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Research article

Interplay of a unit-speed constraint and time-delay in the flocking model with internal variables

Hyunjin Ahn[1] **and Woojoo Shim**[2*]

[1] Department of Data Technology, Myongji University, Seoul 03674, Republic of Korea

[2] Department of Mathematics Education, Kyungpook National University, Daegu 41566, Republic of Korea

* **Correspondence:** wjshim@knu.ac.kr; Tel: +82-053-950-5889.

Abstract: We studied the dynamics of thermodynamic Cucker–Smale (TCS) particles moving with a constant speed constraint. The TCS model describes the collective dynamics of the population of birds with a time varying internal variable, and it was first introduced as the generalization of the Cucker–Smale (CS) model. In this paper, we considered a modification of the TCS model in which each agent moves at a constant speed, such as the Vicsek model, and we additionally considered the effect of time-delays due to the finiteness of the information propagation speed between agents. Then, we presented several sufficient conditions in terms of initial data and system parameters to exhibit asymptotic flocking. We presented two kinds of results for this purpose. One was an estimate of the diameter of the velocity and temperature configuration, and the other was an estimate of the diameter of the configuration within the time-delay bound τ.

Keywords: Cucker-Smale; flocking; multi-agent system; thermodynamic; time-delay effect; unit-speed constraint

1. Introduction

Emergent behaviors of interacting multi-agent systems are often observed in our daily life. For example, aggregation of bacteria [34], flocking of migrating birds [17], schooling of moving fish [18, 33], synchronization of fireflies and pacemaker cells [6, 19, 38], etc. For a brief introduction to this subject, we also refer to the articles and books [1, 4, 20, 29, 30, 32, 36, 37]. In this work, we are interested in self-propelled *flocking* dynamics in which all agents move at a common velocity with limited surroundings and simple rules. After the seminal work done by Vicsek [35], many mathematical models describing flocking behavior have been widely studied in the mathematics community. Among them, the Cucker–Smale (CS) model [17] is one of the most successful models that represents flocking which has been studied from various perspectives, to name a few, local interactions [27], kinetic descriptions [7, 24], hydrodynamic descriptions [21, 26], stochastic descriptions [8], time-delay [10, 13, 14], etc.

In [23], Ha and Ruggeri proposed a CS-type flocking model with internal variable, which was called a *temperature variable* in their context. They considered the standard balance laws (mass, momentum, energy conservation) for the finitely many mixtures of spatially homogeneous ideal gases, together with specific constitutive equations suggested in [31] consistent with the entropy principle in thermodynamics. To fix the idea, let x_i, v_i, T_i be the position, velocity, and temperature of the i-th flocking agent, respectively. Then, *the thermodynamic Cucker-Smale model* (TCS in short) is given by the following second-order ordinary differential equations (ODEs) for *position–velocity–temperature* variables $\{(x_i, v_i, T_i)\}_{i=1}^N$:

$$
\begin{cases}
\dfrac{dx_i}{dt} = v_i, \quad t > 0, \quad i \in [N] := \{1, \dots, N\}, & \text{(1.1a)} \\
\dfrac{dv_i}{dt} = \dfrac{\kappa_1}{N} \displaystyle\sum_{j=1}^{N} \phi(\|x_i - x_j\|) \left(\dfrac{v_j}{T_j} - \dfrac{v_i}{T_i} \right), & \text{(1.1b)} \\
\dfrac{d}{dt} \left(T_i + \dfrac{1}{2} \|v_i\|^2 \right) = \dfrac{\kappa_2}{N} \displaystyle\sum_{j=1}^{N} \zeta(\|x_i - x_j\|) \left(\dfrac{1}{T_i} - \dfrac{1}{T_j} \right), & \text{(1.1c)} \\
(x_i(0), v_i(0), T_i(0)) = (x_i^0, v_i^0, T_i^0) \in \mathbb{R}^d \times \mathbb{R}^d \times (0, \infty), & \text{(1.1d)}
\end{cases}
$$

where κ_1 and κ_2 are nonnegative scale parameters of the velocity and temperature communication weights, respectively. As in the CS model, the authors in [23] assumed that $\phi, \zeta : [0, \infty) \to [0, \infty)$ are nonnegative, locally Lipschitz, and monotonically decreasing, so that (1.1) is locally well-posed and the interaction becomes weaker as the distance between the two agents increases. However, it might be questionable whether the fact that

each bird's motion depends on the temperature of the other birds

is natural in terms of a flocking model. Therefore, when we think of the TCS-type models as the flocking models, we would like to suggest another interpretation for the temperature variable T_i. For instance, we may interpret $1/T_i$ as a time-varying measure of how attractive each bird is relative to other birds in the population. Then, each bird decides which direction to accelerate in proportion to the velocities and attractiveness of others, even for birds with the same velocity, and it becomes a matter of finding the initial conditions of all birds' velocities and attractiveness to converge to the same value. However, out of respect for the first authors' expressions, we decided to continue to refer to it as the TCS model throughout the paper.

Meanwhile, recent experiments on starling flocks [5, 9] indicate that the speed fluctuations of birds are very small during their flights, demonstrating the need for us to study constant speed flocking models as [35]. Since the velocities of CS and TCS models converge to the same value under well-prepared initial conditions, the CS model and the TCS model can be said to have similar asymptotic behavior to their constant speed counterparts. Recently, [2] studied a new TCS-type flocking model in which the speed of each particle is constant. The main idea of [2] to create the new model was the same idea that led to the unit speed CS model [12] from the CS model: replace the right-hand side of (1.1b) by its orthogonal projection onto v_i, i.e.,

$$\text{replace} \quad \phi(\|x_i - x_j\|)\left(\frac{v_j}{T_j} - \frac{v_i}{T_i}\right) \quad \text{to} \quad \frac{\phi(\|x_i - x_j\|)}{T_j}\left(v_j - \frac{\langle v_j, v_i\rangle}{\|v_i\|^2} v_i\right).$$

Note that because it is an orthogonal projection to v_i, the v_i term disappears from the original TCS model, and the only interaction that appears to be asymmetric remains. As a result, the TCS model with a unit-speed constraint was given as

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \quad t > 0, \quad i \in [N], \\ \dfrac{dv_i}{dt} = \dfrac{\kappa_1}{N} \displaystyle\sum_{j=1}^{N} \dfrac{\phi(\|x_i - x_j\|)}{T_j}\left(v_j - \dfrac{\langle v_j, v_i\rangle}{\|v_i\|^2} v_i\right), \\ \dfrac{dT_i}{dt} = \dfrac{\kappa_2}{N} \displaystyle\sum_{j=1}^{N} \zeta(\|x_i - x_j\|)\left(\dfrac{1}{T_i} - \dfrac{1}{T_j}\right), \\ (x_i(0), v_i(0), T_i(0)) = (x_i^0, v_i^0, T_i^0) \in \mathbb{R}^d \times \mathbb{S}^{d-1} \times (0, \infty), \end{cases} \tag{1.2}$$

where the assumptions for κ_1, κ_2, ϕ, and ζ are the same as for the system (1.1), and $\mathbb{S}^{d-1}$ is the $(d-1)$-dimensional unit sphere embedded in $\mathbb{R}^d$, i.e.,

$$\mathbb{S}^{d-1} := \left\{ x = (x^1, \dots, x^d) \;\middle|\; \sum_{i=1}^{d} |x^i|^2 = 1 \right\}.$$

In addition to [2] and [12], there have been several mathematical studies on CS-type models with constant speed constraint. For example, [15] provided a sufficient condition to exhibit a bi-cluster flocking of unit speed CS ensemble, and [22] found a critical coupling strength to exhibit an asymptotic flocking for the same model. In particular, [11] conducted a study on a modified unit speed CS model which also considers the constant *time-delay effects*. Such time-delay effects are in fact prevalent in both physical and biological systems due to the fundamental constraint that information transmission cannot occur instantaneously during mutual communication processes [16, 25]. This limitation on the speed of communication propagation has found significant application within the mathematical biology literature.

In this paper, we study the unit speed TCS model with nonconstant time-delays. More precisely, we define $\tau_{ji}(t)$ as the time-delay for the i-th agent to detect a signal from the j-th agent at time $t \ge 0$, where we assume $\tau_{ii}(t) = 0$ for all $i \in [N]$ to avoid a self-processing time-delay. Furthermore, we assume $\tau_{ji}(t) : [0, \infty) \to [0, \infty)$ nonnegative, continuous, and uniformly bounded by a constant τ for each pair $(i, j) \in [N]^2$, i.e.,

$$\tau_{ji}(\cdot) \in C([0, \infty); [0, \infty)), \quad \sup_{t \ge 0} \max_{i,j \in [N]} \tau_{ji}(t) \le \tau, \quad \tau_{ii}(\cdot) \equiv 0, \quad i, j \in [N].$$

Taking account these time-delay effects to Eq (1.2), we propose the following unit speed TCS model with time-delay (in short TCSUT):

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \quad t > 0, \quad i \in [N], \\ \dfrac{dv_i}{dt} = \dfrac{\kappa_1}{N} \displaystyle\sum_{j=1}^{N} \dfrac{\phi(\|x_i - x_j(t - \tau_{ji}(t))\|)}{T_j(t - \tau_{ji}(t))} \left(v_j(t - \tau_{ji}(t)) - \dfrac{\langle v_j(t - \tau_{ji}(t)), v_i \rangle v_i}{\|v_i\|^2} \right), \\ \dfrac{dT_i}{dt} = \dfrac{\kappa_2}{N} \displaystyle\sum_{j=1}^{N} \zeta(\|x_i - x_j(t - \tau_{ji}(t))\|) \left(\dfrac{1}{T_i} - \dfrac{1}{T_j(t - \tau_{ji}(t))} \right), \\ (x_i(t), v_i(t), T_i(t)) = (x_i^0(t), v_i^0(t), T_i^0(t)) \in \mathbb{R}^d \times \mathbb{S}^{d-1} \times (0, \infty), \quad t \in [-\tau, 0]. \end{cases} \tag{1.3}$$

For this flocking model, our main concern in this paper is to find sufficient conditions on the communication weights and initial data that guarantee the asymptotic flocking to occur. To do this, we first define the asymptotic flocking phenomenon rigorously.

Definition 1.1. (Asymptotic flocking) *Let $Z = \{(x_i, v_i, T_i)\}_{i=1}^N$ be a solution to the system* (1.3)*. The configuration Z exhibits asymptotic flocking if*

$$\sup_{t \geq 0} \max_{i,j \in [N]} \|x_i(t) - x_j(t)\| < \infty,$$
$$\lim_{t \to \infty} \max_{i,j \in [N]} \|v_j(t) - v_i(t)\| = 0,$$
$$\lim_{t \to \infty} \max_{i,j \in [N]} |T_j(t) - T_i(t)| = 0.$$

Note that the nonconstant time-delay in (1.3) means that each i-th bird receives information about the other birds at different points in time to determine its velocity and temperature. Therefore, if we want to analyze the diameters

$$D_X(t) := \max_{i,j \in [N]} \|x_i(t) - x_j(t)\|, \quad D_V(t) := \max_{i,j \in [N]} \|v_i(t) - v_j(t)\|, \quad D_T(t) := \max_{i,j \in [N]} |T_i(t) - T_j(t)| \tag{1.4}$$

to show the asymptotic flocking in Definition 1.1, there are some additional technical difficulties compared to the constant time-delay models. Since it may not be enough to simply use the diameters in (1.4) to fully control the right-hand side of (1.3), we need to measure the upper bound of the error that each velocity and temperature can have due to the time-delay effect. For simplicity, we set

$$X := (x_1, \ldots, x_N), \quad V := (v_1, \ldots, v_N), \quad T := (T_1, \ldots, T_N),$$

and for every $Z \in \{X, V, T\}$, we define the *delayed diameter* D_Z^τ and *perturbation* Δ_Z^τ as

$$D_Z^\tau(t) := \max_{s \in [t-\tau, t]} \max_{i,j \in [N]} \|z_i(s) - z_j(t)\|, \quad \Delta_Z^\tau(t) := \max_{s \in [t-\tau, t]} \max_{i \in [N]} \|z_i(s) - z_i(t)\|, \tag{1.5}$$

which satisfy

$$D_Z(t) \leq D_Z^\tau(t), \quad t > 0, \quad Z \in \{X, V, T\}.$$

Then, the right-hand side of (1.3) can be controlled by using Eq (1.5), since

$$\|v_i(t) - v_j(t - \tau_{ji}(t))\| \leq D_V^\tau(t) \leq D_V(t) + \Delta_V^\tau(t), \quad t > 0, \quad i, j \in [N],$$
$$|T_i(t) - T_j(t - \tau_{ji}(t))| \leq D_T^\tau(t) \leq D_T(t) + \Delta_T^\tau(t), \quad t > 0, \quad i, j \in [N],$$

and the main result of this paper is to find sufficient conditions for D_X to be uniformly bounded and D_V, D_T to converge to zero.

To this end, we will consider two different approaches: constructing a differential inequality for diameters (D_X, D_V, D_T) and constructing a differential inequality for delayed diameters $(D_X^\tau, D_V^\tau, D_T^\tau)$. The analysis of diameters is similar to the method used in [3] and other related works, but the analysis of delayed diameter is, to the best of the authors' knowledge, the *first attempt* on studies of flocking models with time-delays.

The rest of this paper is organized as follows. In Section 2, we provide some basic properties and previous flocking estimate for the unit speed TCS model (1.2) without time-delay. In Section 3, we present several preparatory lemmas which are crucially used to guarantee the global well-posedness of the system (1.3) and also used to construct differential inequalities for the delayed diameters in Section 4. In Section 4, we derive a system of differential inequalities for the delayed-diameters $(D_X^\tau, D_V^\tau, D_T^\tau)$ and present the first main result. In Section 5, we present a system of differential inequalities for the diameters (D_X, D_V, D_T) and the second main result. We also present several numerical experiments in Section 6 to demonstrate our theoretical results. Finally, Section 7 is devoted to summarizing the main results of this paper and some discussion on the remaining issues to be investigated in a future work. In Appendix A and B, we provide the detailed proof of Lemma 3.1 and Lemma 4.1, respectively.

Notation: Throughout this paper, we employ the following notation for simplicity.

$$
\begin{aligned}
&\|\cdot\| := \text{standard } l_2\text{-norm}, \quad \langle\cdot,\cdot\rangle = \text{standard inner product}, \quad [N] := \{1,\dots,N\},\\
&z_j^{\tau_{ji}}(t) := z_j(t-\tau_{ji}(t)) \quad \text{for} \quad Z = (z_1,\dots,z_N) \in \{X, V, T\}, \quad t \geq 0,\\
&\psi_{ji}^{\tau_{ji}}(t) := \psi(\|x_j^{\tau_{ji}}(t) - x_i(t)\|) \quad \text{for} \quad \psi \in \{\phi, \zeta\}, \quad t \geq 0,\\
&T_m = \min_{i\in[N]} T_i, \quad T_M = \max_{i\in[N]} T_i, \quad T_m^\tau(t) = \min_{s\in[t-\tau,t]} T_m(s), \quad T_M^\tau(t) = \max_{s\in[t-\tau,t]} T_M(s).
\end{aligned}
$$

2. Preliminaries

In this section, we briefly review several basic properties and previous results of the TCS model with a unit speed constraint obtained in [2]. Readers with sufficient background knowledge in this subject may skip this section.

2.1. The unit speed TCS model

In this subsection, we introduce the unit speed TCS model in the absence of time-delay. Recall that the unit speed TCS model was given as

$$
\begin{cases}
\dfrac{dx_i}{dt} = v_i, \quad t > 0, \quad i \in [N], \\
\dfrac{dv_i}{dt} = \dfrac{\kappa_1}{N} \displaystyle\sum_{j=1}^{N} \dfrac{\phi(\|x_i - x_j\|)}{T_j} \left(v_j - \dfrac{\langle v_j, v_i \rangle}{\|v_i\|^2} v_i \right), \\
\dfrac{dT_i}{dt} = \dfrac{\kappa_2}{N} \displaystyle\sum_{j=1}^{N} \zeta(\|x_i - x_j\|) \left(\dfrac{1}{T_i} - \dfrac{1}{T_j} \right), \\
(x_i(0), v_i(0), T_i(0)) = (x_i^0, v_i^0, T_i^0) \in \mathbb{R}^d \times \mathbb{S}^{d-1} \times (0, \infty).
\end{cases} \tag{2.1}
$$

Even if we consider T_i as an internal variable other than temperature, the 1st and 2nd laws of thermodynamics still hold for the original TCS model, since it is derived from the standard balance laws for spatially homogeneous gas mixtures. Similarly, in the model with the constant speed condition added, the law of energy conservation and entropy principle hold for T_i's, which are as follows.

Proposition 2.1. *Let* (X, V, T) *be a solution to the system* (2.1). *Then, the following assertions hold:*

1. *(Energy conservation) The total internal energy* $\sum_{i=1}^N T_i$ *is conserved.*
2. *(Entropy principle) The total entropy* $\mathcal{S} := \sum_{i=1}^N \ln T_i$ *is monotonically increasing.*

However, these laws can be considered only as long as each T_i exists as a positive real number, since the solution of (2.1) blows up if any T_i approaches zero. Therefore, in order to see how the solution of the equation behaves for $t \to \infty$, it must be ensured that all T_i's cannot be smaller than some positive real number, which can be seen in the following proposition.

Proposition 2.2. *Let* $\{(x_i^0, v_i^0, T_i^0)\}_{i=1}^N \subset \mathbb{R}^d \times \mathbb{S}^{d-1} \times (0, \infty)$ *be an arbitrary initial configuration. Then, the equation* (2.1) *admits a unique global solution on* $t > 0$ *and*

$$
0 < T_m(0) \le T_m(t) \le T_i(t) \le T_M(t) \le T_M(0), \quad \forall\, i \in [N], \quad t \ge 0.
$$

What Proposition 2.2 means is that we can always be guaranteed that the extreme values of the temperatures will approach each other, even though they may not actually converge to the same value. Therefore, a natural question to ask is whether we can expect a similar behavior for velocity vectors, and the answer to this question can be found in the following proposition.

Proposition 2.3. *Let* (X, V, T) *be a solution to* (2.1) *subject to initial data* $\{(x_i^0, v_i^0, T_i^0)\}_{i=1}^N$ *satisfying*

$$
\min_{i,j \in [N]} \langle v_i^0, v_j^0 \rangle > 0.
$$

Then, the function

$$
A(t) := \min_{i,j \in [N]} \langle v_i(t), v_j(t) \rangle = 1 - \frac{1}{2} D_V(t)^2
$$

is monotonically increasing in t.

2.2. Previous results

In this subsection, we review previous flocking estimates for (2.1) obtained from [2] and discuss whether they can be improved. We begin with a system of differential inequalities in terms of D_X, D_V and D_T.

Proposition 2.4. (Differential inequalities for diameters) *Let* (X, V, T) *be a solution to the system* (2.1) *subject to initial data* $\{(x_i^0, v_i^0, T_i^0)\}_{i=1}^N$ *satisfying* $A(0) > 0$. *Then, we have*

$$\left|\frac{dD_X}{dt}\right| \le D_V, \quad \frac{dD_V}{dt} \le -\frac{\kappa_1 A(0)\phi(D_X)}{T_M(0)} D_V, \quad \frac{dD_T}{dt} \le -\frac{\kappa_2 \zeta(D_X)}{T_M(0)^2} D_T.$$

In particular, the function

$$\mathcal{L}_\pm(D_X, D_V) := D_V \pm \frac{\kappa_1 A(0)}{T_M(0)} \Phi(D_X)$$

is monotonically decreasing with respect to t, where

$$\Phi(a) := \int_0^a \phi(s)ds, \quad \forall\, a \ge 0.$$

Since $\mathcal{L}(D_X, D_V)$ is monotonically decreasing, $\Phi(D_X)$ has a uniform in time upper bound determined by the initial data. Therefore, if this upper bound is smaller than $\Phi(\infty) \in [0, \infty]$, we know that the diameter D_X is uniformly bounded in time. Then, Proposition 2.4 determines the exponential decay rate of the diameters D_V and D_T.

