smoothness of thinned point clouds). *For every $\mathbb{x} \in \mathcal{S}(\mathbb{R}^d)$, both $(\Xi\mathbb{x}^{(n)})^\complement$ and $(\boxminus\mathbb{x}^{(n)})^\complement$ are minimally smooth with $\delta = 1/n$, $M = \sqrt{2nr}$. Furthermore, every connected component of $\Xi\mathbb{x}^{(n)}$ or $\boxminus\mathbb{x}^{(n)}$ has diameter less than $2nr$.*

Proof. It remains to verify the estimate on M. Let $x = o = 0_{\mathbb{R}^d}$ and $y = \left(2r - n^{-1}, 0, \dots, 0\right)$. Then the Lipschitz constant at the intersection of the two balls $\mathbb{B}_r(x)$ and $\mathbb{B}_r(y)$ is less than $\sqrt{2nr}$.

Theorem 26 (Further properties of $\mathtt{F}_n$)**.** *The set $\mathcal{S}_{\mathcal{A}}(\mathbb{R}^d)$ of admissible point clouds is measurable in the vague σ-algebra. Given $n \in \mathbb{N}$, it holds that $\mathtt{F}_n : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is measurable, $\mathcal{S}^{(n)} := \mathtt{F}_n\mathcal{S}(\mathbb{R}^d)$ is compact in the vague topology and the following three properties of $\mathbb{x} \in \mathcal{S}(\mathbb{R}^d)$ are equivalent:*

i) $\mathtt{F}_n\mathbb{x} = \mathbb{x}$,
ii) $\mathbb{x} \in \mathcal{S}^{(n)}$,
iii) Eqs (4.1) and (4.2) hold:

$$\forall x, y \in \mathbb{x}, x \neq y: \quad d(x, y) \notin (0, 1/n) \cup (2r - 1/n,\ 2r + 1/n), \tag{4.1}$$
$$\forall x \in \mathbb{x}: \quad \#C_{\mathbb{x}}(x) \leq n,\ \delta(\mathbb{B}_r(C_{\mathbb{x}}(x))) \geq 1/n. \tag{4.2}$$

Proof. $\mathtt{F}_n\mathbb{x} = \mathbb{x}$ implies $\mathbb{x} \in \mathcal{S}^{(n)}$ since $\mathtt{F}_n\mathbb{x} \in \mathcal{S}^{(n)}$, and vice versa, $\mathbb{x} \in \mathcal{S}^{(n)}$ implies $\mathtt{F}_n\mathbb{x} = \mathbb{x}$ by definition of $\mathtt{F}_n$. By construction of $\mathtt{F}_n$ it follows that Eqs (4.1) and (4.2) hold if, and only if, $\mathbb{x} \in \mathcal{S}^{(n)}$.

Consider the space of (non-simple) counting measures $\mathcal{N}(\mathbb{R}^d) \subset \mathcal{M}(\mathbb{R}^d)$, i.e.,

$$\mathcal{N}(\mathbb{R}^d) := \Big\{\mu \in \mathcal{M}(\mathbb{R}^d) : \mu = \sum_{k \in \mathcal{I} \subset \mathbb{N}} a_k \delta_{x_k} \text{ such that } a_k \in \mathbb{N} \text{ and } x_k \in \mathbb{R}^d\Big\}.$$

We see, e.g., in [2], that

- $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{N}(\mathbb{R}^d)$ are both measurable w.r.t. the Borel-σ-algebra of $\mathcal{M}(\mathbb{R}^d)$.
- $\mathcal{S}(\mathbb{R}^d) \subset \mathcal{N}(\mathbb{R}^d)$ and $\mathcal{N}(\mathbb{R}^d)$ is closed in $\mathcal{M}(\mathbb{R}^d)$. In particular, $\mathcal{N}(\mathbb{R}^d)$ is also complete under the Prokhorov metric.

Now $\mathcal{S}^{(n)}$ is pre-compact because of the characterization of pre-compact sets in the vague topology: For every bounded open $A \subset \mathbb{R}^d$, it holds that $\sup_{\mathbb{x} \in \mathcal{S}^{(n)}} \mathbb{x}(A) \leq C\,(\operatorname{diam} A)^d$ with C depending only on n. It remains to show that $\mathcal{S}^{(n)}$ is closed as a subset of $\mathcal{N}(\mathbb{R}^d)$. Let $\left(\mathbb{x}_j\right)_{j \in \mathbb{N}} \subset \mathcal{S}^{(n)}$ be a converging sequence with limit $\mathbb{x} \in \mathcal{N}(\mathbb{R}^d)$. One checks that (4.1) (namely $d(x, y) \notin (0, 1/n)$) ensures $\mathbb{x} \in \mathcal{S}(\mathbb{R}^d)$, e.g., in a procedure similar to the proof of [2, Lemma 9.1.V]. We observe that for every $x, y \in \mathbb{x}$, there exist $x_j, y_j \in \mathbb{x}_j$ such that $x_j \to x$, $y_j \to y$ as $j \to \infty$. This implies by a limit in Eq (4.1) that $\mathbb{x}$ still satisfies Eq (4.1).

For $x \in \mathbb{x}$, one checks that Eq (4.1) (namely $d(x, y) \notin (2r - 1/n, 2r + 1/n)$) implies $\#C_{\mathbb{x}}(x) \leq n$.

Let $p \in \partial\Xi(\mathbb{x})$ and let $\{x^{(1)}, \dots, x^{(K)}\} = \mathbb{B}_{10r}(p) \cap \mathbb{x}$ with sequences $x_j^{(k)} \to x^{(k)}$, $x_j^{(k)} \in \mathbb{x}_j$. Given $\eta > 0$, let $J \in \mathbb{N}$ such that for all $j > J$ and $k = 1, \dots, K$ it holds $|x^{(k)} - x_j^{(k)}| < \eta$. Then there exists $p_j \in \partial\Xi(\mathbb{x}_j)$ such that $|p_j - p| < \eta$ and $\partial\Xi(\mathbb{x}_j)$ is a Lipschitz graph in the ball $\mathbb{B}_{2\delta}\left(p_j\right)$ for every $\delta < 1/n$. Hence $\partial\Xi(\mathbb{x}_j)$ is a Lipschitz graph in the ball $\mathbb{B}_{2\delta-\eta}(p)$. Because the Lipschitz regularity of $\partial\Xi(\mathbb{x}_j)$ changes continuously under slight shifts of the balls, there exists η_0 such that for $\eta < \eta_0$ and $\partial\Xi(\mathbb{x})$ is a Lipschitz graph in $\mathbb{B}_{2\delta-2\eta}(p)$. Since η is arbitrary, we find $\partial\Xi(\mathbb{x})$ is Lipschitz graph in $\mathbb{B}_{2\delta}(p)$ for every $\delta < 1/n$, implying $\delta(p) \geq 1/n$. Since this holds for every p, we conclude Eq (4.2) and $\mathcal{S}^{(n)}$ is compact.

To see that $\mathcal{S}_{\mathcal{A}}(\mathbb{R}^d)$ is measurable, consider for $\mathbb{x} \in \mathcal{S}(\mathbb{R}^d)$

$$\mathbb{x}_{1,m,R} := \{x \in \mathbb{x} : x \notin \mathbb{B}_R(o) \text{ or } d(x, y) \notin (0, 1/m) \cup (2r - 1/m,\ 2r + 1/m)\ \forall y \in \mathbb{x}\},$$
$$\mathbb{x}_{2,l,R} := \{x \in \mathbb{x} \,|\, x \notin \overline{\mathbb{B}_R(o)} \text{ or } \#C_{\mathbb{x}}(x) \leq l,\ \delta(\mathbb{B}_r(C_{\mathbb{x}}(x))) > 1/l\},$$

and define

$$\mathtt{F}_{1,m,R}\mathbb{x} := \mathbb{x}_{1,m,R} \qquad\qquad \mathcal{S}^{(1,m,R)} := \mathtt{F}_{1,m,R}\mathcal{S}(\mathbb{R}^d)$$
$$\mathtt{F}_{2,l,R}\mathbb{x} := \mathbb{x}_{2,l,R}. \qquad\qquad \mathcal{S}^{(2,m,R)} := \mathtt{F}_{2,m,R}\mathcal{S}(\mathbb{R}^d).$$

We check that $\mathcal{S}^{(1,m,R)}$ is a closed subset inside $\mathcal{S}(\mathbb{R}^d)$ (repeat the arguments above), i.e., $\overline{\mathcal{S}^{(1,m,R)}} \cap \mathcal{S}(\mathbb{R}^d) = \mathcal{S}^{(1,m,R)}$. In particular, $\mathcal{S}^{(1,m,R)}$ is measurable w.r.t. the vague topology of $\mathcal{M}(\mathbb{R}^d)$. Similarly, one shows that $\mathcal{S}(\mathbb{R}^d)\backslash\mathcal{S}^{(2,m,R)}$ is closed as a subset inside $\mathcal{S}(\mathbb{R}^d)$. Again, this shows that $\mathcal{S}^{(2,m,R)}$ is measurable. Consider now the measurable sets

$$\mathcal{S}^{(1,\infty,\infty)} := \bigcap_{R\in\mathbb{N}}\bigcup_{m\in\mathbb{N}}\mathcal{S}^{(1,m,R)} \qquad \text{and} \qquad \mathcal{S}^{(2,\infty,\infty)} := \bigcap_{R\in\mathbb{N}}\bigcup_{m\in\mathbb{N}}\mathcal{S}^{(2,m,R)}.$$

We see that

i) $\mathbb{x} \in \mathcal{S}^{(1,\infty,\infty)}$ if, and only if, for all $x, y \in \mathbb{x}$, it holds $d(x, y) \neq 2r$.
ii) $\mathbb{x} \in \mathcal{S}^{(2,\infty,\infty)}$ if, and only if, for every $x \in \mathbb{x}$, it holds that $\#C_{\mathbb{x}}(x) < \infty$ and $\delta(\mathbb{B}_r(C_{\mathbb{x}}(x))) > 0$.

Therefore,

$$\mathcal{S}_{\mathcal{A}}(\mathbb{R}^d) = \mathcal{S}^{(1,\infty,\infty)} \cap \mathcal{S}^{(2,\infty,\infty)},$$

is measurable.

To see that $\mathtt{F}_n\colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is measurable, recall $\mathtt{F}_n = \mathtt{F}_{2,n} \circ \mathtt{F}_{1,n}$ from Definition 3. It, therefore, suffices to show that the following maps are measurable:

$$\mathtt{F}_{1,n}\colon \mathcal{S}(\mathbb{R}^d) \to \mathtt{F}_{1,n}\mathcal{S}(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d) \qquad \text{and} \qquad \mathtt{F}_{2,n}\colon \mathtt{F}_{1,n}\mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d).$$

For $f \in C_c(\mathbb{R}^d)$, consider the evaluation by f, i.e.,

$$M_f\colon \mathcal{S}(\mathbb{R}^d) \to \mathbb{R}, \qquad \mathbb{x} \mapsto \int_{\mathbb{R}^d} f\,\mathrm{d}\mathbb{x}.$$

If $f \geq 0$, we observe the upper semi-continuity of

$$M_f \circ \mathtt{F}_{1,n}\colon \mathcal{S}(\mathbb{R}^d) \to \mathbb{R} \qquad \text{and} \qquad M_f \circ \mathtt{F}_{2,n}\colon \mathtt{F}_{1,n}\mathcal{S}(\mathbb{R}^d) \to \mathbb{R}.$$

We have lower semi-continuity for $f \leq 0$ since $M_{-f} = -M_f$. Therefore $M_f \circ \mathtt{F}_{i,n}$ with $i \in \{1, 2\}$ is measurable in the cases $f \geq 0$ and $f \leq 0$ and hence, in general. Since the vague topology is generated by $(M_f)_{f\in C_c(\mathbb{R}^d)}$, we conclude that $\mathtt{F}_{1,n}$ and $\mathtt{F}_{2,n}$ are measurable.

Remark 27 (*Fine details of Theorem* 26).

- For $\mathcal{S}^{(n)} := \mathtt{F}_n\mathcal{S}(\mathbb{R}^d)$, we have that

$$\bigcup_{n\in\mathbb{N}}\mathcal{S}^{(n)} \subsetneq \mathcal{S}_{\mathcal{A}}(\mathbb{R}^d) \subsetneq \{\mathbb{x}\colon \lim_{n\to\infty}\mathtt{F}_n\mathbb{x} = \mathbb{x}\} \subsetneq \mathcal{S}(\mathbb{R}^d) \subsetneq \overline{\bigcup_{n\in\mathbb{N}}\mathcal{S}^{(n)}} = \mathcal{N}(\mathbb{R}^d).$$

- $M_f \circ \mathtt{F}_{2,n}$ is *not* upper semi-continuous on $\mathcal{S}(\mathbb{R}^d)$ (in contrast to $\mathtt{F}_{1,n}\mathcal{S}(\mathbb{R}^d)$): The condition that $d(x, y) \notin (2r - 1/n,\ 2r + 1/n)\ \forall x, y \in \mathtt{F}_{1,n}\mathbb{x}$ is crucial to ensure that clusters do not change sizes.

Definition 28. We define the events that the origin is not covered by the filled-up Boolean model, i.e.,

$$\mathbf{G} := \{\mathbb{x} \in \mathcal{S}(\mathbb{R}^d) \,|\, o \notin \boxminus\mathbb{x}\} \qquad \text{and} \qquad \mathbf{G}_n := \mathtt{F}_n{}^{-1}(\mathbf{G}) = \{\mathbb{x} \in \mathcal{S}(\mathbb{R}^d) \,|\, o \notin \boxminus\mathbb{x}^{(n)}\}.$$

This gives us, for $x \in \mathbb{R}^d$, that

$$\mathbb{1}_{\boxminus\mathbb{x}^{\complement}}(x) = \mathbb{1}_{\mathbf{G}}(\tau_x \mathbb{x}).$$

We will later consider the effective conductivities based on these events.

Lemma 29 (Approximation properties)**.** *Let $\mathbb{x} \in \mathcal{S}(\mathbb{R}^d)$ be an admissible point cloud.*

i) For every bounded domain Λ, there exists an $N(\mathbb{x}, \Lambda) \in \mathbb{N}$ such that for every $n \geq N(\mathbb{x}, \Lambda)$

$$\mathbb{x}^{(n)} \cap \Lambda = \mathbb{x} \cap \Lambda, \qquad \textit{in particular} \qquad \mathbb{x} = \bigcup_{n\in\mathbb{N}} \mathbb{x}^{(n)}.$$

ii) For every bounded domain Λ, there exists an $\tilde{N}(\mathbb{x}, \Lambda) \in \mathbb{N}$ such that for every $n \geq \tilde{N}(\mathbb{x}, \Lambda)$

$$\boxminus\mathbb{x}^{(n)} \cap \Lambda = \boxminus\mathbb{x} \cap \Lambda, \qquad \textit{in particular} \qquad \boxminus\,\mathbb{x} = \bigcup_{n\in\mathbb{N}} \boxminus\mathbb{x}^{(n)}.$$

iii) There exists an $N = N(\mathbb{x}) \in \mathbb{N}$ such that for every $n \geq N$:

$$o \notin \boxminus\mathbb{x}^{(n)} \iff o \notin \boxminus\mathbb{x}.$$

In particular, $\bigcap_{n\in\mathbb{N}} \mathbf{G}_n \backslash \mathbf{G}$ only consists of non-admissible point clouds.

Proof. i) Boundedness of Λ implies that there are only finitely many mutually disjoint clusters $C_{\mathbb{x}}(x_i)$, $i = 1, \ldots, N_C$ that intersect with Λ. Furthermore, because $\#\,(\mathbb{x} \cap \mathbb{B}_r(\Lambda)) < \infty$ and because of Property 1 of admissible point clouds, we know

$$\min\{||x - y| - 2r| : x, y \in \mathbb{x} \cap \mathbb{B}_r(\Lambda),\ x \neq y\} > 0$$

and Lemma 24 yields

$$\min\{\delta(\Xi C_{\mathbb{x}}(x_i)) \colon C_{\mathbb{x}}(x_i) \cap \Lambda \neq \emptyset\} > 0.$$

This implies the first statement.

ii) By making Λ larger, we may assume $\Lambda = [-k, k]^d$ for some $k \in \mathbb{N}$. For $n \geq N(\mathbb{B}_r\big([-k, k]^d\big))$,

$$[-k, k]^d \backslash \Xi\mathbb{x}^{(n)} = [-k, k]^d \backslash \Xi\mathbb{x}.$$

$[-k, k]^d \backslash \Xi\mathbb{x}$ only has finitely many connected components C_i. Take one of these connected components C_i and suppose it lies in $\boxminus\mathbb{x}$. Then, it has to be encircled by finitely many balls $\mathbb{B}_r(x)$ in $\Xi\mathbb{x}$. Let n_i be large enough such that all these x lie in $\mathbb{x}^{(n_i)}$. Then, $C_i \subset \boxminus\mathbb{x}^{(n_i)}$. We may do so for every C_i. Take

$$\tilde{N}(\mathbb{x}, \Lambda) := \max\{n_i,\ N(\mathbb{B}_r\big([-k, k]^d\big))\}.$$

For every $n \geq \tilde{N}(\mathbb{x}, \Lambda)$, the connected components C_i of $[-k, k]^d \backslash \Xi\mathbb{x}^{(n)}$ and $[-k, k]^d \backslash \Xi\mathbb{x}$ are identical since $[-k, k]^d \backslash \Xi\mathbb{x}^{(n)} = [-k, k]^d \backslash \Xi\mathbb{x}$. Therefore, we get the claim

$$\begin{aligned} [-k, k]^d \backslash \boxminus\,\mathbb{x}^{(n)} &= \Big([-k, k]^d \backslash \Xi\mathbb{x}^{(n)}\Big) \backslash \bigcup_{C_i \subset \boxminus\mathbb{x}^{(n)}} C_i \\ &= \Big([-k, k]^d \backslash \Xi\mathbb{x}\Big) \backslash \bigcup_{C_i \subset \boxminus\mathbb{x}} C_i \quad = [-k, k]^d \backslash \boxminus\,\mathbb{x}. \end{aligned}$$

iii) This is a direct consequence of Point 2. If $\mathbb{x} \in \bigcap_{n\in\mathbb{N}} \mathbf{G}_n \backslash \mathbf{G}$, then $o \notin ⊟\mathbb{x}^{(n)}$ for every n but $o \in ⊟\mathbb{x}$. Therefore, $\mathbb{x}$ cannot be admissible by Point 2.

Definition 30 (Surface measure of $⊟\mathbb{x}$)**.** We define the *surface measure* for $\mathbb{x} \in \mathcal{S}(\mathbb{R}^d)$

$$\mu_{\mathbb{x}}(A) := \mathcal{H}^{d-1}_{\llcorner \partial ⊟\mathbb{x}}(A) = \mathcal{H}^{d-1}(A \cap \partial ⊟ \mathbb{x}).$$

Note that $\mu_{\mathbb{x}}\Big([0,1]^d\Big) \leq \mathcal{H}^{d-1}(\mathbb{B}_r(o)) \cdot \mathbb{x}\Big(\mathbb{B}_r\Big([0,1]^d\Big)\Big)$.

Definition 31 (Intensity of random measure)**.** Given a stationary random measure $\tilde{\mu}$, we define its intensity

$$\lambda(\tilde{\mu}) := \mathbb{E}[\tilde{\mu}([0,1)^d)].$$

We define the intensity of point processes by identifying them as random measures.

