Networks and Heterogeneous Media

https://www.aimspress.com/journal/nhm

NHM, 19(3): 1116–1132.
DOI: 10.3934/nhm.2024049
Received: 10 August 2024
Revised: 26 September 2024
Accepted: 09 October 2024
Published: 12 October 2024

Research article

Positive solutions for the periodic-parabolic problem with large diffusion

Mingming Fan and Jianwen Sun*

School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China

* **Correspondence:** Email: jianwensun@lzu.edu.cn.

Abstract: In this paper, we study the positive solutions of the periodic-parabolic logistic equation with indefinite weight function and nonhomogeneous diffusion coefficient. By employing sufficient conditions to guarantee negative principal eigenvalue, we obtain the existence, uniqueness, and stability of the positive periodic solutions. Moreover, we prove that the positive periodic solution tends to the unique positive solution of the corresponding non-autonomous logistic equation when the diffusion rate is large.

Keywords: logistic equation; periodic-parabolic; periodic solution

1. Introduction

In this paper, we study the positive solutions of the periodic-parabolic problem

$$\begin{cases} u_t = \mu k(x,t)\Delta u + m(x,t)u - c(x,t)u^p, & \text{in } \Omega\times\mathbb{R}, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega\times\mathbb{R}, \\ u(x,0) = u(x,T), & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N (N \geq 1)$ with smooth boundary $\partial\Omega$, ν is the outward normal vector of $\partial\Omega$, $\mu > 0$ and $p > 1$ is constant, $m(x,t) \in C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}\times\mathbb{R})$ $(0 < \alpha < 1)$ is T-periodic in t, $k(x,t)$, $c(x,t) \in C^{\alpha,1}(\bar{\Omega}\times\mathbb{R})$ are positive and T-periodic in t. It is known that the periodic reaction-diffusion equation (1.1) can be accurately used to describe different diffusion phenomena in infectious diseases, microbial growth, and population ecology, see [1–4]. From a biological point of view, Ω represents the habitat of species u and $\mu k(x,t)$ stands for the diffusion rate, which is time and space dependent. The function $m(x,t)$ represents the growth rate of species. In this situation, in the subset $\{(x,t) \in \Omega\times\mathbb{R} : m(x,t) > 0\}$, the species will increase, while in $\{(x,t) \in \Omega\times\mathbb{R} : m(x,t) < 0\}$, species will decrease. The coefficient $c(x,t)$ means that environment Ω can accommodate species u. There are many interesting conclusions about the study of the reaction-diffusion equation, see [5–8] for the elliptic problems and [9–14] for the periodic problems.

In particular, if $k(x,t) \equiv k(t)$ for $x \in \bar{\Omega}$, problem (1.1) has been well investigated by Hess [2], Cantrell and Cosner [1]. Let $\lambda(\mu)$ be the unique principal eigenvalue of the eigenvalue problem

$$\begin{cases} u_t - \mu k(t)\Delta u - m(x,t)u = \lambda(\mu)u, & \text{in } \Omega \times \mathbb{R}, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega \times \mathbb{R}, \\ u(x,0) = u(x,T), & \text{in } \Omega. \end{cases}$$

It follows from [2, 15] that Eq (1.1) has a positive periodic solution $\theta_\mu(x,t)$ if and only if $\lambda(\mu) < 0$. In addition, Dancer and Hess [16] and Daners and López-Gómez [17] studied the effect of μ on the positive periodic solution of Eq (1.1) with various boundary conditions. The most interesting conclusion of [11, 16, 17] is that

$$\lim_{\mu \to 0+} \theta_\mu(x,t) = \theta(x,t) \ \text{ locally uniformly in } \ \Omega \times [0,T],$$

here $\theta(x,t)$ is the maximum nonnegative periodic solution of

$$\begin{cases} u_t = m(x,t)u - c(x,t)u^p, & t \in \mathbb{R}, \\ u(x,0) = u(x,T). \end{cases}$$

However, there is little result on the associated large diffusion and the effect of large diffusion on positive solutions.

Our goal is to study the existence and uniqueness of positive periodic solutions of Eq (1.1) and the asymptotic behavior of positive periodic solutions when the diffusion rate μ is large. To this end, let $\lambda(\mu; m)$ be the principal eigenvalue of

$$\begin{cases} u_t - \mu k(x,t)\Delta u - m(x,t)u = \lambda(\mu; m)u, & \text{in } \Omega \times \mathbb{R}, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega \times \mathbb{R}, \\ u(x,0) = u(x,T), & \text{in } \Omega. \end{cases} \tag{1.2}$$

It is well known that $\lambda(\mu; m)$ plays a major role in the study of the positive periodic solution of Eq (1.1). The properties of $\lambda(\mu; m)$ will be established in Section 2. In addition, let $W^{2,p}_\nu(\Omega) = \{u \in W^{2,p}(\Omega) : \frac{\partial u}{\partial \nu} = 0\}(N < p < \infty)$. If $u_0 \in W^{2,p}_\nu(\Omega)$, it follows from [2] that the semilinear initial value problem

$$\begin{cases} u_t = \mu k(x,t)\Delta u + m(x,t)u - c(x,t)u^p, & \text{in } \Omega \times \mathbb{R}, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega \times \mathbb{R}, \\ u(x,0) = u_0(x), & \text{in } \Omega, \end{cases}$$

has a unique solution $U(x,t) = U(x,t; u_0)$ satisfying

$$U(x,t) \in C^{1+\alpha, \frac{1+\alpha}{2}}(\bar{\Omega} \times [0,T]) \cap C^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{\Omega} \times (0,T]).$$

Our first result is the existence and uniqueness of positive periodic solutions of Eq (1.1). For simplicity, in the rest of this paper, we use the following notations:

$$k_*(t) = \int_\Omega \frac{1}{k(x,t)}\,dx, \ \ m_*(t) = \int_\Omega \frac{m(x,t)}{k(x,t)}\,dx, \ \ c_*(t) = \int_\Omega \frac{c(x,t)}{k(x,t)}\,dx.$$

Theorem 1.1. *Suppose that* $\int_0^T m_*(t)\,dt > 0$. *Then Eq* (1.1) *admits a unique positive periodic solution* $\theta_\mu(x,t)$ *for all* $\mu > 0$.

Remark 1.1. By the result of Section 3, we know that there exists a unique positive solution to Eq (1.1) if and only if $\lambda(\mu; m) < 0$. In the case $\int_0^T m_*(t)\,dt > 0$, we obtain that $\lambda(\mu; m) < 0$.

Next, we study the asymptotic behavior of positive periodic solutions when the diffusion rate is large.