Proposition 2.5. (Asymptotic flocking) *Let* (X, V, T) *be a solution to the system* (2.1) *subject to initial data* $\{(x_i^0, v_i^0, T_i^0)\}_{i=1}^N$ *satisfying*

$$0 \le \frac{T_M(0)}{\kappa_1 \int_{D_X(0)}^\infty \phi(s)ds} < \frac{A(0)}{D_V(0)} = \frac{1 - \frac{1}{2}D_V(0)^2}{D_V(0)}. \tag{2.2}$$

Then, we have

$$D_X(t) \le D_X^\infty,\ D_V(t) \le D_V(0)\exp\left(-\frac{\kappa_1 A(0)\phi(D_X^\infty)}{T_M(0)} t\right),\ D_T(t) \le D_T(0)\exp\left(-\frac{\kappa_2 \zeta(D_X^\infty)}{T_M(0)^2} t\right),$$

where D_X^∞ *is the unique positive number satisfying*

$$\frac{T_M(0)}{\kappa_1 \int_{D_X(0)}^{D_X^\infty} \phi(s)ds} = \frac{1 - \frac{1}{2}D_V(0)^2}{D_V(0)}.$$

Remark 2.1. *We make a few remarks about the optimality of Proposition 2.5.*

(1) In [2], the author provided two sufficient frameworks for the asymptotic flocking of (2.1). *One is the result stated in Proposition 2.5, which uses the Lyapunov functional* $\mathcal{L}_+$, *and there was another result using 'the bootstrapping argument' according to their context. However, with more precise calculations we can show that the result is weaker under stronger conditions than Proposition 2.5.*

(2) In fact, according to the proof of Proposition 2.4, one can obtain

$$\frac{dD_V(t)}{dt} \le -\frac{\kappa_1 A(t)\phi(D_X(t))}{T_M(0)} D_V(t) = -\frac{\kappa_1\phi(D_X(t))}{T_M(0)}\left(1-\frac{1}{2}D_V^2(t)\right)D_V(t).$$

Then, the function

$$\mathcal{L}(D_X, D_V) := \frac{1}{\sqrt{2}}\ln\left(\frac{\sqrt{2}+D_V}{\sqrt{2}-D_V}\right) + \frac{\kappa_1\Phi(D_X)}{T_M(0)}$$

is monotonically decreasing with respect to t, and we have

$$\frac{D_V(t)^2}{2-D_V(t)^2} \le \frac{D_V(0)^2}{2-D_V(0)^2}\exp\left(-\frac{2\kappa_1\phi(D_X^\infty)}{T_M(0)}t\right),$$

whenever the initial data satisfies

$$0 \le \int_{D_X(0)}^{\infty}\phi(s)ds > \frac{T_M(0)}{\sqrt{2}\kappa_1}\ln\left(\frac{\sqrt{2}+D_V(0)}{\sqrt{2}-D_V(0)}\right).$$

In Section 3, we will show that the generalizations of Proposition 2.2 and Proposition 2.3 hold in the time-delayed model (1.3). These will be crucially used to prove the well-posedness of (1.3) and to find some flocking estimates corresponding to Proposition 2.5.

3. Basic properties of the time-delayed model

In this section, we present the global well-posedness and some basic properties of the system (1.3). To fix the idea, consider a following time-delayed ODE:

$$\frac{dx(t)}{dt} = f(x(t), x(t-\tau(t))), \quad \tau(\cdot)\in C(\mathbb{R};[0,\tau]).$$

Then, $x(t-\tau(t))$ does not play a crucial role to the well-posedness of the solution. Instead, it is more appropriate to view it as an independently given function that does not depend on the value of $x(t)$. Therefore, the local well-posedness of the ODE can be obtained when $f(y,z)$ is Lipschitz in y, continuous in z, and $t\mapsto x(t-\tau(t))$ is continuous. Because of the above reasons, it is clear that the global well-posedness of the system (1.3) can be obtained by the invariance of $\|v_i\|$ and the boundedness of $1/T_i$ for each $i\in[N]$, and the invariance of each $\|v_i\|$ is given by the relation

$$\frac{1}{2}\frac{d\|v_i\|^2}{dt} = \frac{\kappa_1}{N}\sum_{j=1}^N \frac{\phi_{ji}^{\tau_{ji}}}{T_j^{\tau_{ji}}}\left\langle v_j^{\tau_{ji}} - \frac{\langle v_j^{\tau_{ji}}, v_i\rangle v_i}{\|v_i\|^2}, v_i\right\rangle = 0, \quad \forall\, i\in[N].$$

Then, due to the uniqueness of the solution and the unit-speed constraint for initial data, we can rewrite the system (1.3) as the following simplified form:

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \quad t>0, \quad i\in[N], \\ \dfrac{dv_i}{dt} = \dfrac{\kappa_1}{N}\displaystyle\sum_{j=1}^N \frac{\phi_{ji}^{\tau_{ji}}}{T_j^{\tau_{ji}}}\left(v_j^{\tau_{ji}} - \langle v_j^{\tau_{ji}}, v_i\rangle v_i\right), \\ \dfrac{dT_i}{dt} = \dfrac{\kappa_2}{N}\displaystyle\sum_{j=1}^N \zeta_{ji}^{\tau_{ji}}\left(\frac{1}{T_i} - \frac{1}{T_j^{\tau_{ji}}}\right), \\ (x_i(t), v_i(t), T_i(t)) = (x_i^0(t), v_i^0(t), T_i^0(t)) \in \mathbb{R}^d\times\mathbb{S}^{d-1}\times(0,\infty), \quad \forall\, t\in[-\tau,0]. \end{cases} \tag{3.1}$$

In [10], the authors proved the monotonicity of the maximum and minimum temperatures for the *TCS model with time-delay*, which does not have the invariance of speeds as Eq (3.1). Although not rigorously proved in previous works, their basic idea was to use the following lemma to the minimum and maximum temperatures. We here present a rigorous proof for the completeness of this paper and for the use of this lemma in future work.

Lemma 3.1. *Let $f \in C(\mathbb{R};\mathbb{R})$, and define $F = \mathcal{T}[f] : \mathbb{R} \to \mathbb{R}$ as*

$$\mathcal{T}[f](t) := \min_{s\in[t-\tau,t]} f(s), \quad \forall t \in \mathbb{R}.$$

Then, the following assertions hold:

1. *If $f(t_0) > F(t_0)$ for some $t_0 \in \mathbb{R}$, there exists $\delta > 0$ such that*

$$F(t_0) \le F(t), \quad \forall t \in [t_0, t_0 + \delta].$$

2. *F is monotonically increasing if $D_+ f(t_*) \ge 0$ for all $t_* \in \mathbb{R}$ satisfying $F(t_*) = f(t_*)$.*
3. *If $f \in \mathrm{Lip}_L(\mathbb{R},\mathbb{R})$ for some $L > 0$, then $F \in \mathrm{Lip}_L(\mathbb{R},\mathbb{R})$.*

Proof. Since the proof is lengthy and technical, we leave it to Appendix A. □

Remark 3.1. *To preempt some possible misconceptions about the relation between f and $\mathcal{T}[f]$, we present the following examples.*

1. *Convergence of $F = \mathcal{T}[f]$ does not imply the convergence of f. For example, if $f(t) = \cos\left(\frac{2\pi t}{\tau}\right)$, the function $F = \mathcal{T}[f]$ is a constant function $F \equiv -1$. In fact, if F is monotonically increasing, we have*
$$\lim_{t\to\infty} F(t) = \lim_{t\to\infty} \min_{t\le s} F(s) = \lim_{t\to\infty} \inf_{t-\tau\le s} f(s) = \liminf_{t\to\infty} f(t).$$
2. *There may not be a 'first point' that does not satisfy the condition in Lemma 3.1 (2). For example, if $f(t) = -\max\{0,t\}^2$, the function f is monotonically decreasing. Then, $\mathcal{T}[f] \equiv f$ and $f'(t) = -2\max\{0,t\} \ge 0$ for only $t \le 0$, and therefore*

$$F(t) = f(t) \quad \text{for all} \quad t \in \mathbb{R}, \quad \text{but} \quad D_+ f(t) \ge 0 \quad \text{for only} \quad t \le 0.$$

Using Lemma 3.1, one can prove monotonic increase in minimum temperature and monotonic decrease in maximum temperature, similar to the original models without delay [2,23]. More precisely, we can prove that T_m^τ is monotonically increasing and T_M^τ is monotonically decreasing when the initial values of both functions are positive real numbers.

Lemma 3.2. (Monotonicity of T_m^τ and T_M^τ) *Let (X, V, T) be a solution to Eq (3.1), where the initial data*

$$(x_i^0, v_i^0, T_i^0) : [-\tau, 0] \to \mathbb{R}^d \times \mathbb{S}^{d-1} \times (0,\infty)$$

is continuous for every $i \in [N]$. Then, T_m^τ and $-T_M^\tau$ are monotonically increasing, i.e.,

$$0 < T_m^\tau(0) \le T_m^\tau(t) \le T_i(t) \le T_M^\tau(t) \le T_M^\tau(0), \quad \forall\, t \ge 0, \quad i \in [N].$$

Proof. Let us denote by

$$f(t) := \min_{j\in[N]} T_j(t), \quad F := \mathcal{T}[f].$$

Then, the constraint for the initial data can be written as $F(0) > 0$, and we want to prove $F(t) \geq F(0)$ for every $t \geq 0$. If $\varepsilon > 0$ and t_0 is the first time satisfying $F(t) = F(0) - \varepsilon > 0$, then whenever $F(t_*) = f(t_*)$ in $t_* \in [0, t_0]$, we can consider the maximal index set $\mathcal{I} \subset [N]$ such that

$$F(t_*) = f(t_*) = T_j(t_*), \quad \forall\, j \in \mathcal{I}.$$

Then, since each $T_k(t)$ is differentiable in t, we have

$$T_k(t) = T_k(t_*) + \frac{dT_k}{dt}(t_*)(t - t_*) + o(|t - t_*|), \quad \forall\, k \in [N],$$

and by taking the minimum in k, one can obtain

$$f(t) = f(t_*) + \min_{j\in\mathcal{I}} \frac{dT_j}{dt}(t_*)(t - t_*) + o(|t - t_*|), \quad \forall\, 0 < t - t_* \ll 1.$$

Therefore, the Dini derivative $D_+ f$ at $t = t_*$ satisfies

$$\begin{aligned}
D_+ f(t_*) &= \min_{j\in\mathcal{I}} \frac{dT_j}{dt}(t_*) \\
&= \min_{j\in\mathcal{I}} \frac{\kappa_2}{N} \sum_{k=1}^{N} \zeta_{kj}^{\tau_{kj}} \left(\frac{1}{T_j(t_*)} - \frac{1}{T_k^{\tau_{kj}}(t_*)} \right) \\
&= \min_{j\in\mathcal{I}} \frac{\kappa_2}{N} \sum_{k=1}^{N} \zeta_{kj}^{\tau_{kj}} \left(\frac{1}{F(t_*)} - \frac{1}{T_k^{\tau_{kj}}(t_*)} \right) \\
&\geq 0,
\end{aligned}$$

where we used $T_k^{\tau_{kj}}(t_*) \geq F(t_*) > F(0) - \varepsilon > 0$ in the last inequality. By using Lemma 3.1, the function F is therefore monotonically increasing in $[0, t_0]$, which contradicts to $F(t_0) = F(0) - \varepsilon < F(0)$. Therefore, we have

$$0 < T_m^\tau(0) = F(0) \leq F(t) = T_m^\tau(t) \leq T_i(t), \quad \forall\, t \geq 0, \quad i \in [N].$$

In addition, by letting each $s \geq 0$ be a new starting time, this also shows that F is monotonically increasing in $[0, \infty)$:

$$0 < T_m^\tau(s) = F(s) \leq F(t) = T_m^\tau(t) \leq T_i(t), \quad \forall\, t \geq s \geq 0, \quad i \in [N].$$

We can also obtain the monotonic decreasing property of T_M^τ by choosing $f = -T_M$ and $F = \mathcal{T}[f] = -T_M^\tau$. □

What we want to emphasize in this proof is that we assumed the first point t_0 where the value of F is less than or equal to $F(0) - \varepsilon$ and then applied Lemma 3.1 on the interval $[0, t_0]$ to prove its contradiction. This is because, as we pointed out in Remark 3.1. (2), there may not be the first point at which the condition of Lemma 3.1(2) does not hold.

Then, by applying the classical Cauchy-Lipschitz theory, the global existence of the solution to (3.1) can be guaranteed from the invariance of the speed and the existence of positive lower bound for temperatures. However, the use of Lemma 3.1 goes beyond simply showing the monotonicity of extreme temperatures. To be specific, every open hemisphere $U \subset \mathbb{S}^{d-1}$ contains all $v_1(t), \ldots, v_N(t)$ uniformly in $t \geq 0$, whenever U contains all initial velocities.

Lemma 3.3. *Let* (X, V, T) *be a solution to Eq* (3.1)*, where the initial data*

$$(x_i^0, v_i^0, T_i^0) : [-\tau, 0] \to \mathbb{R}^d \times \mathbb{S}^{d-1} \times (0, \infty)$$

is continuous for every $i \in [N]$*. If initial data satisfies*

$$\min_{s \in [-\tau, 0]} \min_{j \in [N]} \langle u, v_j(s) \rangle > 0 \tag{3.2}$$

for some $u \in \mathbb{S}^{d-1}$*, the function*

$$t \mapsto \min_{s \in [t-\tau, t]} \min_{j \in [N]} \langle u, v_j(s) \rangle$$

is monotonically increasing.

Proof. We can employ a similar argument to Lemma 3.2. In this case, we set

$$f(t) := \min_{j \in [N]} \langle u, v_j(t) \rangle, \quad F := \mathcal{T}[f].$$

If $\varepsilon > 0$ and t_0 is the first time satisfying $F(t) = F(0) - \varepsilon > 0$, then whenever $F(t_*) = f(t_*)$ in $t_* \in [0, t_0]$, we can consider the maximal index set $\mathcal{I} \subset [N]$ such that

$$F(t_*) = f(t_*) = \langle u, v_j(t_*) \rangle, \quad \forall\ j \in \mathcal{I}.$$

Then, the Dini derivative $D_+ f$ at $t = t_*$ satisfies

$$\begin{aligned}
D_+ f(t_*) &= \min_{j \in \mathcal{I}} \left\langle \frac{dv_j}{dt}(t_*), u \right\rangle \\
&= \min_{j \in \mathcal{I}} \frac{\kappa_1}{N} \sum_{k=1}^N \frac{\phi_{kj}^{\tau_{kj}}}{T_k^{\tau_{kj}}} \left\langle v_k^{\tau_{kj}}(t_*) - \langle v_k^{\tau_{kj}}(t_*), v_j(t_*) \rangle v_j(t_*), u \right\rangle \\
&\geq \min_{j \in \mathcal{I}} \frac{\kappa_1}{N} \sum_{k=1}^N \frac{\phi_{kj}^{\tau_{kj}}}{T_k^{\tau_{kj}}} F(t_*)(1 - \langle v_k^{\tau_{kj}}(t_*), v_j(t_*) \rangle) \\
&\geq 0,
\end{aligned}$$

where we used $\langle u, v_k^{\tau_{kj}}(t_*) \rangle \geq F(t_*) \geq 0$ in the second inequality. By using Lemma 3.1, the function F is monotonically increasing in $[0, t_0]$, which contradicts to $F(t_0) = F(0) - \varepsilon < F(0)$. Therefore, we have $F(t) \geq F(0)$ for all $t \geq 0$, and by letting each $s \geq 0$ be a new starting time, this also shows that F is monotonically increasing in $[0, \infty)$. □

What should be noted here is that Lemma 3.3 alone cannot prove the desired flocking phenomenon, since the unit vector $u \in \mathbb{S}^{d-1}$ satisfying (3.2) is not unique if it exists. Instead, it is better to use an indicator that can show that individual velocities are getting closer to each other. The most intuitive way to achieve this is to assume that all initial velocities can be used as the vector 'u' in Eq (3.2), so that all velocities approach each other. Under such initial conditions, the lemma below implies that the inner product $\langle v_i(t_1), v_j(t_2)\rangle$ is strictly positive for all $i, j \in [N]$ and $t_1, t_2 \geq -\tau$.

Lemma 3.4. *Let* (X, V, T) *be a solution to* (3.1)*, where the initial data*

$$(x_i^0, v_i^0, T_i^0) : [-\tau, 0] \to \mathbb{R}^d \times \mathbb{S}^{d-1} \times (0, \infty)$$

is continuous for every $i \in [N]$*. If the initial data satisfies*

$$\min_{\substack{s_1 \in [-\tau,0] \\ s_2 \in [-\tau,0]}} \min_{i,j \in [N]} \langle v_i(s_1), v_j(s_2)\rangle > 0, \tag{3.3}$$

then the function

$$A^{\tau,\tau}(t_1, t_2) := \min_{\substack{s_1 \in [t_1-\tau,t_1] \\ s_2 \in [t_2-\tau,t_2]}} \min_{i,j \in [N]} \langle v_i(s_1), v_j(s_2)\rangle$$

is monotonically increasing in both t_1 *and* t_2*.*

Proof. Since $A^{\tau,\tau}(t_1, t_2) = A^{\tau,\tau}(t_2, t_1)$ for all $t_1, t_2 \geq 0$, we only need to show the monotonic increasing property in one variable, which is sufficient to verify

$$A^{\tau,\tau}(0, t_2) \geq A^{\tau,\tau}(0, 0), \quad \forall\, t_2 \geq 0,$$

by taking each positive number as a new starting time as in Lemma 3.2 and Lemma 3.3. For every $s_1 \in [-\tau, 0]$ and $i \in [N]$, we apply Lemma 3.3 to $u = v_i(s_1)$ and obtain

$$t \mapsto \min_{s_2 \in [t-\tau,t]} \min_{j \in [N]} \langle v_i(s_1), v_j(s_2)\rangle$$

is monotonically increasing. Therefore, we have

$$\begin{aligned} A^{\tau,\tau}(0, t_2) &= \min_{s_1 \in [-\tau,0]} \min_{i \in [N]} \min_{s_2 \in [t_2-\tau,t_2]} \min_{j \in [N]} \langle v_i(s_1), v_j(s_2)\rangle \\ &\geq \min_{s_1 \in [-\tau,0]} \min_{i \in [N]} \min_{s_2 \in [-\tau,0]} \min_{j \in [N]} \langle v_i(s_1), v_j(s_2)\rangle \\ &= A^{\tau,\tau}(0, 0), \end{aligned}$$

which is the desired result. □

Before we close this section, we provide several estimates which will play crucial roles in Section 4. Recall that we defined the diameter and perturbation functions affected by the time-delay as follows: for $Z = (z_1, \ldots, z_N) \in \{X, V, T\}$,

$$D_Z^\tau(t) := \max_{s \in [t-\tau,t]} \max_{i,j \in [N]} \|z_i(s) - z_j(t)\|, \quad \Delta_Z^\tau(t) := \max_{s \in [t-\tau,t]} \max_{i \in [N]} \|z_i(s) - z_i(t)\|.$$

Consequently, one can easily obtain $\Delta_Z^\tau \leq D_Z^\tau$ by definition, and Lemma 3.4 yields

$$D_V^\tau(t)^2 = 2 - 2 \min_{s \in [t-\tau,t]} \min_{i,j \in [N]} \langle v_i(s), v_j(t)\rangle \leq 2 - 2A^{\tau,\tau}(t, t) \leq 2 - 2A^{\tau,\tau}(0, 0),$$

provided that Eq (3.3) holds, i.e., $A^{\tau,\tau}(0, 0) > 0$. In addition, we can estimate perturbation functions $\Delta_V^\tau(t)$ and $\Delta_T^\tau(t)$ by using the integration of $\|\dot{z}_i\|$ in $[t - \tau, t]$.