Lemma 32 (Convergence of intensities)**.** *Let $\mathbb{X}$ be an admissible stationary point process with finite intensity $\lambda(\mathbb{X})$. Then,*

$$\lim_{n\to\infty} \lambda(\mathbb{X}^{(n)}) = \lambda(\mathbb{X}) \qquad \text{and} \qquad \lim_{n\to\infty} \lambda(\mu_{\mathbb{X}^{(n)}}) = \lambda(\mu_{\mathbb{X}}).$$

Proof. "Almost surely" is to be understood w.r.t. the distribution $\mathbb{P}$ of $\mathbb{X}$.

i) By Lemma 29, we have almost surely $\mathbb{X}^{(n)}\Big([0,1]^d\Big) \to \mathbb{X}\Big([0,1]^d\Big)$ as $n \to \infty$. Dominated convergence with majorant $\mathbb{X}\Big([0,1]^d\Big)$ yields

$$\lambda(\mathbb{X}^{(n)}) = \mathbb{E}[\mathbb{x}^{(n)}\Big([0,1]^d\Big)] \to \mathbb{E}[\mathbb{x}\Big([0,1]^d\Big)] = \lambda(\mathbb{X}).$$

ii) Again, by Lemma 29, we have almost surely $⊟\mathbb{X}^{(n)} \cap [0,1]^d \to ⊟\mathbb{X} \cap [0,1]^d$, in particular

$$\mu_{\mathbb{X}^{(n)}}\Big([0,1]^d\Big) = \mathcal{H}^{d-1}_{\llcorner \partial ⊟\mathbb{X}^{(n)}}\Big([0,1]^d\Big) \to \mathcal{H}^{d-1}_{\llcorner \partial ⊟\mathbb{X}}\Big([0,1]^d\Big) = \mu_{\mathbb{X}}\Big([0,1]^d\Big).$$

Dominated convergence yields, again, convergence of intensities.

Remark 33 (*Local convergence*). The convergence in Lemma 29 is much stronger than what is actually needed to prove the convergence of intensities. Indeed, we could prove convergence even for so-called tame and local functions $f\colon \mathcal{M}(\mathbb{R}^d) \to \mathbb{R}$, among which the intensity λ is just one special case $f(\mathbb{x}) := \mathbb{x}\Big([0,1]^d\Big)$.

5. Effective conductivity and cell solutions

The structure $(\mathcal{M}(\mathbb{R}^d), \mathcal{B}(\mathcal{M}(\mathbb{R}^d)), \mathbb{P}, \tau)$ as in Definition 8 is a dynamical system:

Definition 34 (Dynamical system, stationarity, ergodicity)**.** Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a separable metric probability space. A *dynamical system* $\tau = (\tau_x)_{x\in\mathbb{R}^d}$ is a family of measurable mappings $\tau_x\colon \Omega \to \Omega$ satisfying

- Group property: $\tau_0 = \mathrm{id}_\Omega$ and $\tau_{x+y} = \tau_x \circ \tau_y$ for any $x,\, y \in \mathbb{R}^d$.
- Measure preserving: For any $x \in \mathbb{R}^d$ and any $F \in \mathcal{F}$, we have $\mathcal{P}(\tau_x(F)) = \mathcal{P}(F)$.

- Continuity: The map $\mathcal{T}\colon \Omega\times\mathbb{R}^d \to \Omega$, $(\omega,\ x) \mapsto \tau_x(\omega)$ is continuous w.r.t. the product topology on $\Omega\times\mathbb{R}^d$.

τ is called *ergodic* if the σ-algebra of τ-invariant sets is trivial under $\mathcal{P}$.

Our practical setting will always be some $\Omega\subset\mathcal{M}(\mathbb{R}^d)$, but we will still work with abstract dynamical systems in Sections 5 and 6. In this context, let us mention that continuity of τ was not needed in the first place [22], but turned out to be very useful in the proofs concerning two-scale convergence [30]. Since we frequently use results from [30] and since we get continuity of τ for free in our applied setting, we will simply rely on this property.

5.1. Potentials and solenoidals

Let $(\Omega,\mathcal{B}(\Omega),\mathcal{P},\tau)$ be a dynamical system. We write $L^2(\Omega) := L^2(\Omega,\ \mathcal{P})$. The dynamical system τ introduces a strongly continuous group action on $L^2(\Omega)\to L^2(\Omega)$ through $T_x f(\omega) := f(\tau_x\omega)$ with the d independent generators

$$\mathrm{D}_i f := \lim_{t\to 0}\frac{1}{t}\left(f - f(\tau_{te_i}\bullet)\right),$$

with domain $\mathcal{D}_i$ where $(e_i)_{i=1,\dots,d}\subset\mathbb{R}^d$ is the canonical Euclidean basis. Introducing

$$H^1(\Omega) := \bigcap_{i=1}^{d}\mathcal{D}_i\subset L^2(\Omega),$$

and the gradient $\nabla_\omega f := (\mathrm{D}_1 f,\dots,\mathrm{D}_d f)^\top$, we can define the space of potential vector fields

$$\mathcal{V}^2_{\mathrm{pot}}(\Omega) := \{\nabla_{\tilde\omega} f \mid f\in H^1(\Omega) \text{ and } \int_\Omega \nabla_{\tilde\omega} f\,\mathrm{d}\mathcal{P}(\omega) = 0_{\mathbb{R}^d}\}.$$

Defining $L^2_{\mathrm{sol}}(\Omega) := \mathcal{V}^2_{\mathrm{pot}}(\Omega)^\perp$, we find with $u_\omega(x) := u(\tau_x\omega)$ that

$$\begin{aligned} L^2_{\mathrm{pot}}(\Omega) &:= \left\{u\in L^2(\Omega;\mathbb{R}^d)\colon\ u_\omega\in L^2_{\mathrm{pot,loc}}(\mathbb{R}^d) \text{ for } \mathcal{P}-\text{a.e. } \omega\in\Omega\right\},\\ L^2_{\mathrm{sol}}(\Omega) &= \left\{u\in L^2(\Omega;\mathbb{R}^d)\colon\ u_\omega\in L^2_{\mathrm{sol,loc}}(\mathbb{R}^d) \text{ for } \mathcal{P}-\text{a.e. } \omega\in\Omega\right\},\\ \mathcal{V}^2_{\mathrm{pot}}(\Omega) &= \left\{u\in L^2_{\mathrm{pot}}(\Omega)\colon\ \int_\Omega u\,\mathrm{d}\mathcal{P} = 0\right\}, \end{aligned} \tag{5.1}$$

because Ω is a separable metric [8] where

$$\begin{aligned} L^2_{\mathrm{pot,loc}}(\mathbb{R}^d) &:= \left\{u\in L^2_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)\colon\ \forall U \text{ bounded domain } \exists\varphi\in W^{1,2}(U)\colon\ u=\nabla\varphi\right\},\\ L^2_{\mathrm{sol,loc}}(\mathbb{R}^d) &:= \left\{u\in L^2_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^d)\colon\ \int_{\mathbb{R}^d} u\cdot\nabla\varphi\,\mathrm{d}x = 0\ \forall\varphi\in C^1_c(\mathbb{R}^d)\right\}. \end{aligned}$$

For $A\subset\Omega$ measurable, we define

$$\mathcal{V}^2_{\mathrm{pot}}(A|\Omega) := cl_{L^2(A)^d}\left\{v_{|A}\colon\ v\in\mathcal{V}^2_{\mathrm{pot}}(\Omega)\right\}.$$

5.2. Cell solutions and effective conductivity

Definition 35 (Cell solutions)**.** Let $(\Omega, \mathcal{B}(\Omega), \mathcal{P})$ be a separable metric probability space with dynamical system τ and let $Q \in \mathcal{B}(\Omega)$. We notice that for every unit vector e_i, the map

$$v \mapsto -\int_Q e_i \cdot v \, \mathrm{d}\mathcal{P}(\omega) = -\langle e_i, v\rangle_{L^2(Q)^d},$$

is a bounded linear functional on the Hilbert space $\mathcal{V}^2_{\mathrm{pot}}(Q|\Omega)$. Using the Riesz representation theorem, we obtain a unique $w_i \in \mathcal{V}^2_{\mathrm{pot}}(Q|\Omega)$ such that $\langle w_i, v\rangle_{\mathcal{V}^2_{\mathrm{pot}}(Q|\Omega)} = -\langle e_i, v\rangle_{L^2(Q)^d}$ or equivalently

$$\forall v \in \mathcal{V}^2_{\mathrm{pot}}(\Omega): \qquad \int_Q [w_i + e_i] \cdot v \, \mathrm{d}\mathcal{P}(\omega) = 0.$$

w_i is called the i-th *cell solution*. The cell solutions satisfy

$$\|w_i\|_{L^2(Q)^d} \leq \sqrt{\mathcal{P}(Q)} \leq 1,$$

and can be grouped in the matrix

$$W_Q := (w_1, \dots, w_d).$$

Remark 36. We observe that by definition, w_i is the minimizer in $\mathcal{V}^2_{\mathrm{pot}}(Q|\Omega)$ of the functional

$$\mathcal{E}_i(w) = \int_Q \frac{1}{2}(w_i + e_i)^2 \, \mathrm{d}\mathcal{P}. \tag{5.2}$$

Definition 37 (Effective conductivity $\mathcal{A}$)**.** Let w_i be the cell solution on $Q \in \mathcal{B}(\Omega)$. The *effective conductivity* $\mathcal{A}$ based on the event Q is defined as

$$\mathcal{A} := \int_Q (I_d + W_Q)^t (I_d + W_Q) \, \mathrm{d}\mathcal{P}(\omega), \tag{5.3}$$

with I_d being the identity matrix. We observe for the entries $\left(\mathcal{A}_{i,j}\right)_{i,j=1,\dots,d}$ of $\mathcal{A}$ that

$$\mathcal{A}_{i,j} = \int_Q [e_i + w_i(\omega)] \cdot \left[e_j + w_j(\omega)\right] \mathrm{d}\mathcal{P}(\omega) = \int_Q [e_i + w_i(\omega)] \cdot e_j \, \mathrm{d}\mathcal{P}(\omega). \tag{5.4}$$

We write $\alpha_{\mathcal{A}} \geq 0$ for its smallest eigenvalue.

Lemma 38 (Convergence of cell solutions)**.** *Let $(Q_n)_{n\in\mathbb{N}} \subset \mathcal{B}(\Omega)$ and $Q \in \mathcal{B}(\Omega)$ such that $\mathbb{1}_{Q_n} \to \mathbb{1}_Q$ $\mathcal{P}$-almost everywhere as $n \to \infty$ and*

- *either $Q_n \supset Q$ for every n, or*
- *there exist $C > 0$, $r \in (1, 2]$ independent from n and continuous operators $\mathcal{U}_n : \mathcal{V}^2_{\mathrm{pot}}(Q|\Omega) \to \mathcal{V}^r_{\mathrm{pot}}(\Omega)$ such that, for every n and every $w \in \mathcal{V}^2_{\mathrm{pot}}(Q_n|\Omega)$, it holds that $\|\mathcal{U}_n w\|_{L^r(\Omega)} \leq C \|w\|_{\mathcal{V}^2_{\mathrm{pot}}(Q_n|\Omega)}$.*

Then, the sequence of cell solutions $w_i^{(n)}$ to the cell problem on Q_n satisfies

$$w_i^{(n)} \rightharpoonup w_i \qquad \text{in } L^2(\Omega)^d \text{ as } n \to \infty,$$

where $w_i \in \mathcal{V}^2_{\mathrm{pot}}(Q|\Omega)$ is the i-th cell solution on Q

$$\forall v \in \mathcal{V}^2_{\mathrm{pot}}(\Omega): \qquad \int_Q [w_i + e_i] \cdot v \, \mathrm{d}\mathcal{P}(\omega) = 0.$$

Proof. We first check that the limit satisfies $\int_Q [w_i + e_i] \cdot v \, d\mathcal{P}(\omega) = 0$ and then $w_i \in \mathcal{V}^2_{\text{pot}}(Q|\Omega)$.

i) In both cases, the a priori estimate yields an L^2-weakly convergent subsequence of $w_i^{(n)} \rightharpoonup w_i \in L^2(\Omega)^d$ after extending $w_i^{(n)}$ to the whole of Ω via 0. Let $v \in \mathcal{V}^2_{\text{pot}}(\Omega)$. We have $\mathbb{1}_{Q_n} \to \mathbb{1}_Q$ $\mathcal{P}$-almost everywhere, so dominated convergence yields

$$\lim_{n\to\infty} \int_{Q_n} e_i \cdot v \, d\mathcal{P}(\omega) = \int_Q e_i \cdot v \, d\mathcal{P}(\omega),$$

while weak convergence yields

$$\lim_{n\to\infty} \int_{Q_n} w_i^{(n)} \cdot v \, d\mathcal{P}(\omega) = \int_\Omega w_i \cdot v \, d\mathcal{P}(\omega).$$

We also have

$$\mathbb{1}_{Q_n} w_i^{(n)} = w_i^{(n)} \xrightharpoonup[n\to\infty]{L^2(\Omega)^d} w_i,$$

which implies

$$\mathbb{1}_Q w_i = w_i.$$

Therefore, with $w_i \in L^2(Q)^d$:

$$0 = \lim_{n\to\infty} \int_{Q_n} [w_i^{(n)} + e_i] \cdot v \, d\mathcal{P}(\omega) = \int_Q [w_i + e_i] \cdot v \, d\mathcal{P}(\omega).$$

ii) In both cases of the lemma, the space $\mathcal{V}^2_{\text{pot}}(Q|\Omega) \subset L^2(Q)^d$ is closed and convex, so it is also weakly closed. We construct a weakly converging sequence in $\mathcal{V}^2_{\text{pot}}(Q|\Omega)$ that converges to w_i. Since $w_i^{(n)} \in \mathcal{V}^2_{\text{pot}}(Q_n|\Omega)$, we find $v^{(n)} \in \mathcal{V}^2_{\text{pot}}(\Omega)$ such that

$$\|w_i^{(n)} - \mathbb{1}_{Q_n} v^{(n)}\|_{L^2(Q)^d} \leq \frac{1}{n}.$$

Since $w_i^{(n)} \rightharpoonup w_i$, we get

$$\mathbb{1}_{Q_n} v^{(n)} \xrightharpoonup[n\to\infty]{L^2(\Omega)^d} w_i.$$

Note that $(\mathbb{1}_{Q_n} - \mathbb{1}_Q) \, v^{(n)}$ is a bounded sequence that is weakly converging to 0 because for every $\phi \in L^2(\Omega)$ we find by dominated convergence that $(\mathbb{1}_{Q_n} - \mathbb{1}_Q) \, \phi \to 0$ strongly in $L^2(\Omega)$. Therefore, we also obtain

$$\mathbb{1}_Q v^{(n)} \xrightharpoonup[n\to\infty]{L^2(\Omega)^d} w_i.$$

In the first case of the lemma, $\mathbb{1}_Q v^{(n)} \in \mathcal{V}^2_{\text{pot}}(Q|\Omega)$, so we get that $w_i \in \mathcal{V}^2_{\text{pot}}(Q|\Omega)$. In the second case, we obtain $\mathbb{1}_Q v^{(n)} \in \mathcal{V}^r_{\text{pot}}(Q|\Omega)$ and hence $w_i \in \mathcal{V}^r_{\text{pot}}(Q|\Omega)$. Since being a L^r_{pot}-function is distinguished from being a pure L^r-function only by the Condition (5.1), the integrability of w_i then yields $w_i \in \mathcal{V}^2_{\text{pot}}(Q|\Omega)$.

Corollary 39 (Convergence of effective conductivities)**.** *Let* $(Q_n)_{n\in\mathbb{N}} \subset \mathcal{B}(\Omega)$ *and* $Q \in \mathcal{B}(\Omega)$ *such that* $\mathbb{1}_{Q_n} \to \mathbb{1}_Q$ *$\mathcal{P}$-almost surely and*

- *either* $Q_n \supset Q$ *for every* n*, or*

- *there exist $C > 0$, $r \in (1,2]$ independent from n and continuous operators $\mathcal{U}_n\colon \mathcal{V}^2_{\mathrm{pot}}(Q|\Omega) \to \mathcal{V}^r_{\mathrm{pot}}(\Omega)$ such that, for every n and every $w \in \mathcal{V}^2_{\mathrm{pot}}(Q_n|\Omega)$, it holds that $\|\mathcal{U}_n w\|_{L^r(\Omega)} \le C\,\|w\|_{\mathcal{V}^2_{\mathrm{pot}}(Q_n|\Omega)}$.*

Let $\mathcal{A}^{(n)}$ be the effective conductivity of Q_n and $\mathcal{A}$ be the effective conductivity of Q. Then,

$$\mathcal{A}^{(n)} \xrightarrow{n\to\infty} \mathcal{A}.$$

Proof. This follows from Eq (5.4) and weak convergence $w_i^{(n)} \rightharpoonup w_i$.

Remark 40 (*Variational formulation*). There is another way to define $\mathcal{A}$: For $\eta \in \mathbb{R}^d$, $W_Q\eta$ (see Definition 35) is the unique minimizer to

$$\min_{\varphi\in\mathcal{V}^2_{\mathrm{pot}}(Q)} \int_Q \left|\eta+\varphi\right|^2 \mathrm{d}\mathcal{P}(\omega),$$

and therefore

$$\eta^t\mathcal{A}\eta = \int_Q \left|(I_d + W_Q)\eta\right|^2 \mathrm{d}\mathcal{P}(\omega) = \min_{\varphi\in\mathcal{V}^2_{\mathrm{pot}}(Q)} \int_Q \left|\eta+\varphi\right|^2 \mathrm{d}\mathcal{P}(\omega).$$

This equality is related to Theorem 57.

5.3. Proof of Theorems 21 and 22

We will prove both theorems at once. The first part follows using the weak formulation, weak convergence of both u_n^ε and ∇u_n^ε as $n \to \infty$ as well as standard PDE arguments. The second part is proved in [30] using our notation and also before in [28]. The convergence of $\mathcal{A}^{(n)}$ in Part three is given by Corollary 39. Hence, it only remains to prove the convergence of u_n.

When it comes to the latter, we have to distinguish between three cases:

i) There exists $C_0 > 0$ such that, for every $\xi \in \mathbb{R}^d \setminus \{0\}$ and every $n \in \mathbb{N}\cup\{\infty\}$, it holds

$$|\xi|^2 \le C_0 \xi\cdot\mathcal{A}^{(n)}\xi.$$

Then, the proof is straightforward using the resulting uniform estimates on $\|\nabla u\|_{L^2(Q)} + \|u\|_{L^2(Q)}$.

ii) $\mathcal{A} = 0$: Then, convergence follows after assuming that $f_n \in H^1(Q)$ is uniformly bounded as well as testing with $\lambda u_n - f_n$. This will eventually lead to an estimate

$$\|\lambda u_n - f_n\|^2_{L^2(Q)} \le \frac{\tilde{C}}{\lambda}\left(\int_Q \nabla f_n\cdot\mathcal{A}^{(n)}\nabla f_n\right) \to 0.$$

Afterward, we may use an approximation argument for f_n bounded in $L^2(Q)$ using a standard mollifier: $f_n^\delta = f_n * \eta_\delta$ with $\|a - a*\eta_\delta\|_{L^2(Q)} \le \delta\|a\|_{L^2(Q)}$. Then,

$$\int_Q \nabla(u_n^\delta - u_n)\mathcal{A}^{(n)}\nabla(u_n^\delta - u_n) + \frac{\lambda}{2}\|u_n^\delta - u_n\|^2_{L^2(Q)} \le \frac{1}{2\lambda}\|f_n^\delta - f_n\|^2_{L^2(Q)}.$$

iii) $\xi\cdot\mathcal{A}\xi = 0$ along some $\xi \in \mathbb{R}^d\setminus\{0\}$. Then, these ξ form a linear subspace and we may restrict our testing functions to a dependence in the orthogonal direction, only. More precisely, we can consider testing functions with support concentrated on the orthogonal plane to ξ as $n \to \infty$.

5.4. Pull-back for thinning maps

In Section 6, we will use two-scale convergence to homogenize Eq (1.1) for fixed n. This process is more convenient to handle if the underlying probability space is compact. Here we show that we may take $\mathtt{F}_n\mathcal{S}(\mathbb{R}^d)$ as the underlying probability space instead of $\mathcal{S}(\mathbb{R}^d)$.