Theorem 1.2. *Suppose that*

$$m_*(t) > 0 \text{ for } t \in [0,T]. \tag{1.3}$$

Let $\theta_\mu(x,t)$ *be the unique positive periodic solution of Eq* (1.1) *for* $\mu > 0$. *Then we have*

$$\lim_{\mu\to\infty} \theta_\mu(x,t) = \omega(t) \text{ in } C^{1,\frac{1}{2}}(\bar{\Omega}\times[0,T]), \tag{1.4}$$

where $\omega(t)$ *is the unique positive periodic solution of*

$$\begin{cases} u_t = \frac{m_*(t)}{k_*(t)}u - \frac{c_*(t)}{k_*(t)}u^p, & t \in \mathbb{R}, \\ u(0) = u(T). \end{cases} \tag{1.5}$$

Remark 1.2. With the approach of local upper-lower solutions developed by Daners and López-Gómez [17] in the study of classical periodic-parabolic logistic equations, we can prove that

$$\lim_{\mu\to 0+} \theta_\mu(x,t) = \theta(x,t) \text{ locally uniformly in } \Omega\times[0,T],$$

provided $\max_\Omega \int_0^T m(x,t)\,dt > 0$. It also shows that when $m_*(t) < 0 < \int_0^T \max_\Omega m(x,t)\,dt$, populations with small dispersal rates survive, while populations with large dispersal rates perish. This means that a small diffusion rate is a better strategy than a large diffusion rate under appropriate circumstances.

The rest of this paper is arranged as follows: In Section 2, we study the properties of principal eigenvalues for the periodic eigenvalue problems. In Section 3, we mainly study the existence, uniqueness and stability of the positive solution to Eq (1.1). Moreover, we investigate the asymptotic profiles of the positive periodic solution to Eq (1.1) as $\mu \to \infty$ in Section 4.

2. Periodic eigenvalue equation

In this section, we consider the principal eigenvalue of Eq (1.2). To this end, we first study the linear initial value problem

$$\begin{cases} u_t - \mu k(x,t)\Delta u - a(x,t)u = 0, & \text{in } \Omega\times(\tau,T], \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega\times(\tau,T], \\ u(x,\tau) = u_0(x), & \text{in } \Omega, \end{cases} \tag{2.1}$$

where $0 \le \tau < T$, $u_0 \in W_\nu^{2,p}(\Omega)(N < p < \infty)$ and $a(x,t) \in C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}\times[\tau,\infty))$. It is well known that there is a one-to-one correspondence between Eq (2.1) and the evolution operator $U_\mu(t,\tau)$. Then we can define that $u(x,t) = U_\mu(t,\tau)u_0$ is the solution of Eq (2.1). For simplicity, let $X = L^p(\Omega)(N < p < \infty)$, $X_1 = W_\nu^{2,p}(\Omega)$ and

$$F = \{u \in C^{\alpha,\frac{\alpha}{2}}(\bar{\Omega}\times\mathbb{R}) : u(\cdot,t+T) = u(\cdot,t) \text{ in } \mathbb{R}\}.$$

Inspired by the classical works of Hess [2], we first give some important results of Eq (2.1), which will be used in the rest of this paper.

Lemma 2.1. *If $u_0 \in X$ is positive, then $U_\mu(t,\tau)u_0 > 0$ in $C^1_\nu(\bar{\Omega})$ for $0 \le \tau < t \le T$.*

Proof. Note that X_1 is compactly embedded in X. The operator $U_\mu(t,\tau)/_{X_1} : X_1 \to X_1$ can be continuously extended to the positive operator $U_\mu(t,\tau) \in \mathcal{L}(X, X_1)$. Thus $U_\mu(t,\tau)u_0 \ge 0$ in X_1. Since $s \mapsto U_\mu(s,\tau)u_0$ is continuous from $[\tau, T]$ to X_1 and $U_\mu(\tau,\tau)u_0 = u_0 \neq 0$, we can get that $U_\mu(s,\tau)u_0 > 0$ in X_1 as $s > \tau$ goes to τ. In addition, we have

$$U_\mu(t,\tau)u_0 = U_\mu(t,s)U_\mu(s,\tau)u_0,$$

for $\tau < s < t$. Thus it can be obtained that $U_\mu(t,\tau)u_0 > 0$ for $0 \le \tau < t \le T$.

We now study the periodic-parabolic eigenvalue problem

$$\begin{cases} u_t - \mu k(x,t)\Delta u - a(x,t)u = \lambda(\mu; a)u, & \text{in } \Omega \times (0,T], \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega \times (0,T], \\ u(x,0) = u(x,T), & \text{in } \Omega. \end{cases} \tag{2.2}$$

If there is a nontrivial solution $u(x,t)$ of Eq (2.2), then $\lambda(\mu; a)$ is called the eigenvalue. In particular, if $u(x,t)$ is positive, then $\lambda(\mu; a)$ is the principal eigenvalue.

Theorem 2.1. *Let $K_\mu := U_\mu(T, 0)$ and r be the spectral radius of K_μ. Then r is the principal eigenvalue of K_μ with positive eigenfunction u_0 if and only if $\lambda(\mu; a) = -\frac{1}{T}\ln r$ is the principal eigenvalue of Eq (2.2) with positive eigenfunction $u(x,t) = e^{\lambda(\mu;a)t}U_\mu(t,0)u_0$.*

Proof. It can be proved by the similar arguments as in [2, Proposition 14.4]. For the completeness, we provide a proof in the following. Suppose that r is the principal eigenvalue of K_μ with positive eigenfunction $u_0 \in X_1$. Let $u(x,t) = e^{\lambda(\mu;a)t}U_\mu(t,0)u_0$. Then $u(x,t)$ satisfies

$$\begin{cases} u_t - \mu k(x,t)\Delta u - a(x,t)u = \lambda(\mu; a)u, & \text{in } \Omega \times (0,T], \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega \times (0,T], \\ u(x,0) = u_0 = \frac{1}{r}K_\mu u_0 = e^{\lambda(\mu;a)T}K_\mu u_0 = u(x,T), & \text{in } \Omega. \end{cases}$$

According to the regularity results, we have $u(x,t) \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{\Omega} \times \mathbb{R})$. This means that $\mu = -\frac{1}{T}\ln r$ is the principal eigenvalue of Eq (2.2), while $u(x,t) = e^{\lambda(\mu;a)t}U_\mu(t,0)u_0$ is the corresponding positive eigenfunction.

On the contrary, suppose that $\lambda(\mu; a) = -\frac{1}{T}\ln r$ is the eigenvalue of Eq (2.2) with positive eigenfunction $u(x,t)$. Set $v(x,t) = e^{-\lambda(\mu;a)t}u(x,t)$. Then $v(x,t)$ is the solution of

$$\begin{cases} v_t - \mu k(x,t)\Delta v - a(x,t)v = 0, & \text{in } \Omega \times (0,T], \\ \frac{\partial v}{\partial \nu} = 0, & \text{on } \partial\Omega \times (0,T], \\ v(x,0) = u(x) =: u_0, & \text{in } \Omega. \end{cases}$$

Thus, we obtain $v(x,t) = U_\mu(t,0)u_0$ for $0 \le t \le T$ and $u_0 \in X_1$ is positive. Hence,

$$v(T) = e^{-\lambda(\mu;a)T}u_0 = K_\mu u_0.$$

It follows from Krein-Rutman theorem that $e^{-\lambda(\mu;a)T} = r$.

Remark 2.1. For $\tau < t$, it follows that $U_\mu(t,\tau)$ is a compact and strongly positive operator on X_1. Moreover, by Krein-Rutman theorem, we obtain $r > 0$, and r is the unique principal eigenvalue of K_μ. This implies that Eq (2.2) has the unique principal eigenvalue $\lambda(\mu; a)$ for any $\mu > 0$.