Lemma 3.5. *Let* (X, V, T) *be a solution to* (3.1)*, where the initial data*

$$(x_i^0, v_i^0, T_i^0) : [-\tau, 0] \to \mathbb{R}^d \times \mathbb{S}^{d-1} \times (0, \infty)$$

is Lipschitz continuous for every $i \in [N]$*. Then, for every* $t \geq 0$*, we have*

$$\Delta_V^\tau(t) \leq \max\{\tau - t, 0\} \max_{i\in[N]} \|v_i^0\|_{\mathrm{Lip}} + \frac{(N-1)\kappa_1\phi(0)}{NT_m^\tau(0)} \int_{\max\{t-\tau,0\}}^t D_V^\tau(s)ds,$$

$$\Delta_T^\tau(t) \leq \max\{\tau - t, 0\} \max_{i\in[N]} \|T_i^0\|_{\mathrm{Lip}} + \frac{(N-1)\kappa_2\zeta(0)}{NT_m^\tau(0)^2} \int_{\max\{t-\tau,0\}}^t D_T^\tau(s)ds.$$

Proof. For given $Z \in \{V, T\}$ and $t \geq 0$, suppose we have

$$\Delta_Z^\tau(t) = \|z_i(s) - z_i(t)\|, \quad s \in [t-\tau, t].$$

Then, by using the triangle inequality, one has

$$\begin{aligned}
\|z_i(s) - z_i(t)\| &\leq \int_s^{\max\{s,0\}} \|\dot{z}_i(u)\|\, du + \int_{\max\{s,0\}}^t \|\dot{z}_i(u)\|\, du \\
&\leq \max\{\tau - t, 0\}\|z_i^0\|_{\mathrm{Lip}} + \int_{\max\{t-\tau,0\}}^t \|\dot{z}_i(u)\|\, du.
\end{aligned}$$

Below, we estimate the upper bound of $\|\dot{z}_i(u)\|$ in $u \geq 0$ one by one. For the velocity derivative, we have

$$\begin{aligned}
\left\|\frac{dv_i(t)}{dt}\right\| &= \left\|\frac{\kappa_1}{N}\sum_{j=1}^N \frac{\phi_{ji}^{\tau_{ji}}}{T_j^{\tau_{ji}}}\left(v_j^{\tau_{ji}} - \langle v_j^{\tau_{ji}}, v_i\rangle v_i\right)\right\| \\
&\leq \frac{\kappa_1\phi(0)}{NT_m^\tau(0)} \sum_{\substack{j\in[N]\\ j\neq i}} \left\|v_j^{\tau_{ji}} - \langle v_j^{\tau_{ji}}, v_i\rangle v_i\right\| \\
&= \frac{\kappa_1\phi(0)}{NT_m^\tau(0)} \sum_{\substack{j\in[N]\\ j\neq i}} \sqrt{1 - \langle v_j^{\tau_{ji}}, v_i\rangle^2} \\
&\leq \frac{\kappa_1\phi(0)}{NT_m^\tau(0)} \sum_{\substack{j\in[N]\\ j\neq i}} \sqrt{2 - 2\langle v_j^{\tau_{ji}}, v_i\rangle} \\
&\leq \frac{(N-1)\kappa_1\phi(0)}{NT_m^\tau(0)} D_V^\tau(t), \quad \forall\, t \geq 0,
\end{aligned}$$

where we used

$$\begin{aligned}
&\phi(\|x_i(t) - x_j(t - \tau_{ji}(t))\|) \leq \phi(0), \quad T_j(t - \tau_{ji}(t)) \geq T_m^\tau(0), \\
&2 - 2\langle v_j(t - \tau_{ji}(t)), v_i(t)\rangle = \|v_j(t - \tau_{ji}(t)) - v_i(t)\|^2 \leq D_V^\tau(t)^2,
\end{aligned}$$

in the first and the last inequality, respectively. Similarly, for the temperature derivative, we apply Lemma 3.2 to Eq (3.1) to obtain

$$\begin{aligned}
\left|\frac{dT_i(t)}{dt}\right| &= \frac{\kappa_2}{N} \sum_{\substack{j\in[N]\\ j\neq i}} \zeta(\|x_i(t) - x_j(t - \tau_{ji}(t))\|) \left|\frac{T_i(t) - T_j(t - \tau_{ji}(t))}{T_i(t)T_j(t - \tau_{ji}(t))}\right| \\
&\leq \frac{(N-1)\kappa_2\zeta(0)}{NT_m^\tau(0)^2} D_T^\tau(t), \quad \forall\, t \geq 0,
\end{aligned}$$

where we used

$$\zeta(\|x_i(t) - x_j(t - \tau_{ji}(t))\|) \le \zeta(0), \quad T_i(t)T_j(t - \tau_{ji}(t)) \ge T_m^\tau(0)^2,$$
$$|T_i(t) - T_j(t - \tau_{ji}(t))| \le D_T^\tau(t), \quad \forall\, t \ge 0,$$

in the last inequality. □

Remark 3.2. *Since $\dot{z}_i^0(s)$ is not given in the differential equation* (3.1)*, we need to use the Lipschitz constant of the initial data $\{z_i^0\}_{i=1}^N$ to evaluate $\Delta_Z^\tau(t)$ in $t \in [0, \tau)$. In addition, the integration of $\|\dot{v}_i(u)\|$ in $u \in [s, t]$ is greater than or equal to the 'geodesic distance' between $v_i(s)$ and $v_i(t)$, since velocities are moving on the unit sphere $\mathbb{S}^{d-1}$. Therefore, it is possible to modify Lemma 3.5 to*

$$\cos^{-1}\left(1 - \frac{1}{2}\Delta_V^\tau(t)^2\right) \le \frac{(N-1)\kappa_1\phi(0)}{NT_m^\tau(0)} \int_{t-\tau}^t D_V^\tau(s)ds, \quad t \ge \tau.$$

Finally, the last thing we need to check is that the diameter functions D_Z^τ are absolutely continuous, so it satisfies the fundamental theorem of calculus. If we verify this, we can use the system of differential inequalities for D_X^τ, D_V^τ, and D_T^τ, which we will prove in the next section, to find sufficient framework that guarantees the desired flocking phenomenon.

Lemma 3.6. *Let (X, V, T) be a solution to* (3.1)*, where the initial data*

$$(x_i^0, v_i^0, T_i^0) : [-\tau, 0] \to \mathbb{R}^d \times \mathbb{S}^{d-1} \times (0, \infty)$$

is Lipschitz continuous for every $i \in [N]$. Then, D_X^τ, D_V^τ, and D_T^τ are globally Lipschitz in $t \ge 0$.

Proof. By using the uniform boundedness of $\frac{dz_i}{dt}$ obtained from Lemma 3.2, one can see that

$$s_1 \mapsto -\|z_i(s_1) - w\|$$

is globally Lipschitz. Since the minimum of finitely many Lipschitz continuous functions is Lipschitz by the relation

$$\min\{f_1, f_2\} = \frac{f_1 + f_2 - |f_1 - f_2|}{2},$$

we have the global Lipschitz continuity of the function

$$s \mapsto -\max_{i \in [N]} \|z_i(s) - w\|,$$

for every fixed w. In addition, by using Lemma 3.1(3), one can also obtain the Lipschitz continuity of

$$t \mapsto \max_{s \in [t-\tau, t]} \max_{i \in [N]} \|z_i(s) - w\|.$$

Now, let us denote a nonempty, compact set $C_Z(t)$ by

$$C_Z(t) := \{z_i(s) : s \in [t - \tau, t],\ i \in [N]\}.$$

Then, the function

$$t \mapsto \max_{z \in C_Z(t)} \|z - w\|$$

is globally Lipschitz, and the function

$$s_2 \mapsto \max_{z\in C_Z(t_1)} \|z - z_j(s_2)\|$$

is also globally Lipschitz since $z_j(s_2)$ is Lipschitz in s_2 and the Lipschitz constant of the function $w \mapsto \max_{z\in C_Z(t)} \|z-w\|$ is not greater than 1. Therefore, all delayed diameter functions are globally Lipschitz, i.e.,

$$D_Z^\tau(t) = \max_{j\in[N]} \max_{z\in C_Z(t)} \|z - z_j(t)\|$$

is globally Lipschitz in t. □

4. Asymptotic behavior of the delayed diameters

In this section, we present a system of differential inequalities on delayed diameters D_X^τ, D_V^τ, and D_T^τ to deduce suitable sufficient frameworks for the asymptotic flocking of the system (3.1). Unlike the commonly used diameter $D_Z(t) = \max_{i,j\in[N]} \|z_i(t) - z_j(t)\|$, each delayed diameter does not specify at which point in time the distance between two vectors are evaluated. Therefore, in order to estimate the derivative of the delayed diameter, we need to know the behavior of the function given by a maximum value of a differentiable function over a certain range.

Lemma 4.1. *Let $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a continuous function, and define*

$$\mathcal{S}(t) := [t - \tau, t] \times \{t\} \cup \{t\} \times [t - \tau, t], \quad m[f](t) := \max_{(s_1,s_2)\in\mathcal{S}(t)} f(s_1, s_2), \quad t \in \mathbb{R}.$$

If there is a continuous function $\lambda : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $t_0 \in \mathbb{R}$ such that

$$\limsup_{h\to 0+} \frac{f(t_1 + h, t_2 + h) - f(t_1, t_2)}{h} \le \lambda(t_1, t_2), \quad \forall\ (t_1, t_2) \in \mathbb{R}^2,$$
$$\lambda(t_1, t_2) \le c \quad \text{whenever} \quad m[f](t_0) = f(t_1, t_2) \text{ and } (t_1, t_2) \in \mathcal{S}(t_0),$$

we have

$$D^+ m[f](t_0) \le c.$$

Proof. Basically, the main idea to prove this lemma is finding a uniform upper bound of the Dini derivative of $h \mapsto f(t_1 + h, t_2 + h)$ for all points (t_1, t_2) which maximizes f on $\mathcal{S}(t)$. However, since we are considering the values of f on an infinite set $\mathcal{S}(t)$ and analyzing the temporal evolution of their maximum, we want the Dini derivative to be uniformly bounded by a reasonable value in the neighborhood of every maximum point in $\mathcal{S}(t)$. This is why we required a continuous upper bound λ, and the technical difficulties of the proof can be resolved with some simple preparation and with the help of *the Berge maximum theorem* (see [28] for details). We leave the detailed proof to Appendix B. □

Now, we present the differential inequalities of the delayed diameters. For every $Z \in \{X, V\}$, $D_Z^\tau(t)^2$ can be represented as the maximum value of

$$\mathcal{S}(t) \ni (t_1, t_2) \mapsto f_Z(t_1, t_2) := \max_{i,j\in[N]} \|z_i(t_1) - z_j(t_2)\|^2.$$

Once we find a continuous upper bound λ of the Dini derivative of the function f_Z in the $(1,1)$ direction, we can use the maximum of λ over the maximizing set of f_Z as the upper bound of the Dini derivative of $(D_Z^\tau)^2$. Therefore, by expressing the value of the upper bound λ in terms of the value of f_Z, we can obtain a differential inequality for the delayed diameters.

Lemma 4.2. *Let* (X, V, T) *be a solution to* (3.1)*, where the initial data*

$$(x_i^0, v_i^0, T_i^0) : [-\tau, 0] \to \mathbb{R}^d \times \mathbb{S}^{d-1} \times (0, \infty)$$

are Lipschitz continuous functions satisfying $\frac{dx_i^0}{dt} = v_i^0$ *for all* $i \in [N]$ *and*

$$A^{\tau,\tau}(0,0) = \cos\delta, \quad \delta \in (0, \frac{\pi}{2}).$$

In addition, we define a continuous function S_X^τ *as*

$$S_X^\tau(t) = D_X^\tau(0) + \int_0^t D_V^\tau(s)ds, \quad t \geq 0.$$

Then, the following differential inequalities hold:

1. $\dfrac{dD_X^\tau(t)}{dt} \leq D_V^\tau(t), \quad a.e. \quad t > 0.$
2. $\dfrac{dD_V^\tau(t)}{dt} \leq -\dfrac{\kappa_1\phi(S_X^\tau(t))\cos\delta}{T_M^\tau(0)} D_V^\tau(t) + \dfrac{3\kappa_1\phi(0)}{T_m^\tau(0)} \max_{s\in[t-\tau,t]} \Delta_V^\tau(s), \quad a.e. \quad t > \tau.$
3. $\dfrac{dD_T^\tau(t)}{dt} \leq -\dfrac{\kappa_2\zeta(S_X^\tau(t))}{T_M^\tau(0)^2} D_T^\tau(t) + \dfrac{3\kappa_2\zeta(0)}{T_m^\tau(0)^2} \max_{s\in[t-\tau,t]} \Delta_T^\tau(s), \quad a.e. \quad t > \tau.$

Proof. (1) For every $i, j \in N$ and $t_1, t_2 \geq 0$, define

$$f_{ij}(t_1, t_2) := \|x_i(t_1) - x_j(t_2)\|^2.$$

Then, for the continuous function $f_X := \max_{i,j\in[N]} f_{ij}$,

$$\begin{aligned}\limsup_{h\to 0+} \frac{f_X(t_1+h, t_2+h) - f_X(t_1,t_2)}{h} &\leq \max_{i,j\in[N]} \lim_{h\to 0+} \frac{f_{ij}(t_1+h, t_2+h) - f_{ij}(t_1,t_2)}{h} \\ &= \max_{i,j\in[N]} 2\langle x_i(t_1) - x_j(t_2), v_i(t_1) - v_j(t_2)\rangle \\ &=: \lambda(t_1, t_2).\end{aligned}$$

In particular, for every $(t_1, t_2) \in \mathcal{S}(t)$, we have

$$\lambda(t_1, t_2) \leq 2D_X^\tau(t) D_V^\tau(t).$$

Therefore, by using Lemma 4.1, one can obtain

$$D^+[(D_X^\tau)^2](t) = D^+m[f_X](t) \leq 2D_X^\tau(t)D_V^\tau(t),$$

which implies

$$\frac{dD_X^\tau(t)}{dt} \leq D_V^\tau(t)$$

whenever D_X^τ is differentiable at t. Finally, from the Lipschitz continuity of D_X^τ in Lemma 3.6, we have the desired result.

(2) For the velocity diameter D_V^τ, we set

$$f_{ij}(t_1,t_2) = \|v_i(t_1) - v_j(t_2)\|^2, \quad f_V = \max_{i,j\in[N]} f_{ij}.$$

Then, we have

$$\begin{aligned}
&\lim_{h\to 0+} \frac{f_{ij}(t_1+h,t_2+h) - f_{ij}(t_1,t_2)}{h} \\
&= 2\langle v_i(t_1) - v_j(t_2), \dot v_i(t_1) - \dot v_j(t_2)\rangle \\
&= -2\langle v_i(t_1), \dot v_j(t_2)\rangle - 2\langle v_j(t_2), \dot v_i(t_1)\rangle \\
&= -\frac{2\kappa_1}{N}\sum_{k\in[N]} \frac{\phi_{kj}^{\tau_{kj}}(t_2)}{T_k^{\tau_{kj}}(t_2)} \Big\langle v_i(t_1), v_k^{\tau_{kj}}(t_2) - \langle v_k^{\tau_{kj}}(t_2), v_j(t_2)\rangle v_j(t_2)\Big\rangle \\
&\quad - \frac{2\kappa_1}{N}\sum_{k\in[N]} \frac{\phi_{ki}^{\tau_{ki}}(t_1)}{T_k^{\tau_{ki}}(t_1)} \Big\langle v_j(t_2), v_k^{\tau_{ki}}(t_1) - \langle v_k^{\tau_{ki}}(t_1), v_i(t_1)\rangle v_i(t_1)\Big\rangle \\
&=: -\frac{2\kappa_1}{N}\sum_{k\in[N]} \frac{\phi_{kj}^{\tau_{kj}}(t_2)}{T_k^{\tau_{kj}}(t_2)} \mathcal{I}_{ijk}(t_1,t_2) - \frac{2\kappa_1}{N}\sum_{k\in[N]} \frac{\phi_{ki}^{\tau_{ki}}(t_1)}{T_k^{\tau_{ki}}(t_1)} \mathcal{I}_{jik}(t_2,t_1),
\end{aligned}$$

where each $\mathcal{I}_{ijk}(t_1,t_2)$ satisfies

$$\begin{aligned}
\mathcal{I}_{ijk}(t_1,t_2) &= \Big\langle v_i(t_1), v_k^{\tau_{kj}}(t_2) - \langle v_k^{\tau_{kj}}(t_2), v_j(t_2)\rangle v_j(t_2)\Big\rangle \\
&= \Big\langle v_i(t_1) - v_j(t_2), v_k^{\tau_{kj}}(t_2) - v_k(t_2) - \langle v_k^{\tau_{kj}}(t_2) - v_k(t_2), v_j(t_2)\rangle v_j(t_2)\Big\rangle \\
&\quad + \Big\langle v_i(t_1), v_k(t_2) - \langle v_k(t_2), v_j(t_2)\rangle v_j(t_2)\Big\rangle \\
&\ge -\|v_i(t_1) - v_j(t_2)\|\Delta_V^\tau(t_2) + \Big\langle v_i(t_1), v_k(t_2) - \langle v_k(t_2), v_j(t_2)\rangle v_j(t_2)\Big\rangle.
\end{aligned}$$

In particular, if $f_{ij}(t_1,t_2) = f_V(t_1,t_2)$ for some $i,j\in[N]$ and $t_1,t_2\ge 0$,

$$\begin{aligned}
&\lim_{h\to 0+} \frac{f_{ij}(t_1+h,t_2+h) - f_{ij}(t_1,t_2)}{h} \\
&= -\frac{2\kappa_1}{N}\sum_{k\in[N]} \frac{\phi_{kj}^{\tau_{kj}}(t_2)}{T_k^{\tau_{kj}}(t_2)} \mathcal{I}_{ijk}(t_1,t_2) - \frac{2\kappa_1}{N}\sum_{k\in[N]} \frac{\phi_{ki}^{\tau_{ki}}(t_1)}{T_k^{\tau_{ki}}(t_1)} \mathcal{I}_{jik}(t_2,t_1) \\
&\le \left(\frac{2\kappa_1\phi(0)}{T_m^\tau(t_2)}\Delta_V^\tau(t_2) + \frac{2\kappa_1\phi(0)}{T_m^\tau(t_1)}\Delta_V^\tau(t_1)\right)\|v_i(t_1) - v_j(t_2)\| \\
&\quad - \frac{2\kappa_1}{N}\frac{\phi(D_X^\tau(t_2))}{T_M^\tau(t_2)}\sum_{k\in[N]}\Big\langle v_i(t_1), v_k(t_2) - \langle v_k(t_2), v_j(t_2)\rangle v_j(t_2)\Big\rangle \\
&\quad - \frac{2\kappa_1}{N}\frac{\phi(D_X^\tau(t_1))}{T_M^\tau(t_1)}\sum_{k\in[N]}\Big\langle v_j(t_2), v_k(t_1) - \langle v_k(t_1), v_i(t_1)\rangle v_i(t_1)\Big\rangle,
\end{aligned}$$

where we used $\langle v_i(t_1), v_j(t_2)\rangle \geq A^{\tau,\tau}(t_1,t_2) > 0$ and

$$\|x_i(t_1) - x_k^{\tau_{ki}}(t_1)\| \leq D_X^\tau(t_1), \quad \|x_j(t_2) - x_k^{\tau_{kj}}(t_2)\| \leq D_X^\tau(t_2),$$
$$\left\langle v_i(t_1), v_k(t_2) - \langle v_k(t_2), v_j(t_2)\rangle v_j(t_2)\right\rangle \geq \left\langle v_i(t_1), v_j(t_2)\right\rangle (1 - \langle v_k(t_2), v_j(t_2)\rangle) \geq 0,$$
$$\left\langle v_j(t_2), v_k(t_1) - \langle v_k(t_1), v_i(t_1)\rangle v_i(t_1)\right\rangle \geq \left\langle v_i(t_1), v_j(t_2)\right\rangle (1 - \langle v_k(t_1), v_i(t_1)\rangle) \geq 0,$$

in the last inequality. In addition, by using the inequality

$$\begin{aligned}
&\left\langle v_i(t_1), v_k(t_2) - \langle v_k(t_2), v_j(t_2)\rangle v_j(t_2)\right\rangle + \left\langle v_j(t_2), v_k(t_1) - \langle v_k(t_1), v_i(t_1)\rangle v_i(t_1)\right\rangle \\
&= (\langle v_i(t_1), v_k(t_2)\rangle + \langle v_k(t_1), v_j(t_2)\rangle)(1 - \langle v_i(t_1), v_j(t_2)\rangle) \\
&\quad - \langle v_k(t_1) - v_k(t_2), v_i(t_1) - v_j(t_2)\rangle\langle v_i(t_1), v_j(t_2)\rangle \\
&\geq \langle v_i(t_1), v_j(t_2)\rangle \|v_i(t_1) - v_j(t_2)\| \left(\|v_i(t_1) - v_j(t_2)\| - \|v_k(t_2) - v_k(t_1)\|\right),
\end{aligned}$$

we have

$$\begin{aligned}
&\lim_{h\to 0+} \frac{f_V(t_1+h, t_2+h) - f_V(t_1,t_2)}{h} \\
&\leq \left(\frac{2\kappa_1\phi(0)}{T_m^\tau(t_2)}\Delta_V^\tau(t_2) + \frac{2\kappa_1\phi(0)}{T_m^\tau(t_1)}\Delta_V^\tau(t_1)\right)\sqrt{f_V(t_1,t_2)} \\
&\quad - 2\kappa_1 \min\left\{\frac{\phi(D_X^\tau(t_1))}{T_M^\tau(t_1)}, \frac{\phi(D_X^\tau(t_2))}{T_M^\tau(t_2)}\right\}\times \\
&\qquad\qquad \left(1 - \frac{1}{2}f_V(t_1,t_2)\right)\sqrt{f_V(t_1,t_2)}\left(\sqrt{f_V(t_1,t_2)} - \max_{k\in[N]}\|v_k(t_2) - v_k(t_1)\|\right) \\
&=: \lambda(t_1,t_2).
\end{aligned}$$

Therefore, by using Lemma 4.1, one can obtain

$$\begin{aligned}
D^+[(D_V^\tau)^2](t) = D^+m[f_V](t) &\leq \left(\frac{2\kappa_1\phi(0)}{T_m^\tau(0)}\Delta_V^\tau(t) + \frac{2\kappa_1\phi(0)}{T_m^\tau(0)}\max_{s\in[t-\tau,t]}\Delta_V^\tau(s)\right)D_V^\tau(t) \\
&\quad - \frac{2\kappa_1\phi(S_X^\tau(t))}{T_M^\tau(0)}\left(1 - \frac{1}{2}D_V^\tau(t)^2\right)D_V^\tau(t)\left(D_V^\tau(t) - \Delta_V^\tau(t)\right),
\end{aligned}$$

which implies the following inequalities whenever D_V^τ is differentiable at t:

$$\begin{aligned}
\frac{dD_V^\tau(t)}{dt} &\leq \left(\frac{\kappa_1\phi(0)}{T_m^\tau(0)}\Delta_V^\tau(t) + \frac{\kappa_1\phi(0)}{T_m^\tau(0)}\max_{s\in[t-\tau,t]}\Delta_V^\tau(s)\right) \\
&\quad - \frac{\kappa_1\phi(S_X^\tau(t))}{T_M^\tau(0)}\left(1 - \frac{1}{2}D_V^\tau(t)^2\right)(D_V^\tau(t) - \Delta_V^\tau(t)) \\
&\leq -\frac{\kappa_1\phi(S_X^\tau(t))}{T_M^\tau(0)}\left(1 - \frac{1}{2}D_V^\tau(t)^2\right)D_V^\tau(t) + \frac{3\kappa_1\phi(0)}{T_m^\tau(0)}\max_{s\in[t-\tau,t]}\Delta_V^\tau(s), \\
&\leq -\frac{\kappa_1\phi(S_X^\tau(t))\cos\delta}{T_M^\tau(0)}D_V^\tau(t) + \frac{3\kappa_1\phi(0)}{T_m^\tau(0)}\max_{s\in[t-\tau,t]}\Delta_V^\tau(s).
\end{aligned}$$

Similar to the case of D_X^τ, we reach the desired result due to the Lipschitz continuity of Lemma 3.6.