Lemma 41. *Let $\mathcal{P}$ be a distribution on $\mathcal{S}(\mathbb{R}^d)$ and let $\mathcal{S}^{(n)} := \mathtt{F}_n\mathcal{S}(\mathbb{R}^d)$ with the push-forward measure $\tilde{\mathcal{P}}_n := \mathcal{P} \circ \mathtt{F}_n^{-1}$. Recall $\mathbf{G}_n := \left\{\mathbb{x} \in \mathcal{S}(\mathbb{R}^d) : o \notin \boxminus\mathbb{x}^{(n)}\right\}$ and let $\tilde{\mathbf{G}}_n := \left\{\mathbb{x} \in \mathcal{S}^{(n)} : o \notin \boxminus\mathbb{x}\right\} = \mathtt{F}_n\mathbf{G}_n$. Let $w_i^{(n)}$ be the cell solutions on $\mathbf{G}_n$ and $\tilde{w}_i^{(n)}$ the cell solutions on $\tilde{\mathbf{G}}_n$ for their respective dynamical systems. Then, for every $i, j \in \{1, \dots, d\}$, it holds that*

$$\int_{\mathbf{G}_n} [w_i^{(n)} + e_i] \cdot e_j \,\mathrm{d}\mathcal{P} = \int_{\tilde{\mathbf{G}}_n} [\tilde{w}_i^{(n)} + e_i] \cdot e_j \,\mathrm{d}\tilde{\mathcal{P}}_n. \tag{5.5}$$

Lemma 42 (Properties of pull-back functions)**.** *Let $(\Omega, \mathcal{F}, \mathcal{P}, \tau)$, $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{P}}, \tilde{\tau})$ be dynamical systems, $\phi \colon \Omega \to \tilde{\Omega}$ measurable such that $\tilde{\mathcal{P}} = \mathcal{P} \circ \phi^{-1}$ and such that for every $x \in \mathbb{R}^d$*

$$\phi \circ \tau_x = \tilde{\tau}_x \circ \phi. \tag{5.6}$$

Then, the following holds: For every $\tilde{f} \in L^2(\tilde{\Omega})^d$, we have $f := \tilde{f} \circ \phi \in L^2(\Omega)^d$ with $\|f\|_{L^2(\Omega)^d} = \|\tilde{f}\|_{L^2(\tilde{\Omega})^d}$. If $\tilde{f} \in \mathcal{V}^2_{\mathrm{pot}}(\tilde{\Omega})$, then $f \in \mathcal{V}^2_{\mathrm{pot}}(\Omega)$. If $\tilde{f} \in L^2_{\mathrm{sol}}(\tilde{\Omega})$, then $f \in L^2_{\mathrm{sol}}(\Omega)$. f is called the pull-back *of $\tilde{f}$.*

Proof. Due to $\tilde{\mathcal{P}} = \mathcal{P} \circ \phi^{-1}$, we immediately obtain for arbitrary measurable $\tilde{g} \in L^1(\tilde{\Omega})^d$ and its pull-back g

$$\int_{\tilde{\Omega}} \tilde{g} \,\mathrm{d}\tilde{\mathcal{P}} = \int_{\Omega} \tilde{g} \circ \phi \,\mathrm{d}\mathcal{P} = \int_{\Omega} g \,\mathrm{d}\mathcal{P}. \tag{5.7}$$

Therefore $\|f\|_{L^2(\Omega)^d} = \|\tilde{f}\|_{L^2(\tilde{\Omega})^d}$ and (5.6) yields $\tilde{f} \in \mathcal{V}^2_{\mathrm{pot}}(\tilde{\Omega}) \implies f \in \mathcal{V}^2_{\mathrm{pot}}(\Omega)$. For $\tilde{f} \in L^2_{\mathrm{sol}}(\tilde{\Omega})$, $f \in L^2_{\mathrm{sol}}(\Omega)$, follows from $\phi \circ \tau_x = \tilde{\tau}_x \circ \phi$ and checking

$$\int_{\Lambda} f(\tau_x\omega) \cdot \nabla\varphi(x) \,\mathrm{d}x = \int_{\Lambda} \tilde{f}(\tilde{\tau}_x\omega) \cdot \nabla\varphi(x) \,\mathrm{d}x = 0,$$

for $\mathcal{P}$-almost every ω and every $\varphi \in C^1_c(\Lambda)$ on a bounded domain $\Lambda \subset \mathbb{R}^d$.

Proof of Lemma 41. We use Lemma 42 for $\phi = \mathtt{F}_n$ with $(\Omega, \mathcal{F}, \mathcal{P}, \tau) = (\mathcal{S}(\mathbb{R}^d), \mathcal{B}(\mathcal{S}(\mathbb{R}^d)), \mathbb{P}, \tau)$, where τ_x is the shift-operator on $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{B}(\mathcal{S}(\mathbb{R}^d))$ is the Borel-σ-algebra generated by the vague topology, and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{P}}, \tilde{\tau}) = (\mathcal{S}^{(n)}), \mathcal{B}(\mathcal{S}^{(n)}), \mathbb{P} \circ \mathtt{F}_n^{-1}, \tau)$ again with shift-operator and Borel-σ-algebra. Let $\overline{w}_i$ be the pull-back of $\tilde{w}_i^{(n)}$ according to Lemma 42. We see $\mathtt{F}_n^{-1}\tilde{\mathbf{G}}_n = \mathbf{G}_n$, so $\overline{w}_i$ has support in $\mathbf{G}_n$. Let $\tilde{v}_k \in \mathcal{V}^2_{\mathrm{pot}}(\mathcal{S}^{(n)})$ with $\|\tilde{w}_i^{(n)} - \tilde{v}_k\|_{L^2(\tilde{\mathbf{G}}_n)^d} \leq \frac{1}{k}$. The pull-back $v_k \in \mathcal{V}^2_{\mathrm{pot}}(\Omega)$ of $\tilde{v}_k$ satisfies $\|\overline{w}_i - v_k\|_{L^2(\mathbf{G}_n)^d} \leq \frac{1}{k}$ and hence, $\overline{w}_i \in \mathcal{V}^2_{\mathrm{pot}}(\mathbf{G}_n|\mathcal{S}(\mathbb{R}^d))$. We observe $(\tilde{w}_i^{(n)} + e_i)\mathbb{1}_{\tilde{\mathbf{G}}_n} \in L^2_{\mathrm{sol}}(\mathcal{S}^{(n)})$ with the pull-back $(\overline{w}_i + e_i)\mathbb{1}_{\mathbf{G}_n} \in L^2_{\mathrm{sol}}(\mathcal{S}(\mathbb{R}^d))$. This implies $\overline{w}_i = w_i^{(n)}$. Eq (5.7) yields Eq (5.5).

6. Proof of Lemmas 7 and 10

We first collect all the tools needed to prove the homogenization result for minimally smooth domains (Lemma 10).

6.1. Extensions and traces for thinned point clouds

Theorem 43 (Extending beyond holes and trace operator)**.** *There exists a constant $C > 0$ depending only on $n \in \mathbb{N}$ and $M_0 > 1$ such that the following holds: Assume that $Q \subset \mathbb{R}^d$ is a bounded Lipschitz domain with Lipschitz constant M_0, $\mathbb{x} \in \mathtt{F}_n\mathcal{S}(\mathbb{R}^d)$ and $\mathbb{x}_Q \subset \mathbb{x}$ such that for every $x \in \mathbb{x}_Q$ it holds $\mathbb{B}_{2r}(x) \subset Q$. Then, there exists an extension operator*

$$\mathcal{U}_{\mathbb{x}_Q} \colon W^{1,2}(Q \setminus \boxminus\mathbb{x}_Q) \to W^{1,2}(Q),$$

such that $(\mathcal{U}_{\mathbb{x}_Q} u)_{|Q\setminus\boxminus\mathbb{x}_Q} = u$ and

$$\left\|\mathcal{U}_{\mathbb{x}_Q} u\right\|_{L^2(Q)} \le C \left\|u\right\|_{L^2(Q\setminus\boxminus\mathbb{x}_Q)}, \qquad \left\|\nabla\mathcal{U}_{\mathbb{x}_Q} u\right\|_{L^2(Q)} \le C \left\|\nabla u\right\|_{L^2(Q\setminus\boxminus\mathbb{x}_Q)}. \tag{6.1}$$

Furthermore, there exists a trace operator

$$\mathcal{T}_{\mathbb{x}_Q} \colon W^{1,2}(Q \setminus \boxminus\mathbb{x}_Q) \to L^2(\partial \boxminus \mathbb{x}_Q) := L^2(\partial \boxminus \mathbb{x}_Q, \mathcal{H}^{d-1}),$$

such that $\mathcal{T}_{\mathbb{x}_Q} u = u|_{\partial\boxminus\mathbb{x}_Q}$ for every $u \in C_c^1(Q)$ and

$$\|\mathcal{T}_{\mathbb{x}_Q} u\|_{L^2(\partial\boxminus\mathbb{x}_Q)} \le C\left(\|u\|_{L^2(Q\setminus\boxminus(\mathbb{x}_Q))} + \|\nabla u\|_{L^2(Q\setminus\boxminus\mathbb{x}_Q)}\right). \tag{6.2}$$

Proof. For every $\mathbb{x}_Q \subset \mathbb{x}$ with $\mathbb{x} \in \mathtt{F}_n\mathcal{S}(\mathbb{R}^d)$, the set $Q \setminus \boxminus\mathbb{x}_Q$ is minimally smooth with $\delta = \min\{1/n, r\}$ and, $M = \max\left\{\sqrt{2nr}, M_0\right\}$. Furthermore, the connected components of $\boxminus\mathbb{x}_Q$ have a diameter less than $2nr$. The existence of $\mathcal{U}_{\mathbb{x}_Q}$ satisfying Eq (6.1) follows from [5, Lemma 2.4] (actually, this was pointed out before by [7, Section 3], but the implications there are not obvious). The existence of $\mathcal{T}_{\mathbb{x}_Q}$ satisfying Eq (6.2) is provided in [9].

6.2. Stochastic two-scale convergence

Definition 44 (Stationary and ergodic random measures)**.** A random measure $\mu_\bullet \colon \Omega \to \mathcal{M}(\mathbb{R}^d)$ with underlying dynamical system $(\Omega, \mathcal{F}, \mathcal{P}, \tau)$ is called *stationary* if

$$\mu_{\tau_x\omega}(A) = \mu_\omega(A + x),$$

for every measurable $A \subset \mathbb{R}^d$, $x \in \mathbb{R}^d$, and $\mathcal{P}$-almost every $\omega \in \Omega$. $\mu_\bullet$ is called *ergodic* if it is stationary and τ is ergodic.

Definition 44 is compatible with Definition 8 given in Section 3 by considering the canonical underlying probability space $(\mathcal{M}(\mathbb{R}^d), \mathcal{B}(\mathcal{M}(\mathbb{R}^d)), \mathbb{P}_\mu, \tau)$ with $\mathbb{P}_\mu$ being the distribution of μ.

Theorem 45 (Palm theorem (for finite intensity) [17])**.** *Let $\mu_\bullet$ be a stationary random measure with underlying dynamical system $(\Omega, \mathcal{F}, \mathcal{P}, \tau)$ of finite intensity $\lambda(\mu_\bullet)$.*

Then, there exists a unique finite measure $\mu_\mathcal{P}$ on $(\Omega, \mathcal{F})$ such that for every $g \colon \mathbb{R}^d \times \Omega \to \mathbb{R}$ measurable and either $g \ge 0$ or $g \in L^1\left(\mathbb{R}^d \times \Omega,\ \mathcal{L}^d \otimes \mu_\mathcal{P}\right)$:

$$\int_\Omega \int_{\mathbb{R}^d} g(x, \tau_x\omega)\, \mathrm{d}\mu_\omega(x)\, \mathrm{d}\mathcal{P}(\omega) = \int_{\mathbb{R}^d} \int_\Omega g(x, \omega)\, \mathrm{d}\mu_\mathcal{P}(\omega)\, \mathrm{d}x.$$

For arbitrary $f \in L^1(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} f \,\mathrm{d}x = 1$, we have that

$$\mu_{\mathcal{P}}(A) = \int_\Omega \int_{\mathbb{R}^d} f(x) \mathbb{1}_A(\tau_x \omega) \,\mathrm{d}\mu_\omega(x) \,\mathrm{d}\mathcal{P}(\omega),$$

in particular, $\mu_{\mathcal{P}}(\Omega) = \lambda(\mu)$. Furthermore, for every $\phi \in C_c(\mathbb{R}^d)$ and $g \in L^1(\Omega; \mu_{\mathcal{P}})$, the ergodic limit

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \phi(x)\, g\left(\tau_{\frac{x}{\varepsilon}} \omega\right) \mathrm{d}\mu_\omega(x) = \int_{\mathbb{R}^d} \int_\Omega \phi(x)\, g(\tilde{\omega}) \,\mathrm{d}\mu_{\mathcal{P}}(\tilde{\omega}) \,\mathrm{d}x, \tag{6.3}$$

holds for $\mathcal{P}$-almost every ω. We call $\mu_{\mathcal{P}}$ the Palm measure of $\mu_\bullet$.

For the rest of this subsection, we use the following assumptions.

Assumption 46. *Ω is a compact metric space with a probability measure $\mathcal{P}$ and continuous dynamical system $(\tau_x)_{x \in \mathbb{R}^d}$. Furthermore, $\mu_\bullet : \Omega \to \mathcal{M}(\mathbb{R}^d)$ is a stationary ergodic random measure with Palm measure $\mu_{\mathcal{P}}$. We define $\mu_\omega^\varepsilon(A) := \varepsilon^d \mu_\omega(\varepsilon^{-1} A)$.*

According to [30] (by an application of Eq (6.3)), almost every $\omega \in \Omega$ is *typical*, i.e., for such an ω, it holds for every $\phi \in C(\Omega)$ that

$$\lim_{\varepsilon \to 0} |Q|^{-1} \int_Q \phi\left(\tau_{\frac{x}{\varepsilon}} \omega\right) \mathrm{d}x = \int_\Omega \phi \,\mathrm{d}\mathcal{P}.$$

Definition 47 (Two-scale convergence)**.** Let Assumption 46 hold and let $\omega \in \Omega$ be typical. Let $(u^\varepsilon)_{\varepsilon > 0}$ be a sequence $u^\varepsilon \in L^2(Q, \mu_\omega^\varepsilon)$ and let $u \in L^2(Q; L^2(\Omega, \mu_{\mathcal{P}}))$ such that

$$\sup_{\varepsilon > 0} \|u^\varepsilon\|_{L^2(Q, \mu_\omega^\varepsilon)} < \infty,$$

and such that for every $\varphi \in C_c^\infty(Q)$, $\psi \in C(\Omega)$

$$\lim_{\varepsilon \to 0} \int_Q u^\varepsilon(x) \varphi(x) \psi\left(\tau_{\frac{x}{\varepsilon}} \omega\right) \mathrm{d}\mu_\omega^\varepsilon(x) = \int_Q \int_\Omega u(x, \tilde{\omega}) \varphi(x) \psi(\tilde{\omega}) \,\mathrm{d}\mu_{\mathcal{P}}(\tilde{\omega}) \,\mathrm{d}x. \tag{6.4}$$

Then, u^ε is said to be (weakly) *two-scale convergent* to u, written $u^\varepsilon \xrightharpoonup{2s} u$.

Remark 48 (*Extending the space of test functions*).

- For $K \in \mathbb{N}$, let $\chi_1, \ldots, \chi_K \in L^\infty(\Omega, \mu_{\mathcal{P}})$. The original proof of the following Lemma 49 in [30] shows that we can equally define two-scale convergence additionally claiming Eq (6.4) has to hold for every $\psi \in C(\Omega)$ and $\psi = \chi_k \tilde{\psi}$, where $\tilde{\psi} \in C(\Omega)$. In particular, given a fixed $\chi \in L^\infty(\Omega, \mu_{\mathcal{P}})$, we can w.l.o.g. say that two-scale convergence of $u^\varepsilon \xrightharpoonup{2s} u$ implies $u^\varepsilon \chi(\tau_{\frac{\cdot}{\varepsilon}} \omega) \xrightharpoonup{2s} u\chi$.
- Using a standard approximation argument, we can extend the class of test functions from $\varphi \in C_c^\infty(Q)$ to $\varphi \in L^2(Q)$, provided μ_ω is uniformly continuous w.r.t. the Lebesgue measure. Then, strong $L^2(Q)$-convergence implies two-scale convergence for $\mu_\omega \equiv \mathcal{L}^d$.

Lemma 49 ([30, Lemma 5.1])**.** *Let Assumption 46 hold. Let $\omega \in \Omega$ be typical and $u^\varepsilon \in L^2(Q, \mu_\omega^\varepsilon)$ be a sequence of functions such that $\|u^\varepsilon\|_{L^2(Q, \mu_\omega^\varepsilon)} \le C$ for some $C > 0$ independent of ε. Then, there exists a subsequence of $(u^{\varepsilon'})_{\varepsilon' \to 0}$ and $u \in L^2(Q; L^2(\Omega, \mu_{\mathcal{P}}))$ such that $u^{\varepsilon'} \xrightharpoonup{2s} u$ and*

$$\|u\|_{L^2(Q; L^2(\Omega, \mu_{\mathcal{P}}))} \le \liminf_{\varepsilon' \to 0} \|u^{\varepsilon'}\|_{L^2(Q, \mu_\omega^\varepsilon)}. \tag{6.5}$$

Theorem 50 (Two-scale convergence in $W^{1,2}(Q)$ [30])**.** *Under Assumption 46, for every typical $\omega \in \Omega$ the following holds: If $u^\varepsilon \in W^{1,2}(Q;\mathbb{R}^d)$ for all ε and if*

$$\sup_{\varepsilon>0} \left(\|u^\varepsilon\|_{L^2(Q)} + \|\nabla u^\varepsilon\|_{L^2(Q)} \right) < \infty,$$

then, there exists a $u \in W^{1,2}(Q)$ with $u^\varepsilon \rightharpoonup u$ weakly in $W^{1,2}(Q)$ and there exists $v \in L^2(Q;\mathcal{V}^2_{\mathrm{pot}}(\Omega))$ such that $\nabla u^\varepsilon \overset{2s}{\rightharpoonup} \nabla u + v$ weakly in two scales.

6.3. Two-scale convergence on perforated domains

Due to Theorem 26, the set $\mathtt{F}_n\mathcal{S}(\mathbb{R}^d) \subset \mathcal{M}(\mathbb{R}^d)$ is compact, hence the above two-scale convergence method can be applied for the stationary ergodic point process $\mathbb{X}^{(n)}$ taking values in $\mathtt{F}_n\mathcal{S}(\mathbb{R}^d)$ only. To be more precise, we consider the compact metric probability space $\Omega = \mathtt{F}_n\mathcal{S}(\mathbb{R}^d)$ and a random variable $\mathbb{X}^n \colon \Omega \to \mathcal{S}(\mathbb{R}^d)$ such that $\mathbb{X}^n$ and $\mathbb{X}^{(n)}$ have the same distribution. By the considerations made in Subsection 5.4, they will both result in the same partial differential equation.