Lemma 2.2. *Let $a_1(x,t)$, $a(x,t) \in F$ satisfy*

$$a_1(x,t) < a_2(x,t) \ \ in \ \ \bar{\Omega} \times [0,T].$$

Then $\lambda(\mu; a_2) < \lambda(\mu; a_1)$ for any $\mu > 0$.

Proof. Assume that there exists $\mu_1 > 0$ such that $\lambda(\mu_1; a_2) \geq \lambda(\mu_1; a_1)$. Let $u_1(x,t)$ and $u_2(x,t)$ be corresponding positive eigenfunctions, chosen in such a way that

$$0 < u_1(x,t) < u_2(x,t) \ \text{ in } \ \bar{\Omega} \times [0,T].$$

Then $\omega(x,t) = u_2(x,t) - u_1(x,t)$ satisfies

$$\begin{cases} \omega_t - \mu_1 k(x,t)\Delta\omega - a_1(x,t)\omega > \lambda(\mu_1; a_1)\omega, & \text{in } \Omega \times (0,T], \\ \frac{\partial \omega}{\partial \nu} = 0, & \text{on } \partial\Omega \times (0,T], \\ \omega(x,0) = \omega(x,T), & \text{in } \Omega. \end{cases}$$

Set $\phi(x,t) = e^{-\lambda(\mu_1;a_1)t}\omega(x,t)$, then we have

$$\begin{cases} \phi_t - \mu_1 k(x,t)\Delta\phi - a_1(x,t)\phi > 0, & \text{in } \Omega \times (0,T], \\ \frac{\partial \phi}{\partial \nu} = 0, & \text{on } \partial\Omega \times (0,T], \\ \phi(x,0) = \omega(x,0) = \omega(x,T), & \text{in } \Omega. \end{cases}$$

Thus, for any $x \in \Omega$, we can obtain

$$\phi(x,T) > K_{\mu_1}\omega(x,0) \ \text{ and } \ \phi(x,T) = e^{-\lambda(\mu_1;a_1)T}\omega(x,0).$$

Hence, we obtain

$$(e^{-\lambda(\mu_1;a_1)T} - K_{\mu_1})\omega(x,0) > 0 \ \text{ in } \ X_1.$$

Note that $\omega(x,0) > 0$, it follows from [2, Theorem 7.3] that

$$e^{-\lambda(\mu_1;a_1)T} = r_{\mu_1} < e^{-\lambda(\mu_1;a_1)T},$$

where r_{μ_1} is the principal eigenvalue of K_{μ_1}. This is a contradiction.

Lemma 2.3. *Suppose that for any $n \in \mathbb{N}$, $a_n(x,t) \in F$ satisfies*

$$\lim_{n\to\infty} a_n(x,t) = a(x,t) \ \ in \ \ C^1(\bar{\Omega} \times [0,T]).$$

Then for fixed $\mu > 0$, we have

$$\lim_{n\to\infty} \lambda(\mu; a_n) = \lambda(\mu; a).$$

Proof. For any given $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that for any $n > n_\varepsilon$, there holds

$$a(x,t) - \varepsilon < a_n(x,t) < a(x,t) + \varepsilon \quad \text{in} \quad \bar{\Omega} \times [0,T].$$

Notice that $\lambda(\mu; a \pm \varepsilon) = \lambda(\mu; a) \mp \varepsilon$. From Lemma 2.2, we have

$$\lambda(\mu; a) - \varepsilon < \lambda(\mu; a_n) < \lambda(\mu; a) + \varepsilon,$$

for any $n > n_\varepsilon$.

Lemma 2.4. *Let $\lambda(\mu; a)$ be the principal eigenvalue of Eq* (2.2) *for $\mu > 0$. Then we have*

$$\lambda(\mu; a) \leq -\frac{\int_0^T a_*(t)\,dt}{\int_0^T k_*(t)\,dt}, \tag{2.3}$$

here $a_*(t) = \int_\Omega \frac{a(x,t)}{k(x,t)}\,dx$.

Proof. First, we consider the case

$$\int_0^T \int_\Omega \frac{k_t(x,t)}{k^2(x,t)}\,dxdt \neq 0.$$

Let $\varphi(x,t)$ be the positive eigenfunction corresponding to the principal eigenvalue $\lambda(\mu; a)$. Taking $\alpha > 0$ satisfies

$$\ln \alpha = \frac{-\int_0^T \int_\Omega \frac{k_t(x,t)\ln\varphi(x,t)}{k^2(x,t)}\,dxdt}{\int_0^T \int_\Omega \frac{k_t(x,t)}{k^2(x,t)}\,dxdt}.$$

Then $\varphi_\alpha := \alpha\varphi(x,t)$ is also the principal eigenfunction of Eq (2.2). It is easy to obtain

$$\begin{aligned} &\lambda(\mu; a) \int_0^T \int_\Omega \frac{1}{k(x,t)}\,dxdt \\ &= -\int_0^T \int_\Omega \frac{a(x,t)}{k(x,t)}\,dxdt - \mu \int_0^T \int_\Omega \frac{\Delta\varphi_\alpha}{\varphi_\alpha}\,dxdt \\ &= -\int_0^T \int_\Omega \frac{a(x,t)}{k(x,t)}\,dxdt - \mu \int_0^T \int_\Omega \frac{|D\varphi_\alpha|^2}{\varphi_\alpha^2}\,dxdt. \end{aligned} \tag{2.4}$$

This implies that Eq (3.2) holds.

Next, we consider the case of

$$\int_0^T \int_\Omega \frac{k_t(x,t)}{k^2(x,t)}\,dxdt = 0.$$

We can find smooth T-periodic functions $\{k_n(x,t)\}$ such that

$$\lim_{n\to\infty} k_n(x,t) = k(x,t) \;\text{ in }\; C(\bar{\Omega} \times [0,T]),$$

and

$$\int_0^T \int_\Omega \frac{(k_n(x,t))_t}{k_n^2(x,t)}\,dxdt \neq 0.$$

It follows from Lemma 2.3 that

$$\lim_{n\to\infty} \lambda_n(\mu; a) = \lambda(\mu; a),$$

where $\lambda_n(\mu; a)$ is the principal eigenvalue of Eq (2.2) with $k(x,t)$ replaced by $k_n(x,t)$. It is clear from Eq (2.4) that

$$\lambda_n(\mu; a) = -\int_0^T \int_\Omega \frac{a(x,t)}{k_n(x,t)}\, dxdt - \mu \int_0^T \int_\Omega \frac{|D\varphi_\alpha|^2}{\varphi_\alpha^2}\, dxdt.$$

Letting $n \to \infty$, we have Eq (3.2).

Remark 2.2. In Eq (3.2), we obtain upper estimates for the principal eigenvalue of the Neumann problem Eq (2.2). Indeed, let λ_D be the principal eigenvalue of the eigenvalue problem

$$\begin{cases} u_t - \mu k(x,t)\Delta u - a(x,t)u = \lambda_D u, & \text{in } \Omega \times (0,T], \\ u = 0, & \text{on } \partial\Omega \times (0,T], \\ u(x,0) = u(x,T), & \text{in } \Omega. \end{cases}$$

By a similar way as in [2], we can show

$$\lambda_D \leq -\frac{1}{T}\int_0^T [\mu k(x,s) + a(s)]\, ds,$$

for any $\mu > 0$.