(3) In this case, we define

$$f_{ij}(t_1,t_2) = T_i(t_1) - T_j(t_2), \quad f_T = \max_{i,j\in[N]} f_{ij}.$$

Then, whenever $f_{ij}(t_1,t_2) = f_T(t_1,t_2)$ and $t_1, t_2 \le t$, we get

$$\begin{aligned}
&\lim_{h\to 0+} \frac{f_{ij}(t_1+h,t_2+h) - f_{ij}(t_1,t_2)}{h} \\
&\quad = \dot{T}_i(t_1) - \dot{T}_j(t_2) \\
&\quad = \frac{\kappa_2}{N}\sum_{k\in[N]} \zeta_{ki}^{\tau_{ki}}(t_1)\left(\frac{1}{T_i(t_1)} - \frac{1}{T_k^{\tau_{ki}}(t_1)}\right) - \frac{\kappa_2}{N}\sum_{k\in[N]} \zeta_{kj}^{\tau_{kj}}(t_2)\left(\frac{1}{T_j(t_2)} - \frac{1}{T_k^{\tau_{kj}}(t_2)}\right) \\
&\quad \le \frac{\kappa_2}{N}\sum_{k\in[N]} \zeta_{ki}^{\tau_{ki}}(t_1)\left(\frac{1}{T_i(t_1)} - \frac{1}{T_k(t_1)}\right) - \frac{\kappa_2}{N}\sum_{k\in[N]} \zeta_{kj}^{\tau_{kj}}(t_2)\left(\frac{1}{T_j(t_2)} - \frac{1}{T_k(t_2)}\right) \\
&\qquad + \kappa_2\zeta(0)\left(\frac{\Delta_T^\tau(t_1)}{T_m^\tau(t_1)^2} + \frac{\Delta_T^\tau(t_2)}{T_m^\tau(t_2)^2}\right).
\end{aligned}$$

By using the relations

$$\frac{1}{T_i(t_1)} - \frac{1}{T_k(t_1)} = \frac{1}{T_M(t_1)} - \frac{1}{T_k(t_1)} \le 0 \le \frac{1}{T_m(t_2)} - \frac{1}{T_k(t_2)} = \frac{1}{T_j(t_2)} - \frac{1}{T_k(t_2)},$$

the above inequality reduces to

$$\begin{aligned}
&\lim_{h\to 0+} \frac{f_{ij}(t_1+h,t_2+h) - f_{ij}(t_1,t_2)}{h} \\
&\quad \le \frac{\kappa_2}{N}\sum_{k\in[N]} \zeta(S_X^\tau(t_1))\left(\frac{1}{T_M(t_1)} - \frac{1}{T_k(t_1)}\right) - \frac{\kappa_2}{N}\sum_{k\in[N]} \zeta(S_X^\tau(t_2))\left(\frac{1}{T_m(t_2)} - \frac{1}{T_k(t_2)}\right) \\
&\qquad + \kappa_2\zeta(0)\left(\frac{\Delta_T^\tau(t_1)}{T_m^\tau(t_1)^2} + \frac{\Delta_T^\tau(t_2)}{T_m^\tau(t_2)^2}\right) \\
&\quad \le \frac{\kappa_2}{N}\sum_{k\in[N]} \zeta(S_X^\tau(t))\left(\frac{1}{T_M(t_1)} - \frac{1}{T_k(t_1)} - \frac{1}{T_m(t_2)} + \frac{1}{T_k(t_2)}\right) \\
&\qquad + \kappa_2\zeta(0)\left(\frac{\Delta_T^\tau(t_1)}{T_m^\tau(t_1)^2} + \frac{\Delta_T^\tau(t_2)}{T_m^\tau(t_2)^2}\right) \\
&\quad \le -\kappa_2\zeta(S_X^\tau(t))\left(\frac{1}{T_m(t_2)} - \frac{1}{T_M(t_1)}\right) + \frac{\kappa_2\zeta(0)}{T_m^\tau(t_1)T_m^\tau(t_2)}\max_{k\in[N]}|T_k(t_1) - T_k(t_2)| \\
&\qquad + \kappa_2\zeta(0)\left(\frac{\Delta_T^\tau(t_1)}{T_m^\tau(t_1)^2} + \frac{\Delta_T^\tau(t_2)}{T_m^\tau(t_2)^2}\right) \\
&\quad =: \lambda(t_1,t_2).
\end{aligned}$$

Therefore, by using Lemma 4.1, one can show that

$$D^+[D_T^\tau](t) \le -\frac{\kappa_2\zeta(S_X^\tau(t))}{T_M^\tau(0)^2}D_T^\tau(t) + \frac{\kappa_2\zeta(0)}{T_m^\tau(0)^2}\left(2\Delta_T^\tau(t) + \max_{s\in[t-\tau,t]}\Delta_T^\tau(s)\right),$$

which implies

$$\frac{dD_T^\tau(t)}{dt} \le -\frac{\kappa_2\zeta(S_X^\tau(t))}{T_M^\tau(0)^2}D_T^\tau(t) + \frac{3\kappa_2\zeta(0)}{T_m^\tau(0)^2}\max_{s\in[t-\tau,t]}\Delta_T^\tau(s),$$

whenever D_T^τ is differentiable at t. □

Note that the differential inequality in Lemma 4.2 holds for (almost every) $t > \tau$. This is because we need to estimate the derivative of f_Z at all points on $\mathcal{S}(t)$. Since the ODE (3.1) only gives the derivative of $\dot{z}_i(s)$ for $s > 0$, we require $t - \tau > 0$ in Lemma 4.2.

The next lemma allows us to present the upper bound of the objective function when the differential inequalities as Lemma 3.5 and Lemma 4.2 are given.

Lemma 4.3. *Let $y : [0,\infty) \to [0,\infty)$ be a Lipschitz continuous function and $f : [0,\infty) \to \mathbb{R}$ be a continuous function. Assume that y and f satisfy*

$$\dot{y}(t) \le -ay(t) + \max_{s\in[t-\tau,t]} f(s), \quad t > \tau, \tag{4.1a}$$

$$f(t) \le d\int_{t-\tau}^{t} y(s)ds, \quad t > \tau, \tag{4.1b}$$

for some constants $a, d > 0$ and $\tau \ge 0$. If τ is sufficiently small to satisfy

$$\tau < \frac{a}{d},$$

and the following assertions hold.

(1) There are two constants $b, c > 0$ such that

$$\begin{aligned} &L(c) \le \frac{1}{d}, \quad L : (0,a) \to \mathbb{R}, \quad L(x) := \frac{e^{x\tau}(e^{x\tau}-1)}{x(a-x)}, \\ &y(\tau) \le \frac{be^{c\tau}}{a-c}, \quad \max_{t\in[0,2\tau]} f(t)e^{c(t-\tau)} < b. \end{aligned} \tag{4.2}$$

(2) Whenever $b, c > 0$ satisfy (4.2), we have

$$\begin{aligned} &y(t) < y(\tau)e^{-a(t-\tau)} + \frac{be^{c\tau}}{a-c}\left(e^{-c(t-\tau)} - e^{-a(t-\tau)}\right), \quad t > \tau, \\ &f(t) < be^{-c(t-\tau)}, \quad t \ge 0. \end{aligned} \tag{4.3}$$

Proof. Since L is continuous and

$$\lim_{x\to 0+} L(x) = \frac{\tau}{a} < \frac{1}{d}, \quad \lim_{x\to a-} L(x) = +\infty,$$

one can find a positive number $c \in (0,a)$ satisfying $L(c) \le \frac{1}{d}$, and one can also choose a sufficiently large b to satisfy (4.2). Now, assume there exists the first time $t_* \in (2\tau,\infty)$ such that

$$f(t_*)e^{c(t_*-\tau)} = b. \tag{4.4}$$

Then, from the definition of t_*, we have

$$f(t) < be^{-c(t-\tau)}, \quad t \in [0, t_*), \tag{4.5}$$

and we substitute (4.5) to (4.1) to obtain

$$\dot{y}(t) < -ay(t) + be^{-c(t-2\tau)}, \quad t \in (\tau, t_*). \tag{4.6}$$

Consequently, the Grönwall inequality (4.6) yields

$$\begin{aligned} y(t) &< y(\tau)e^{-a(t-\tau)} + \frac{be^{c\tau}}{a-c}\left(e^{-c(t-\tau)} - e^{-a(t-\tau)}\right) \\ &= \frac{be^{c\tau}}{a-c}e^{-c(t-\tau)} + \left(y(\tau) - \frac{be^{c\tau}}{a-c}\right)e^{-a(t-\tau)} \\ &\le \frac{be^{c\tau}}{a-c}e^{-c(t-\tau)}, \quad \forall\, t \in (\tau, t_*], \end{aligned} \tag{4.7}$$

where we used Eq (4.2) in the last inequality. On the other hand, by using Eq (4.7) to Eq (4.1b) at $t = t_*$, one can also obtain

$$\begin{aligned} f(t_*) &\le d\int_{t_*-\tau}^{t_*} y(t)dt \\ &< d\int_{t_*-\tau}^{t_*} \frac{be^{c\tau}}{a-c}e^{-c(t-\tau)}dt \\ &= \frac{bde^{c\tau}}{c(a-c)}\left(e^{-c(t_*-2\tau)} - e^{-c(t_*-\tau)}\right) \\ &= bde^{-c(t_*-\tau)}L(c) \\ &\le be^{-c(t_*-\tau)}, \end{aligned}$$

which leads to a contradiction in Eq (4.4). Therefore, we have

$$f(t) < be^{-c(t-\tau)}$$

for all $t \ge 0$, which implies the desired result for Eq (4.3).

□

Remark 4.1. *For the case when* $\tau = 0$*, the function* L *becomes a constant function* 0*. Then, we can choose any number* c *from* $(0, a)$ *to satisfy* $L(c) \le \frac{1}{d}$*, and every positive* b *satisfying* $b \ge (a-c)y(0)$ *also satisfies the condition* (4.2)*. Therefore, we have*

$$y(t) < y(0)e^{-at} + (y(0) + \frac{1}{n})(e^{-(a-\frac{1}{n})t} - e^{-at}), \quad \forall\, t > 0, \quad n \in \mathbb{N},$$

which implies

$$y(t) \le y(0)e^{-at}, \quad \forall\, t > 0.$$

Since inequality (4.1) *for* $\tau = 0$ *is*

$$\dot{y}(t) \le -ay(t) + f(t), \quad f(t) \le 0, \quad \forall\, t > 0,$$

we can say that Lemma 4.3 gives the optimal upper bound for $\tau = 0$*.*

Now, we introduce the first sufficient framework for the emergence of flocking.

• ($\mathcal{F}1$): $A^{\tau,\tau}(0,0) = \cos\delta > 0, \quad \delta \in (0, \frac{\pi}{2})$.
• ($\mathcal{F}2$): There exists a constant $D_X^{\tau,\infty} > 0$ satisfying

$$\Phi(D_X^{\tau,\infty}) > \Phi\left(D_X^\tau(0)\right) + \frac{T_M^\tau(0)}{\kappa_1 \cos\delta} \cdot 2\sin\frac{\delta}{2}, \quad \Phi(x) := \int_0^x \phi(u)du.$$

• ($\mathcal{F}3$): The time-delay bound $\tau \geq 0$ is sufficiently small to satisfy

$$\begin{aligned} &\Phi(D_X^{\tau,\infty}) > \Phi\left(D_X^\tau(0) + 2\tau\sin\frac{\delta}{2}\right) + \frac{T_M^\tau(0)}{\kappa_1\cos\delta} \cdot \left(2\sin\frac{\delta}{2} + d_1\tau \cdot \frac{\beta e^{c_1\tau}}{c_1}\right), \\ &a_1 = c_1 + d_1\tau e^{c_1\tau} \cdot \frac{e^{c_1\tau} - 1}{c_1\tau}, \quad a_2 = c_2 + d_2\tau e^{c_2\tau} \cdot \frac{e^{c_2\tau} - 1}{c_2\tau}, \end{aligned} \tag{4.8}$$

for some $c_1 \in (0, a_1], c_2 \in (0, a_2]$, where the constants a_1, a_2, d_1, d_2, and β are given by

$$\begin{aligned} &a_1 = \frac{\kappa_1\phi(D_X^{\tau,\infty})\cos\delta}{T_M^\tau(0)}, \quad a_2 = \frac{\kappa_2\zeta(D_X^{\tau,\infty})}{T_M^\tau(0)^2}, \\ &d_1 = \frac{3(N-1)}{N}\left(\frac{\kappa_1\phi(0)}{T_m^\tau(0)}\right)^2, \quad d_2 = \frac{3(N-1)}{N}\left(\frac{\kappa_2\zeta(0)}{T_m^\tau(0)^2}\right)^2, \\ &\beta := \max\left\{2\sin\frac{\delta}{2}, \frac{\max_{i\in[N]}\|v_i^0\|_{\mathrm{Lip}}}{\frac{(N-1)\kappa_1\phi(0)}{NT_m^\tau(0)}}\right\}. \end{aligned}$$

Now, we are ready to provide the first main result on the asymptotic flocking of time-delayed unit speed TCS model (3.1), by showing that D_X^τ is uniformly bounded and D_V^τ, D_T^τ converge to zero.

Theorem 4.1. *Let* (X, V, T) *be a solution to* (3.1)*, where the initial data*

$$(x_i^0, v_i^0, T_i^0) : [-\tau, 0] \to \mathbb{R}^d \times \mathbb{S}^{d-1} \times (0, \infty), \quad i \in [N]$$

are Lipschitz continuous functions satisfying $\frac{dx_i^0}{dt} = v_i^0$ *and* ($\mathcal{F}1$)–($\mathcal{F}3$)*. Then, we estimate*

$$\begin{aligned} &D_X^\tau(t) \leq S_X^\tau(t) < D_X^{\tau,\infty} \quad \forall\, t > 0, \\ &D_V^\tau(t) \leq \frac{\beta c_1\tau}{1 - e^{-c_1\tau}} \cdot e^{-c_1(t-\tau)} \quad \forall\, t > \tau, \quad \Delta_V^\tau(t) \leq \frac{(N-1)\kappa_1\phi(0)}{NT_m^\tau(0)}\beta\tau e^{-c_1(t-2\tau)} \quad \forall\, t \geq 0, \\ &D_T^\tau(t) \leq \frac{\gamma c_2\tau}{1 - e^{-c_2\tau}} \cdot e^{-c_2(t-\tau)} \quad \forall\, t > \tau, \quad \Delta_T^\tau(t) \leq \frac{(N-1)\kappa_2\zeta(0)}{NT_m^\tau(0)^2}\gamma\tau e^{-c_2(t-2\tau)} \quad \forall\, t \geq 0, \end{aligned}$$

where the constant γ *is given by*

$$\gamma := \max\left\{T_M^\tau(0) - T_m^\tau(0), \frac{\max_{i\in[N]}\|T_i^0\|_{\mathrm{Lip}}}{\frac{(N-1)\kappa_2\zeta(0)}{NT_m^\tau(0)^2}}\right\}.$$

Therefore, the solution (X, V, T) *exhibits the asymptotic flocking.*

Proof. (Step 1) First, we find the range of t such that D_V^τ and D_T^τ satisfy the inequalities playing the role of $y(t)$ in Lemma 4.3. Assume there exists a minimum $t^* < \infty$ among all t satisfying $S_X^\tau(t) \geq D_X^{\tau,\infty}$. Then, by using Lemma 3.4, we have

$$\begin{aligned}\Phi(D_X^{\tau,\infty}) &= \Phi(S_X^\tau(t^*)) = \Phi\left(D_X^\tau(0) + \int_0^{t^*} D_V^\tau(s)ds\right)\\ &\leq \Phi\left(D_X^\tau(0) + \sqrt{2-2A^{\tau,\tau}(0,0)}t^*\right) = \Phi\left(D_X^\tau(0) + 2t^*\sin\frac{\delta}{2}\right),\end{aligned}$$

which implies $t^* > \tau$ from the condition (4.8). We then use the monotone decreasing property of ϕ, ζ and Lemma 3.5 and Lemma 4.2 to obtain

$$\begin{aligned}&\frac{dD_V^\tau(t)}{dt} \leq -a_1 D_V^\tau(t) + \max_{s\in[t-\tau,t]} \frac{3\kappa_1\phi(0)}{T_m^\tau(0)}\Delta_V^\tau(s), \quad \text{a.e.} \quad t \in (\tau, t^*),\\ &\frac{3\kappa_1\phi(0)}{T_m^\tau(0)}\Delta_V^\tau(t) \leq d_1\int_{t-\tau}^t D_V^\tau(s)ds, \quad t\in(\tau,t^*),\end{aligned} \tag{4.9}$$

for the delayed diameter D_V^τ, and similarly,

$$\begin{aligned}&\frac{dD_T^\tau(t)}{dt} \leq -a_2 D_T^\tau(t) + \max_{s\in[t-\tau,t]} \frac{3\kappa_2\zeta(0)}{T_m^\tau(0)^2}\Delta_T^\tau(s), \quad \text{a.e.} \quad t \in (\tau, t^*),\\ &\frac{3\kappa_2\zeta(0)}{T_m^\tau(0)^2}\Delta_T^\tau(t) \leq d_2\int_{t-\tau}^t D_T^\tau(s)ds, \quad t\in(\tau,t^*),\end{aligned} \tag{4.10}$$

for the delayed diameter D_T^τ.
(Step 2) Now, we will show that t^* cannot be a finite number so that Eqs (4.9) and (4.10) hold for all $t \in (\tau, \infty)$. For every positive number b_1, b_2 satisfying

$$b_1 > \beta e^{c_1\tau}\cdot d_1\tau, \quad b_2 > \gamma e^{c_2\tau}\cdot d_2\tau, \tag{4.11}$$

one can verify that (4.2) holds for (b_1, c_1) and (b_2, c_2). More precisely, by using Eqs (4.8) and (4.11) and Lemma 3.4, we show that

$$\frac{b_1 e^{c_1\tau}}{a_1 - c_1} = \frac{b_1}{d_1\tau}\cdot\frac{c_1\tau}{e^{c_1\tau}-1} > \beta\cdot\frac{c_1\tau}{1-e^{-c_1\tau}} \geq \beta \geq 2\sin\frac{\delta}{2} \geq D_V^\tau(\tau).$$

In addition, we use the monotonic increasing property of $A^{\tau,\tau}(t,t)$ (see Lemma 3.4) to verify

$$\begin{aligned}\frac{3\kappa_1\phi(0)}{T_m^\tau(0)}\Delta_V^\tau(t) &\leq \frac{3\kappa_1\phi(0)}{T_m^\tau(0)}\left[\max\{\tau-t,0\}\max_{i\in[N]}\|v_i^0\|_{\text{Lip}} + \frac{(N-1)\kappa_1\phi(0)}{NT_m^\tau(0)}\int_{\max\{0,t-\tau\}}^t D_V^\tau(u)du\right]\\ &\leq \frac{3\kappa_1\phi(0)}{T_m^\tau(0)}\left[\max\{\tau-t,0\}\max_{i\in[N]}\|v_i^0\|_{\text{Lip}} + \frac{(N-1)\kappa_1\phi(0)}{NT_m^\tau(0)}\int_{\max\{0,t-\tau\}}^t \sqrt{2-2A^{\tau,\tau(0,0)}}du\right]\\ &= \frac{3\kappa_1\phi(0)}{T_m^\tau(0)}\left[\max\{\tau-t,0\}\max_{i\in[N]}\|v_i^0\|_{\text{Lip}} + \frac{(N-1)\kappa_1\phi(0)}{NT_m^\tau(0)}(t-\max\{0,t-\tau\})\left(2\sin\frac{\delta}{2}\right)\right]\\ &= \frac{3\kappa_1\phi(0)}{T_m^\tau(0)}\left[\max\{\tau-t,0\}\max_{i\in[N]}\|v_i^0\|_{\text{Lip}} + \frac{(N-1)\kappa_1\phi(0)}{NT_m^\tau(0)}(\min\{t,\tau\})\left(2\sin\frac{\delta}{2}\right)\right]\end{aligned}$$

$$
\begin{aligned}
&\le \frac{3\kappa_1\phi(0)}{T_m^\tau(0)}\left[\max\{\tau-t,0\}\frac{(N-1)\kappa_1\phi(0)}{NT_m^\tau(0)}\beta+\frac{(N-1)\kappa_1\phi(0)}{NT_m^\tau(0)}(\min\{t,\tau\})\beta\right]\\
&=\frac{3\kappa_1\phi(0)}{T_m^\tau(0)}\cdot\frac{(N-1)\kappa_1\phi(0)}{NT_m^\tau(0)}\cdot\beta\left[\max\{\tau-t,0\}+\min\{t,\tau\}\right]\\
&=\frac{3\kappa_1\phi(0)}{T_m^\tau(0)}\cdot\frac{(N-1)\kappa_1\phi(0)}{NT_m^\tau(0)}\cdot\beta\left[\max\{\tau,t\}+\min\{t,\tau\}-t\right]\\
&=\frac{3\kappa_1\phi(0)}{T_m^\tau(0)}\cdot\frac{(N-1)\kappa_1\phi(0)}{NT_m^\tau(0)}\cdot\beta\left[\tau+t-t\right]\\
&=\beta\cdot d_1\tau,\quad \forall\, t\in[0,2\tau],
\end{aligned}
$$

which implies

$$
\max_{t\in[0,2\tau]}\frac{3\kappa_1\phi(0)}{T_m^\tau(0)}\Delta_V^\tau(t)e^{c_1(t-\tau)}\le\beta e^{c_1\tau}\cdot d_1\tau<b_1.
$$