Theorem 51 (Extension and trace estimates on $Q^\varepsilon_{\mathbb{x}}$ for $\mathbb{x} \in \mathtt{F}_n\mathcal{S}(\mathbb{R}^d)$)**.** *Let $Q \subset \mathbb{R}^d$ be a bounded domain, $n \in \mathbb{N}$ be fixed. Let $\mathbb{X}$ be an admissible point process with values in $\mathtt{F}_n\mathcal{S}(\mathbb{R}^d)$. For almost every realization $\mathbb{x}$ of $\mathbb{X}$, we have: Let $Q^\varepsilon_{\mathbb{x}}$ and $G^\varepsilon_{\mathbb{x}}$ be defined according to Definition 4.*

i) There exists a $C > 0$ depending only on Q and n and a family of extension and trace operators

$$\mathcal{U}_{\varepsilon,\mathbb{x}} \colon W^{1,2}(Q^\varepsilon_{\mathbb{x}}) \to W^{1,2}(Q), \qquad \mathcal{T}_{\varepsilon,\mathbb{x}} \colon W^{1,2}(Q^\varepsilon_{\mathbb{x}}) \to L^2(\partial G^\varepsilon_{\mathbb{x}}),$$

such that for every $u \in W^{1,2}(Q^\varepsilon_{\mathbb{x}})$ it holds

$$\begin{aligned} \left\|\mathcal{U}_{\varepsilon,\mathbb{x}} u\right\|_{W^{1,2}(Q)} &\le C \|u\|_{W^{1,2}(Q^\varepsilon_{\mathbb{x}})}, \\ \varepsilon \left\|\mathcal{T}_{\varepsilon,\mathbb{x}} u\right\|^2_{L^2(\partial G^\varepsilon_{\mathbb{x}})} &\le C \left(\|u\|^2_{L^2(Q^\varepsilon_{\mathbb{x}})} + \varepsilon^2 \|\nabla u\|^2_{L^2(Q^\varepsilon_{\mathbb{x}})} \right). \end{aligned}$$

ii) If $u^\varepsilon \in W^{1,2}(Q^\varepsilon_{\mathbb{x}})$ is a sequence satisfying $\sup_\varepsilon \|u^\varepsilon\|_{W^{1,2}(Q^\varepsilon_{\mathbb{x}})} < \infty$, then, there exists a $u \in W^{1,2}(Q)$ and a subsequence still indexed by ε such that $\mathcal{U}_{\varepsilon,\mathbb{x}} u^\varepsilon \rightharpoonup u$ weakly in $W^{1,2}(Q)$ and there exists $v \in L^2(Q;\mathcal{V}^2_{\mathrm{pot}}(\Omega))$ such that

$$\nabla \mathcal{U}_{\varepsilon,\mathbb{x}} u^\varepsilon \overset{2s}{\rightharpoonup} \nabla u + v, \qquad \nabla u^\varepsilon \overset{2s}{\rightharpoonup} \mathbb{1}_{\mathbf{G}_n} (\nabla u + v),$$

where $\mathbf{G}_n := \left\{ \mathbb{x} \in \mathtt{F}_n\mathcal{S}(\mathbb{R}^d) \,|\, o \notin \boxminus\mathbb{x} \right\}$. Furthermore, for some $C > 0$ depending only on Q and n

$$\varepsilon \left\|\mathcal{T}_{\varepsilon,\mathbb{x}}(u^\varepsilon - u)\right\|^2_{L^2(\partial G^\varepsilon_{\mathbb{x}})} \le C \left(\left\|\mathcal{U}_{\varepsilon,\mathbb{x}} u^\varepsilon - u\right\|^2_{L^2(Q)} + \varepsilon^2 \left\|\nabla \mathcal{U}_{\varepsilon,\mathbb{x}} u^\varepsilon - \nabla u\right\|^2_{L^2(Q)} \right). \tag{6.6}$$

Proof. i) follows from using Theorem 43 on $\varepsilon^{-1} G^\varepsilon_{\mathbb{x}}$ and rescaling the inequalities (6.1) and (6.2).

ii) is a bit more lengthy. The existence of a subsequence and $u \in W^{1,2}(Q)$ and $v \in L^2(Q;L^2_{\mathrm{pot}}(\Omega))$ such that $\mathcal{U}_{\varepsilon,\mathbb{x}} u^\varepsilon \rightharpoonup u$ and $\nabla \mathcal{U}_{\varepsilon,\mathbb{x}} u^\varepsilon \overset{2s}{\rightharpoonup} \nabla u + v$ follows from Theorem 50. We observe that $\mathbb{1}_{\mathbf{G}_n}(\tau_x \mathbb{x}) = \mathbb{1}_{\boxminus\mathbb{x}^\complement}(x)$ and $\mathbb{1}_{\boxminus\mathbb{x}^\complement}(\frac{x}{\varepsilon}) = \mathbb{1}_{\varepsilon\boxminus\mathbb{x}^\complement}(x)$. Therefore, $\mathbb{1}_{\varepsilon\boxminus\mathbb{x}^\complement} \nabla \mathcal{U}_{\varepsilon,\mathbb{x}} u^\varepsilon \overset{2s}{\rightharpoonup} (\nabla u + v)\mathbb{1}_{\mathbf{G}_n}$

(Remark 48(i)). Furthermore, we observe with $Q^{\varepsilon}_{n,r} := \{x \in Q \colon \operatorname{dist}(x, \partial Q) \le \varepsilon n r\}$ such that we have $(\varepsilon \boxminus \mathbb{X} \cap Q) \backslash G^{\varepsilon}_{\mathbb{X}} \subset Q^{\varepsilon}_{n,r}$ and

$$\left|\mathbb{1}_{\varepsilon\boxminus\mathbb{X}} - \mathbb{1}_{G^{\varepsilon}_{\mathbb{X}}}\right| \le \mathbb{1}_{Q^{\varepsilon}_{n,r}} \xrightarrow{\varepsilon\to 0} 0 \text{ pointwise a.e..}$$

Therefore, $\mathbb{1}_{\varepsilon\boxminus\mathbb{X}} - \mathbb{1}_{G^{\varepsilon}_{\mathbb{X}}} \to 0$ strongly in $L^p(Q)$, $p \in [1, \infty)$, and hence, taking any arbitrary $\phi \in C(\Omega)$, $\psi \in C(\overline{Q})$, we find

$$\int_Q (\mathbb{1}_{\varepsilon\boxminus\mathbb{X}} - \mathbb{1}_{G^{\varepsilon}_{\mathbb{X}}})\nabla \mathcal{U}_{\varepsilon,\mathbb{X}} u^{\varepsilon} \phi(\tau_{\frac{x}{\varepsilon}}\mathbb{X})\psi \,\mathrm{d}x \le \left\|\mathbb{1}_{\varepsilon\boxminus\mathbb{X}} - \mathbb{1}_{G^{\varepsilon}_{\mathbb{X}}}\right\|_{L^2(Q)} \left\|\nabla \mathcal{U}_{\varepsilon,\mathbb{X}} u^{\varepsilon}\right\|_{L^2(Q)} \|\phi\|_\infty \|\psi\|_\infty \to 0,$$

which means that $\mathbb{1}_{\varepsilon\boxminus\mathbb{X}}\nabla \mathcal{U}_{\varepsilon,\mathbb{X}} u^{\varepsilon}$ and $\nabla u^{\varepsilon} = \mathbb{1}_{G^{\varepsilon}_{\mathbb{X}}}\nabla \mathcal{U}_{\varepsilon,\mathbb{X}} u^{\varepsilon}$ have the same two-scale limit

$$\nabla u^{\varepsilon} \xrightarrow[\varepsilon\to 0]{2s} \mathbb{1}_{\mathbf{G}_n}(\nabla u + v).$$

Due to the absolutely bounded diameter of the connected components of $\boxminus\mathbb{X}$, there exists a domain $B \supset Q$ big enough such that, with the notation of Definition 4,

$$Q \cap \varepsilon \boxminus (J_{\varepsilon}(\mathbb{X}, B)) = Q \cap \varepsilon \boxminus \mathbb{X} \quad \forall \varepsilon \in (0, 1).$$

Now let $\mathcal{U}_Q \colon W^{1,2}(Q) \to W^{1,2}(B)$ be the canonical extension operator satisfying

$$\left\|\mathcal{U}_Q u\right\|_{L^2(B)} \le C \|u\|_{L^2(Q)} \qquad \text{and} \qquad \left\|\nabla \mathcal{U}_Q u\right\|_{L^2(B)} \le C \|\nabla u\|_{L^2(Q)}.$$

Reapplying Theorem 43 to the trace on $\varepsilon^{-1}(B \backslash \varepsilon \boxminus (J_{\varepsilon}(\mathbb{X}, B)))$, we find for some constant C independent from ε and $\mathbb{X}$ but depending on Q, B, and n and varying from line to line:

$$\begin{aligned}
\varepsilon \left\|\mathcal{T}_{\varepsilon,\mathbb{X}}(u^{\varepsilon} - u)\right\|^2_{L^2(\partial G^{\varepsilon}_{\mathbb{X}})} &\le \varepsilon \left\|\mathcal{T}_{\varepsilon,\mathbb{X}}(\mathcal{U}_Q \mathcal{U}_{\varepsilon,\mathbb{X}} u^{\varepsilon} - \mathcal{U}_Q u)\right\|^2_{L^2(\varepsilon\partial\boxminus(J_{\varepsilon}(\mathbb{X},B)))} \\
&\le C\left(\left\|\mathcal{U}_Q \mathcal{U}_{\varepsilon,\mathbb{X}} u^{\varepsilon} - \mathcal{U}_Q u\right\|^2_{L^2(B)} + \varepsilon^2 \left\|\nabla \mathcal{U}_Q \mathcal{U}_{\varepsilon,\mathbb{X}} u^{\varepsilon} - \nabla \mathcal{U}_Q u\right\|^2_{L^2(B)}\right) \\
&\le C\left(\left\|\mathcal{U}_{\varepsilon,\mathbb{X}} u^{\varepsilon} - u\right\|^2_{L^2(Q)} + \varepsilon^2 \left\|\nabla \mathcal{U}_{\varepsilon,\mathbb{X}} u^{\varepsilon} - \nabla u\right\|^2_{L^2(Q)}\right),
\end{aligned}$$

as desired.

6.4. Existence of solution on perforated domains (Lemma 7)

Due to the perforations, $\partial_t u^{\varepsilon}$ cannot be embedded in a common space in a convenient way for the application of the Aubin–Lions theorem. Hence, we use the following general characterization of compact sets.

Theorem 52 (Characterization of compact sets in $L^p(I; V)$ [24, Theorem 1]). *Let V be a Banach space, $p \in [1, \infty)$ and $\Lambda \subset L^p(I; V)$. Λ is relatively compact in $L^p(I; V)$ if, and only if,*

$$\left\{\int_{t_1}^{t_2} v(t)\,\mathrm{d}t \,\middle|\, v \in \Lambda\right\} \text{ is relatively compact in } V \;\forall\, 0 < t_1 < t_2 < T, \tag{6.7}$$

$$\sup_{v\in\Phi} \|\mathfrak{s}_h[v] - v\|_{L^p(0,T-h;V)} \to 0 \text{ as } h \to 0, \tag{6.8}$$

where $\mathfrak{s}_h[v(\,\cdot\,)] := v(\,\cdot\, + h)$ is the shift by $h \in (0, T)$.

We can now establish the existence of a solution for fixed $\varepsilon > 0$ to our partial differential equation.

Theorem 53 (Existence of solution on perforated domains and a priori estimate). *Let $\mathbb{x} \in \mathtt{F}_n\mathcal{S}(\mathbb{R}^d)$. Under Assumption 6 and with $Q^\varepsilon_{\mathbb{x}}$ as defined in Definition 4, we have that there exists a solution $u^\varepsilon \in L^2(I; W^{1,2}(Q^\varepsilon_{\mathbb{x}}))$, with generalized time derivative $\partial_t u^\varepsilon \in L^2(I; W^{1,2}(Q^\varepsilon_{\mathbb{x}})^*)$, to Eq (1.1), i.e.,*

$$\begin{aligned} \partial_t u^\varepsilon - \nabla \cdot (A(u^\varepsilon)\, \nabla u^\varepsilon) &= f && \text{in } I \times Q^\varepsilon_{\mathbb{x}} \\ A(u^\varepsilon)\, \nabla u^\varepsilon \cdot \nu &= 0 && \text{on } I \times \partial Q \\ A(u^\varepsilon)\, \nabla u^\varepsilon \cdot \nu &= \varepsilon h(u^\varepsilon) && \text{on } I \times \partial Q^\varepsilon_{\mathbb{x}} \setminus \partial Q \\ u^\varepsilon(0, x) &= u_0(x) && \text{in } Q^\varepsilon_{\mathbb{x}}, \end{aligned} \tag{6.9}$$

which satisfies the following a priori estimates for ε small enough

$$\begin{aligned} \operatorname*{ess\,sup}_{t \in I} \|u^\varepsilon(t)\|^2_{L^2(Q^\varepsilon_{\mathbb{x}})} &\leq \exp(C_1)[\|u_0\|^2_{L^2(Q)} + C_2] \\ \|\nabla u^\varepsilon\|^2_{L^2(I; L^2(Q^\varepsilon_{\mathbb{x}}))} &\leq \frac{1}{\inf(A)}(1 + C_1 \exp(C_1))[\|u_0\|^2_{L^2(Q)} + C_2] \\ \|\partial_t u^\varepsilon\|^2_{L^2(I; W^{1,2}(Q^\varepsilon_{\mathbb{x}})^*)} &\leq \tilde{C}, \end{aligned} \tag{6.10}$$

where

$$C_1 := T(1 + 3CL_h) \qquad \text{and} \qquad C_2 := TL_h h(0)^2 \mathcal{L}^d(Q) + \|f\|^2_{L^1(I; L^2(Q))},$$

C is from Theorem 51 depending only on Q and n and where $\tilde{C} > 0$ is independent of ε.

Proof. We will only sketch the proof. There are 3 main steps: Deriving a priori estimates, existence of Galerkin solutions and the limit passing.

i) Testing Eq (6.9) with u^ε and using

$$\langle \partial_t u^\varepsilon,\, u^\varepsilon \rangle_{W^{1,2}(Q^\varepsilon_{\mathbb{x}})^*, W^{1,2}(Q^\varepsilon_{\mathbb{x}})} = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|u^\varepsilon\|^2_{L^2(Q^\varepsilon_{\mathbb{x}})},$$

yields

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|u^\varepsilon\|^2_{L^2(Q^\varepsilon_{\mathbb{x}})} + A(u^\varepsilon)\, \|\nabla u^\varepsilon\|^2_{L^2(Q^\varepsilon_{\mathbb{x}})} - \varepsilon \big(h(u^\varepsilon),\, u^\varepsilon\big)_{L^2(\partial G^\varepsilon_{\mathbb{x}})} = (f,\, u^\varepsilon)_{L^2(Q^\varepsilon_{\mathbb{x}})}.$$

The a priori estimate then follows from the Gronwall inequality and the trace estimate in Theorem 51. For the a priori estimate in $\partial_t u^\varepsilon$, one simply uses

$$\langle \partial_t u^\varepsilon,\, \varphi \rangle = \big(A(u^\varepsilon)\, \nabla u^\varepsilon,\, \nabla \varphi\big)_{L^2(Q^\varepsilon_{\mathbb{x}})} + \varepsilon \big(h(u^\varepsilon),\, \varphi\big)_{L^2(\partial G^\varepsilon_{\mathbb{x}})} + (f,\, \varphi)_{L^2(Q^\varepsilon_{\mathbb{x}})}.$$

ii) Let $(V_m)_{m \in \mathbb{N}}$ be a family of finite-dimensional vector spaces, $V_m \nearrow W^{1,2}(Q^\varepsilon_{\mathbb{x}})$. Solutions to Eq (1.1) exist in V_m, that is,

$$\begin{aligned} \partial_t u^\varepsilon_{(m)} - \nabla \cdot \Big(A(u^\varepsilon_{(m)})\, \nabla u^\varepsilon_{(m)}\Big) &= \mathbf{P}_m f && \text{in } I \times Q^\varepsilon \\ A(u^\varepsilon_{(m)})\, \nabla u^\varepsilon_{(m)} \cdot \nu &= 0 && \text{on } I \times \partial Q \\ A(u^\varepsilon_{(m)})\, \nabla u^\varepsilon_{(m)} \cdot \nu &= \varepsilon h(u^\varepsilon_{(m)}) && \text{on } I \times \partial Q^\varepsilon \setminus \partial Q \\ u^\varepsilon_{(m)}(0, x) &= \mathbf{P}_m u_0(x) && \text{in } Q^\varepsilon, \end{aligned}$$

where $\mathbf{P}_m\colon W^{1,2}(Q^\varepsilon_{\mathbb{x}}) \to V_m$ is the orthogonal projection. This can be shown via, e.g., [21, Theorem 3.7] which yields solutions for some $I' := [0,T'] \subset [0,T]$. In this reference, their $y \in W^{1,p}(I')$ corresponds to the parameters of the Galerkin approximation, i.e., $u^\varepsilon_{(m)} = \sum_{i=1}^m y_i v_i$ with $v_i \in V_m$ being fixed orthonormal base vectors. These solutions $u^\varepsilon_{(m)}$ satisfy the a priori estimate in Eq (6.10) as well as

$$\sup_{m\in\mathbb{N}} \|\partial_t u^\varepsilon_{(m)}\|_{L^2(I';V_m^*)} < \infty, \tag{6.11}$$

in particular, we may extend the solution $u^\varepsilon_{(m)}$ to the whole time interval I (with still uniform bounds in m).

iii) The a priori estimates yield a $L^2(I;L^2(Q^\varepsilon_{\mathbb{x}}))$-weakly convergent subsequence to some $u^\varepsilon \in L^2(I;L^2(Q^\varepsilon_{\mathbb{x}}))$. Theorem 52 and Eq (6.11) imply pre-compactness of $(u^\varepsilon_{(m)})_{m\in\mathbb{N}} \subset L^2(I;L^2(Q^\varepsilon_{\mathbb{x}}))$ as well as pre-compactness of $(\mathcal{T}_{\varepsilon,\mathbb{x}} u^\varepsilon_{(m)})_{m\in\mathbb{N}} \subset L^2(I;L^2(\partial G^\varepsilon_{\mathbb{x}}))$, see Remark 54. Testing with functions in $L^2(I;V_m)$ and passing to the limit $m \to \infty$ finishes the proof since $\bigcup_{m\in\mathbb{N}} V_m$ is dense in $W^{1,2}(Q^\varepsilon_{\mathbb{x}})$.

Remark 54 (*Procedure of Simon's theorem*). We will use Simon's theorem (Theorem 52) on multiple occasions. The general procedure will always be the same. We will exemplaily prove the following result: Let $I = [0,T]$, $U \subset \mathbb{R}^d$ be some bounded Lipschitz-domain and $\mathcal{T}\colon W^{1,2}(U) \to L^2(\partial U)$ the trace operator. For each $k \in \mathbb{N}$, let $u_k \in L^2(I;W^{1,2}(U))$ with generalized time-derivative $\partial_t u_k \in L^2(I;W^{1,2}(U)^*)$ via $W^{1,2}(U) \hookrightarrow L^2(U) \hookrightarrow W^{1,2}(U)^*$. Assume that

$$C := \sup_{k\in\mathbb{N}} \|u_k\|_{L^2(I;W^{1,2}(U))} < \infty \qquad \text{and} \qquad \tilde{C} := \sup_{k\in\mathbb{N}} \|\partial_t u_k\|_{L^2(I;V_k^*)} < \infty,$$

for either the situation that $W^{1,2}(U) \subset V_k \subset L^2(U)$ with $\|\cdot\|_{V_k} \leq \|\cdot\|_{W^{1,2}(U)}$ and uniformly continuous injective maps $\mathcal{U}_k\colon V_k \to W^{1,2}(U)$ or for the situation that $V_k \subset W^{1,2}(U)$. We further claim $u_k(t) \in V_k$ for almost every $t \in I$. Then,

$$(\mathcal{U}_k u_k)_{k\in\mathbb{N}} \subset L^2(U) \text{ resp. } (u_k)_{k\in\mathbb{N}} \subset L^2(U) \quad \text{and} \quad (\mathcal{T} u_k)_{k\in\mathbb{N}} \subset L^2(\partial U),$$

are relatively compact.