3. Positive solutions of periodic-parabolic equation

In this section, we study the existence and uniqueness of positive solutions of Eq (1.1). First, we show that if Eq (1.2) has negative principal eigenvalues, then Eq (1.1) has a unique positive solution. To this end, we recall the upper-lower solutions of Eq (1.1). For the sake of convenience, let

$$Q_T = \Omega \times (0,T], \quad Q_1 = \partial\Omega \times (0,T].$$

Definition 3.1. *The continuous function $\bar{u}(x,t)$ is called the upper-solution of Eq* (1.1)*, if*

$$\begin{cases} u_t \geq \mu k(x,t)\Delta u + m(x,t)u - c(x,t)u^p, & \text{in } Q_T, \\ \frac{\partial u}{\partial \nu} \geq 0, & \text{on } Q_1, \\ u(x,0) \geq u(x,T), & \text{in } \Omega, \end{cases}$$

is satisfied.

The definition of the lower-solution is similar. We then can prove the following result, see [2,4,5,15].

Theorem 3.1. *Suppose that $\bar{u}(x,t)$, $\underline{u}(x,t)$ are a pair of ordered bounded upper-lower solutions of Eq* (1.1)*. Then Eq* (1.1) *has a unique positive periodic solution $\theta_\mu(x,t) \in C^{1+\alpha,(1+\alpha)/2}(\bar{Q}_T)$ that satisfies*

$$\underline{u}(x,t) \leq \theta_\mu(x,t) \leq \bar{u}(x,t) \ \text{ in } \ \bar{Q}_T.$$

Proof. Let

$$f(x,t,u) = m(x,t)u - c(x,t)u^p.$$

Then there exists a constant $K > 0$ such that

$$|f(x,t,u_2) - f(x,t,u_1)| \leq K|u_2 - u_1|,$$

for any $(x,t,u_i) \in \bar{Q}_T \times [\underline{u}(x,t), \bar{u}(x,t)]$, $i = 1,2$. It follows from L^p theory that for any $u \in C^{\alpha,\frac{\alpha}{2}}(\bar{Q}_T)$ satisfying $[\underline{u}(x,t), \bar{u}(x,t)]$, the linear initial value problem

$$\begin{cases} v_t - \mu k(x,t)\Delta v + Kv = Ku + f(x,t,u), & \text{in } Q_T, \\ \frac{\partial v}{\partial \nu} = 0, & \text{on } Q_1, \\ v(x,0) = u(x,T), & \text{in } \Omega, \end{cases}$$

admits a unique solution v. Thus, a nonlinear operator $v = \mathcal{F}u$ is defined. We will prove that there is a fixed point for $\mathcal{F}$ in four steps.

Step 1. In this step, we prove that if $\underline{u} \leq u_1 \leq u_2 \leq \bar{u}$, then $\underline{u} \leq v_1 = \mathcal{F}u_1 \leq v_2 = \mathcal{F}u_2 \leq \bar{u}$.

Take $\omega_1 = v_2 - v_1$, then ω_1 satisfies

$$\begin{cases} [\omega_1]_t - \mu k(x,t)\Delta\omega_1 + K\omega_1 = K(u_2 - u_1) + f(x,t,u_2) - f(x,t,u_1) \geq 0, & \text{in } Q_T, \\ \frac{\partial \omega_1}{\partial \nu} = 0, & \text{on } Q_1, \\ \omega_1(x,0) = u_2(x,T) - u_1(x,T) \geq 0, & \text{in } \Omega. \end{cases}$$

By the comparison principle, we obtain $\omega_1 \geq 0$. This implies $\mathcal{F}u_2 \geq \mathcal{F}u_1$. Similarly, let $\omega_2 = v_1 - \underline{u}$, then ω_2 satisfies

$$\begin{cases} [\omega_2]_t - \mu k(x,t)\Delta\omega_2 + K\omega_2 = K(u_1 - \underline{u}) + f(x,t,u_1) - f(x,t,\underline{u}) \geq 0, & \text{in } Q_T, \\ \frac{\partial \omega_2}{\partial \nu} = 0, & \text{on } Q_1, \\ \omega_1(x,0) = u_1(x,T) - \underline{u}(x,T) \geq 0, & \text{in } \Omega. \end{cases}$$

Thus, $\underline{u} \leq v_1$. Similarly, $v_2 \leq \bar{u}$.

Step 2. In this step, we construct a convergent monotone sequence.

The iterative sequences $\{u_n\}$ and $\{v_n\}$ are constructed as follows:

$$u_1 = \mathcal{F}\bar{u},\ u_2 = \mathcal{F}u_1,\ \cdots,\ u_n = \mathcal{F}u_{n-1} \cdots,$$

$$v_1 = \mathcal{F}\underline{u},\ v_2 = \mathcal{F}v_1,\ \cdots,\ v_n = \mathcal{F}v_{n-1} \cdots.$$

Since $\underline{u} \leq \bar{u}$ and $\mathcal{F}$ is monotonically non-decreasing, then

$$\underline{u} \leq v_1 \leq u_1 \leq \bar{u}.$$

Similarly, we obtain

$$\underline{u} \leq v_n \leq u_n \leq \bar{u}.$$

And since $\underline{u} \le u_1 \le \bar{u}$,

$$\underline{u} \le u_2 \le u_1 \le \bar{u}.$$

By induction, we have $u_{n+1} \le u_n$. In the same way, we obtain $v_n \le v_{n+1}$. Thus, we have

$$\underline{u} \le v_1 \le v_2 \le \cdots \le u_2 \le u_1 \le \bar{u}.$$

This also implies that $\{u_n\}$ and $\{v_n\}$ are monotonically bounded sequences, so there are $u_0(x,t)$ and $v_0(x,t)$ such that

$$\lim_{n\to\infty} u_n(x,t) = u_0(x,t), \quad \lim_{n\to\infty} v_n(x,t) = v_0(x,t).$$

Thus

$$\underline{u}(x,t) \le v_0(x,t) \le u_0(x,t) \le \bar{u}(x,t) \ \text{ in } \ \bar{Q}_T.$$

Step 3. In this step, we prove that $u_0(x,t)$ and $v_0(x,t)$ are solutions of Eq (1.1).

Take $E = W_p^{2,1}(Q_T)(p > n+2)$. First, we prove that $\mathcal{F} : D \to C(\bar{Q}_T)$ is a compact operator, where

$$D = \{u(x,t) \in E : \underline{u}(x,t) \le u(x,t) \le \bar{u}(x,t) \ \text{ in } \ \bar{Q}_T\}.$$

For $u_1, u_2 \in E$, let $v_1 = \mathcal{F}u_1$ and $v_2 = \mathcal{F}u_2$, then $\omega_3 = v_2 - v_1$ satisfies

$$\begin{cases} [\omega_3]_t - \mu k(x,t)\Delta\omega_3 + K\omega_3 = K(u_2 - u_1) + f(x,t,u_2) - f(x,t,u_1), & \text{in } Q_T, \\ \frac{\partial \omega_3}{\partial \nu} = 0, & \text{on } Q_1, \\ \omega_1(x,0) = u_2(x,T) - u_1(x,T), & \text{in } \Omega. \end{cases}$$

By the L^p estimate and embedding theorem, it follows that

$$\|\omega_3\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(\bar{Q}_T)} \le C\|\omega_3\|_{W_p^{2,1}(Q_T)} \le C_1(\|u_2 - u_1\|_{L^p(Q_T)} + \|u_2(x,T) - u_1(x,T)\|_{L^p(\Omega)}),$$

here C and C_1 are positive constants. Thus $\mathcal{F} : D \to C(\bar{Q}_T)$ is continuous. It is known from the embedding theorem that if u is bounded in $W_p^{2,1}(Q_T)$, then $\mathcal{F}u$ is bounded in $C^{1+\alpha,(1+\alpha)/2}(\bar{Q}_T)$. This means that $\mathcal{F} : D \to C(\bar{Q}_T)$ is a compact operator.