Similarly, by using Eqs (4.8) and (4.11) and Lemma 3.2, we have

$$
\frac{b_2e^{c_2\tau}}{a_2-c_2}=\frac{b_2}{d_2\tau}\cdot\frac{c_2\tau}{e^{c_2\tau}-1}>\gamma\cdot\frac{c_2\tau}{1-e^{-c_2\tau}}\ge\gamma\ge T_M^\tau(0)-T_m^\tau(0),
$$

and we use the monotonic decreasing property of $D_T^\tau(t)=T_M^\tau(t)-T_m^\tau(t)$ (see Lemma 3.2) to derive

$$
\begin{aligned}
\frac{3\kappa_2\zeta(0)}{T_m^\tau(0)^2}\Delta_T^\tau(t)&\le\frac{3\kappa_2\zeta(0)}{T_m^\tau(0)^2}\left[\max\{\tau-t,0\}\max_{i\in[N]}\|T_i^0\|_{\mathrm{Lip}}+\frac{(N-1)\kappa_2\zeta(0)}{NT_m^\tau(0)^2}\int_{\max\{0,t-\tau\}}^t D_T^\tau(u)du\right]\\
&\le\frac{3\kappa_2\zeta(0)}{T_m^\tau(0)^2}\left[\max\{\tau-t,0\}\max_{i\in[N]}\|T_i^0\|_{\mathrm{Lip}}+\frac{(N-1)\kappa_2\zeta(0)}{NT_m^\tau(0)^2}\int_{\max\{0,t-\tau\}}^t D_T^\tau(0)du\right]\\
&=\frac{3\kappa_2\zeta(0)}{T_m^\tau(0)^2}\left[\max\{\tau-t,0\}\max_{i\in[N]}\|T_i^0\|_{\mathrm{Lip}}+\frac{(N-1)\kappa_2\zeta(0)}{NT_m^\tau(0)^2}D_T^\tau(0)(\min\{t,\tau\})\right]\\
&\le\frac{3\kappa_2\zeta(0)}{T_m^\tau(0)^2}\left[\max\{\tau-t,0\}\cdot\frac{(N-1)\kappa_2\zeta(0)}{NT_m^\tau(0)^2}\cdot\gamma+\frac{(N-1)\kappa_2\zeta(0)}{NT_m^\tau(0)^2}\cdot\gamma\cdot\min\{t,\tau\}\right]\\
&=\frac{3\kappa_2\zeta(0)}{T_m^\tau(0)^2}\cdot\frac{(N-1)\kappa_2\zeta(0)}{NT_m^\tau(0)^2}\cdot\gamma\left[\max\{\tau-t,0\}+\min\{t,\tau\}\right]\\
&=\gamma\cdot d_2\tau,
\end{aligned}
$$

which implies

$$
\max_{t\in[0,2\tau]}\frac{3\kappa_2\zeta(0)}{T_m^\tau(0)^2}\Delta_T^\tau(t)e^{c_2(t-\tau)}\le\gamma e^{c_2\tau}\cdot d_2\tau<b_2.
$$

Therefore, we apply Lemma 4.3 to (4.9) and (4.10) for all b_1, b_2 satisfying (4.11) to obtain

$$
\frac{3\kappa_1\phi(0)}{T_m^\tau(0)}\Delta_V^\tau(t)\le d_1\tau\beta e^{-c_1(t-2\tau)},\quad\frac{3\kappa_2\zeta(0)}{T_m^\tau(0)^2}\Delta_T^\tau(t)\le d_2\tau\gamma e^{-c_2(t-2\tau)},\quad\forall\, t\in(0,t^*).\tag{4.12}
$$

In particular, we employ the result of Lemma 4.2 to (4.12) to get

$$
\frac{d}{dt}\left[D_V^\tau(t)+\frac{\kappa_1\Phi(S_X^\tau(t))\cos\delta}{T_M^\tau(0)}\right]\le d_1\tau\beta e^{-c_1(t-2\tau)},\quad\forall\, t\in(\tau,t^*).\tag{4.13}
$$

However, from direct calculation, (4.8) and (4.13) yield

$$\begin{aligned} D_V^\tau(t^*) + \frac{\kappa_1 \Phi(S_X^\tau(t^*))\cos\delta}{T_M^\tau(0)} &\le D_V^\tau(\tau) + \frac{\kappa_1 \Phi(S_X^\tau(\tau))\cos\delta}{T_M^\tau(0)} + d_1\tau\cdot\frac{\beta e^{c_1\tau}}{c_1}(1-e^{-c_1(t^*-\tau)}) \\ &\le 2\sin\frac{\delta}{2} + \frac{\kappa_1\Phi(D_X^\tau(0)+2\tau\sin\frac{\delta}{2})\cos\delta}{T_M^\tau(0)} + d_1\tau\cdot\frac{\beta e^{c_1\tau}}{c_1} \\ &< \frac{\kappa_1\Phi(D_X^{\tau,\infty})\cos\delta}{T_M^\tau(0)}, \end{aligned}$$

which contradicts to $S_X^\tau(t^*) = D_X^{\tau,\infty}$ obtained from the existence of $t^* < \infty$. Therefore, we have $S_X^\tau(t) < D_X^{\tau,\infty}$ for all $t > \tau$, and we apply Lemma 4.3 to (4.9) and (4.10) for all b_1, b_2 satisfying (4.11) to reach the desired results. □

Remark 4.2. *Theorem 4.1 is also applicable when time-delay τ is set to 0. If $\tau = 0$, the framework ($\mathcal{F}1$)–($\mathcal{F}3$) becomes*

$$A(0) > 0, \qquad \int_{D_X(0)}^\infty \phi(s)ds > \frac{T_M(0)D_V(0)}{\kappa_1 A(0)},$$

which coincides with the condition (2.2)*. In addition, the condition* (4.8) *reduces to*

$$c_1 = a_1, \quad c_2 = a_2, \quad \beta = D_V(0), \quad \gamma = D_T(0).$$

Therefore, Theorem 4.1 exactly coincides with Proposition 2.5 for $\tau = 0$.

5. Asymptotic behavior of the diameters

In this section, we provide a system of differential inequalities on diameters D_X, D_V, and D_T. In this case, the Lipschitz continuity of the diameters can be easily obtained from the Lipschitz continuity of X, V, T. The goal of this section is to find a sufficient framework without the condition ($\mathcal{F}1$), so that the initial data might allow $\langle v_i(t_1), v_j(t_2)\rangle < 0$ for some $i, j \in [N]$ and $t_1, t_2 \in [-\tau, 0]$. More precisely, we will replace ($\mathcal{F}1$) to the weaker condition that

$$\max_{i,j\in[N]} \langle v_i(0), v_j(0)\rangle > 0.$$

In this case, since we cannot apply the result of Lemma 3.4, the proof of Theorem 4.1 cannot be applied as is, and we must consider the possibility that $A^{\tau,\tau}$ may oscillate at the beginning rather than increasing monotonically. The overall flow of this section is that we can still control the diameter D_V under the condition that τ is sufficiently small.

Lemma 5.1. *Let (X, V, T) be a solution to Eq* (3.1) *subject to the continuous initial data $\{(x_i^0, v_i^0, T_i^0)\}_{i=1}^N$ satisfying $\frac{dx_i^0}{dt} = v_i^0$ for all $i \in [N]$. Then, the following differential inequalities hold:*

1. *Whenever D_X is differentiable at time $t > -\tau$, we have*

$$\frac{dD_X(t)}{dt} \le D_V(t).$$

2. *Whenever D_V is differentiable at time $t > 0$ and $D_V(t) \leq \sqrt{2}$, we have*

$$\frac{dD_V(t)}{dt} \leq -\frac{\kappa_1\phi(D_X(t)+\tau)}{T_M^\tau(t)}\left[1-\frac{N-1}{2N}D_V(t)^2\right]D_V(t)+\frac{2(N-1)\kappa_1\phi(0)}{NT_m^\tau(t)}\Delta_V^\tau(t).$$

3. *Whenever D_T is differentiable at time $t > 0$, we have*

$$\frac{dD_T(t)}{dt} \leq -\kappa_2\zeta(D_X(t)+\tau)\left(\frac{1}{T_m(t)}-\frac{1}{T_M(t)}\right)+\frac{2(N-1)\kappa_2\zeta(0)}{N(T_m^\tau(t))^2}\Delta_T^\tau(t).$$

Proof. We can prove this lemma by using a similar argument to Lemma 4.2.
(1) For every $i, j \in [N]$ and $t > -\tau$, we have

$$\frac{d}{dt}\|x_i(t)-x_j(t)\|^2 = 2\langle x_i(t)-x_j(t), v_i(t)-v_j(t)\rangle \leq 2D_X(t)D_V(t).$$

Therefore, one can obtain

$$D^+[(D_X)^2](t) \leq 2\max_{i,j\in[N]}\langle x_i(t)-x_j(t), v_i(t)-v_j(t)\rangle \leq 2D_X(t)D_V(t),$$

which implies

$$2D_X(t)\frac{dD_X(t)}{dt} \leq 2D_X(t)D_V(t),$$

whenever D_X is differentiable at time t.

(2) Let $\mathcal{J}(t)$ be the set of all index pairs (i, j) which maximize the difference between two velocities at time t, i.e.,

$$(i, j) \in \mathcal{J}(t) \quad \Longleftrightarrow \quad \|v_i(t)-v_j(t)\| = D_V(t) \leq \sqrt{2}.$$

Then, for every $(i, j) \in \mathcal{J}(t)$, we can split the derivative of $\|v_i - v_j\|^2$ at time t into four parts as follows:

$$
\begin{aligned}
&\frac{d}{dt}\left\|v_i(t) - v_j(t)\right\|^2 \\
&= 2\left\langle v_i(t) - v_j(t), \frac{dv_i}{dt}(t) - \frac{dv_j}{dt}(t)\right\rangle \\
&= -\frac{2\kappa_1}{N}\sum_{k\in[N]-\{i\}}\frac{\phi_{ki}^{\tau_{ki}}}{T_k^{\tau_{ki}}}(t)\left\langle v_j(t), v_k^{\tau_{ki}}(t) - \langle v_k^{\tau_{ki}}(t), v_i(t)\rangle v_i(t)\right\rangle \\
&\quad - \frac{2\kappa_1}{N}\sum_{k\in[N]-\{j\}}\frac{\phi_{kj}^{\tau_{kj}}}{T_k^{\tau_{kj}}}(t)\left\langle v_i(t), v_k^{\tau_{kj}}(t) - \langle v_k^{\tau_{kj}}(t), v_j(t)\rangle v_j(t)\right\rangle \\
&= -\frac{2\kappa_1}{N}\sum_{k\in[N]-\{i\}}\frac{\phi_{ki}^{\tau_{ki}}}{T_k^{\tau_{ki}}}(t)\left\langle v_j(t), v_k(t) - \langle v_k(t), v_i(t)\rangle v_i(t)\right\rangle \\
&\quad - \frac{2\kappa_1}{N}\sum_{k\in[N]-\{j\}}\frac{\phi_{kj}^{\tau_{kj}}}{T_k^{\tau_{kj}}}(t)\left\langle v_i(t), v_k(t) - \langle v_k(t), v_j(t)\rangle v_j(t)\right\rangle \\
&\quad - \frac{2\kappa_1}{N}\sum_{k\in[N]-\{i\}}\frac{\phi_{ki}^{\tau_{ki}}}{T_k^{\tau_{ki}}}(t)\left\langle v_j(t), v_k^{\tau_{ki}}(t) - v_k(t) - \langle v_k^{\tau_{ki}}(t) - v_k(t), v_i(t)\rangle v_i(t)\right\rangle \\
&\quad - \frac{2\kappa_1}{N}\sum_{k\in[N]-\{j\}}\frac{\phi_{kj}^{\tau_{kj}}}{T_k^{\tau_{kj}}}(t)\left\langle v_i(t), v_k^{\tau_{kj}}(t) - v_k(t) - \langle v_k^{\tau_{kj}}(t) - v_k(t), v_j(t)\rangle v_j(t)\right\rangle \\
&=: \mathcal{L}_{11} + \mathcal{L}_{12} + \mathcal{L}_{13} + \mathcal{L}_{14}.
\end{aligned}
$$

Below, we estimate these four parts separately into two parts.

⋄ (Estimate of $\mathcal{L}_{11} + \mathcal{L}_{12}$): In this case, one can verify the following inequalities at time t:

$$
\begin{aligned}
&\left\langle v_j, v_k - \langle v_k, v_i\rangle v_i\right\rangle \geq \langle v_j, v_i\rangle - \langle v_j, v_i\rangle\langle v_k, v_i\rangle = \langle v_j, v_i\rangle\,(1 - \langle v_k, v_i\rangle) \geq 0, \\
&\left\langle v_i, v_k - \langle v_k, v_j\rangle v_j\right\rangle \geq \langle v_i, v_j\rangle - \langle v_i, v_j\rangle\langle v_k, v_j\rangle = \langle v_i, v_j\rangle\left(1 - \langle v_k, v_j\rangle\right) \geq 0,
\end{aligned}
$$

where we used $\langle v_i, v_j\rangle = 1 - \frac{1}{2}D_V^2 \geq 0$ in the last inequality. This implies that both $\mathcal{L}_{11}$ and $\mathcal{L}_{12}$ are nonpositive numbers, and therefore

$$
\begin{aligned}
\mathcal{L}_{11} + \mathcal{L}_{12} &\leq -\frac{2\kappa_1}{N}\sum_{k\in[N]-\{i\}}\frac{\phi(D_X+\tau)}{T_M^\tau}\left\langle v_j, v_k - \langle v_k, v_i\rangle v_i\right\rangle \\
&\quad - \frac{2\kappa_1}{N}\sum_{k\in[N]-\{j\}}\frac{\phi(D_X+\tau)}{T_M^\tau}\left\langle v_i, v_k - \langle v_k, v_j\rangle v_j\right\rangle \\
&= -\frac{2\kappa_1\phi(D_X+\tau)}{NT_M^\tau}\sum_{k\in[N]-\{i,j\}}\left[\left\langle v_j, v_k - \langle v_k, v_i\rangle v_i\right\rangle + \left\langle v_i, v_k - \langle v_k, v_j\rangle v_j\right\rangle\right] \\
&\quad - \frac{2\kappa_1\phi(D_X+\tau)}{NT_M^\tau}\sum_{k\in\{i,j\}}[1 - \langle v_i, v_j\rangle^2],
\end{aligned}
\tag{5.1}
$$

where we used the monotonic decreasing property of ϕ and

$$\|x_i - x_k^{\tau_{ki}}\| \le \|x_i - x_k\| + \|x_k - x_k^{\tau_{ki}}\| \le D_X + \tau, \quad T_i(t-\tau_{ki}(t)) \le T_M^\tau(t),$$
$$\|x_j - x_k^{\tau_{kj}}\| \le \|x_j - x_k\| + \|x_k - x_k^{\tau_{kj}}\| \le D_X + \tau, \quad T_j(t-\tau_{kj}(t)) \le T_M^\tau(t),$$

in the first inequality. Then, we use $\langle v_i + v_j, v_k\rangle \ge 2\langle v_i, v_j\rangle$ and

$$\sum_{k\in\{i,j\}} (1-\langle v_i, v_j\rangle) = \begin{cases} 2(1-\langle v_i, v_j\rangle) & (i \ne j) \\ 0 & (i = j) \end{cases} = 2(1-\langle v_i, v_j\rangle) \tag{5.2}$$

in Eq (5.1) to obtain

$$\begin{aligned}
&\mathcal{L}_{11} + \mathcal{L}_{12} \\
&\quad \le -\frac{2\kappa_1\phi(D_X+\tau)}{NT_M^\tau}\Big(1-\langle v_i, v_j\rangle\Big)\left[\sum_{k\in[N]-\{i,j\}} 2\langle v_i, v_j\rangle + \sum_{k\in\{i,j\}}(1+\langle v_i, v_j\rangle)\right] \\
&\quad = -\frac{2\kappa_1\phi(D_X+\tau)}{NT_M^\tau}\Big(1-\langle v_i, v_j\rangle\Big)\left[2N\langle v_i, v_j\rangle + \sum_{k\in\{i,j\}}(1-\langle v_i, v_j\rangle)\right] \\
&\quad = -\frac{2\kappa_1\phi(D_X+\tau)}{T_M^\tau}\Big(1-\langle v_i, v_j\rangle\Big)\left[2\langle v_i, v_j\rangle + \frac{2}{N}(1-\langle v_i, v_j\rangle)\right].
\end{aligned} \tag{5.3}$$

⋄ (Estimate of $\mathcal{L}_{13} + \mathcal{L}_{14}$): From direct calculation, we have

$$\begin{aligned}
\mathcal{L}_{13} + \mathcal{L}_{14} &= -\frac{2\kappa_1}{N}\sum_{k\in[N]-\{i\}} \frac{\phi_{ki}^{\tau_{ki}}}{T_k^{\tau_{ki}}}\Big\langle v_j - v_i, v_k^{\tau_{ki}} - v_k - \langle v_k^{\tau_{ki}} - v_k, v_i\rangle v_i\Big\rangle \\
&\qquad - \frac{2\kappa_1}{N}\sum_{k\in[N]-\{j\}} \frac{\phi_{kj}^{\tau_{kj}}}{T_k^{\tau_{kj}}}\Big\langle v_i - v_j, v_k^{\tau_{kj}} - v_k - \langle v_k^{\tau_{kj}} - v_k, v_j\rangle v_j\Big\rangle \\
&\le \frac{4(N-1)\kappa_1\phi(0)}{NT_m^\tau}\|v_i - v_j\|\Delta_V^\tau,
\end{aligned} \tag{5.4}$$

where we used

$$\phi_{ki}^{\tau_{ki}} \le \phi(0), \quad T_m^\tau \le T_k^{\tau_{ki}}, \quad \Big\|v_k^{\tau_{ki}} - v_k - \langle v_k^{\tau_{ki}} - v_k, v_i\rangle v_i\Big\| \le \|v_k^{\tau_{ki}} - v_k\| \le \Delta_V^\tau,$$
$$\phi_{kj}^{\tau_{kj}} \le \phi(0), \quad T_m^\tau \le T_k^{\tau_{kj}}, \quad \Big\|v_k^{\tau_{kj}} - v_k - \langle v_k^{\tau_{kj}} - v_k, v_j\rangle v_j\Big\| \le \|v_k^{\tau_{kj}} - v_k\| \le \Delta_V^\tau,$$

in the last inequality.

Therefore, we combine Eqs (5.3) and (5.4) to obtain

$$D^+[(D_V)^2](t) \le -\frac{\kappa_1\phi(D_X+\tau)}{T_M^\tau}D_V^2\left[2 - \left(1-\frac{1}{N}\right)D_V^2\right] + \frac{4(N-1)\kappa_1\phi(0)}{NT_m^\tau}D_V\Delta_V^\tau,$$

which implies

$$\frac{dD_V}{dt} \le -\frac{\kappa_1\phi(D_X+\tau)}{T_M^\tau}D_V\left[1 - \frac{N-1}{2N}D_V^2\right] + \frac{2(N-1)\kappa_1\phi(0)}{NT_m^\tau}\Delta_V^\tau,$$

whenever D_V is differentiable and $D_V \le \sqrt{2}$.

(3) Let $M(t), m(t)$ be the set of all indices such that

$$i \in M(t),\ j \in m(t) \quad \Longleftrightarrow \quad T_i(t) = T_M(t),\ T_j(t) = T_m(t).$$

Then, for every $i \in M(t)$ and $j \in m(t)$, we can split the derivative of $T_i - T_j$ at time t into two parts as follows:

$$\begin{aligned}
\frac{d}{dt}(T_i - T_j) &= \frac{\kappa_2}{N}\sum_{k=1}^{N}\zeta_{ki}^{\tau_{ki}}\left(\frac{1}{T_i} - \frac{1}{T_k^{\tau_{ki}}}\right) - \frac{\kappa_2}{N}\sum_{k=1}^{N}\zeta_{kj}^{\tau_{kj}}\left(\frac{1}{T_j} - \frac{1}{T_k^{\tau_{kj}}}\right)\\
&= \frac{\kappa_2}{N}\sum_{k=1}^{N}\zeta_{ki}^{\tau_{ki}}\left(\frac{1}{T_i} - \frac{1}{T_k}\right) - \frac{\kappa_2}{N}\sum_{k=1}^{N}\zeta_{kj}^{\tau_{kj}}\left(\frac{1}{T_j} - \frac{1}{T_k}\right)\\
&\quad + \frac{\kappa_2}{N}\sum_{k=1}^{N}\zeta_{ki}^{\tau_{ki}}\left(\frac{1}{T_k} - \frac{1}{T_k^{\tau_{ki}}}\right) - \frac{\kappa_2}{N}\sum_{k=1}^{N}\zeta_{kj}^{\tau_{kj}}\left(\frac{1}{T_k} - \frac{1}{T_k^{\tau_{kj}}}\right)\\
&=: \mathcal{L}_{21} + \mathcal{L}_{22}.
\end{aligned}$$

Then, we estimate these two parts separately.