Exemplarily proof for the procedure of Simon's theorem. We need to show Conditions (6.7) and (6.8) from Theorem 52.

i) Condition (6.7) usually relies on compactness results for the stationary setting. Since

$$\sup_{k\in\mathbb{N}} \left\| \int_{t_1}^{t_2} u_k \,\mathrm{d}t \right\|_{W^{1,2}(Q)} \leq \sup_{k\in\mathbb{N}} \sqrt{T} \|u_k\|_{L^2(I;W^{1,2}(U))} < \infty,$$

compactness of $\mathcal{T}$ yields pre-compactness of $\left(\int_{t_1}^{t_2} \mathcal{T} u_k \,\mathrm{d}t\right)_{k\in\mathbb{N}} = \left(\mathcal{T} \int_{t_1}^{t_2} u_k \,\mathrm{d}t\right)_{k\in\mathbb{N}} \subset L^2(\partial U)$, so we have shown Condition (6.7).

ii) Condition (6.8) will additionally require some a priori estimate on $\partial_t u_k$. We have

$$u_k(t_2) = u_k(t_1) + \int_{t_1}^{t_2} \partial_t u_k \,\mathrm{d}s,$$

as elements of $W^{1,2}(U)^*$. Using the Cauchy–Schwarz inequality twice, we get for $h \in (0,T)$:

$$\|\mathfrak{s}_h[u_k] - u_k\|^2_{L^2((0,T-h);L^2(U))} = \int_0^{T-h} \big(u_k(t+h) - u_k(t),\, u_k(t+h) - u_k(t)\big)_{L^2(U)} \,\mathrm{d}t$$

$$
\begin{aligned}
&= \int_0^{T-h} \langle \int_t^{t+h} \partial_t u_k(s)\, \mathrm{d}s,\ u_k(t+h) - u_k(t) \rangle_{W^{1,2}(U)^*, W^{1,2}(U)}\, \mathrm{d}t \\
&\le \int_0^{T-h} \| \int_t^{t+h} \partial_t u_k(s)\, \mathrm{d}s \|_{L^2(V_k^*)}\, \| \mathcal{U}_k u_k(t+h) - \mathcal{U}_k u_k(t) \|_{W^{1,2}(U)}\, \mathrm{d}t \\
&\le h \|\partial_t u_k\|_{L^2(I;V_k^*)}\, 2\|u_k\|_{L^2(I;W^{1,2}(U))} \le 2hC\tilde{C}.
\end{aligned}
$$

Compactness of $\mathcal{T}$ implies that for every $\delta > 0$, there exists a $C_\delta > 0$ such that

$$
\|\mathcal{T} v\|^2_{L^2(\partial U)} \le C_\delta \|v\|^2_{L^2(U)} + \delta \|\nabla v\|^2_{L^2(U)}\ \forall v \in W^{1,2}(U).
$$

Therefore,

$$
\begin{aligned}
\|\mathfrak{s}_h[\mathcal{T} u_k] - \mathcal{T} u_k\|^2_{L^2((0,T-h);L^2(\partial U))} &= \|\mathcal{T}[\mathfrak{s}_h u_k - u_k]\|^2_{L^2((0,T-h);L^2(\partial U))} \\
&\le C_\delta \|\mathfrak{s}_h u_k - u_k\|^2_{L^2(\partial U)} + \delta \|\nabla \mathfrak{s}_h u_k - \nabla u_k\|^2_{L^2(U)} \\
&\le 2hC_\delta C\tilde{C} + 2\delta\tilde{C}.
\end{aligned}
$$

The estimate is independent of the chosen u_k, and Condition (6.8) holds.

We have shown both conditions and concluded.

6.5. *Homogenization for minimally smooth domains (Lemma 10)*

We can now pass to the limit $\varepsilon \to 0$ for the homogenized system. Some extra care has to be taken since $Q^\varepsilon_{\mathbb{x}} \neq Q \backslash \varepsilon \boxminus \mathbb{x}$, especially in the boundary term. However, we show that the difference becomes negligible for the two-scale convergence as $\varepsilon \to 0$.

Theorem 55 (Homogenized system for $\boxminus\mathbb{X}^{(n)}$)**.** *Let $\mathbb{X}$ be a stationary ergodic point process with values in $\mathtt{F}_n\mathcal{S}(\mathbb{R}^d)$. Recall the surface measure $\mu_{\mathbb{x}}$ from Definition 30*

$$
\mu_{\mathbb{x}} := \mathcal{H}^{d-1}_{\llcorner \partial \boxminus \mathbb{x}}.
$$

Under Assumption 6, we have for almost every realization $\mathbb{x}$ of $\mathbb{X}$ and with $Q^\varepsilon_{\mathbb{x}}$ as defined in Definition 4: Let u^ε be a solution to Eq (6.9) and let $\mathcal{U}_{\varepsilon,\mathbb{x}}$ be given as in Theorem 51. There exists a $u_n \in L^2(I; W^{1,2}(Q))$ with generalized time derivative $\partial_t u_n \in L^2(I; W^{1,2}(Q)^*)$ such that for a subsequence

$$
\mathcal{U}_{\varepsilon,\mathbb{x}} u^\varepsilon \xrightarrow[\varepsilon\to 0]{L^2(I;L^2(Q))} u_n \qquad \text{and} \qquad \partial_t u^\varepsilon \xrightarrow[\varepsilon\to 0]{L^2(I;W^{1,2}(Q)^*)} \mathbb{P}(\mathbf{G}_n)\partial_t u_n,
$$

and u_n is a (not necessarily unique) solution to

$$
\begin{aligned}
\mathbb{P}(\mathbf{G}_n)\partial_t u_n - \nabla \cdot (A(u_n)\mathcal{A}\nabla u_n) - \lambda(\mu_{\mathbb{X}})h(u_n) &= \mathbb{P}(\mathbf{G}_n) f && \text{in } I \times Q \\
A(u_n)\mathcal{A}^{(n)}\nabla u_n \cdot \nu &= 0 && \text{on } I \times \partial Q \\
u_n(0,x) &= \mathbb{P}(\mathbf{G}_n)u_0(x) && \text{in } Q,
\end{aligned}
\tag{6.12}
$$

with $\mathcal{A}^{(n)}$ being the effective conductivity based on the event $\mathbf{G}_n = \{\mathbb{x} \in \mathtt{F}_n\mathcal{S}(\mathbb{R}^d) \,|\, o \notin \boxminus\mathbb{x}\}$ defined in Definition 37. Furthermore, u_n satisfies the following a priori estimates

$$
\operatorname*{ess\,sup}_{t\in I} \|u_n(t)\|^2_{L^2(Q)} \le \exp(C_1^{(n)})[\|u_0\|^2_{L^2(Q)} + C_2^{(n)}]
$$

$$\|\nabla u_n\|^2_{L^2(I;L^2(Q))} \leq \frac{\mathbb{P}(\mathbf{G}_n)}{2\alpha_{\mathcal{A}^{(n)}} \inf(A)}(1 + C_1^{(n)} \exp(C_1^{(n)}))[\|u_0\|^2_{L^2(Q)} + C_2^{(n)}],$$

for

$$C_1^{(n)} := T(1 + \frac{\lambda(\mu_{\mathbb{X}})}{\mathbb{P}(\mathbf{G}_n)}(1 + 2L_h)) \qquad \text{and} \qquad C_2^{(n)} := \|f\|^2_{L^2(I;L^2(Q))} + 2T\frac{\lambda(\mu_{\mathbb{X}})}{\mathbb{P}(\mathbf{G}_n)}\left|h(0)\right|^2.$$

Proof. The a priori estimates in Eq (6.10) and Theorem 51 tell us that

$$\mathcal{U}_{\varepsilon,\mathbb{X}} u^\varepsilon \overset{2s}{\rightharpoonup} u_n, \qquad \nabla\mathcal{U}_{\varepsilon,\mathbb{X}} u^\varepsilon \overset{2s}{\rightharpoonup} \nabla u_n + v, \qquad u^\varepsilon \overset{2s}{\rightharpoonup} \mathbb{1}_{\mathbf{G}_n} u_n, \qquad \nabla u^\varepsilon \overset{2s}{\rightharpoonup} \mathbb{1}_{\mathbf{G}_n}(\nabla u_n + v),$$

for some $u_n \in L^2(I; W^{1,2}(Q))$ and $v \in L^2(I; L^2(Q; \mathcal{V}^2_{\mathrm{pot}}(\Omega)))$ where the two-scale convergence is with respect to the Lebesgue measure $\mathcal{L}^d$. The uniform bound for $\partial_t u^\varepsilon$ in Eq (6.10) together with Theorem 52 yields (for yet another subsequence)

$$\mathcal{U}_{\varepsilon,\mathbb{X}} u^\varepsilon \xrightarrow[\varepsilon\to 0]{L^2(I;L^2(Q))} u_n, \tag{6.13}$$

compared to, e.g., Remark 54.

For $\varphi_1, \varphi_2 \in C^1([0,T] \times \overline{Q})$ with $\varphi_1(T,\cdot) = 0$ and $\psi \in H^1(\Omega)$ with $\int_\Omega \nabla_{\tilde{\omega}}\psi \, \mathrm{d}\mathbb{P} = 0$, we use $\varphi^\varepsilon(t,x) := \varphi_1(t,x) + \varepsilon\varphi_2(t,x)\psi(\tau_{\frac{x}{\varepsilon}}\mathbb{X})$ as a test function and pass to the limit using two-scale convergence. Furthermore, we use

$$A(u^\varepsilon)\nabla u^\varepsilon \overset{2s}{\rightharpoonup} A(u_n)\mathbb{1}_{\mathbf{G}_n}(\nabla u_n + v) \qquad \text{and} \qquad h(u^\varepsilon) \xrightarrow{2s,\mu_{\mathbb{X}}} h(u_n), \tag{6.14}$$

which we will prove below. We then obtain the two equations

$$-\int_0^T \int_Q u_n \partial_t \varphi_1 \, \mathrm{d}x \, \mathrm{d}t + \int_Q u_0 \varphi_1 \, \mathrm{d}x + \int_0^T \int_Q \int_{\mathbf{G}_n} \nabla\varphi_1 \cdot A(u_n)(\nabla u_n + v) \, \mathrm{d}\mathbb{P}(\mathbb{x}) \, \mathrm{d}x \, \mathrm{d}t$$
$$+ \int_0^T \int_Q h(u_n)\varphi_1 \, \mathrm{d}x \, \mathrm{d}t \int_\Omega \mathrm{d}\mu_{\mathcal{P}}(x) = \int_0^T \int_Q f\varphi_1 \, \mathrm{d}x \, \mathrm{d}t, \tag{6.15}$$

$$\int_0^T \int_Q \int_{\mathbf{G}_n} \varphi_2 \nabla_{\tilde{\omega}}\psi \cdot A(u_n)(\nabla u_n + v) \, \mathrm{d}\mathbb{P}(\mathbb{x}) \, \mathrm{d}x \, \mathrm{d}t = 0. \tag{6.16}$$

The second equation holds true for every choice of φ_2 and ψ as above if we make the standard ansatz $v = \sum_{i=1}^d \partial_i u_n w_i^{(n)}$ (see the remark following this proof), where $w_i^{(n)}$ are the cell solutions from Definition 35 for $\Omega = \mathtt{F}_n\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{P} = \mathbb{P}$ being the distribution of $\mathbb{X}$. Plugging this information into the first equation yields (6.12). The a priori estimate follows from testing Eq (6.12) with u_n and the Gronwall inequality (see, e.g., the proof of Theorem 53). It only remains to prove Eq (6.14).

Now, we show the first part of Eq (6.14). By Remark 48, we know that

$$A(u_n)\nabla u^\varepsilon \xrightarrow[\varepsilon\to 0]{2s} \mathbb{1}_{\mathbf{G}_n} A(u_n)(\nabla u_n + v).$$

Using dominated convergence and Eq (6.13) yields a subsequence such that $A(\mathcal{U}_{\varepsilon,\mathbb{X}} u^\varepsilon) \to A(u_n)$ in $L^p(0,T; L^p(Q))$ for every $1 \leq p < \infty$. Using test functions $\phi \in C(\overline{Q})$ and $\psi \in C(\Omega)$, we observe that $(A(\mathcal{U}_{\varepsilon,\mathbb{X}} u^\varepsilon) - A(u_n))\nabla u^\varepsilon \overset{2s}{\rightharpoonup} 0$, so $A(u^\varepsilon)\nabla u^\varepsilon \overset{2s}{\rightharpoonup} \mathbb{1}_{\mathbf{G}_n} A(u_n)(\nabla u_n + v)$.

The second part of Eq (6.14) is more difficult. Given $\varphi \in C^1(\overline{Q})$ and $\psi \in C(\Omega)$, we set $\psi^{\varepsilon,\mathbb{x}}(x) := \psi(\tau_{\frac{x}{\varepsilon}}\mathbb{x})$ and find

$$\begin{aligned}
&\left|\varepsilon \int_{\partial G^\varepsilon_{\mathbb{x}}} h(u^\varepsilon(x))\varphi(x)\psi^{\varepsilon,\mathbb{x}}(x)\mathrm{d}\mathcal{H}^{d-1}(x) - \int_Q \int_\Omega h(u_n(x))\,\varphi(x)\,\psi(\mathbb{x})\,\mathrm{d}\mu_{\mathcal{P}}(\mathbb{x})\,\mathrm{d}x\right| \\
\leq\ & \left|\varepsilon \int_{\partial G^\varepsilon_{\mathbb{x}}} h(u^\varepsilon(x))\varphi(x)\psi^{\varepsilon,\mathbb{x}}(x)\mathrm{d}\mathcal{H}^{d-1}(x) - \varepsilon \int_{\partial G^\varepsilon_{\mathbb{x}}} h(u_n(x))\varphi(x)\psi^{\varepsilon,\mathbb{x}}(x)\mathrm{d}\mathcal{H}^{d-1}(x)\right| \\
& + \left|\varepsilon \int_{\partial G^\varepsilon_{\mathbb{x}}} h(u_n(x))\varphi(x)\psi^{\varepsilon,\mathbb{x}}(x)\mathrm{d}\mathcal{H}^{d-1}(x) - \varepsilon \int_{Q\cap\varepsilon\partial\boxminus\mathbb{x}} h(u_n(x))\varphi(x)\psi^{\varepsilon,\mathbb{x}}(x)\mathrm{d}\mathcal{H}^{d-1}(x)\right| \\
& + \left|\varepsilon \int_{Q\cap\varepsilon\partial\boxminus\mathbb{x}} h(u_n(x))\varphi(x)\psi^{\varepsilon,\mathbb{x}}(x)\mathrm{d}\mathcal{H}^{d-1}(x) - \int_Q \int_\Omega h(u_n(x))\,\varphi(x)\,\psi(\mathbb{x})\,\mathrm{d}\mu_{\mathcal{P}}(\mathbb{x})\,\mathrm{d}x\right|.
\end{aligned}$$

We will show that all these terms go to 0 as $\varepsilon \to 0$. Due to the Lipschitz continuity of h and Stampacchia's lemma, we find

$$\begin{aligned}
\|\nabla h(u^\varepsilon)\|_{L^2(Q^\varepsilon_{\mathbb{x}})} &\leq \|h\|_{C^{0,1}}\,\|\nabla u^\varepsilon\|_{L^2(Q^\varepsilon_{\mathbb{x}})}, & \|h(u^\varepsilon)\|_{L^2(Q^\varepsilon_{\mathbb{x}})} &\leq \|h\|_{C^{0,1}}\left(\|u^\varepsilon\|_{L^2(Q^\varepsilon_{\mathbb{x}})} + 1\right), \\
\|\nabla h(u_n)\|_{L^2(Q)} &\leq \|h\|_{C^{0,1}}\,\|\nabla u_n\|_{L^2(Q)}, & \|h(u_n)\|_{L^2(Q)} &\leq \|h\|_{C^{0,1}}\left(\|u_n\|_{L^2(Q)} + 1\right).
\end{aligned}$$

Furthermore, $\mathcal{U}_{\varepsilon,\mathbb{x}}u^\varepsilon \to u_n$ strongly in $L^2(I;L^2(Q))$ and weakly in $L^2(I;W^{1,2}(Q))$ implies that $h\left(\mathcal{U}_{\varepsilon,\mathbb{x}}u^\varepsilon\right) \to h(u_n)$ in the same topologies. $G^\varepsilon_{\mathbb{x}}$ as in Definition 4 fulfills $\partial G^\varepsilon_{\mathbb{x}} = \partial Q^\varepsilon_{\mathbb{x}} \backslash \partial Q$. Eq (6.6) together with the strong convergence of $\mathcal{U}_{\varepsilon,\mathbb{x}}u^\varepsilon$ tells us

$$\varepsilon\left\|h(\mathcal{T}_{\varepsilon,\mathbb{x}}u^\varepsilon) - h(\mathcal{T}_{\varepsilon,\mathbb{x}}u_n)\right\|^2_{L^2(I;L^2(\partial G^\varepsilon_{\mathbb{x}}))} \leq L_h\varepsilon\left\|\mathcal{T}_{\varepsilon,\mathbb{x}}(u^\varepsilon - u_n)\right\|^2_{L^2(I;L^2(\partial G^\varepsilon_{\mathbb{x}}))} \to 0,$$

which already shows convergence in the first summand. Similar considerations to the proof of Eq (6.6) tell us that $\mathcal{T}_{\varepsilon,\mathbb{x}}\colon W^{1,2}(Q^\varepsilon_{\mathbb{x}}) \to L^2(\varepsilon\partial\boxminus\mathbb{x} \cap Q)$ is a bounded linear operator, so we can consider the trace not only on $\partial G^\varepsilon_{\mathbb{x}}$ but even for clusters close to the boundary. We have, with $C > 0$ changing from line to line but independent of ε,

$$\begin{aligned}
&\left|\varepsilon \int_I \int_Q (\mathbb{1}_{\varepsilon\partial\boxminus\mathbb{x}} - \mathbb{1}_{\partial G^\varepsilon_{\mathbb{x}}})h(u^\varepsilon)\varphi\psi^{\varepsilon,\mathbb{x}}\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t\right|^2 \\
&\qquad \leq C\varepsilon\|h(u^\varepsilon)\|^2_{L^2(I;L^2((\varepsilon\partial\boxminus\mathbb{x})\backslash\partial G^\varepsilon_{\mathbb{x}}))} \cdot \varepsilon\|1\|^2_{L^2(I;L^2((\varepsilon\partial\boxminus\mathbb{x})\backslash\partial G^\varepsilon_{\mathbb{x}}))} \\
&\qquad \leq C\{\|\mathbb{1}_{Q^\varepsilon_{n,r}}h(u^\varepsilon)\|^2_{L^2(I;L^2(Q))} + \varepsilon\|\mathbb{1}_{Q^\varepsilon_{n,r}}\nabla h(u^\varepsilon)\|^2_{L^2(I;L^2(Q))}\} \cdot \mathcal{L}^d(Q^\varepsilon_{n,r}),
\end{aligned}$$

where

$$Q^\varepsilon_{n,r} := \{x \in Q \,|\, \mathrm{dist}(x,\partial Q) \leq \varepsilon n r\}.$$

We observe that $\mathcal{L}^d(Q^\varepsilon_{n,r}) \to 0$ as $\varepsilon \to 0$, i.e., $\mathbb{1}_{Q^\varepsilon_{n,r}} \to 0$ point-wise $\mathcal{L}^d$-almost everywhere. We know that $h(u^\varepsilon) \to h(u)$ strongly in $L^2(I;L^2(Q))$, so dominated convergence yields that the second summand also converges to 0.

The third summand follows from two-scale convergence, i.e., $h(u_n) \xrightarrow{2s,\mu_{\mathbb{x}}} h(u_n)$ for almost every $\mathbb{x}$.