Since u_n is bounded, $u_n = \mathcal{F}u_{n-1}$ has a convergent subsequence in $C(\bar{Q}_T)$. By the monotonicity of u_n,

$$\lim_{n\to\infty} u_n(x,t) = u_0(x,t) \ \text{ in } \ C(\bar{Q}_T).$$

Thus $u_0(x,t)$ is the solution of Eq (1.1) in $W_p^{2,1}(Q_T)$. The embedding theorem is used again to get $u_0(x,t) \in C^{1+\alpha,(1+\alpha)/2}(\bar{Q}_T)$. In the same way, we get that $v_0(x,t)$ is the classical solution of Eq (1.1).

Step 4. In this step, we prove the uniqueness and periodicity of the solution of Eq (1.1).

Since $k(x,t)$, $m(x,t)$ and $c(x,t)$ are periodic about t, then $\tau(x,t) = u_0(x,t+T) - u_0(x,t)$ satisfies

$$\begin{cases} \tau_t(x,t) - \mu k(x,t)\Delta\tau(x,t) = m(x,t)\tau(x,t) - pc(x,t)\tilde{u}^{p-1}(x,t)\tau(x,t), & \text{in } Q_T, \\ \frac{\partial \tau}{\partial \nu} = 0, & \text{on } Q_1, \\ \tau(x,0) = 0, & \text{in } \Omega, \end{cases} \tag{3.1}$$

here $\tilde{u}(x,t)$ is between $u_0(x,t+T)$ and $u_0(x,t)$. It is well known that the solution of Eq (3.1) is unique, thus $u_0(x,t+T) \equiv u_0(x,t)$ in $\bar{Q}_T$.

The uniqueness of the solution is based on the results of [2,4] and can also be found in recent research results [11, 15]. Assume that v_1 and v_2 are two positive periodic solutions of Eq (1.1). We first prove that there exists a large constant $M > 1$ such that

$$M^{-1}v_1 < v_2 < Mv_1 \ \text{ in } \ Q_T.$$

Indeed, it is clear that there exists $M_1 > 1$ such that

$$v_2(x,0) = v_2(x,T) < M_1 v_1(x,T) = M_1 v_1(x,0) \ \text{ in } \ \Omega.$$

This implies $v_2(x,0) \not\equiv M_1 v_1(x,0)$ on $\bar{\Omega}$. Let $\eta(x,t) := M_1 v_1(x,t) - v_2(x,t)$, then $\eta(x,t)$ satisfies

$$\begin{aligned} \eta_t - \mu k(x,t)\Delta\eta =& m(x,t)\eta - c(x,t)[M_1 v_1^p - v_2^p] \\ >& m(x,t)\eta - c(x,t)[(M_1 v_1)^p - v_2^p] \\ =& m(x,t)\eta - pc(x,t)\varsigma^{p-1}(x,t)\eta, \end{aligned}$$

where $\varsigma(x,t)$ is lying between $M_1 v_1(x,t)$ and $v_2(x,t)$. Notice that $\frac{\partial\eta}{\partial\nu} = 0$ on Q_1. By the maximum principle, we have $\eta > 0$ in $\bar{Q}_T$. Similarly, we can obtain that there exists $M_2 > 0$ such that $v_1 < M_2 v_2$ in $\bar{Q}_T$. Take $M = \max\{M_1, M_2\}$, then we have

$$M^{-1}v_1 < v_2 < Mv_1 \ \text{ in } \ Q_T.$$

We know that $Mv_1(x,t)$ and $M^{-1}v_1(x,t)$ are a pair of upper-lower solutions of Eq (1.1). According to the second step, Eq (1.1) has a minimum solution u_* and a maximum solution u^*, which satisfies $u_* \le v \le u^*$ in $\bar{Q}_T$ for all solution v satisfying $M^{-1}v_1 \le v \le Mv_1$. Thus, we obtain $u_* \le v_i \le u^*$ for $i = 1, 2$. Hence, $u_* \equiv u^*$ implies the uniqueness of the solution to Eq (1.1). Set

$$\alpha_* = \inf\left\{\alpha > 0 \,|\, u^*(x,t) \le \alpha u_*(x,t) \ \text{ in } \bar{Q}_T\right\}.$$

It is clear that $\alpha_* \ge 1$. Note that if $\alpha_* = 1$, then $u_*(x,t) \equiv u^*(x,t)$ in $\bar{Q}_T$. Assume that $\alpha_* > 1$. Let $\pi(x,t) = \alpha_* u_* - u^*$. It is known from the maximum principle that $\pi(x,t) > 0$ in $\bar{Q}_T$. On the other hand, we know that

$$\pi(x,0) = \pi(x,T) \ge \alpha_1 u_*(x,T) = \alpha_1 u_*(x,0) \ \text{ on } \ \bar{\Omega},$$

for some small $\alpha_1 > 0$. We can use the previous method to prove the existence of M to show that

$$\pi(x,t) \ge \alpha_1 u_*(x,t) \ \text{ on } \ \bar{Q}_T.$$

Then we have $u^*(x,t) \le (\alpha_* - \alpha)u_*(x,t)$. This is in contradiction with the definition of α_*. Thus, we obtain $\alpha_* = 1$. The uniqueness is proved.

Lemma 3.1. *If $\lambda(\mu; m) < 0$, then Eq (1.1) admits a unique positive periodic solution $\theta_\mu(x,t) \in C^{1+\alpha,(1+\alpha)/2}(\bar{Q}_T)$. Moreover, $\theta_\mu(x,t)$ is globally asymptotically stable.*

Proof. Let $\theta(x,t)$ be a principal eigenfunction of Eq (1.2) normalized by $\|\theta(x,t)\|_{C(\bar{Q}_T)} = 1$. Then $\underline{u} = \varepsilon\theta(x,t)$ is a lower-solution of Eq (1.1) for any

$$0 < \varepsilon \le \left[\frac{-\lambda(\mu;m)}{\max_{\bar{Q}_T} c(x,t)}\right]^{1-p}.$$

Take

$$M > \max\left\{1, \left[\frac{-\lambda(\mu;m)}{\min_{\bar{Q}_T} c(x,t)}\right]^{1-p}\right\}.$$

Then we have $\bar{u} = M\theta(x,t)$ is an upper-solution of Eq (1.1). From Theorem 3.1, we get that Eq (1.1) has a unique positive solution $\theta_\mu(x,t)$.