$\diamond$ (Estimate of $\mathcal{L}_{21}$): From a direct calculation, it follows that

$$\begin{aligned}
\mathcal{L}_{21} &\le -\frac{\kappa_2\zeta(D_X+\tau)}{N}\left[\sum_{k\in[N]-\{i\}}\left(\frac{1}{T_k} - \frac{1}{T_i}\right) + \sum_{k\in[N]-\{j\}}\left(\frac{1}{T_j} - \frac{1}{T_k}\right)\right]\\
&= -\frac{\kappa_2\zeta(D_X+\tau)}{N}\left[\sum_{k\in[N]-\{i,j\}}\left(\frac{1}{T_k} - \frac{1}{T_i}\right) + \sum_{k\in[N]-\{i,j\}}\left(\frac{1}{T_j} - \frac{1}{T_k}\right) + \sum_{k\in\{i,j\}}\left(\frac{1}{T_j} - \frac{1}{T_i}\right)\right]\\
&= -\kappa_2\zeta(D_X+\tau)\left(\frac{1}{T_j} - \frac{1}{T_i}\right),
\end{aligned}$$

where we used $T_j(t) \le T_k(t) \le T_i(t)$ for all $i \in M(t)$ and $j \in m(t)$ in the first inequality.

$\diamond$ (Estimate of $\mathcal{L}_{22}$): In this case, we use Lemma 3.2 to obtain

$$\begin{aligned}
\mathcal{L}_{22} &\le \frac{\kappa_2\zeta(0)}{N}\sum_{k\in[N]-\{i\}}\frac{|T_k - T_k^{\tau_{ki}}|}{T_kT_k^{\tau_{ki}}} + \frac{\kappa_2\zeta(0)}{N}\sum_{k\in[N]-\{j\}}\frac{|T_k - T_k^{\tau_{kj}}|}{T_kT_k^{\tau_{kj}}}\\
&\le \frac{2(N-1)\kappa_2\zeta(0)}{N}\cdot\frac{\Delta_T^\tau}{(T_m^\tau)^2}.
\end{aligned}$$

Therefore, we combine these two estimates to obtain

$$D^+[D_T](t) \le -\kappa_2\zeta(D_X(t)+\tau)\left(\frac{1}{T_m(t)} - \frac{1}{T_M(t)}\right) + \frac{2(N-1)\kappa_2\zeta(0)}{N(T_m^\tau(t))^2}\Delta_T^\tau(t),$$

which implies the desired result. □

Remark 5.1. *One thing to keep in mind during the proof of Lemma 5.1 is that the acceleration may not be zero even if there is a point at which the velocity diameter D_V is zero, due to the time-delay effect. Therefore, even if we find i, j with $D_V = \|v_i - v_j\|$, we cannot exclude the possibility that i and j are equal. Another thing to keep in mind is that the condition $D_V(t) \le \sqrt{2}$ is necessary to obtain a meaningful inequality for D_V because of the technical reason, otherwise the values*

$$\langle v_i, v_k - \langle v_k, v_j\rangle v_j\rangle = \langle v_i, v_k\rangle - \langle v_k, v_j\rangle\langle v_i, v_j\rangle,$$
$$\langle v_j, v_k - \langle v_k, v_i\rangle v_i\rangle = \langle v_j, v_k\rangle - \langle v_k, v_i\rangle\langle v_j, v_i\rangle,$$

can be negative when $\langle v_i, v_k\rangle = \langle v_j, v_k\rangle = \langle v_i, v_j\rangle < 0$.

Similar to what we have done in Section 4, we prepare a lemma to present the upper bound of (D_X, D_V, D_T) and (Δ_V, Δ_T) by using Lemma 3.5 and Lemma 5.1.

Lemma 5.2. *Let $y : [0, \infty) \to [0, \infty)$ be a Lipschitz continuous function and $f : [0, \infty) \to \mathbb{R}$ be a continuous function. Assume that y and f satisfy*

$$\dot{y}(t) \le -\chi y(t) + 2\theta f(t), \quad t > 0, \tag{5.5a}$$
$$f(t) \le L \max\{\tau - t, 0\} + \theta \int_{\max\{t-\tau,0\}}^{t} (y(s) + f(s))ds, \quad t > 0, \tag{5.5b}$$

for some constants $\chi, \theta, L > 0$ and $\tau \ge 0$. If τ is sufficiently small to satisfy

$$\tau\theta\left(\frac{2\theta}{\chi} + 1\right) < 1, \tag{5.6}$$

the following assertions hold.

(1) There are two constants $\xi > 0$ and $\mu \in (0, \chi)$ such that

$$\left[\frac{Le^{\mu\tau}}{\xi} + \theta\left(\frac{2\theta}{\chi - \mu} + 1\right) \cdot \frac{e^{\mu\tau} - 1}{\mu\tau}\right]\tau \le 1, \quad y(0) \le \frac{2\theta\xi}{\chi - \mu}, \quad f(0) < \xi. \tag{5.7}$$

(2) Whenever $\xi, \mu > 0$ satisfy Eq (5.7), we have

$$\begin{aligned} &y(t) < y(0)e^{-\chi t} + \frac{2\xi\theta}{\chi - \mu}\left(e^{-\mu t} - e^{-\chi t}\right), \quad t > 0, \\ &f(t) < \xi e^{-\mu t}, \quad t \ge 0. \end{aligned} \tag{5.8}$$

Proof. From Eqs (5.5) and (5.6), we can choose a sufficiently large ξ satisfying

$$f(0) \le L\tau < \frac{L\tau}{1 - \tau\theta\left(\frac{2\theta}{\chi} + 1\right)} < \xi, \quad y(0) \le \frac{2\theta\xi}{\chi}.$$

Then, the condition (5.7) holds for sufficiently small positive number $\mu \in (0, \chi)$. Now, assume there exists a minimum $t_* \in (0, \infty)$ among all t satisfying

$$f(t) = \xi e^{-\mu t} \tag{5.9}$$

Then, we have

$$f(t) < \xi e^{-\mu t}, \quad t \in [0, t_*), \tag{5.10}$$

and we substitute (5.10) to (5.5) to obtain

$$\dot{y}(t) < -\chi y(t) + 2\theta\xi e^{-\mu t}, \quad t \in (0, t_*).$$

Consequently, this Grönwall's inequality yields

$$\begin{aligned} y(t) &< y(0)e^{-\chi t} + \frac{2\theta\xi}{\chi - \mu}\left(e^{-\mu t} - e^{-\chi t}\right) \\ &\le \frac{2\theta\xi}{\chi - \mu} e^{-\mu t} \quad t \in (0, t_*], \end{aligned} \tag{5.11}$$

where we used Eq (5.7) in the last inequality. On the other hand, by using Eqs (5.10) and (5.11) to (5.5b) at $t = t_*$, we have

$$\begin{aligned} f(t_*) &\le L \max\{\tau - t_*, 0\} + \theta \int_{\max\{t_* - \tau, 0\}}^{t_*} (y(s) + f(s))ds \\ &< L \max\{\tau - t_*, 0\} + \theta \int_{\max\{t_* - \tau, 0\}}^{t_*} \left(\frac{2\theta\xi}{\chi - \mu} + \xi\right) e^{-\mu s} ds \\ &= L \max\{\tau - t_*, 0\} + \frac{\theta}{\mu}\left(\frac{2\theta\xi}{\chi - \mu} + \xi\right)\left(e^{-\mu \max\{t_* - \tau, 0\}} - e^{-\mu t_*}\right) \\ &= \left[L \max\{\tau - t_*, 0\} e^{\mu t_*} + \frac{\theta}{\mu}\left(\frac{2\theta\xi}{\chi - \mu} + \xi\right)\left(e^{\mu \min\{t_*, \tau\}} - 1\right)\right] e^{-\mu t_*} \\ &\le \left[L\tau e^{\mu\tau} + \frac{\theta}{\mu}\left(\frac{2\theta\xi}{\chi - \mu} + \xi\right)(e^{\mu\tau} - 1)\right] e^{-\mu t_*} \\ &= \left[L e^{\mu\tau} + \theta\xi\left(\frac{2\theta}{\chi - \mu} + 1\right)\left(\frac{e^{\mu\tau} - 1}{\mu\tau}\right)\right] \tau e^{-\mu t_*} \\ &\le \xi e^{-\mu t_*}, \end{aligned}$$

where we used Eq (5.7) in the last inequality. However, this leads a contradiction to Eq (5.9), and therefore

$$f(t) < \xi e^{-\mu t}, \quad \forall\, t \ge 0,$$

which implies the desired inequality (5.8). □

Unlike in Section 4, we need to guarantee $D_V(t) \le \sqrt{2}$ to apply Lemma 5.2 to Lemma 5.1. From Eqs (5.7) and (5.8), one can find a time-invariant upper bound for $y(t)$. Since the function

$$t \mapsto y(0)e^{-\chi t} + \frac{2\xi\theta}{\chi - \mu}(e^{-\mu t} - e^{-\chi t})$$

has a maximum at $t = t_0$ satisfying

$$-\mu \frac{2\theta}{\chi - \mu} e^{-\mu t_0} + \chi\left(\frac{2\xi\theta}{\chi - \mu} - y(0)\right) e^{-\chi t_0} = 0,$$

we have

$$\begin{aligned} y(0)e^{-\chi t_0} + \frac{2\xi\theta}{\chi-\mu}(e^{-\mu t_0} - e^{-\chi t_0}) &= \frac{2\xi\theta}{\chi-\mu}e^{-\mu t_0} - \left(\frac{2\xi\theta}{\chi-\mu} - y(0)\right)e^{-\chi t_0} \\ &= \left(1-\frac{\mu}{\chi}\right)\frac{2\xi\theta}{\chi-\mu}e^{-\mu t_0} \\ &= \frac{2\xi\theta}{\chi}e^{-\mu t_0} \le \frac{2\xi\theta}{\chi}. \end{aligned}$$

Therefore, we have the following time-invariant upper bound for $y(t)$:

$$y(t) < \frac{2\xi\theta}{\chi}, \quad \forall\, t > 0.$$

Now, by using Lemma 5.1 and 5.2, we can construct another sufficient framework leading to the asymptotic flocking. In this case, we assume $D_X(0)$ and $D_V(0)$ sufficiently small, instead of assuming the smallness of $D_X^\tau(0)$ and $D_V^\tau(0)$.

- $(\mathcal{F}1)'$: $D_V(0) = 2\sin\frac{\delta}{2}, \quad \delta \in (0, \frac{\pi}{2})$.
- $(\mathcal{F}2)'$: There exists a constant $D_X^\infty > 0$ satisfying

$$\Phi(D_X^\infty + \tau) > \Phi(D_X(0) + \tau) + \frac{T_M^\tau(0)}{\kappa_1 \cos\delta} \cdot 2\sin\frac{\delta}{2}.$$

- $(\mathcal{F}3)'$: The time-delay bound $\tau > 0$ is sufficiently small to satisfy

$$\Phi(D_X^\infty + \tau) > \Phi(D_X(0) + \tau) + \frac{T_M^\tau(0)}{\kappa_1 \cos\delta} \cdot \left(2\sin\frac{\delta}{2} + \frac{2(\chi_1 - \mu_1)}{\mu_1}\right), \tag{5.12a}$$

$$\left[\frac{\theta_1 L_1 e^{\mu_1 \tau}}{\chi_1 - \mu_1} + \theta_1\left(\frac{2\theta_1}{\chi_1 - \mu_1} + 1\right) \cdot \frac{e^{\mu_1\tau} - 1}{\mu_1 \tau}\right]\tau = 1, \tag{5.12b}$$

$$\left[\frac{2\theta_2 L_2 e^{\mu_2 \tau}}{(\chi_2 - \mu_2)(T_M^\tau(0) - T_m^\tau(0))} + \theta_2\left(\frac{2\theta_2}{\chi_2 - \mu_2} + 1\right) \cdot \frac{e^{\mu_2\tau} - 1}{\mu_2 \tau}\right]\tau = 1, \tag{5.12c}$$

$$\frac{\chi_1 - \mu_1}{\chi_1} < \frac{1}{2}\min\left\{\sqrt{2}, \sqrt{\frac{N}{N-1}} D_V(0)\right\}, \tag{5.12d}$$

for some $\mu_1 \in (0, \chi_1)$ and $\mu_2 \in (0, \chi_2)$, where the constants $\chi_1, \chi_2, \theta_1, \theta_2, L_1, L_2$ are defined as

$$\begin{aligned} \chi_1 &= \frac{\kappa_1 \phi(D_X^\infty + \tau)}{T_M^\tau(0)} \cdot \left(1 - \frac{1}{2}D_V(0)^2\right), \quad \chi_2 = \frac{\kappa_2 \zeta(D_X^\infty + \tau)}{T_M^\tau(0)^2}, \\ \theta_1 &= \frac{(N-1)\kappa_1\phi(0)}{NT_m^\tau(0)}, \quad \theta_2 = \frac{(N-1)\kappa_2\zeta(0)}{NT_m^\tau(0)^2}, \\ L_1 &= \max_{i\in[N]} \|v_i^0\|_{\mathrm{Lip}}, \quad L_2 = \max_{i\in[N]} \|T_i^0\|_{\mathrm{Lip}}. \end{aligned}$$

Note that from the equality conditions in (5.12), μ_1 tends to χ_1 during $\tau \to +0$. Therefore, $(\mathcal{F}3)'$ is indeed a condition that holds for sufficiently small τ.

Finally, we are ready to provide the second main result, which also shows that D_X is uniformly bounded and D_V, D_T converge to zero. Although the proof might not be intuitive and its implications are hard to grasp at first glance, we wrote it down in detail for the completeness.

Theorem 5.1. *Let* (X, V, T) *be a solution to Eq* (3.1)*, where the initial data*

$$(x_i^0, v_i^0, T_i^0) : [-\tau, 0] \to \mathbb{R}^d \times \mathbb{S}^{d-1} \times (0, \infty), \quad i \in [N]$$

are Lipschitz continuous functions satisfying $\frac{dx_i^0}{dt} = v_i^0$ *and* $(\mathcal{F}1)'$*–*$(\mathcal{F}3)'$*. Then, the following inequalities hold for all* $t > 0$*:*

$$\begin{aligned}
&D_X(t) < D_X^\infty,\\
&D_V(t) < D_V(0)e^{-\chi_1 t} + 2\left(e^{-\mu_1 t} - e^{-\chi_1 t}\right),\\
&D_T(t) < D_T(0)e^{-\chi_2 t} + (T_M^\tau(0) - T_m^\tau(0))\left(e^{-\mu_2 t} - e^{-\chi_2 t}\right),\\
&\Delta_V(t) < \frac{\chi_1 - \mu_1}{\theta_1} e^{-\mu_1 t}, \quad \Delta_T(t) < \frac{(\chi_2 - \mu_2)(T_M^\tau(0) - T_m^\tau(0))}{2\theta_2} e^{-\mu_2 t}.
\end{aligned}$$

Therefore, the solution (X, V, T) *exhibits the asymptotic flocking.*

Proof. (Step 1) Assume there exists a minimum $t^* < \infty$ among all t satisfying

$$D_X(t) \geq D_X^\infty \quad \text{or} \quad 1 - \frac{N-1}{2N} D_V(t)^2 \leq 1 - \frac{1}{2} D_V(0)^2 \quad \text{or} \quad D_V(t) \geq \sqrt{2}. \tag{5.13}$$

Then, we will show that, in fact, only the first condition among the three can be satisfied, i.e., $D_X(t^*) = D_X^\infty$. First, one can obtain $t^* > 0$ by using $(\mathcal{F}1)', (\mathcal{F}2)'$, and the continuity of D_V, and we use the monotonic decreasing property of ϕ, ζ, Lemma 3.5, and Lemma 5.1 to obtain

$$\begin{aligned}
&\frac{dD_V(t)}{dt} \leq -\chi_1 D_V(t) + 2\theta_1 \Delta_V^\tau(t), \quad \text{a.e.} \quad t \in (0, t^*),\\
&\Delta_V^\tau(t) \leq L_1 \max\{\tau - t, 0\} + \theta_1 \int_{\max\{t-\tau,0\}}^t (D_V(s) + \Delta_V^\tau(s))ds, \quad t \in (0, t^*),
\end{aligned} \tag{5.14}$$

for the diameter D_V, and similarly,

$$\begin{aligned}
&\frac{dD_T(t)}{dt} \leq -\chi_2 D_T(t) + 2\theta_2 \Delta_T^\tau(t), \quad \text{a.e.} \quad t \in (0, t^*),\\
&\Delta_T^\tau(t) \leq L_2 \max\{\tau - t, 0\} + \theta_2 \int_{\max\{t-\tau,0\}}^t (D_T(s) + \Delta_T^\tau(s))ds, \quad t \in (0, t^*),
\end{aligned} \tag{5.15}$$

for the diameter D_T. Now, for the two constants ξ_1, ξ_2 defined as

$$\begin{aligned}
&\xi_1 = \frac{\chi_1 - \mu_1}{\theta_1} = \frac{L_1 e^{\mu_1 \tau} + 2\theta_1}{1 - \theta_1 \tau \left(\frac{e^{\mu_1 \tau} - 1}{\mu_1 \tau}\right)} \tau,\\
&\xi_2 = \frac{(\chi_2 - \mu_2)(T_M^\tau(0) - T_m^\tau(0))}{2\theta_2} = \frac{L_2 e^{\mu_2 \tau} + (T_M^\tau(0) - T_m^\tau(0))\theta_2}{1 - \theta_2 \tau \left(\frac{e^{\mu_2 \tau} - 1}{\mu_2 \tau}\right)} \tau,
\end{aligned}$$

one can verify that (5.7) holds for (L_1,χ_1,μ_1,ξ_1) and (L_2,χ_2,μ_2,ξ_2). More precisely, the first condition of (5.7) is immediately obtained from (5.12b) and (5.12c), and the third condition is obtained from

$$\Delta_V^\tau(0) \le L_1\tau, \quad \Delta_T^\tau(0) \le L_2\tau,$$

which is a consequence of Lemma 3.5 for $t = 0$. The second condition of (5.7) can also be verified by the relation

$$D_V(0) \le 2 = \frac{2\xi_1\theta_1}{\chi_1-\mu_1}, \quad D_T(0) \le T_M^\tau(0) - T_m^\tau(0) = \frac{2\theta_2\xi_2}{\chi_2-\mu_2}.$$

As a consequence, we apply Lemma 5.2 to Eqs (5.14) and (5.15) to get the following inequalities for $t \in (0,t^*)$:

$$\begin{aligned} &D_V(t) < D_V(0)e^{-\chi_1 t} + 2\left(e^{-\mu_1 t} - e^{-\chi_1 t}\right), \\ &\Delta_V(t) < \frac{\chi_1-\mu_1}{\theta_1}e^{-\mu_1 t}, \\ &D_T(t) < D_T(0)e^{-\chi_2 t} + (T_M^\tau(0) - T_m^\tau(0))\left(e^{-\mu_2 t} - e^{-\chi_2 t}\right), \\ &\Delta_T(t) < \frac{(\chi_2-\mu_2)(T_M^\tau(0)-T_m^\tau(0))}{2\theta_2}e^{-\mu_2 t}. \end{aligned} \tag{5.16}$$

Then, we use Eq (5.12d) to obtain

$$D_V(t) < \frac{2\xi_1\theta_1}{\chi_1} = \frac{2(\chi_1-\mu_1)}{\chi_1} < \min\left\{\sqrt{2}, \sqrt{\frac{N}{N-1}D_V(0)}\right\}, \quad t \in (0,t^*),$$

and therefore, the only possible case to satisfy (5.13) is the first case, i.e., $D_X(t^*) = D_X^\infty$.
(Step 2) Therefore, we can apply Lemma 5.1 (2) to $t \in (0,t^*)$ to obtain

$$\begin{aligned} \frac{dD_V(t)}{dt} &\le -\frac{\kappa_1\phi(D_X(t)+\tau)}{T_M^\tau(t)}\left[1 - \frac{N-1}{2N}D_V(t)^2\right]D_V(t) + \frac{2(N-1)\kappa_1\phi(0)}{NT_m^\tau(t)}\Delta_V^\tau(t) \\ &\le -\frac{\kappa_1\phi(D_X(t)+\tau)}{T_M^\tau(0)}\left[1 - \frac{1}{2}D_V(0)^2\right]D_V(t) + 2\xi_1\theta_1 e^{-\mu_1 t}, \quad t \in (0,t^*), \end{aligned}$$

which implies that

$$\frac{d}{dt}\left[D_V(t) + \frac{\kappa_1\left(1-\frac{1}{2}D_V(0)^2\right)}{T_M^\tau(0)}\Phi(D_X(t)+\tau) + \frac{2\xi_1\theta_1}{\mu_1}e^{-\mu_1 t}\right] \le 0, \quad t \in (0,t^*). \tag{5.17}$$

However, (5.17) yields

$$\frac{\kappa_1\left(1-\frac{1}{2}D_V(0)^2\right)}{T_M^\tau(0)}\Phi(D_X(t^*)+\tau) \le D_V(0) + \frac{\kappa_1\left(1-\frac{1}{2}D_V(0)^2\right)}{T_M^\tau(0)}\Phi(D_X(0)+\tau) + \frac{2\xi_1\theta_1}{\mu_1},$$

which leads a contradiction to (5.12a). Thus, there is no $t^* \in (0,\infty)$ satisfying (5.13), and the inequalities in (5.16) hold for all $t > 0$. □

Remark 5.2. *Similar to the case in Remark 4.2, Theorem 5.1 is also applicable when time-delay τ is set to* 0*. If $\tau = 0$, the framework $(\mathcal{F}1)' - (\mathcal{F}3)'$ becomes*

$$A(0) > 0, \quad \int_{D_X(0)}^{\infty} \phi(s)ds > \frac{T_M(0)D_V(0)}{\kappa_1 A(0)},$$

which coincides with the condition (2.2)*. In addition, the condition* (5.12) *reduces to*

$$\mu_1 = \chi_1, \quad \mu_2 = \chi_2.$$

Therefore, Theorem 5.1 also exactly coincides with Proposition 2.5 for $\tau = 0$.