Remark (*On the special solution of Eq (6.16)*). Our special ansatz $v = \sum_{i=1}^{d} \partial_i u_n w_i^{(n)}$ as a solution to Eq (6.16) for a fixed, given u has a long tradition, where we mention [5] for perforated domains or the more "historical" papers [22, 30]. For readers familiar with periodic homogenization, let us mention that due to v being $\mathcal{V}^2_{\mathrm{pot}}(\Omega)$ problem (6.16) is the probabilistic equivalent of

$$\forall \varphi \in C_c(Q), \psi \in H^1_{per}([0,1)^d): \qquad \int_{[0,1)^d} \int_Q \varphi(x) \nabla_y \psi(y) A(\nabla u + \nabla_y u_1) \,\mathrm{d}x\,\mathrm{d}y = 0.$$

Now given a $u_n \in L^2(I; W^{1,2}(Q))$, we are looking for a solution $v \in L^2(I; L^2(Q; \mathcal{V}^2_{\mathrm{pot}}(\Omega)))$ to

$$\int_0^T \int_Q \int_{\mathbf{G}_n} \varphi_2 \nabla_{\tilde{\omega}} \psi \cdot A(u_n)\,(\nabla u_n + v)\; \mathrm{d}\mathbb{P}(\mathbb{x})\,\mathrm{d}x\,\mathrm{d}t = 0,$$

for arbitrary $\varphi_2 \in C^1([0,T] \times \overline{Q})$ and $\psi \in H^1(\Omega)$ with $\int_\Omega \nabla_{\tilde{\omega}} \psi \,\mathrm{d}\mathbb{P} = 0$. Since functions of the form $\nabla_{\tilde{\omega}} \psi|_{\mathbf{G}_n}$are dense in $\mathcal{V}^2_{\mathrm{pot}}(\mathbf{G}_n|\Omega)$ for $\psi \in H^1(\Omega)$ with $\int_\Omega \nabla_{\tilde{\omega}} \psi = 0$ (by definition of $\mathcal{V}^2_{\mathrm{pot}}(\Omega)$), this equation actually has to hold for arbitrary $\phi \in \mathcal{V}^2_{\mathrm{pot}}(\mathbf{G}_n|\Omega)$. By the fundamental lemma of calculus of variations, this solution must satisfy for almost every $t \in I$ and almost every $x \in Q$

$$\phi \in \mathcal{V}^2_{\mathrm{pot}}(\mathbf{G}_n|\Omega): \qquad \int_{\mathbf{G}_n} \phi \cdot A(u_n)\,(\nabla u_n + v)\; \mathrm{d}\mathbb{P} = 0.$$

Next, we see that $v := \sum_{i=1}^{d} \partial_i u_n w_i^{(n)} \in L^2(I; L^2(Q; \mathcal{V}^2_{\mathrm{pot}}(\mathbf{G}_n|\Omega)))$ is indeed a solution since the $w_i^{(n)}$ are cell solutions

$$\int_{\mathbf{G}_n} \phi \cdot A(u_n)\,(\nabla u_n + v)\; \mathrm{d}\mathbb{P} = \sum_{i=1}^{d} \partial_i u_n\, A(u_n) \int_{\mathbf{G}_n} [e_i + w_i] \cdot \phi \,\mathrm{d}\mathbb{P} = 0.$$

On the other hand, such a solution is unique: First, observe that functions of the form $\varphi_2 \cdot \nabla_{\tilde{\omega}} \psi$ are dense in $L^2(I; L^2(Q; \mathcal{V}^2_{\mathrm{pot}}(\Omega)))$. Then, we consider the coercive bilinear form on $L^2(I; L^2(Q; \mathcal{V}^2_{\mathrm{pot}}(\Omega)))$:

$$B(v_1, v_2) := \int_0^T \int_Q \int_{\mathbf{G}_n} v_1 \cdot A(u_n) v_2 \,\mathrm{d}\mathbb{P}(\mathbb{x})\,\mathrm{d}x\,\mathrm{d}t.$$

Lax–Milgram tells us that the solution $v \in L^2(I; L^2(Q; \mathcal{V}^2_{\mathrm{pot}}(\mathbf{G}_n|\Omega)))$ to

$$B(\,\cdot\,, v) = \Big[v_1 \mapsto -\int_0^T \int_Q \int_{\mathbf{G}_n} v_1 \cdot A(u_n) \nabla u_n \,\mathrm{d}\mathbb{P}(\mathbb{x})\,\mathrm{d}x\,\mathrm{d}t\Big],$$

is unique, which is equivalent to

$$0 = B(\phi, v) + \int_0^T \int_Q \int_{\mathbf{G}_n} \phi \cdot A(u_n) \nabla u_n = \int_0^T \int_Q \int_{\mathbf{G}_n} \phi \cdot A(u_n)[\nabla u_n + v],$$

which justifies the standard ansatz.

7. Proof of main theorem (Theorem 17)

Theorem 17 is a consequence of the following.

Theorem 56 (Main theorem: homogenized limit of admissible point processes)**.** *Let $\mathbb{X}$ be a stationary ergodic admissible point process with distribution $\mathbb{P}$ such that $⊟\mathbb{X}^{\complement}$ is statistically connected. Under Assumption 6, let $u_n \in L^2(I; W^{1,2}(Q))$ be a homogenized solution from Theorem 55 for the thinned point process $\mathbb{X}^{(n)}$.*

For any subsequence of $(u_n)_{n\in\mathbb{N}}$, we are able to extract yet another subsequence that converges to a $u \in L^2(I; W^{1,2}(Q))$. This u is a (not necessarily unique) weak solution to the initial value problem

$$\begin{aligned} \mathbb{P}(\mathbf{G})\partial_t u - \nabla\cdot(A(u)\mathcal{A}\nabla u) - \lambda(\mu_{\mathbb{X}})h(u) &= \mathbb{P}(\mathbf{G})f && \text{in } I\times Q,\\ A(u)\mathcal{A}\nabla u\cdot\nu &= 0 && \text{on } I\times\partial Q,\\ u(0,x) &= \mathbb{P}(\mathbf{G})u_0(x) && \text{in } Q. \end{aligned}$$

Here $\mathcal{A}$ is the effective conductivity defined in Definition 37 based on the event $Q = \mathbf{G} = \{\mathbb{x} \in \mathcal{S}(\mathbb{R}^d) \,|\, o \notin ⊟\mathbb{x}\}$, $\Omega = \mathcal{S}(\mathbb{R}^d)$, $\mathcal{P} = \mathbb{P}$ and $\lambda(\mu_{\mathbb{X}})$ is the intensity of $\mu_{\mathbb{X}} := \mathcal{H}^{d-1}_{\llcorner\partial⊟\mathbb{X}}$. Furthermore, with $\alpha_{\mathcal{A}} > 0$ being the smallest eigenvalue of $\mathcal{A}$ and L_h being the Lipschitz constant of h

$$\operatorname*{ess\,sup}_{t\in I} \|u(t)\|^2_{L^2(Q)} \le \exp(C_1)[\|u_0\|^2_{L^2(Q)} + C_2]$$

$$\|\nabla u\|^2_{L^2(I;L^2(Q))} \le \frac{\mathbb{P}(\mathbf{G})}{2\alpha_{\mathcal{A}}\inf(A)}(1+\exp(C_1))[\|u_0\|^2_{L^2(Q)} + C_2],$$

where

$$C_1 := T\Big(1 + \frac{\lambda(\mu_{\mathbb{X}})}{\mathbb{P}(\mathbf{G})}(1+2L_h)\Big) \qquad \text{and} \qquad C_2 := \|f\|^2_{L^2(I;L^2(Q))} + 2T\frac{\lambda(\mu_{\mathbb{X}})}{\mathbb{P}(\mathbf{G})}\big|h(0)\big|^2.$$

Proof. We note that $\mathcal{A}^{(n)}$ from Theorem 55 is defined with cell solutions on $\Omega = \mathtt{F}_n\mathcal{S}(\mathbb{R}^d)$ and the push-forward measure $\mathbb{P}\circ\mathtt{F}_n^{-1}$. We use the pull-back result from Lemma 41 to obtain a representation of $\mathcal{A}^{(n)}$ in terms of $\Omega = \mathcal{S}(\mathbb{R}^d)$ and the original probability distribution.

Lemmas 32 and 29 and Corollary 39 yield respectively

$$\lambda(\mu_{\mathbb{X}^{(n)}}) \to \lambda(\mu_{\mathbb{X}}), \qquad \mathbb{P}(\mathbf{G}_n) \to \mathbb{P}(\mathbf{G}) > 0, \qquad \mathcal{A}^{(n)} \to \mathcal{A}, \qquad \alpha_{\mathcal{A}^{(n)}} \to \alpha_{\mathcal{A}} > 0,$$

for $\mathbf{G}_n := \{\mathbb{x} \,|\, o \notin ⊟\mathbb{x}^{(n)}\}$ and $\mathbf{G} := \{\mathbb{x} \,|\, o \notin ⊟\mathbb{x}\}$. From the a priori estimates in Theorem 55, we furthermore find

$$\limsup_{n\to\infty}\Big(\|u_n(t)\|^2_{L^\infty(0,T;L^2(Q))} + \|\nabla u_n\|^2_{L^2(I;L^2(Q))}\Big) < \infty,$$

$$\limsup_{n\to\infty} \|\partial_t u_n\|_{L^2(I;W^{1,2}(Q)^*)} < \infty,$$

and Aubin–Lions (or more general, Theorem 52) yields pre-compactness. These uniform bounds together with compactness arguments yield the existence of $u \in L^2(I; W^{1,2}(Q))$ with generalized time derivative $\partial_t u \in L^2(I; W^{1,2}(Q))^*$, such that for a subsequence

$$u_n \xrightarrow[n\to\infty]{L^2(I;W^{1,2}(Q))} u, \quad \partial_t u_n \xrightarrow[n\to\infty]{L^2(I;W^{1,2}(Q))^*} \partial_t u, \quad u_n \xrightarrow[n\to\infty]{L^2(I;L^2(Q))} u, \quad h(u_n) \xrightarrow[n\to\infty]{L^2(I;L^2(Q))} h(u),$$

as well as

$$A(u_n)\mathcal{A}^{(n)}\nabla u_n \xrightarrow[n\to\infty]{L^2(I;L^2(Q))} A(u)\mathcal{A}\nabla u.$$

From here we conclude.

8. Criterion for non-degeneracy of effective conductivity

In this chapter, we will establish a criterion for $⊟\mathbb{X}^{\complement}$ to be statistically connected (Definition 15), that is Theorem 60. To be precise, we will show that

$$e_1^t \mathcal{A} e_1 > 0,$$

as all other directions $\eta \in \mathbb{R}^d$ can be shown analogously via rotation. The procedure will be based on [13, Chapter 9]. The matrix $\mathcal{A}$ corresponds to the matrix $\mathcal{A}^0$ there. We will also see that $⊟\mathbb{X}^{\complement}$ is statistically connected if $\Xi\mathbb{X}^{\complement}$ is statistically connected.

Notation. Given a fixed admissible point process $\mathbb{X}$, we write in this section

$$⊟ := ⊟\mathbb{X} \qquad \Xi := \Xi\mathbb{X}.$$

Most arguments work for more general random perforations Ξ and their filled-up versions as long as Ξ has no infinite connected component (Theorem 58 needs additionally that almost surely, the bounded connected components of $\mathbb{R}^d \backslash \Xi$ have non-zero distance to the infinite connected components). We refrain from doing so since we would need to introduce the notion of stationary random sets and the main focus here lies on point processes.

8.1. Variational formulation

The following theorem gives us a different point of view on the effective conductivity $\mathcal{A}$:

Theorem 57 (Variational formulation [13, Theorem 9.1])**.** *For every ergodic admissible point process, we have almost surely and for every $\eta \in \mathbb{R}^d$:*

$$\eta^t \mathcal{A} \eta = \lim_{n\to\infty} n^{-d} \inf_{v \in C_0^\infty([0,n]^d)} \int_{[0,n]^d \backslash \Xi} \left|\eta - \nabla v\right|^2 \mathrm{d}x,$$

where $\mathcal{A}$ is the effective conductivity based on the event $\{o \notin \Xi\}$.

The first observation we can make is that the effective conductivity depends monotonously on the domain: The larger the set of holes, the lower the effective conductivity. The question arises in which cases this term becomes 0. This should only happen if $\mathbb{R}^d \backslash \Xi$ is "insufficiently connected". Intuitively, we want $v \approx -\eta \cdot x + \text{const}$, but at the same time, v needs to be 0 at the boundaries. If our region is badly connected, we can hide large gradients inside the holes, see, e.g., Figure 3. As in [13], we will see that the existence of sufficiently many "channels" connecting the left to the right side of a box $[0, n]^d$ will ensure $e_1^t \mathcal{A} e_1 > 0$. Before we do that, we establish an important fact:

We have defined statistical connectedness (Definition 15) via the filled-up Boolean model $⊟$. Unfortunately, filling up holes is non-local (depending on the size of holes), which is troublesome on the stochastic side. However, an analogue of [13, Lemma 9.7] tells us that the effective conductivity of both the Boolean model Ξ and its filled-up version $⊟$ are the same.

Theorem 58 (Filling up holes preserves the effective conductivity)**.** *For every ergodic admissible point process, we have almost surely*

$$\eta^t \mathcal{A} \eta = \lim_{n\to\infty} n^{-d} \inf_{v} \int_{[0,n]^d \backslash ⊟} \left|\eta - \nabla v\right|^2 \mathrm{d}x = \lim_{n\to\infty} n^{-d} \inf_{v} \int_{[0,n]^d \backslash \Xi} \left|\eta - \nabla v\right|^2 \mathrm{d}x,$$

where the infimum is over $v \in C_0^\infty([0, n]^d)$.

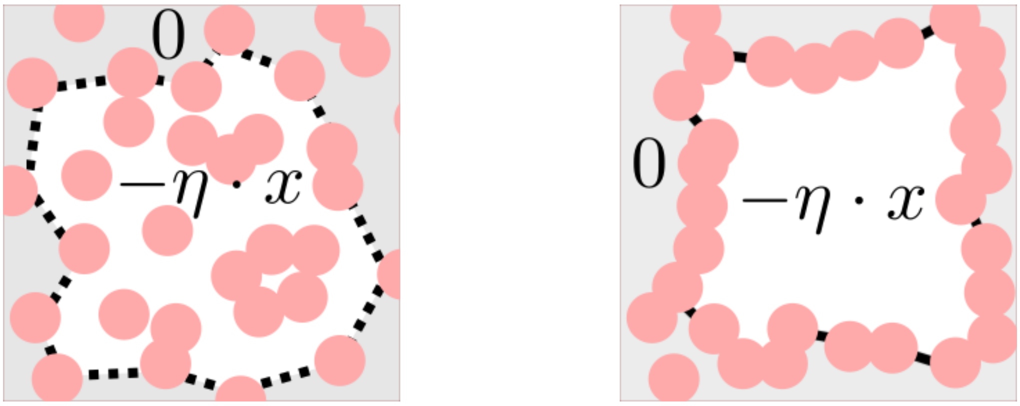

Figure 3. High versus low conductivity. The balls represent Ξ. The white area corresponds to $v \approx -\eta \cdot x + const$. Black lines indicate large contributions to $\int_{[0,n]^d \setminus \Xi} \left|\eta + \nabla v\right|^2 \mathrm{d}x$.

Proof. As mentioned before, this is a variation of [13, Lemma 9.7] fitted to our purpose. Let

- K_n^s be the set of islands (i.e., connected components in $\mathbb{R}^d \setminus \Xi$ of finite diameter) of diameter $\leq s$ that intersect but do not lie inside $[0,n]^d$.
- L_n^s be the set of islands of diameter $> s$ that do not completely lie inside $[0,n]^d$ and that are encircled by a Ξ-cluster of size larger than s.

All the islands in K_n^s and L_n^s belong to connected components of $\mathbb{R}^d \setminus \Xi$ different from $\boxminus^{\complement}$ (the unique unbounded connected component). Since $\mathbb{X}$ is admissible, almost surely they all have non-zero distance to $\boxminus^{\complement}$. Therefore, the following infimum decomposes, with all the infima being over $v \in C_0^\infty([0,n]^d)$

$$\begin{aligned}
\inf_v \int_{[0,n]^d \setminus \Xi} \left|\eta - \nabla v\right|^2 \mathrm{d}x &= \inf_v \int_{([0,n]^d \setminus \Xi) \setminus (K_n^s \cup L_n^s)} \left|\eta - \nabla v\right|^2 \mathrm{d}x + \inf_v \int_{K_n^s \cup L_n^s} \left|\eta - \nabla v\right|^2 \mathrm{d}x \\
&= \inf_v \int_{([0,n]^d \setminus \boxminus) \setminus (K_n^s \cup L_n^s)} \left|\eta - \nabla v\right|^2 \mathrm{d}x + \inf_v \int_{K_n^s \cup L_n^s} \left|\eta - \nabla v\right|^2 \mathrm{d}x \\
&= \inf_v \int_{[0,n]^d \setminus \boxminus} \left|\eta - \nabla v\right|^2 \mathrm{d}x + C,
\end{aligned}$$

with

$$\left|C\right| \leq \left|\eta\right|^2 \left[\mathcal{L}^d(K_n^s) + \mathcal{L}^d(L_n^s)\right],$$

and where the second equality comes from the fact that filling up islands that lie completely inside $[0,n]^d$ does not change the value of the infimum. Now, we observe

- $\mathcal{L}^d(K_n^s) \sim O(n^{d-1})$ for fixed s, so $\lim_{n\to\infty} n^{-d} \mathcal{L}^d(K_n^s) = 0$ and
- denoting by L^s all islands of diameter greater than s, we have almost surely

$$\lim_{n\to\infty} n^{-d} \mathcal{L}^d(L_n^s) \leq \lim_{n\to\infty} n^{-d} \mathcal{L}^d(L^s \cap [0,n]^d) =: \mathrm{density}(L^s) = \mathbb{P}(o \in L^s).$$

All islands are of finite size, yielding $\bigcap_{s\in\mathbb{N}} L^s = \emptyset$. Hence, $\lim_{s\to\infty} \mathrm{density}(L^s) = 0$.

Choosing s sufficiently large finishes the proof.

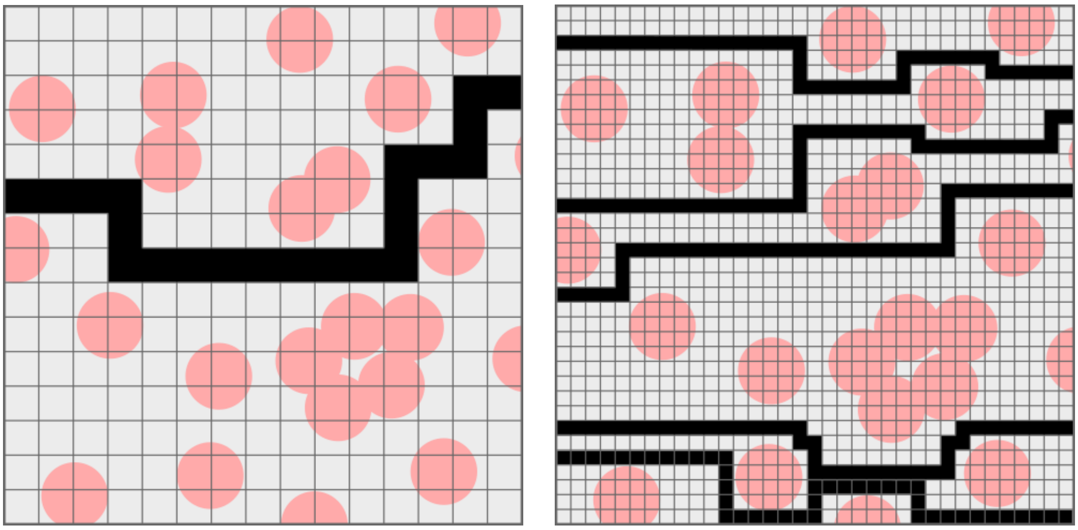

Figure 4. Percolation channels for different k_{scale}.

8.2. *Percolation channels*

Definition 59 (Percolation channels (see Figure 4))**.** Fix a $k_{\text{scale}} \in \mathbb{N}$. We consider the lattice $\mathbb{Z}^d_n \subset \mathbb{Z}^d$ and the cube with corner $z = (z_1, \ldots, z_d) \in \mathbb{Z}^d$

$$\mathbb{Z}^d_n := \mathbb{Z}^d \cap [0, n)^d \qquad \text{and} \qquad \mathcal{K}_z := \bigtimes_{i=1}^{d} [z_i, z_i + 1]$$

and call two vertices z, z' *neighbors* if their l^1-distance is equal to 1.