Since $\theta_\mu(x,t)$ is the solution of Eq (1.1), then $\lambda(\mu; m - c\theta_\mu^{p-1}) = 0$. Let λ_1 be the eigenvalue of the linear problem

$$\begin{cases} u_t - \mu k(x,t)\Delta u - \left[m(x,t) - pc(x,t)\theta_\mu^{p-1}\right] u = \lambda_1 u, & \text{in } \Omega \times (0,T], \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega \times (0,T], \\ u(x,0) = u(x,T), & \text{in } \Omega. \end{cases}$$

Due to

$$m(x,t) - c(x,t)\hat{u}^{p-1} > m(x,t) - pc(x,t)\hat{u}^{p-1},$$

for $u > 0$. Then $\lambda_1 > 0$. It follows from Theorem 2.1 that $r < 1$. Thus, $\theta_\mu(x,t)$ is locally asymptotically stable. In addition, we can choose ε small enough and M large enough such that $\varepsilon\theta(x,t)$ and $M\theta(x,t)$ are a pair of ordered bounded upper-lower solutions of Eq (1.1). Then we know that $\theta_\mu(x,t)$ is globally asymptotically stable by the standard iteration argument as in [2].

Lemma 3.2. *If* (1.1) *has a positive periodic solution, then* $\lambda(\mu;m) < 0$.

Proof. Let $\theta_\mu(x,t)$ be a positive periodic solution of Eq (1.1). Thanks to [2], we can get that Eq (1.1) is equivalent to

$$(I - K_\mu)\theta_\mu(x,0) = -\int_0^T U_\mu(T,\tau)c(x,\tau)\theta_\mu^p(x,\tau)\,d\tau \ \text{ in } \ X_1.$$

Notice that $\theta_\mu(x,t) > 0$. We now apply [2, Theorem 7.3] to obtain

$$e^{-\lambda(\mu;m)T} > 1.$$

Thus, $\lambda(\mu;m) < 0$.

Proposition 3.1. *If* $\int_0^T m_*(t)\,dt > 0$*, then Eq* (1.1) *admits a unique positive periodic solution* $\theta_\mu(x,t)$ *for all* $\mu > 0$.

Proof. According to Lemma 2.4, we know that

$$\lambda(\mu;m) \le -\frac{\int_0^T m_*(t)\,dt}{\int_0^T k_*(t)\,dt}. \tag{3.2}$$

Due to $\int_0^T m_*(t)\,dt > 0$, $\lambda(\mu;m) < 0$. This together with Lemma 3.1 implies that Eq (1.1) admits a unique positive periodic solution for all $\mu > 0$.

4. Positive solutions with large diffusion rate

In this section, we study the asymptotic behavior of the positive periodic solution of Eq (1.1) when the diffusion rate is large. Here we use regularity estimates together with the perturbation technique to prove our main result. To do this, we first consider the perturbation equation

$$\begin{cases} u_t = \mu k(x,t)\Delta u + m(x,t)(u+\varepsilon) - c(x,t)u^p, & \text{in } Q_T, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } Q_1, \\ u(x,0) = u(x,T), & \text{in } \Omega, \end{cases} \tag{4.1}$$

where the parameter $\varepsilon > 0$.

Lemma 4.1. *Assume that Eq* (1.3) *holds. Then Eq* (4.1) *has a positive periodic solution* $\theta_\mu^\varepsilon(x,t)$ *for* $\mu > 0$*, provided* $\varepsilon > 0$ *is small. Moreover, we can find* $\mu_1 > 0$ *such that*

$$\lim_{\varepsilon \to 0+} \theta_\mu^\varepsilon(x,t) = \theta_\mu(x,t) \ \ in \ \ C^{1,\frac{1}{2}}(\bar{Q}_T), \tag{4.2}$$

for $\mu \geq \mu_1$.

Proof. Through a similar argument as in Theorem 3.1, we know the existence of the positive solution $\theta_\mu^\varepsilon(x,t)$ to Eq (4.1). We only need to prove Eq (4.2). Let $\sigma(x,t) = \theta_\mu^\varepsilon(x,\frac{t}{\mu})$. Then $\sigma(x,t)$ satisfies

$$\begin{cases} \sigma_t = k(x,\frac{t}{\mu})\Delta\sigma + \frac{1}{\mu}[m(x,\frac{t}{\mu})(\sigma+\varepsilon) - c(x,\frac{t}{\mu})\sigma^p], & \text{in } Q_T, \\ \frac{\partial \sigma}{\partial \nu} = 0, & \text{on } Q_1, \\ \sigma(x,0) = \theta_\mu^\varepsilon(x,0), & \text{in } \Omega. \end{cases}$$

It is known from the L^p estimate that there exists $\mu_1 > 0$ such that $\sigma(x,t)$ is bounded in $W_p^{2,1}(\Omega \times (0,\mu T])$ for any $\mu > \mu_1$. It then follows that $\theta_\mu^\varepsilon(x,t)$ is bounded in $W_p^{2,1}(Q_T)$ for any $\mu > \mu_1$. Then, taking p large enough, we know from the embedding theorem that $\theta_\mu^\varepsilon(x,t)$ is compact in $C^{1,\frac{1}{2}}(\bar{Q}_T)$. Thus there is a subsequence, still denoted by $\theta_\mu^\varepsilon(x,t)$, such that

$$\lim_{\varepsilon \to 0+} \theta_\mu^\varepsilon(x,t) = \nu(x,t) \ \ \text{in} \ \ C^{1,\frac{1}{2}}(\bar{Q}_T), \tag{4.3}$$

for some nonnegative periodic function $\nu(x,t) \in C(\bar{Q}_T)$. It follows from the argument of Lemma 3.1 that $\varepsilon\theta(x,t)$ is a lower-solution of Eq (4.1). Thus we have $\nu(x,t) > 0$ for $(x,t) \in \bar{\Omega} \times [0,T]$. Since $\theta_\mu^\varepsilon(x,t)$ is bounded and Eq (4.3), ν satisfies

$$\nu(x,t) = \nu(x,0) + \mu \int_0^t [k(x,s)\Delta\nu(x,s) + m(x,s)\nu - c(x,s)\nu^p]\, ds.$$

It is easy to obtain

$$\begin{cases} \nu_t = \mu k(x,t)\Delta\nu + m(x,t)\nu - c(x,t)\nu^p, & \text{in } Q_T, \\ \frac{\partial \nu}{\partial \nu} = 0, & \text{on } Q_1, \\ \nu(x,0) = \nu(x,T), & \text{in } \Omega. \end{cases}$$

By standard parabolic regularity, we know that $\nu(x,t) \in C^{1+\alpha,(1+\alpha)/2}(\bar{Q}_T)$. The uniqueness of the solution means that Eq (4.2) holds.

At the end of this section, we prove Theorem 1.2.

Proof of Theorem 1.2. We divide the proof into the following three steps.

Step 1. In this step, we prove that $\theta_\mu(x,t)$ has a convergent subsequence as $\mu \to \infty$.

It follows from a similar argument to Lemma 4.1 that there exists $\mu_1 > 0$ such that $\theta_\mu(x,t)$ is compact in $C^{1,\frac{1}{2}}(\bar{Q}_T)$ for any $\mu > \mu_1$. Thus, by passing to a subsequence, there is a nonnegative periodic function $\theta \in C(\bar{Q}_T)$ such that

$$\lim_{\mu\to\infty} \theta_\mu(x,t) = \theta(x,t) \;\text{ in }\; C^{1,\frac{1}{2}}(\bar{Q}_T).$$

Step 2. In this step, we show that $\theta(x,t)$ is independent of t.