6. Numerical simulation

In this section, we perform numerical simulations on (1.3) to verify the results of Theorem 4.1 and Theorem 5.1, and to explore whether there are other properties that we could not prove due to technical reasons. For the numerical implementation, we used the *Euler method* with time step $\Delta t = 0.05$, and we fixed N, ϕ, ζ as

$$N = 50, \quad \phi(r) = \frac{1}{1+r^2}, \quad \zeta(r) = \frac{1}{1+r^{3/2}}, \quad r \geq 0. \tag{6.1}$$

In addition, for the time delay τ_{ij}, we set

$$\tau_{ij}(t) := \tau \times \left[\frac{2+\sin(0.1(i+j)t)}{3}\right], \quad t \geq 0, \tag{6.2}$$

for each $i, j \in [N]$. Thus, we compare how the dynamics of (1.3) vary with respect to κ_1, κ_2, τ, and initial data.

In particular, we prepare two types of initial data for equation (1.3), which we will refer to as 'good initial' and 'bad initial' based on the velocity range. The way we set the initial conditions for X and T is common to both types and is as follows. For each $i \in [N]$ and $n \in [-\tau/\Delta t, 0]$, we randomly select $x_i(n)$ from a uniform distribution on $(0,1)^3$ and $T_i(n)$ from a uniform distribution on $(1, 3.5)$. Then, the 'good initial' refers to the initial data for which

Lemma 3.4 and Theorem 4.1 can be applied.

To achieve this, for each $i \in [N]$ and $n \in [-\tau/\Delta t, 0]$, we randomly select $\tilde{v}_i(n)$ from a uniform distribution on $(0,1)^3$, just like the initial positions $x_i(n)$. Then, we normalize their norms to 1 as follows:

$$v_i(n) := \frac{\tilde{v}_i(n)}{\|\tilde{v}_i(n)\|}, \quad i \in [N], \quad n \in [-\tau/\Delta t, 0]. \tag{6.3}$$

If so, the discretized version of $A^{\tau,\tau}(0,0)$, i.e.,

$$\min_{\substack{n_1\Delta t \in [-\tau,0] \\ n_2 \Delta t \in [-\tau,0]}} \min_{i,j\in[N]} \langle v_i(n_1), v_j(n_2)\rangle$$

is strictly positive, and we can expect that (the discrete analogue of) $A^{\tau,\tau}(t_1,t_2)$ will be monotonically increasing, as proven in Lemma 3.4. On the other hand, by 'bad initial,' we refer to the initial data for which

Lemma 3.4 and Theorem 4.1 cannot be applied, but Theorem 5.1 can be applied.

In other words, it refers to the initial conditions that do not satisfy the condition $(\mathcal{F}1)$ but satisfy the condition $(\mathcal{F}1)'$. We meet these requirements by randomly selecting $\tilde{v}_i(n)$ from a uniform distribution on $(-1,1)^3$ for each $i \in [N]$ and $n \in [-\tau/\Delta t, -1]$, and also select $\tilde{v}_i(0)$ randomly from a uniform distribution on $(0,1)^3$. Then, we normalize their norms to 1 as in Eq (6.3) to satisfy the unit speed constraint. In this case, we can expect that flocking can occur when τ is sufficiently small, according to Theorem 5.1. However, since we cannot apply Lemma 3.4, $A^{\tau,\tau}$ may not be increasing monotonically at the beginning. Thus, one of the goals of this section is to confirm that the initial behavior of $A^{\tau,\tau}$ differs between good initial and bad initial based on the applicability of Lemma 3.4. However, when we actually ran the simulations, we found that observing the behavior of $D_V(t)$ is sufficient to clearly distinguish between the two cases. Therefore, to save computation time, we only display $D_X(t), D_V(t), T_M(t)$, and $T_m(t)$ over t (time) in all figures of this section.

6.1. *Simulations for good initial data*

In Figures 1 and 2, we fix $\tau = 1.8$, $\kappa_1 = \kappa_2 = 8$ and used the good initial data to simulate the solution of (1.3) for $t \in [0, 50]$. The first two plots in Figure 1 are the temporal evolution of $D_X(t)$ and $D_V(t)$ over t, and Figure 2 shows the temporal evolution of $T_M(t)$ and $T_m(t)$. These plots show that the solution exhibited asymptotic flocking under the given setting. The last two plots in Figure 1 confirm that the speeds of all particles remained at 1 throughout the simulation. In the case of D_V, it exhibits a monotonically decreasing behavior over time, while we can also observe an inflection point around $t = 2.4$. In fact, such inflection points were consistently observed in D_V throughout the simulations, and we found that their position was always approximately at $\frac{4\tau}{3}$. We suspect that the value $\frac{4}{3}$ itself is not particularly important, but rather that the average of τ_{ij} set in Eq (6.2) is $\frac{2\tau}{3}$, and something special seems to occur around twice that time.

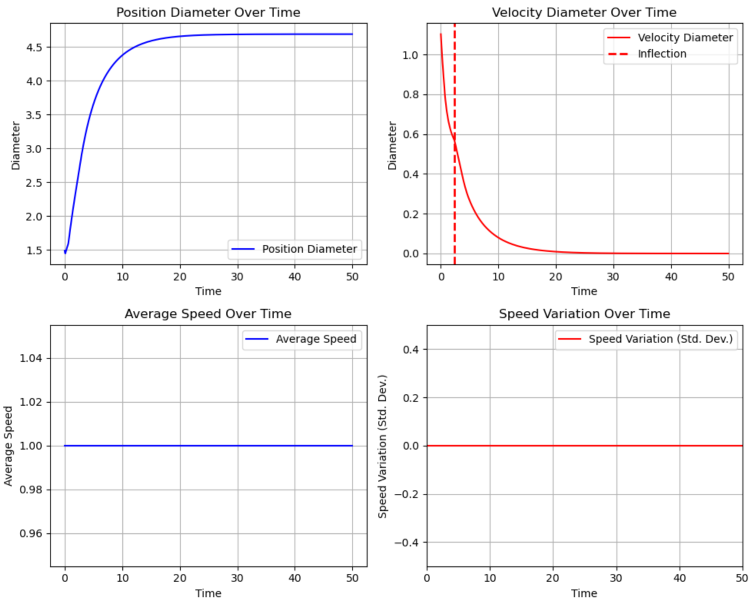

Figure 1. Good initial data, $\tau = 1.8$, $\kappa_1 = \kappa_2 = 8$.

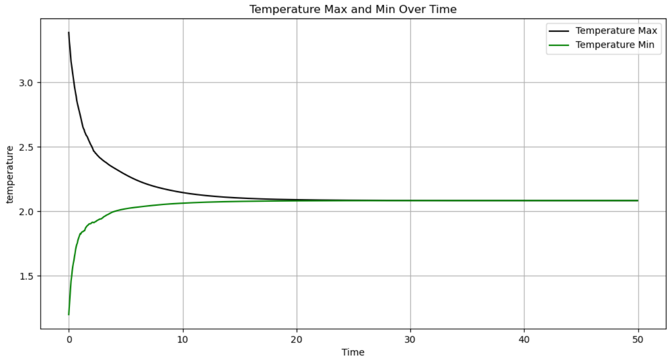

Figure 2. Good initial data, $\tau = 1.8$, $\kappa_1 = \kappa_2 = 8$.

In Figure 3, we fix $\tau = 6$, $\kappa_1 = \kappa_2 = 8$ and used the good initial data to simulate the solution of Eq (1.3) for $t \in [0, 500]$. Once again, asymptotic flocking occurred, but the rate of convergence was significantly slower compared to the previous examples. Additionally, we observed an inflection point in D_V around $t = \frac{4\tau}{3} = 8$.

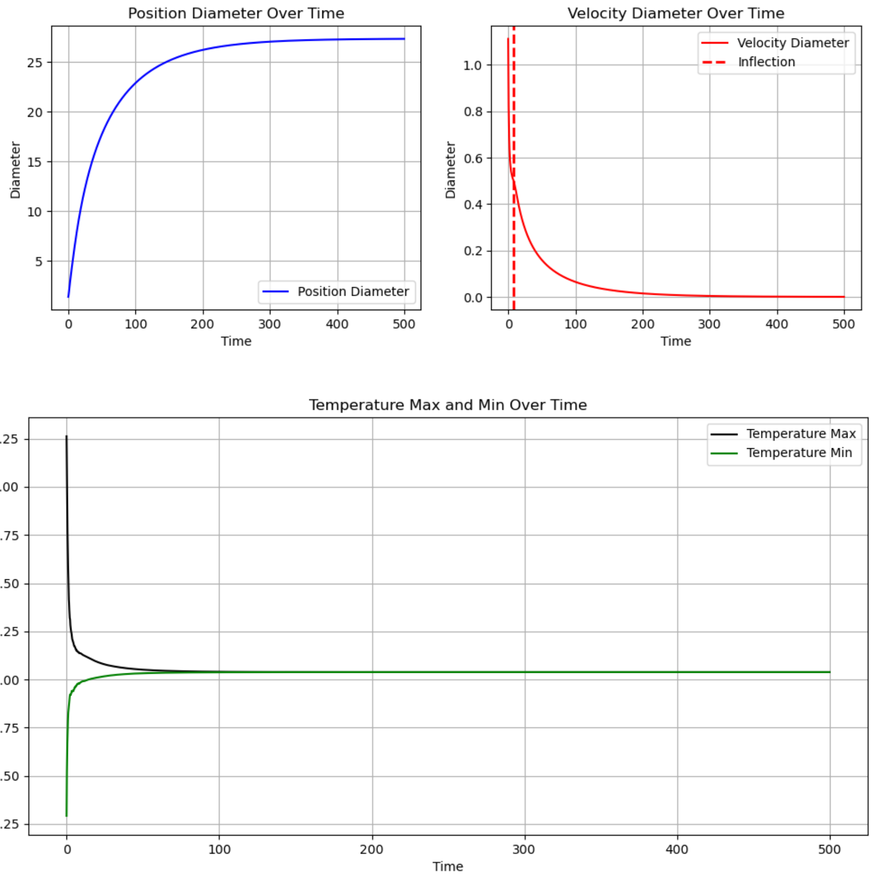

Figure 3. Good initial data, $\tau = 6$, $\kappa_1 = \kappa_2 = 8$.

In Figure 4, we fix $\tau = 30$, $\kappa_1 = \kappa_2 = 8$ and used the good initial data to simulate the solution of (1.3) for $t \in [0, 10000]$. As τ increases, the time it takes for D_V to saturate becomes much longer. Therefore, to verify whether flocking occurs, we had to observe the behavior of diameters for such a long time. For this reason, the inflection point is not clearly visible; however, the position of the dashed line set in this figure is also at $t = \frac{4\tau}{3} = 40$. Additionally, in this figure, we can see that D_V does not converge to 0, indicating that the solution did not exhibit flocking. This means that even if the condition $(\mathcal{F}1)$ is satisfied and κ_1, κ_2 are fixed, flocking will no longer occur if τ becomes sufficiently large.

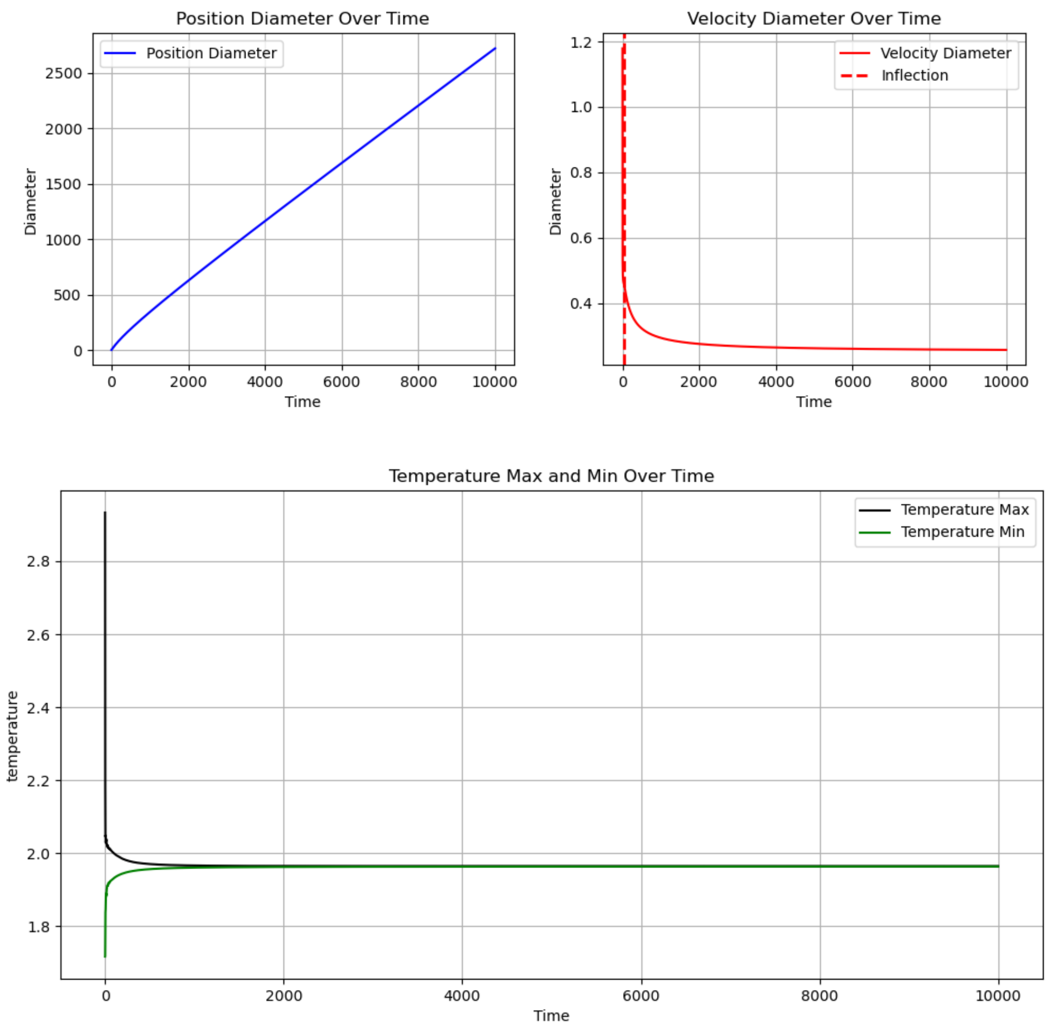

Figure 4. Good initial data, $\tau = 30$, $\kappa_1 = \kappa_2 = 8$.

In Figure 5, we fix $\tau = 1.8$, $\kappa_1 = \kappa_2 = 1$ and used the good initial data to simulate the solution of (1.3) for $t \in [0, 2500]$. In this case, although τ is small as in Figures 1 and 2, the small values of κ_1, κ_2 result in a situation where flocking does not occur. Notably, due to the small value of κ_2, D_T did not converge to 0 in this case.

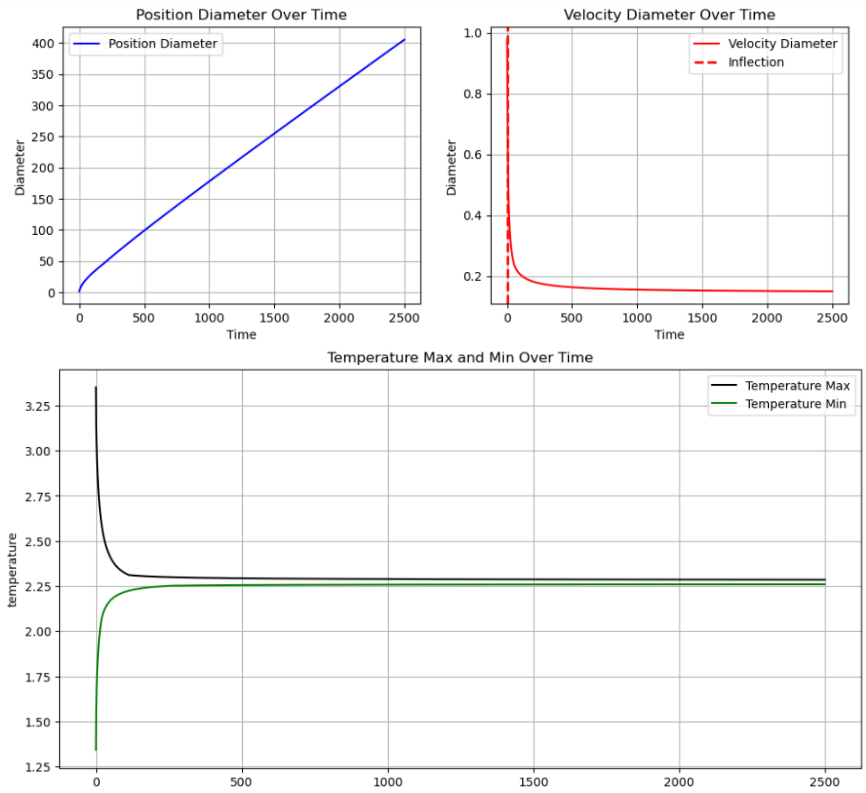

Figure 5. Good initial data, $\tau = 1.8$, $\kappa_1 = \kappa_2 = 1$.

6.2. *Simulations for bad initial data*

In Figures 6 and 7, we fix $\tau = 1.8$, $\kappa_1 = \kappa_2 = 8$ and used the bad initial data to simulate (1.3).

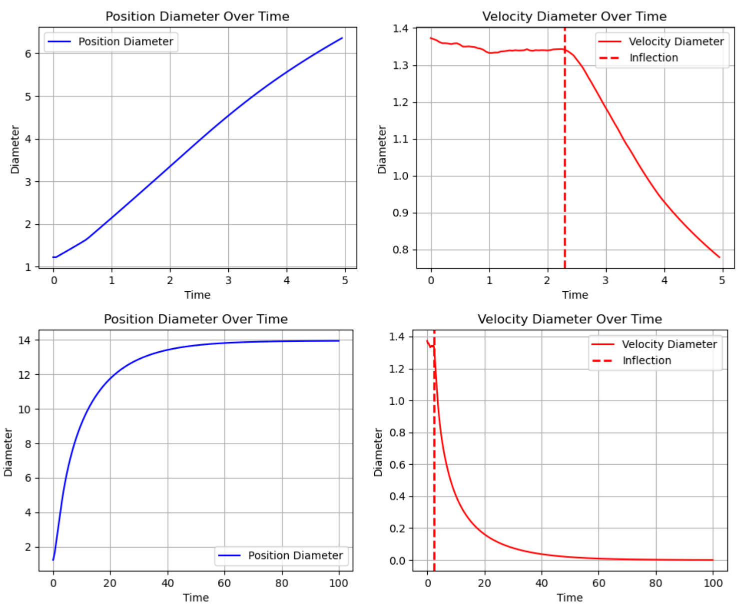

Figure 6. Bad initial data, $\tau = 1.8$, $\kappa_1 = \kappa_2 = 8$.

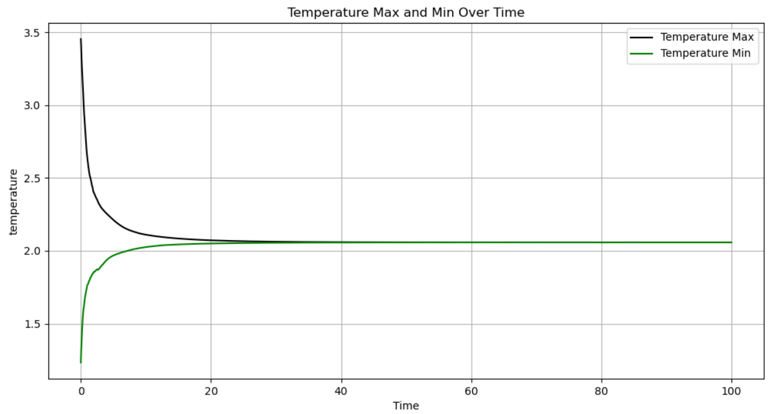

Figure 7. Bad initial data, $\tau = 1.8$, $\kappa_1 = \kappa_2 = 8$.