We call z open if

$$\Xi \cap k_{\text{scale}}^{-1} \mathcal{K}_z = \emptyset.$$

An open left-right crossing $\gamma = (z^{(1)}, \ldots, z^{(l)})$ of $\mathbb{Z}^d_n$ is called a *percolation channel* in $\mathbb{Z}^d_n$, i.e.,

i) $z^{(i)}, z^{(i+1)}$ are neighbors for each $i < l$,
ii) all the $z^{(i)}$ are open, and
iii) $z^{(1)}_1 = 0$ and $z^{(l)}_1 = n - 1$.

We define the quantity (depending on the random Ξ and on k_{scale})

$$\begin{aligned} \mathbf{N}(n) :&= \max\{j \,|\, \gamma_1, \ldots, \gamma_j \text{ are disjoint percolation channels in } \mathbb{Z}^d_n\} \\ &= \text{“maximal number of disjoint percolation channels in } \mathbb{Z}^d_n\text{”} \end{aligned}$$

and the tube $L(\gamma)$ corresponding to the path $\gamma = (z^{(1)}, \ldots, z^{(l)})$ as

$$L(\gamma) := \bigcup_{i \le l} k_{\text{scale}}^{-1} \mathcal{K}_{z^{(i)}}.$$

Statistical connectedness of $\mathbb{R}^d \backslash \Xi$ then reads as follows:

Theorem 60 (Percolation channels imply conductivity)**.** *For almost every realization* $\mathbb{x}$ *of an ergodic admissible point process, we have for* $\Xi = \Xi_{\mathbb{x}}$

$$\lim_{n\to\infty} n^{-d} \inf_{v\in C_0^\infty([0,n]^d)} \int_{[0,n]^d\setminus\Xi} \left|e_1 - \nabla v\right|^2 \mathrm{d}x \geq \limsup_{n\to\infty} \Big(\frac{\mathbf{N}(n)}{n^{d-1}}\Big)^2.$$

In particular, the effective conductivity is strictly positive if almost surely

$$\limsup_{n\to\infty} \frac{\mathbf{N}(n)}{n^{d-1}} > 0. \tag{8.1}$$

Proof. This is an analogue of [13, Theorem 9.11] and relies on defining a suitable vector field $\overrightarrow{F}_\gamma \colon [0, k_{\mathrm{scale}}^{-1} n]^d \to \mathbb{R}^d$ inside channels $\gamma = (z^{(1)}, \ldots, z^{(l)})$ on $\mathbb{Z}_n^d$. We want $\overrightarrow{F}_\gamma$ to satisfy the following

- $\left|\overrightarrow{F}_\gamma(x)\right| = 1$ for every x inside the tube $L(\gamma)$ and $\overrightarrow{F}_\gamma(x) = 0$ outside.
- $\overrightarrow{F}_\gamma$ is parallel to $\partial L(\gamma)$ except on corners as well as $\partial L(\gamma)_- := \mathcal{K}_{z^{(1)}} \cap \{x_1 = 0\}$ and $\partial L(\gamma)_+ := \mathcal{K}_{z^{(l)}} \cap \{x_1 = k_{\mathrm{scale}}^{-1} n\}$.
- $\overrightarrow{F}_\gamma(x) = e_1$ for $x \in \partial L(\gamma)_- \cup \partial L(\gamma)_+$.
- For the standard normal vector ν to $\partial L(\gamma)$:

$$\int_{L(\gamma)} (e_1 - \nabla v) \cdot \overrightarrow{F}_\gamma \, \mathrm{d}x = \int_{\partial L(\gamma)} (x_1 - v) \overrightarrow{F}_\gamma \cdot \nu \, \mathrm{d}\mathcal{H}^{d-1}(x). \tag{8.2}$$

Thus, the vector field was deliberately chosen such that the Gauß divergence theorem only yields contributions from the "starting" and "ending" surfaces. Figure 5 illustrates how $\overrightarrow{F}_\gamma$ can be chosen to satisfy these properties.

The rest is simple. Take $\gamma_1, \ldots, \gamma_{\mathbf{N}(n)}$ disjoint nonself-intersecting channels in $\mathbb{Z}_n^d$. Set

$$T := \bigcup_{i\leq \mathbf{N}(n)} L(\gamma_i) \subset [0, k_{\mathrm{scale}}^{-1} n]^d, \qquad \overrightarrow{F} := \sum_{i\leq \mathbf{N}(n)} \overrightarrow{F}_{\gamma_i}.$$

Then,

$$\begin{aligned} \int_{[0,k_{\mathrm{scale}}^{-1} n]^d\setminus\Xi} \left|e_1 - \nabla v\right|^2 \mathrm{d}x &\geq \int_T \left|e_1 - \nabla v\right|^2 \mathrm{d}x \geq \int_T \left|(e_1 - \nabla v) \cdot \overrightarrow{F}\right|^2 \mathrm{d}x \\ &\geq \frac{1}{\mathcal{L}^d(T)} \left(\int_T (e_1 - \nabla v) \cdot \overrightarrow{F} \, \mathrm{d}x\right)^2 \geq \frac{k_{\mathrm{scale}}^d}{n^d} \left(\int_T (e_1 - \nabla v) \cdot \overrightarrow{F} \, \mathrm{d}x\right)^2. \end{aligned}$$

For a fixed tube $L = L(\gamma_i)$, we have

$$\begin{aligned} \int_L (e_1 - \nabla v) \cdot \overrightarrow{F} \, \mathrm{d}x &= \int_{\partial L} (x_1 - v) \overrightarrow{F} \cdot \nu \, \mathrm{d}\mathcal{H}^{d-1}(x) = \int_{\partial L_- \cup \partial L_+} (x_1 - v) \overrightarrow{F} \cdot \nu \, \mathrm{d}\mathcal{H}^{d-1}(x) \\ &= \int_{\partial L_+} k_{\mathrm{scale}}^{-1} n e_1 \cdot e_1 \, \mathrm{d}\mathcal{H}^{d-1}(x) = k_{\mathrm{scale}}^{-1} n \mathcal{H}^{d-1}(\partial L_+) = k_{\mathrm{scale}}^{-d} n. \end{aligned}$$

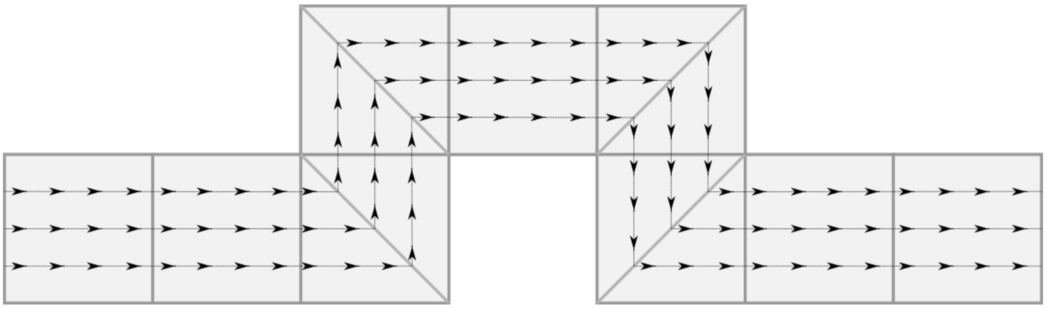

Figure 5. The vector field $\overrightarrow{F}$ simply follows along the direction of the path γ. Whenever the tube makes a turn, we divide the corresponding cube diagonally as depicted with the vector field remaining constant in each respective half. The resulting $\overrightarrow{F}$ is piecewise constant and Eq (8.2) holds via the Gauß divergence theorem as contributions on the diagonal surfaces cancel out: We have a positive contribution from the incoming arrows and a negative contribution from the outgoing one.

Therefore,

$$\int_{[0,k_{\text{scale}}^{-1}n]^d\setminus\Xi)} \left|e_1 - \nabla v\right|^2 \mathrm{d}x \geq \frac{k_{\text{scale}}^d}{n^d}(k_{\text{scale}}^{-d} n\mathbf{N}(n))^2,$$

and so

$$(k_{\text{scale}}^{-1}n)^{-d}\int_{[0,k_{\text{scale}}^{-1}n]^d\setminus\Xi} \left|e_1 - \nabla v\right|^2 \mathrm{d}x \geq \left(\frac{\mathbf{N}(n)}{n^{d-1}}\right)^2.$$

Passing to the lim sup finishes the proof.

Remark 61 ($d = 2$ *and bottom-top crossings*). Let $\mathbf{L}(n)$ be the minimal number of open vertices that a l^∞-bottom-top crossing of $\mathbb{Z}_n^2$ must have. It turns out that in $d = 2$

$$\mathbf{L}(n) = \mathbf{N}(n)$$

(see Lemma 73). We will use this to show Eq (8.1) for the Poisson point process $\mathbb{X}_{\text{poi}}$.

9. Example: Poisson point processes

The driving force behind this work has been a stationary Poisson point process $\mathbb{X}_{\text{poi}}$. It is known that the Poisson point process is ergodic (even mixing) and its high spatial independence makes it *the* canonical random point process. As pointed out before though, $\Xi\mathbb{X}_{\text{poi}}$ gives rise to numerous analytical issues which prevent the usage of the usual homogenization tools.

The main theorem (Theorem 56) tells us that homogenization is still reasonable for highly irregular filled-up Boolean models $\boxminus\mathbb{X}$ driven by admissible point processes $\mathbb{X}$.

It is known for $\mathbb{X}_{\text{poi}}$ that there exists some critical radius $r_c := r_c[\lambda(\mathbb{X}_{\text{poi}})] \in (0, \infty)$ such that

- $\boxminus\mathbb{X}_{\text{poi}}$ only consists of finite clusters for $r < r_c$ (subcritical regime) and
- $\boxminus\mathbb{X}_{\text{poi}}$ has a unique infinite cluster for $r > r_c$ (supercritical regime).

The behavior at criticality $r = r_c$ is still a point of research. For details, we refer to [14] for the Poisson point process $\mathbb{X}_{\text{poi}}$ and [18, Chapter 3] for the Boolean model $\Xi\mathbb{X}_{\text{poi}}$.

We will see in the subcritical regime that

i) $\mathbb{X}_{\text{poi}}$ is an ergodic admissible point process and
ii) $\Xi\mathbb{X}_{\text{poi}}{}^{\complement}$ is statistically connected, which is equivalent to $\boxminus\mathbb{X}_{\text{poi}}{}^{\complement}$ being statistically connected (see Theorem 58).

We therefore make the following assumption for the rest of this section:

Assumption 62 (subcritical regime)**.** *We assume that*

$$r < r_c.$$

Remark 63 (*Scaling relation*). r_c has the following scaling relation

$$r_c[k^d \cdot \lambda(\mathbb{X}_{\text{poi}})] = r_c[\lambda(k^{-1}\mathbb{X}_{\text{poi}})] = k^{-1} r_c[\lambda(\mathbb{X}_{\text{poi}})].$$

9.1. Admissibility of Poisson point processes

The Mecke–Slivnyak theorem tells us that the Palm probability measure (Theorem 45) of a stationary Poisson point process is just a Poisson point process with a point added in the origin. This gives us the following lemma:

Lemma 64 (Equidistance property)**.** *The stationary Poisson point process $\mathbb{X}_{\text{poi}}$ satisfies the equidistance property for arbitrary $r > 0$, i.e.,*

$$\mathbb{P}\Big(\exists x, y \in \mathbb{X}_{\text{poi}} \,|\, d(x, y) = 2r\Big) = 0.$$

Proof. This follows from using the Palm theorem Theorem 45 on

$$f(x, \mathbb{x}) := \sum_{x_i \in \mathbb{x}} \mathbb{1}\left\{d(x, x_i) = 2r\right\},$$

and the Mecke–Slivnyak theorem [14, Theorem 9.4].

Corollary 65 ($\mathbb{X}_{\text{poi}}$ is admissible)**.** *Under Assumption 62, $\mathbb{X}_{\text{poi}}$ is an admissible point process.*

Proof. $\mathbb{X}_{\text{poi}}$ is not just ergodic, but even mixing (see [14, Theorem 8.13]). The equidistance property has been proven in Corollary 64. Finiteness of clusters follows from the subcritical regime (Assumption 62).

9.2. Statistical connectedness for Poisson point processes

Proving the statistical connectedness of $\Xi\mathbb{X}_{\text{poi}}{}^{\complement}$ (Definition 15) is much harder and does not immediately follow from readily available results. Our procedure is as follows:

i) We employ the criterion from Section 8. Therefore, we will check that there are sufficiently many percolation channels for $\Xi\mathbb{X}_{\text{poi}}$.
ii) Using the spatial independence of the Poisson point process $\mathbb{X}_{\text{poi}}$, we show that it is sufficient to only consider 2-dimensional slices.

iii) We show the statement in $d = 2$ using ideas in [11, Chapter 11]. There, the result has been proven for certain iid fields on planar graphs, including $\mathbb{Z}^2$.

Additionally to Assumption 62, we need sufficient discretization for the percolation channels:

Assumption 66 (Sufficient scaling)**.** *Let $k_{\text{scale}} \in \mathbb{N}$ be large enough such that, for the critical radius r_c,*

$$(r_c - r)/2 > \sqrt{d}k_{\text{scale}}^{-1},$$

e.g., $k_{\text{scale}} := \lceil 2\frac{\sqrt{d}}{r_c - r}\rceil + 1$.

Definition 67 (Recap and random field $(X_z)_{z\in\mathbb{Z}^d}$)**.** Recall Definition 59, most importantly

$$\mathbb{Z}_n^i : = \mathbb{Z}^i \cap [0, n)^i \qquad \text{and} \qquad \mathcal{K}_z := \bigtimes_{i=1}^d [z_i, z_i + 1],$$

as well as the notion of percolation channels for k_{scale} and

$$\mathbf{N}(n) : = \text{“maximal number of disjoint percolation channels in } \mathbb{Z}_n^d\text{”}.$$

We define the *random field*

$$(X_z)_{z\in\mathbb{Z}^d} := \Big(\mathbb{1}\{\Xi\mathbb{X}_{\text{poi}} \cap k_{\text{scale}}^{-1}\mathcal{K}_z = \emptyset\}\Big)_{z\in\mathbb{Z}^d}.$$

We say that $z \in \mathbb{Z}^d$ is *blocked* if $X_z = 0$ and *open* if $X_z = 1$ (this is consistent with Definition 59).

Theorem 68 (Percolation channels of the Poisson point process)**.** *Under Assumptions 62 and 66, there is a $C > 0$ such that Eq (8.1) holds, i.e.,*

$$\mathbb{P}(\limsup_{n\to\infty} n^{1-d}\mathbf{N}(n) \geq C) = 1.$$

In particular, $\boxminus\mathbb{X}_{\text{poi}}{}^{\complement}$ is statistically connected (see Theorem 60).

The rest of the section deals with the proof of Theorem 68. It will follow as a direct consequence of Lemma 70 (reduction to $d = 2$) and Lemma 74 (main result for $d = 2$) which are given later.

9.2.1. Spatial independence and moving to $d = 2$

For disjoint $U_1, U_2, \ldots \subset \mathbb{R}^d$ and events A_i only depending on $\mathbb{X}_{\text{poi}}$ inside U_i, we know that $(A_i)_i$ is an independent family. This is one of the striking properties of a Poisson point process and we will heavily make use of it. The Boolean model $\Xi\mathbb{X}_{\text{poi}}$ for radius r still retains this property in a slightly weaker form and correspondingly the random field $(X_z)_{z\in\mathbb{Z}^d}$:

Lemma 69 (Independence in large distances)**.** *Let $A, B \subset \mathbb{Z}^d$ such that*

$$d^\infty(A, B) := \min_{z_a\in A, z_b\in B} \|z_b - z_a\|_\infty \geq 2rk_{\text{scale}} + 1. \tag{9.1}$$

Then, $(X_z)_{z\in A}$ and $(X_z)_{z\in B}$ are independent.

Proof. $(X_z)_{z\in A}$ is only affected by points of $\mathbb{X}_{\text{poi}}$ inside

$$U_A := \bigcup_{z\in A} \mathbb{B}_r\Big(k_{\text{scale}}^{-1}\mathcal{K}_z\Big).$$

The same holds for $(X_z)_{z\in B}$ and we check that Eq (9.1) implies $U_A \cap U_B = \emptyset$.

Lemma 70 (2-dimensional percolation channels imply channel property for $d > 2$)**.** *For $\tilde{z} \in \mathbb{Z}^{d-2}$, we define (compare to Definition 59)*

$$\mathbf{N}_{\tilde{z}}^{(2)}(n) : = \text{“}\textit{maximal number of disjoint percolation channels in } \mathbb{Z}_n^2 \times \tilde{z}\text{”}.$$

If there are $\tilde{C},\ p_0 > 0$ such that for some $\tilde{z} \in \mathbb{Z}^{d-2}$,

$$\limsup_{n\to\infty} \mathbb{P}\Big(\mathbf{N}_{\tilde{z}}^{(2)}(n) \geq \tilde{C}n\Big) > p_0 > 0, \tag{9.2}$$

then there exists a $C > 0$ such that

$$\limsup_{n\to\infty} \mathbb{P}\Big(\mathbf{N}(n) \geq Cn^{d-1}\Big) = \mathbb{P}(\limsup_{n\to\infty} n^{1-d}\mathbf{N}(n) \geq C > 0) = 1.$$

(This proof heavily relies on the independence structure of the Poisson point process, i.e., Lemma 69.)

Proof. $\mathbb{X}_{\text{poi}}$ is stationary, so for distinct $\tilde{z}_1,\ \tilde{z}_2 \in \mathbb{Z}^{d-2}$,

$$p(n) := \mathbb{P}\Big(\mathbf{N}_{\tilde{z}_1}^{(2)}(n) \geq \tilde{C}n\Big) = \mathbb{P}\Big(\mathbf{N}_{\tilde{z}_2}^{(2)}(n) \geq \tilde{C}n\Big).$$

Let $k := \lceil 2rk_{\text{scale}}\rceil + 1$. By Lemma 69, the events on $\mathbb{Z}^2 \times (k\tilde{z}_1)$ are independent from the events on $\mathbb{Z}^2 \times (k\tilde{z}_2)$. Therefore, $(\mathbb{1}\{\mathbf{N}_{k\tilde{z}}^{(2)}(kn) \geq \tilde{C}n\})_{\tilde{z}\in\mathbb{Z}^{d-2}}$ is an iid family of Bernoulli random variables with parameter $p(n)$. Then,

$$\begin{aligned}\mathbb{P}\Big(\mathbf{N}(kn) \geq \frac{\tilde{C}p_0}{2k^{d-2}}(kn)^{d-1}\Big) &\geq \mathbb{P}(\text{For at least } p_0/2 \text{ of the } \tilde{z} \in \mathbb{Z}_n^{(d-2)} \colon \mathbf{N}_{\tilde{z}}^{(2)}(kn) \geq \tilde{C}kn)\\ &= \mathbb{P}\Big(\frac{1}{\#\mathbb{Z}_n^{(d-2)}} \sum_{\tilde{z}\in\mathbb{Z}_n^{(d-2)}} \mathbb{1}\{\mathbf{N}_{k\tilde{z}}^{(2)}(kn) \geq \tilde{C}n\} \geq \frac{1}{2}p_0\Big).\end{aligned}$$

By Eq (9.2) and the law of large numbers, we get

$$\limsup_{n\to\infty} \mathbb{P}\Big(\mathbf{N}(n) \geq \frac{\tilde{C}p_0}{2k^{d-2}}n^{d-1}\Big) \geq \limsup_{n\to\infty} \mathbb{P}\Big(\mathbf{N}(kn) \geq \frac{\tilde{C}p_0}{2k^{d-2}}(kn)^{d-1}\Big) = 1.$$

Setting $C = \frac{\tilde{C}p_0}{2k^{d-2}}$, we obtain Eq (8.1) after checking

$$\mathbb{P}(\limsup_{n\to\infty} n^{1-d}\mathbf{N}(n) \geq C) = \limsup_{n\to\infty} \mathbb{P}(n^{1-d}\mathbf{N}(n) \geq C) = 1,$$

which finishes the proof.