Let $f(t)$ be a smooth T-periodic function, then we have

$$\begin{aligned}-\int_0^T \theta_\mu(x,t) f_t(t)\,dt =& \mu\int_0^T k(x,t)f(t)\Delta\theta_\mu(x,t)\,dt\\ &+\int_0^T m(x,t)\theta_\mu(x,t)f(t)\,dt - \int_0^T c(x,t)\theta_\mu^p(x,t)f(t)\,dt.\end{aligned}$$

By dividing μ and making $\mu \to \infty$, we have

$$\int_0^T k(x,t)f(t)\Delta\theta(x,t)\,dt = 0.$$

Since $f(t)$ is arbitrary, we obtain

$$\Delta\theta(x,t) = 0.$$

Then we derive

$$\int_\Omega |\nabla\theta(x,t)|^2\,dx = 0.$$

Thus we have $\theta(x,t) \equiv \theta(t)$ for $x \in \bar{\Omega}$.

Step 3. In this step, we show that $\theta(t) = \omega(t)$ in $[0,T]$.

First, we assert that $\theta(t) \in C^1((0,\infty))$. Indeed, it is easy to obtain from Eq (1.1) that

$$\int_t^{t+\varepsilon}\int_\Omega \frac{u_s(x,s)}{k(x,s)}\,dxds = \int_t^{t+\varepsilon}\int_\Omega \frac{m(x,s)}{k(x,s)}u(x,s)\,dxds - \int_t^{t+\varepsilon}\int_\Omega \frac{c(x,s)}{k(x,s)}u^p(x,s)\,dxds.$$

Then we have

$$\begin{aligned}&\int_\Omega \frac{u(x,t+\varepsilon)}{k(x,t+\varepsilon)}\,dx - \int_\Omega \frac{u(x,t)}{k(x,t)}\,dx + \int_t^{t+\varepsilon}\int_\Omega \frac{k_t(x,s)}{k^2(x,s)}u(x,s)\,dxds\\ =&\int_t^{t+\varepsilon}\int_\Omega \frac{m(x,s)}{k(x,s)}u(x,s)\,dxds - \int_t^{t+\varepsilon}\int_\Omega \frac{c(x,s)}{k(x,s)}u^p(x,s)\,dxds.\end{aligned}$$

Taking $\mu \to \infty$, we obtain

$$\begin{aligned}&\int_\Omega \frac{\theta(t+\varepsilon)}{k(x,t+\varepsilon)}\,dx - \int_\Omega \frac{\theta(t)}{k(x,t)}\,dx + \int_t^{t+\varepsilon}\int_\Omega \frac{k_t(x,s)}{k^2(x,s)}\theta(s)\,dxds\\ =&\int_t^{t+\varepsilon}\int_\Omega \frac{m(x,s)}{k(x,s)}\theta(s)\,dxds - \int_t^{t+\varepsilon}\int_\Omega \frac{c(x,s)}{k(x,s)}\theta^p(s)\,dxds.\end{aligned}$$

Thus, we derive

$$\left[\theta(t)\int_\Omega \frac{1}{k(x,t)}\,dx\right]_t = \int_\Omega \frac{m(x,t)}{k(x,t)}\,dx\theta(t) - \int_\Omega \frac{c(x,t)}{k(x,t)}\,dx\theta^p(t) - \int_\Omega \frac{k_t(x,t)}{k^2(x,t)}\,dx\theta(t).$$

Hence,

$$\begin{aligned}\theta_t(t) =& \frac{1}{k_*(t)}\int_\Omega \frac{m(x,t)}{k(x,t)}\,dx\theta(t) - \frac{1}{k_*(t)}\int_\Omega \frac{c(x,t)}{k(x,t)}\,dx\theta^p(t)\\ &- \frac{1}{k_*(t)}\int_\Omega \frac{k_t(x,t)}{k^2(x,t)}\,dx\theta(t) - \frac{[k_*(t)]_t}{k_*(t)}\theta(t),\end{aligned}$$

for $t > 0$. Thus $\theta(t) \in C^1((0,\infty))$ holds.

We then prove that $\theta(t)$ satisfies Eq (1.5). It is obvious from Eq (1.1) that

$$\int_\Omega \frac{u_t(x,t)}{k(x,t)}\,dx = \int_\Omega \frac{m(x,t)}{k(x,t)}u(x,t)\,dx - \int_\Omega \frac{c(x,t)}{k(x,t)}u^p(x,t)\,dx. \tag{4.4}$$

Similarly, suppose that $f(t)$ is a smooth T-periodic function. Multiplying $f(t)$ on both sides of Eq (4.4) and integrating over $[0,T]$ gives

$$\begin{aligned}&-\int_0^T\int_\Omega u(x,t)\left[\frac{f(t)}{k(x,t)}\right]_t\,dxdt\\ =&\int_0^T\int_\Omega \frac{m(x,t)}{k(x,t)}u(x,t)f(t)\,dxdt - \int_0^T\int_\Omega \frac{c(x,t)}{k(x,t)}u^p(x,t)f(t)\,dxdt.\end{aligned}$$

Letting $\mu \to \infty$, we know

$$-\int_0^T\int_\Omega \theta(t)\left[\frac{f(t)}{k(x,t)}\right]_t\,dxdt = \int_0^T\int_\Omega \frac{m(x,t)}{k(x,t)}\theta(t)f(t)\,dxdt - \int_0^T\int_\Omega \frac{c(x,t)}{k(x,t)}\theta^p(t)f(t)\,dxdt.$$

This implies

$$\int_0^T\int_\Omega \frac{f(t)}{k(x,t)}\theta_t(t)\,dxdt = \int_0^T\int_\Omega \frac{m(x,t)}{k(x,t)}\theta(t)f(t)\,dxdt - \int_0^T\int_\Omega \frac{c(x,t)}{k(x,t)}\theta^p(t)f(t)\,dxdt.$$

By the arbitrary of $f(t)$, it follows that

$$\int_\Omega \frac{1}{k(x,t)}\,dx\theta_t(t) = \int_\Omega \frac{m(x,t)}{k(x,t)}\,dx\theta(t) - \int_\Omega \frac{c(x,t)}{k(x,t)}\,dx\theta^p(t).$$

Thus, we have

$$\begin{cases}\theta_t = \frac{\tilde{M}(t)}{\tilde{k}(t)}\theta - \frac{\tilde{C}(t)}{\tilde{k}(t)}\theta^p, & t \in \mathbb{R},\\ \theta(0) = \theta(T).\end{cases}$$

Finally, we prove that $\theta(t) > 0$ in $t \in [0,T]$. We define $\theta_\mu^\varepsilon(x,t)$ as the unique positive periodic solution of Eq (4.1) for small $\varepsilon > 0$ and large μ. Similarly to the previous argument, it can be shown that

$$\lim_{\mu\to\infty}\theta_\mu^\varepsilon(x,t) = \omega^\varepsilon(t) \text{ in } C^{1,\frac{1}{2}}(\bar{Q}_T), \tag{4.5}$$

where $\omega^{\varepsilon}(t)$ satisfies

$$\begin{cases}\omega_t^{\varepsilon} = \frac{M_*(t)}{k_*(t)}(\omega^{\varepsilon} + \varepsilon) - \frac{c_*(t)}{k_*(t)}(\omega^{\varepsilon})^p, & t \in \mathbb{R}, \\ \omega^{\varepsilon}(0) = \omega^{\varepsilon}(T). \end{cases} \tag{4.6}$$