In other words, we are using the same τ and κ_1, κ_2 as in Figures 1 and 2, but with the bad initial data. The first two plots in Figure 6 illustrate the behavior of D_X and D_V from the given bad initial data up to time $t = 5$, while the last two plots show the behavior up to $t = 100$ for the same initial data. A notable feature observed in this figure is that, similar to Figure 1, the behavior of D_V changes around the time $t = \frac{4\tau}{3} = 2.4$. However, the behavior of $D_V(t)$ for $t \in [0, 2.4]$ becomes unstable rather than just monotonically decreasing, which is a result not observed in any simulations using good initial data. Nevertheless, as demonstrated in Theorem 5.1, even with bad initial data, the asymptotic flocking can still occur if τ is sufficiently small and κ_1, κ_2 are sufficiently large.

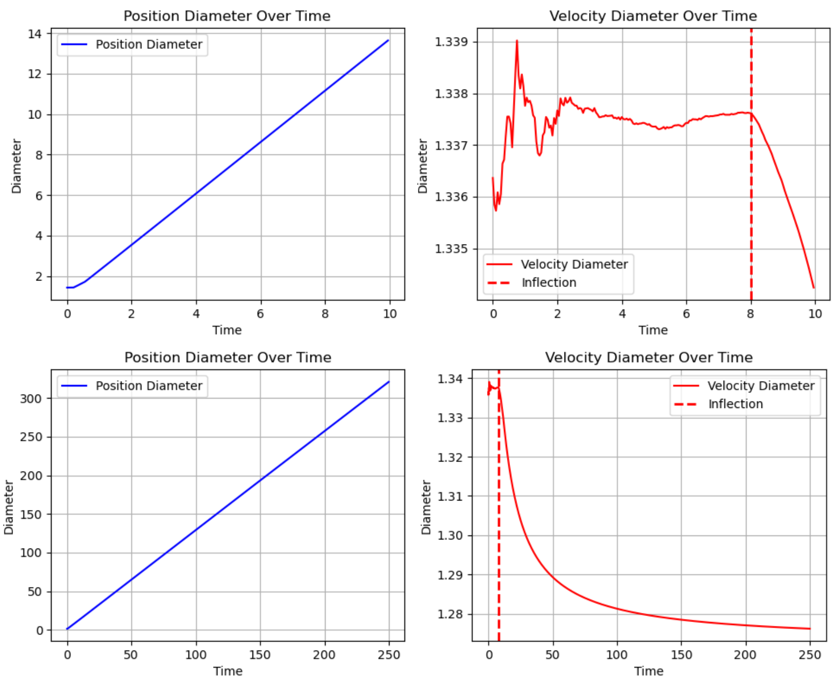

Figure 8. Bad initial data, $\tau = 6$, $\kappa_1 = \kappa_2 = 8$.

In Figures 8 and 9, we fix $\tau = 6$, $\kappa_1 = \kappa_2 = 8$ and used the bad initial data to simulate (1.3). In other words, we are using the same τ and κ_1, κ_2 as in Figure 3, but with bad initial data. The first two plots in Figure 6 illustrate the behavior of D_X and D_V from the given bad initial data up to time $t = 10$, while the last two plots show the behavior up to $t = 250$ for the same initial data. In this case, D_V shows a very unstable oscillation before $t = \frac{4\tau}{3} = 8$, and it even becomes larger than $D_V(0)$ at $t = 8$. However, this effect disappears after $t = 8$ and D_V begins to decrease monotonically. Nevertheless, since both D_V and D_T do not converge to 0 in this case, asymptotic flocking does not occur. This contrasts with the occurrence of asymptotic flocking in Figure 3, demonstrating that the condition ($\mathcal{F}1$) also affects the occurrence of asymptotic flocking.

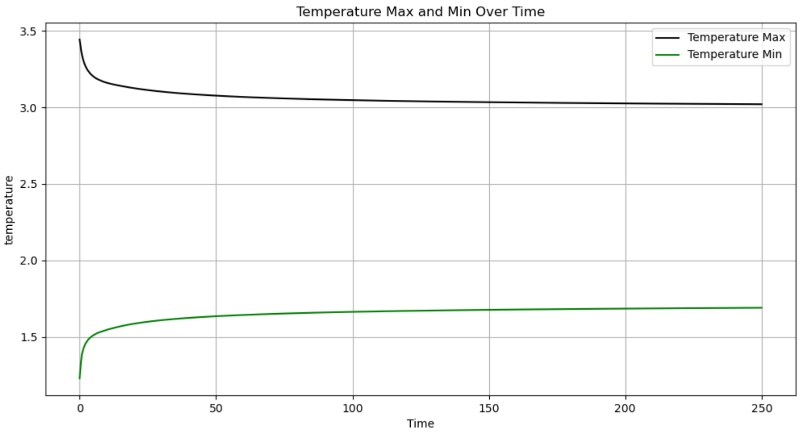

Figure 9. Bad initial data, $\tau = 6$, $\kappa_1 = \kappa_2 = 8$.

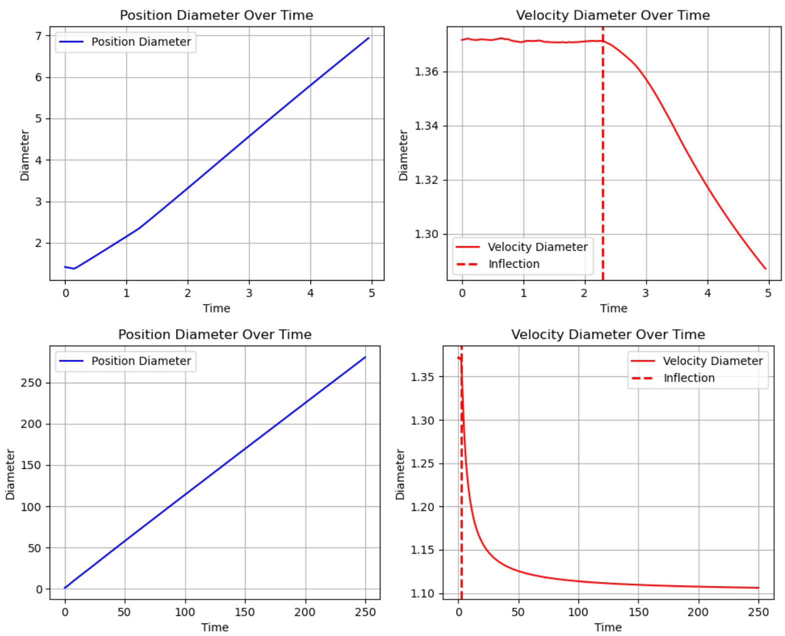

Figure 10. Bad initial data, $\tau = 1.8$, $\kappa_1 = \kappa_2 = 1$.

Finally, in Figures 10 and 11, we fix $\tau = 1.8$, $\kappa_1 = \kappa_2 = 1$ and used the bad initial data to simulate (1.3).

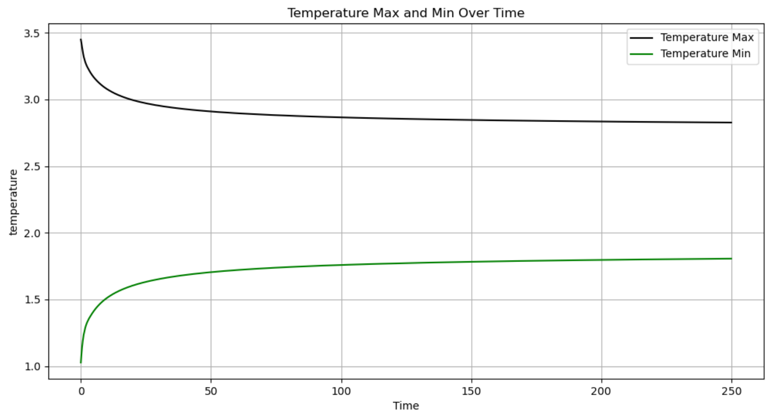

Figure 11. Bad initial data, $\tau = 1.8$, $\kappa_1 = \kappa_2 = 1$.

That is, we are using the same τ and κ_1, κ_2 as in Figures 5, but with the bad initial data. The first two plots in Figure 10 illustrate the behavior of D_X and D_V from the given bad initial data up to time $t = 5$, while the last two plots show the behavior up to $t = 150$ for the same initial data. In this case, D_V shows a small oscillation before $t = \frac{4\tau}{3} = 2.4$, but decrease monotonically after $t = 2.4$. However, as with the use of good initial data in Figure 5, asymptotic flocking did not occur in this case either.

7. Conclusion

In this paper, we have demonstrated several sufficient frameworks for the asymptotic flocking dynamics of the thermodynamic CS model with a unit-speed constraint and time-delay. To do this, we first proved the monotonic property of extreme temperatures and maximal angle between velocities, and then we provided basic estimates concerning position–velocity–temperature diameters and perturbation functions. Then, we derived dissipative inequalities with respect to the diameters and delayed diameters, and proposed suitable ansatz for the decay rate of the perturbation function to find the sufficient framework to exhibit asymptotic flocking of (1.3). However, there are still some interesting topics that might be studied in the future. For instance, we wonder if it is possible to find differential inequalities for $A^{\tau,\tau}$ to show the asymptotic flocking. Since we have already shown that $A^{\tau,\tau}$ is monotonically increasing, we expect that if we succeed in obtaining a differential inequality for the $A^{\tau,\tau}$ itself, we will not need to use perturbation functions Δ_V^τ and suggest an ansatz on its exponential decay rate. We leave this issue for future work.

Authors contribution

Hyunjin Ahn: Investigation, Funding acquisition, Writing, Validation; Woojoo Shim: Investigation, Methodology, Writing, Supervision.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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Appendix A Proof of Lemma 3.1

Proof. (1) Suppose we have $f(t_0) > F(t_0)$ for some $t_0 \in \mathbb{R}$. Then, there exists $t_1 \in [t_0 - \tau, t_0)$ such that $f(t_1) = F(t_0) < f(t_0)$. Since f is continuous, one can find $\delta > 0$ such that

$$t_1 + \delta \in [t_0 - \tau, t_0), \qquad \max_{s \in [t_1, t_1+\delta]} f(s) < \min_{s \in [t_0, t_0+\delta]} f(s).$$

Therefore, we have

$$\begin{aligned} F(t) &= \min_{s\in[t-\tau,t]} f(s) \\ &= \min\left\{ \min_{s\in[t-\tau,t_0]} f(s), \min_{s\in[t_0,t]} f(s) \right\} \\ &\geq \min\left\{ \min_{s\in[t_0-\tau,t_0]} f(s), \max_{s\in[t_1,t_1+\delta]} f(s) \right\} \\ &= \min_{s\in[t_0-\tau,t_0]} f(s) = F(t_0), \quad \forall t \in [t_0, t_0+\delta]. \end{aligned}$$

(2) To show that F is monotonically increasing, we claim the following:

for every point $t \in \mathbb{R}$, we have $D^+F(t) \geq 0$.

Once we prove the above claim, we can show $F(t_0) \leq F(t_1)$ for all $t_0 < t_1$. To see this, consider a function

$$G(t) := F(t) - \frac{F(t_1) - F(t_0)}{t_1 - t_0}(t - t_0), \quad t \in [t_0, t_1].$$

This function has a maximum point $c \in [t_0, t_1]$ as a consequence of the Weierstrass extreme value theorem. In addition, since $G(t_0) = G(t_1) = F(t_0)$, we may assume $c \in [t_0, t_1)$. Therefore, we have

$$\begin{aligned} 0 &\geq \limsup_{y\to c+} \frac{G(y) - G(c)}{y - c} \\ &= \limsup_{y\to c+} \frac{F(y) - F(c)}{y - c} - \frac{F(t_1) - F(t_0)}{t_1 - t_0} \\ &= D^+F(c) - \frac{F(t_1) - F(t_0)}{t_1 - t_0} \\ &\geq -\frac{F(t_1) - F(t_0)}{t_1 - t_0}, \end{aligned}$$

which implies $F(t_0) \leq F(t_1)$.

Now, suppose we have $D^+F(t_*) < 0$ for some $t_* \in \mathbb{R}$. Then, one can obtain $f(t_*) = F(t_*)$ from the result (1), and we have

$$\begin{aligned} \liminf_{h\to 0+} \frac{f(t_*+h) - F(t_*+h)}{h} &= \liminf_{h\to 0+} \frac{f(t_*+h) - f(t_*) + f(t_*) - F(t_*+h)}{h} \\ &= \liminf_{h\to 0+} \frac{f(t_*+h) - f(t_*) + F(t_*) - F(t_*+h)}{h} \\ &\geq \liminf_{h\to 0+} \frac{f(t_*+h) - f(t_*)}{h} - D^+F(t_*) \\ &= D_+f(t_*) - D^+F(t_*) > 0, \end{aligned}$$

where we used $f(t_*) = F(t_*)$ in the second equality. Therefore, there exist two constants $\delta, \varepsilon > 0$ such that

$$f(t_*+h) > F(t_*+h) + \varepsilon h, \quad \forall h \in (0, \delta). \tag{A.1}$$

From (A.1), we can employ Lemma 3.1(1) to $t_0 := t_* + h$ for each $h \in (0, \delta)$ and obtain

$$F(t_1) \le F(t_2), \quad \forall t_* < t_1 \le t_2 < t_* + \delta,$$

and since F is continuous ($\because$ Berge's maximum theorem), we have

$$F(t_*) \le F(t), \quad \forall t \in [t_*, t_* + \delta),$$

which leads to a contradiction in $D^+F(t_*) < 0$.

(3) It is sufficient to prove that

$$\forall\, t \in \mathbb{R}, \quad \exists\, \delta > 0 \quad \text{such that} \quad |F(t) - F(s)| \le L|t - s|, \quad \forall\, s \in [t, t + \delta].$$

If $f(t_0) > F(t_0)$, one can find $\delta > 0$ such that

$$F(t_0) + L\delta < \min_{s \in [t_0, t_0 + \delta]} f(s),$$

due to the continuity of f. Therefore, we have

$$\begin{aligned} F(t) &= \min_{s \in [t-\tau, t]} f(s) \\ &= \min \left\{ \min_{s \in [t-\tau, t_0]} f(s), \min_{s \in [t_0, t]} f(s) \right\} \\ &\le \min \left\{ \min_{s \in [t_0-\tau, t_0]} f(s) + L|t - t_0|, \min_{s \in [t_0, t]} f(s) \right\} \\ &\le F(t_0) + L|t - t_0|, \quad \forall\, t \in [t_0, t_0 + \delta], \end{aligned}$$

where we used the Lipschitz continuity of f. On the other hand, if $f(t_0) = F(t_0)$, then

$$\begin{aligned} F(t_0 + h) &= \min_{s \in [t_0 - \tau + h, t_0 + h]} f(s) \\ &= \min \left\{ \min_{s \in [t_0 - \tau + h, t_0]} f(s), \min_{s \in [t_0, t_0 + h]} f(s) \right\} \\ &= \min \left\{ F(t_0), \min_{s \in [t_0, t_0 + h]} f(s) \right\}, \quad \forall\, h \in (0, \tau), \end{aligned}$$

where we used

$$\min_{s \in [t_0 - \tau + h, t_0]} f(s) = f(t_0) = F(t_0)$$

in the last equality. Therefore, we have

$$\begin{aligned} |F(t_0 + h) - F(t_0)| &= \left| \min \left\{ 0, \min_{s \in [t_0, t_0 + h]} f(s) - F(t_0) \right\} \right| \\ &= \left| \min \left\{ 0, \min_{s \in [t_0, t_0 + h]} f(s) - f(t_0) \right\} \right| \\ &\le Lh, \end{aligned}$$

where we used the Lipschitz continuity of f in the last inequality. □

Appendix B Proof of Lemma 4.1

Proof. From the Berge maximum theorem, the set valued map

$$t \mapsto C(t) := \arg\max\{f(t_1, t_2) : t_1, t_2 \in \mathcal{S}(t)\}$$

is upper hemi-continuous with nonempty and compact values. This means that for every $t \geq 0$, if $V(\subset \mathbb{R}^2)$ contains $C(t)$, there exists a neighborhood U of t such that for all $t \in U$, $C(t)$ is a subset of V.

Now, fix arbitrary $\varepsilon > 0$, and define $g(t_1, t_2) := f(t_1, t_2) - (c + \varepsilon)\max\{t_1, t_2\}$. Then, we have

$$\limsup_{h \to 0+} \frac{g(s_1 + h, s_2 + h) - g(s_1, s_2)}{h} \leq \lambda(s_1, s_2) - (c + \varepsilon), \quad \forall\ (s_1, s_2) \in \mathcal{S}(t).$$

Since λ is continuous, there exists an open neighborhood

$$V(s, t_0) := \{(s + h_1 + h_2, t_0 + h_2) : |h_1| < \delta_{(s,t_0)} < t_0 - s, |h_2| < \delta_{(s,t_0)}\}$$

for each $(s, t_0) \in C(t_0)$, $s < t_0$, such that $\lambda(s_1, s_2) < c + \frac{\varepsilon}{2}$ for all $(s_1, s_2) \in V(s, t_0)$. Similarly, for each $(t_0, s) \in C(t_0)$, $s < t_0$, there exists an open neighborhood

$$V(t_0, s) := \{(t_0 + h_2, s + h_1 + h_2) : |h_1| < \delta_{(t_0,s)} < t_0 - s, |h_2| < \delta_{(t_0,s)}\}$$

such that $\lambda(s_1, s_2) < c + \frac{\varepsilon}{2}$ for all $(s_1, s_2) \in V(t_0, s)$. Finally, there exists an open set

$$V(t_0, t_0) := \{(t_1 + h, t_2 + h) : \max\{t_1, t_2\} = t_0, \min\{t_1, t_2\} > t_0 - \delta_{(t_0,t_0)},\ |h| < \delta_{(t_0,t_0)}\}$$

such that $\lambda(s_1, s_2) < c + \frac{\varepsilon}{2}$ for all $(s_1, s_2) \in V(t_0, t_0)$. Then, $\{V(s_1, s_2) : (s_1, s_2) \in C(t_0)\}$ is an open cover of $C(t_0)$. In addition, since $C(t_0)$ is compact, we can find its finite subcover $\{V_1, \ldots, V_n\}$. Therefore, we can find an open subset V_0 of $S(t_0)$ such that

$$C(t_0) \subset V_0, \quad V := \{(s_1 + h, s_2 + h) : (s_1, s_2) \in V_0,\ |h| < \delta\} \subset \bigcup_{i=1}^{n} V_i.$$

Therefore, by using the Berge maximum theorem, there exists a positive number $\delta_0 < \delta$ such that $C(t) \subset V$ for all $t \in (t_0 - \delta_0, t_0 + \delta_0)$. This means that the Dini derivative $D^+G_{t_1,t_2}(h)$ of

$$G_{t_1,t_2}(h) := g(t_1 + h, t_2 + h)$$

is less than or equal to $-\frac{\varepsilon}{2}$ for all $(t_1, t_2) \in V_0$ and $|h| < \delta_0$, and therefore $G_{t_1,t_2}(h)$ is monotonically decreasing in $|h| < \delta_0$ for all $(t_1, t_2) \in V_0$. By using this result, we have

$$\begin{aligned} m[g](t_0 + h) &= \max_{(s_1,s_2)\in\mathcal{S}(t_0+h)} g(s_1, s_2) \\ &= \max_{(s_1,s_2)\in V_0} g(s_1 + h, s_2 + h) \\ &\leq \max_{(s_1,s_2)\in V_0} g(s_1, s_2) \\ &= \max_{(s_1,s_2)\in\mathcal{S}(t)} g(s_1, s_2) \\ &= m[g](t_0), \quad \forall\ |h| < \delta_0, \end{aligned}$$

which implies that for every $|h| < \delta_0$,

$$\begin{aligned} m[f](t_0+h) &= \max_{(t_1,t_2)\in\mathcal{S}(t_0+h)} f(t_1,t_2) \\ &= \max_{(t_1,t_2)\in\mathcal{S}(t_0+h)} [g(t_1,t_2)+(c+\varepsilon)\max\{t_1,t_2\}] \\ &= \max_{(t_1,t_2)\in\mathcal{S}(t_0+h)} g(t_1,t_2)+(c+\varepsilon)(t_0+h) \\ &\le \max_{(t_1,t_2)\in\mathcal{S}(t_0)} g(t_1,t_2)+(c+\varepsilon)(t_0+h) \\ &= \max_{(t_1,t_2)\in\mathcal{S}(t_0)} f(t_1,t_2)+(c+\varepsilon)h \\ &= m[f](t_0)+(c+\varepsilon)h. \end{aligned}$$

Since ε can be any positive number, we have

$$D^+m[f](t_0) \le c.$$

□

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