Remark 71. Spatial independence is needed to move from $d = 2$ to $d \geq 3$. The strong independence properties of $\mathbb{X}_{\text{poi}}$ allow far weaker conditions on $\mathbf{N}^{(2)}(n)$ (positive probability) than on $\mathbf{N}(n)$ (probability 1). Either way, Lemma 74 shows that $\mathbb{P}(\mathbf{N}^{(2)}(n) < Cn)$ drops exponentially in n.

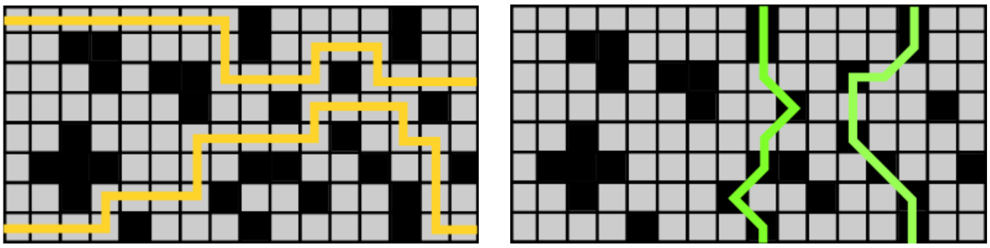

Figure 6. Disjoint percolation channels vs. vertical crossings. On the left side, we see that we can only have at most two l^1-channels. The right figure shows that any l^∞-vertical crossing must contain at least two open vertices.

9.2.2. $d = 2$: Definitions and preliminary results

As shown before, we may limit ourselves to a fixed lattice $\mathbb{Z}^2 \times 0_{\mathbb{Z}^{d-2}} \simeq \mathbb{Z}^2$. Therefore, we will often suppress the "anchor point" $0_{\mathbb{Z}^{d-2}}$ and just act like we are in $\mathbb{Z}^2$. Our random field from Definition 67 is then by abuse of notation

$$(X_z)_{z\in\mathbb{Z}^2} \simeq (X_z)_{z\in\mathbb{Z}^2\times 0_{\mathbb{Z}^{d-2}}}.$$

Definition 72 (Vertical crossings)**.** Consider the $(\mathbb{Z}^2, l^\infty)$-lattice, that is z, z' are *neighbors* if $\|z - z'\|_\infty = 1$.

An l^∞-bottom-top crossing in $\mathbb{Z}_n^2$ is called a *vertical crossing*. We call a path *blocked* if all its vertices are blocked. We define the quantity

$$\mathbf{L}(n) := \text{"minimal number of open vertices in a vertical crossing in } \mathbb{Z}_n^2\text{"}.$$

(The percolation channels lie on the l^1-graph, while the vertical crossings lie on the l^∞-graph.)

We may work with single vertical crossings instead of collections of percolation channels:

Lemma 73 (Percolation channels vs vertical crossings (see Figure 6))**.** *It holds that*

$$\mathbf{N}(n) = \mathbf{L}(n).$$

Proof. See the proof of [11, Theorem 11.1] based on Menger's theorem and [11, Proposition 2.2].

The main work is proving the following equivalent of [11, Proposition 11.1]:

Lemma 74 (Open vertices in vertical crossings)**.** *Under Assumptions 62 and 66, there are $C_i > 0$ such that*

$$\mathbb{P}(\exists o \rightsquigarrow \mathbb{Z} \times \{n\} \textit{ with at most } C_1 n \textit{ open vertices}) \le C_2 \exp(-C_3 n),$$

in particular,

$$\mathbb{P}(\mathbf{N}(n) \ge C_1 n) \ge 1 - C_2 n \exp(-C_3 n).$$

The proof relies on a reduction scheme of the path $\gamma\colon o \rightsquigarrow \mathbb{Z} \times \{n\}$. We divide γ into several segments which must either contain an open vertex or contain a blocked path of large diameter. Since we are in the subcritical regime, the probability of such paths decreases exponentially in their diameter:

Lemma 75 (Diameter of blocked paths)**.** *Let $z \in \mathbb{Z}^2$. Under Assumption 62 and $k_{\text{scale}} \in \mathbb{N}$ as in Assumption 66, there are $C_i > 0$ such that*

$$\mathbb{P}(\exists \textit{blocked path } \gamma,\, z \in \gamma,\, \operatorname{diam}(\gamma) \geq n) \leq C_1 \exp\big(-C_2 n\big),$$

where

$$\operatorname{diam}(\gamma) := \max_{z_1, z_2 \in \gamma} \|z_1 - z_2\|_2.$$

Proof. Consider the Boolean model for radius $R := \frac{1}{2}(r + r_c) < r_c$, i.e.,

$$\Xi^{(R)}\mathbb{X}_{\text{poi}} := \mathbb{B}_R\Big(\mathbb{X}_{\text{poi}}\Big).$$

Let $\gamma = (z^{(1)}, \ldots, z^{(l)})$ be a blocked path in $\mathbb{Z}^2$ containing z with diameter $\geq n$. Since γ is blocked and $R - r > \sqrt{d}k_{\text{scale}}^{-1}$ (Assumption 66), we find for every $1 \leq i \leq l$ some $x_i \in \mathbb{X}_{\text{poi}}$ such that

$$\mathcal{K}_{z^{(i)}} \cap k_{\text{scale}}\mathbb{B}_r(x_i) \neq \emptyset,$$

and therefore

$$\mathcal{K}_{z^{(i)}} \subset k_{\text{scale}}\mathbb{B}_R(x_i).$$

Connecting all the $z^{(i)}$ by a straight line, we obtain a continuous path inside $k_{\text{scale}}\Xi^{(R)}\mathbb{X}_{\text{poi}}$. In particular, they all belong to the same $k_{\text{scale}}\Xi^{(R)}\mathbb{X}_{\text{poi}}$-cluster. Then,

$$\begin{aligned}
&\mathbb{P}(\exists \text{closed path } \gamma,\, z \in \gamma \text{ and } \operatorname{diam}(\gamma) \geq n)\\
&\quad \leq \mathbb{P}(z \text{ lies in a cluster in } k_{\text{scale}}\Xi^{(R)}\mathbb{X}_{\text{poi}} \text{ of diameter } \geq n)\\
&\quad \leq C_1 \exp(-C_2 n),
\end{aligned}$$

since the occurrence of large clusters drops exponentially in the diameter ([18, Lemma 2.4]).

9.2.3. Proof of Lemma 74 (open vertices in vertical crossings)

Let $n \in \mathbb{N}$. As pointed out before, we follow the procedure in [11, Proposition 11.1] adjusted to the continuum setting. We define $A(z, k)$ for $z \in \mathbb{Z}^2$ and $k \in \mathbb{N}$ as

$$A(z, k) := \{\exists l^\infty\text{-path } z \rightsquigarrow \mathbb{Z} \times \{n\} \text{ with at most } k \text{ open vertices}\}.$$

The idea is to break up the path $o \rightsquigarrow \mathbb{Z} \times \{n\}$ into multiple segments (see Figure 7). In each segment, we can either reduce k by 1 or employ Lemma 75. We set

$$\begin{aligned}
\tilde{s} &:= \lceil 2rk_{\text{scale}} \rceil + 1\\
B_1^\infty(z, s) &:= \{v \in \mathbb{Z}^2 \mid \|z - v\|_\infty \leq s\}\\
B_2^\infty(z, s) &:= \{v \in \mathbb{Z}^2 \mid \|z - v\|_\infty \leq s + \tilde{s}\}\\
D^\infty(z, s) &:= \{v \in \mathbb{Z}^2 \mid \|z - v\|_\infty = s + \tilde{s} + 1\} = \text{"boundary of } B_2^\infty(z, s)\text{"}.
\end{aligned}$$

These boxes are defined so that the following holds: For fixed $z \in \mathbb{Z}^2$, we have by Lemma 69 that the random variables $(X_v)_{v \in B_1^\infty(z,s)}$ and $(X_v)_{v \in \mathbb{Z}^2 \setminus B_2^\infty(z,s)}$ are independent. That means the state of the vertices in $B_1^\infty(z,s)$ is independent from the state of the vertices in $\mathbb{Z}^2 \backslash B_2^\infty(z,s) = B_2^\infty(z,s)^\complement$. Additionally, we define the probability

$$g(z,s) := \mathbb{P}(\exists z \rightsquigarrow B_1^\infty(z,s)^\complement \text{ blocked inside } B_1^\infty(z,s)).$$

The key inequality for the iteration in k is the following

$$\mathbb{P}(A(z,k)) \le \sum_{v \in D^\infty(z,s)} \Big[g(z,s)\mathbb{P}(A(v,k)) + \mathbb{P}(A(v,k-1)) \Big], \tag{9.3}$$

for $z = (z_1, z_2) \in \mathbb{Z}^2$ whenever $z_2 < n - (s + \tilde{s})$.

Proof of Eq (9.3). Consider the event that for some $v \in D^\infty(z,s)$, we find a path $v \rightsquigarrow \mathbb{Z} \times \{n\}$ that has at most $k-1$ open vertices, i.e.,

$$E := \bigcup_{v \in D^\infty(z,s)} A(v, k-1).$$

Now assume that the event $A(z,k) \backslash E$ happens. Take a path $\gamma = (z, v^{(1)}, \dots, v^{(j)})$ with $v_2^{(j)} = n$ and at most k of the $v^{(i)}$ being open. Let i_1 be the last index with $v^{(i_1)} \in D^\infty(z,s)$. This i_1 exists since $z_2 < n - (s + \tilde{s})$, so γ has to pass by $D^\infty(z,s)$ to reach $\mathbb{Z} \times \{n\}$. For this i_1, we know that $(v^{(i_1)}, \dots, v^{(j)})$ completely lies in $B_2^\infty(z,s)^\complement$. Since E does not happen, it must have k open vertices. $(z, v^{(1)}, \dots, v^{(i_1)})$ is a path from z to $B_1^\infty(z,s)^\complement$ that is blocked everywhere except its end. Therefore,

$$\begin{aligned} A(z,k) \backslash E \subset \bigcup_{v \in D^\infty(z,s)} \{ & \exists z \rightsquigarrow B_1^\infty(z,s)^\complement \text{ blocked in } B_1^\infty(z,s) \text{ and} \\ & \exists v \rightsquigarrow \mathbb{Z} \times \{n\} \text{ in } B_2^\infty(z,s)^\complement \text{ with at most } k \text{ open vertices}\}. \end{aligned}$$

As previously mentioned, the events in $B_1^\infty(z,s)$ and $B_2^\infty(z,s)^\complement$ are independent from each other. This gives us

$$\begin{aligned} \mathbb{P}(A(z,k) \backslash E) &\le \sum_{v \in D^\infty(z,s)} \mathbb{P}(\exists z \rightsquigarrow B_1^\infty(z,s)^\complement \text{ blocked in } B_1^\infty(z,s) \text{ and} \\ &\qquad \exists v \rightsquigarrow \mathbb{Z} \times \{n\} \text{ in } B_2^\infty(z,s)^\complement \text{ with at most } k \text{ open vertices}) \\ &= \sum_{v \in D^\infty(z,s)} \mathbb{P}(\exists z \rightsquigarrow B_1^\infty(z,s)^\complement \text{ blocked in } B_1^\infty(z,s)) \\ &\qquad \times \mathbb{P}(\exists v \rightsquigarrow \mathbb{Z} \times \{n\} \text{ in } B_2^\infty(z,s)^\complement \text{ with at most } k \text{ open vertices}) \\ &\le \sum_{v \in D^\infty(z,s)} g(z,s)\mathbb{P}(A(v,k)). \end{aligned}$$

Now,

$$\mathbb{P}(A(z,k)) \le \mathbb{P}(A(z,k) \backslash E) + \mathbb{P}(E) \le \sum_{v \in D^\infty(z,s)} \Big[g(z,s)\mathbb{P}(A(v,k)) + \mathbb{P}(A(v,k-1)) \Big],$$

concludes the proof of Eq (9.3).

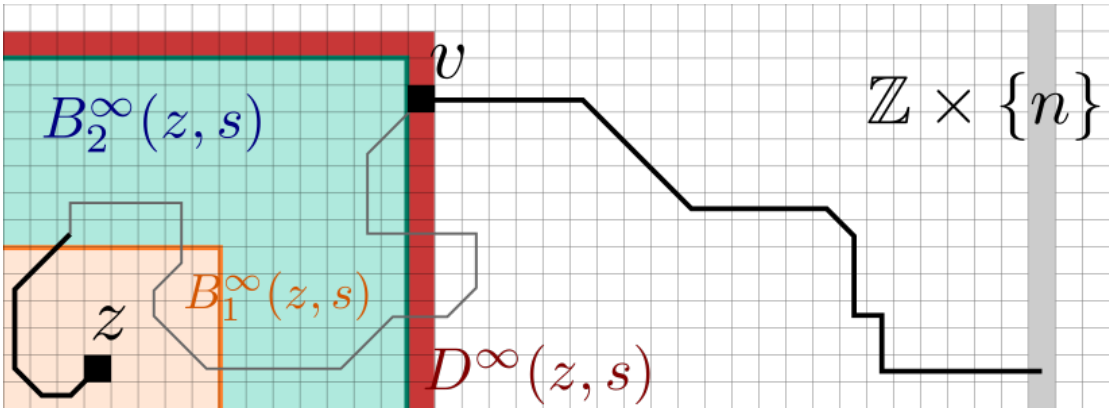

Figure 7. B_1^∞, B_2^∞, D^∞ and decomposition of paths (left-right crossing instead of top-bottom).

Observe that the reduction in k can only happen until $k = 0$, so more $g(z,s)$-terms have to show up at some point. Since any path $z \rightsquigarrow B_1^\infty(z,s)^\complement$ has diameter of at least s, Lemma 75 tells us that $g(z,s) \leq C_1 \exp(-C_2 s)$ for $C_1, C_2 > 0$ independent of z and s. Choose s large such that

$$g(z,s) \cdot \#D^\infty(z,s) \leq C_1 \exp(-C_2 s) \cdot 8(s + \tilde{s} + 1) \leq 1/4.$$

For simplicity, we introduce

$$D^\infty := D^\infty(o,s) \qquad \text{and} \qquad h(z,y) := \begin{cases} g(z,s) & \text{if } y = 0 \\ 1 & \text{if } y = 1, \end{cases}$$

and rewrite Eq (9.3) into

$$\mathbb{P}(A(z,k)) \leq \sum_{v^{(1)} \in D^\infty,\, y_1 \in \{0,1\}} h(z,y_1) \mathbb{P}(A(z + v^{(1)}, k - y_1)). \tag{9.4}$$

We now iteratively use Eq (9.4) up to l times as it is only applicable when $z_2 < n - (s + \tilde{s})$. All the $v^{(i)}$ are summed over D^∞ and all the y_i over $\{0,1\}$.

$$\begin{aligned}
&\mathbb{P}(A(o,k)) \\
&\leq \sum_{\substack{i \leq l, v^{(i)}, y_i \\ v_2^{(1)} + \cdots + v_2^{(l)} < n - (s+\tilde{s}) \\ y_1 + \cdots + y_l \leq k}} \mathbb{P}(A(v^{(1)} + \cdots + v^{(l)}, k - y_1 - \cdots - y_l)) \prod_{m \leq l} h(v^{(1)} + \cdots + v^{(m-1)}, y_m) \\
&\quad + \sum_{\frac{n-(s+\tilde{s})}{s+\tilde{s}+1} \leq j \leq l} \sum_{\substack{i \leq j, v^{(i)}, y_i \\ v_2^{(1)} + \cdots + v_2^{(j-1)} < n - (s+\tilde{s}) \\ v_2^{(1)} + \cdots + v_2^{(j)} \geq n - (s+\tilde{s}) \\ y_1 + \cdots + y_j \leq k}} \mathbb{P}(A(v^{(1)} + \cdots + v^{(j)}, k - y_1 - \cdots - y_j)) \prod_{m \leq j} h(v^{(1)} + \cdots + v^{(m-1)}, y_m) \\
&\leq \sum_{\substack{y_1, \ldots, y_l \\ y_1 + \cdots + y_l \leq k}} \Big(\#D^\infty \sup_{\substack{v \in \mathbb{Z}^2 \\ v_2 < n - (s+\tilde{s})}} h(v, y_m) \Big)^l + \sum_{\frac{n-(s+\tilde{s})}{s+\tilde{s}+1} \leq j \leq l} \sum_{\substack{y_1, \ldots, y_j \\ y_1 + \cdots + y_j \leq k}} \Big(\#D^\infty \sup_{\substack{v \in \mathbb{Z}^2 \\ v_2 < n - (s+\tilde{s})}} h(v, y_m) \Big)^j.
\end{aligned}$$

We iterate as long as $0 + v_2^{(1)} + \cdots + v_2^{(j)} < n - (s + \tilde{s})$; otherwise, we stop for $0 + v_2^{(1)} + \cdots + v_2^{(j)}$ and land in the second summand. Only $y_1 + \cdots + y_j \leq k$ matters since $A(z, m) = 0$ whenever $m < 0$. Also, observe that $v_2^{(1)} + \cdots + v_2^{(j)} \geq n - (s + \tilde{s})$ can only happen if

$$j \geq \frac{n - (s + \tilde{s})}{s + \tilde{s} + 1},$$

since we "gain" at most $s + \tilde{s} + 1$ to the second component in each $v^{(i)}$.

Let $\alpha > 0$ be large enough such that

$$\phi(\alpha) := \sum_{y \in \{0,1\}} \sup_{\substack{z \in \mathbb{Z}^2 \\ z_2 < n - (s + \tilde{s})}} \#D^\infty \cdot h(z, y) \cdot e^{-\alpha y} \leq \frac{1}{4} + \#D^\infty \cdot e^{-\alpha} \leq \frac{1}{2}.$$

Then,

$$\begin{aligned}
\mathbb{P}(A(o, k)) \leq\ & e^{\alpha k} \sum_{\substack{y_1, \ldots, y_l \\ y_1 + \cdots + y_l \leq k}} \Big(\#D^\infty \sup_{\substack{v \in \mathbb{Z}^2 \\ v_2 < n - (s + \tilde{s})}} h(v, y_m) \Big)^l \prod_{i \leq l} e^{-\alpha y_i} \\
& + e^{\alpha k} \sum_{\frac{n - (s + \tilde{s})}{s + \tilde{s} + 1} \leq j \leq l} \sum_{\substack{y_1, \ldots, y_j \\ y_1 + \cdots + y_j \leq k}} \Big(\#D^\infty \sup_{\substack{v \in \mathbb{Z}^2 \\ v_2 < n - (s + \tilde{s})}} h(v, y_m) \Big)^j \prod_{i \leq j} e^{-\alpha y_i} \\
\leq\ & e^{\alpha k} \phi(\alpha)^l + e^{\alpha k} \sum_{j \geq \frac{n - (s + \tilde{s})}{s + \tilde{s} + 1}}^{l} \phi(\alpha)^j \leq e^{\alpha k} [2^{-l} + 2^{-\frac{n - (s + \tilde{s})}{s + \tilde{s} + 1} + 1}].
\end{aligned}$$

Since $l \in \mathbb{N}$ was arbitrary, we get

$$\mathbb{P}(A(o, k)) \leq e^{\alpha k} \cdot 2^{-\frac{n - (s + \tilde{s})}{s + \tilde{s} + 1} + 1} = e^{\alpha k - \left[\frac{n - (s + \tilde{s})}{s + \tilde{s} + 1} - 1\right] \ln 2} = C_3 e^{\alpha k - C_4 n}.$$

Now we finally make use of k. Setting $C_5 := \frac{C_4}{2\alpha}$ and $k := C_5 n$, the claim follows:

$$\mathbb{P}(A(o, C_5 n)) \leq C_3 \exp\big(- C_4 n / 2 \big).$$

Authors contribution

All results of the publication have resulted from joint discussions. They can therefore not be attributed separately to the authors, but are to be regarded as joint achievements.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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