Since

$$\int_{\Omega} \frac{m(x,t)}{k(x,t)} dx > 0 \text{ in } [0,T],$$

we can obtain that Eq (1.5) admits a unique periodic positive solution $\omega(t)$. Note that $\omega(t)$ is a lower solution of Eq (4.6). Then there exists a unique positive periodic solution $\omega^{\varepsilon}(t)$ to Eq (4.6). In addition, $\omega^{\varepsilon}(t)$ is monotonically increasing about ε, and $\omega_{\varepsilon}(t) \geq \omega(t) > 0$. We obtain that there exists a positive continuous function $\omega_0(t)$ such that

$$\lim_{\varepsilon \to 0+} \omega^{\varepsilon}(t) = \omega_0(t) \text{ in } [0,T].$$

The uniqueness of the positive solution of Eq (4.6) implies that

$$\lim_{\varepsilon \to 0+} \omega^{\varepsilon}(t) = \omega(t) \text{ in } [0,T].$$

It follows from Lemma 4.1 that

$$\lim_{\varepsilon \to 0+} \theta_{\mu}^{\varepsilon}(x,t) = \theta_{\mu}(x,t) \text{ in } C^{1,\frac{1}{2}}(\bar{Q}_T).$$

This means that $\theta(t)$ is positive, together with (4.4)–(4.6). Thus, we must have

$$\theta(t) \equiv \omega(t) \text{ in } [0,T].$$

This also implies that

$$\lim_{\mu \to \infty} \theta_{\mu}(x,t) = \omega(t) \text{ in } C^{1,\frac{1}{2}}(\bar{Q}_T),$$

holds for the entire sequence.

5. Conclusions

We consider the positive solutions of the periodic-parabolic logistic equation with indefinite weight function and nonhomogeneous diffusion coefficient. When the dispersal rate is small, we can obtain a similar result as in the homogeneous diffusion coefficient. Here we are interested in the case of large diffusion coefficient with nonhomogeneous diffusion coefficient.

In Theorem 1.1, we obtain the condition of $m(x,t)$ to guarantee a positive periodic solution for all $\mu > 0$. Then we investigate the effect of large μ on the positive solution and establish that the limiting profile is determined by the positive solution of Eq (1.5). More precisely, we prove that the positive periodic solution tends to the unique positive solution of the corresponding non-autonomous logistic equation when the diffusion rate is large.

Author contributions

M. Fan was responsible for writing the original draft. J. Sun handled the review and supervision.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors would like to thank the anonymous reviewers for the careful reading and several valuable comments to revise the paper. Fan was supported by Gansu postgraduate scientifc research (20230XZX-055) and Sun was supported by NSF of China (12371170) and NSF of Gansu Province of China (21JR7RA535, 21JR7RA537).

Conflict of interest

The authors declare there is no conflict of interest.

References

1. R. S. Cantrell, C. Cosner, *Spatial Ecology via Reaction-Diffusion Equations*, John Wiley & Sons, Chichester, UK, 2003. https://doi.org/10.1002/0470871296
2. P. Hess, Periodic-parabolic boundary value problems and positivity, *Bull. London Math. Soc.*, **247** (1991). https://doi.org/10.1112/blms/24.6.619
3. Y. Du, R. Peng, The periodic logistic equation with spatial and temporal degeneracies, *Trans. Am. Math. Soc.*, **364** (2012), 6039–6070. https://doi.org/10.1090/S0002-9947-2012-05590-5
4. Y. Du, R. Peng, Sharp spatiotemporal patterns in the diffusive time-periodic logistic equation, *J. Differ. Equations*, **254** (2013), 3794–3816. https://doi.org/10.1016/j.jde.2013.02.004
5. H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM Rev.*, **18** (1976), 620–709. https://doi.org/10.1137/1018114
6. H. Brézis, L. Oswald, Remarks on sublinear elliptic equations, *Nonlinear Anal. Theory Methods Appl.*, **10** (1986), 55–64. https://doi.org/10.1016/0362-546X(86)90011-8
7. W. T. Li, J. López-Gómez, J. W. Sun, Sharp blow-up profiles of positive solutions for a class of semilinear elliptic problems, *Adv. Nonlinear Stud.*, **21** (2021), 751–765. https://doi.org/10.1515/ans-2021-2149
8. W. T. Li, J. López-Gómez, J. W. Sun, Sharp patterns of positive solutions for some weighted semilinear elliptic problems, *Calc. Var. Partial Differ. Equations*, **60** (2021), 1–36. https://doi.org/10.1007/s00526-021-01993-9
9. I. Antón, J. López-Gómez, Principal eigenvalues of weighted periodic-parabolic problems, *Rend. Istit. Mat. Univ. Trieste*, **49** (2017), 287–318. https://doi.org/10.13137/2464-8728/16217
10. I. Antón, J. López-Gómez, Principal eigenvalue and maximum principle for cooperative periodic-parabolic systems, *Nonlinear Anal.*, **178** (2019), 152–189. https://doi.org/10.1016/j.na.2018.07.014
11. S. Cano-Casanova, S. Fernández-Rincón, J. López-Gómez, A singular perturbation result for a class of periodic-parabolic BVPs, *Open Math.*, **22** (2024), 20240020. https://doi.org/10.1515/math-2024-0020

12. J. López-Gómez, Protection zones in periodic-parabolic problems, *Adv. Nonlinear Stud.*, **20** (2020), 253–276. https://doi.org/10.1515/ans-2020-2084

13. J. López-Gómez, E. Muñoz-Hernández, Global structure of subharmonics in a class of periodic predator-prey models, *Nonlinearity*, **33** (2020), 34–71. https://doi.org/10.1088/1361-6544/ab49e1

14. W. Shen, Y. Yi, Dynamics of almost periodic scalar parabolic equations, *J. Differential Equations*, **122** (1995), 114–136. https://doi.org/10.1006/jdeq.1995.1141

15. D. Aleja, I. Antón, J. López-Gómez, Global structure of the periodic positive solutions for a general class of periodic-parabolic logistic equations with indefinite weights, *J. Math. Anal. Appl.*, **487** (2020), 123961. https://doi.org/10.1016/j.jmaa.2020.123961

16. E. N. Dancer, P. Hess, Behaviour of a semi-linear periodic-parabolic problem when a parameter is small, in *Functional-Analytic Methods for Partial Differential Equations: Proceedings of a Conference and a Symposium held in Tokyo, Japan, July 3–9, 1989*, Heidelberg: Springer Berlin Heidelberg, Berlin, (2006), 12–19. https://doi.org/10.1007/BFb0084895

17. D. Daners, J. López-Gómez, The singular perturbation problem for the periodic-parabolic logistic equation with indefinite weight functions, *J. Dyn. Differ. Equations*, **6** (1994), 659–670. https://doi.org/10.1007/BF02218853