Networks and Heterogeneous Media

http://www.aimspress.com/journal/nhm

NHM, 19(3): 1013–1037.
DOI: 10.3934/nhm.2024045
Received: 22 August 2024
Revised: 18 September 2024
Accepted: 23 September 2024
Published: 27 September 2024

Research article

A variational MAX ensemble numerical algorism for a transient heat model with random inputs

Tingfu Yao[1,2], **Changlun Ye** [3], **Xianbing Luo** [1,*] **and Shuwen Xiang**[1,2]

[1] School of Mathematics and Statistics, Guizhou University, Guiyang 550025, China.

[2] College of Science, Guiyang University, Guiyang 550005, China.

[3] School of Mathematical Sciences, Guizhou Normal University, Guiyang 550001, China

* **Correspondence:** Email: xbluo1@gzu.edu.cn.

Abstract: A variational MAX ensemble-based time-stepping numerical method is proposed to simulate a transient heat equation with uncertain Robin boundary and diffusion coefficients. Instead of employing ensemble means for Robin coefficients as well as diffusion coefficients, the maximums of these coefficients are utilized at per time step. This is a new variational ensemble Monte Carlo (MC) numerical method, which we call the variational MAX ensemble Monte Carlo (VMEMC) method. In contrast with related methodologies, the novelty of this algorithm is that it is unconditionally stable. And also, the error estimates are proved. Numerical tests illustrate the theoretical properties for the VMEMC method.

Keywords: transient heat equation; Robin coefficient; diffusion coefficient; uncertain inputs; variational MAX ensemble; MC sampling

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, where Ω is a sample space, $\mathcal{F} \subset 2^{\Omega}$ is the σ-algebra of events, and $P : \mathcal{F} \to [0, 1]$ is a probability measure. $[0, T]$ is the temporal interval. $D \subseteq \mathbb{R}^2$ is an open, bounded, Lipschitz physical space. $\partial D = \partial D_0 \bigcup \partial D_1$ is the union of disjoint portions for the boundary. The unit outward normal vector to ∂D is denoted by $\mathbf{n}$. We carry out the numerical simulations of a transient heat model with random Robin coefficients and diffusion coefficients. That's to find a random function $y : [0, T] \times D \times \Omega \longrightarrow \mathbb{R}$ conforming to equations almost surely $\omega \in \Omega$,

$$\begin{cases} y_t - \nabla \cdot [a(t, \mathbf{x}, \omega)\nabla y] = f(t, \mathbf{x}, \omega), & \text{in } [0, T] \times D \times \Omega, \\ \qquad\qquad y(0, \mathbf{x}, \omega) = y^0(\mathbf{x}, \omega), & \text{in } D \times \Omega, \\ a\nabla y(t, \mathbf{x}, \omega) \cdot \mathbf{n} = 0, & \text{on } [0, T] \times \partial D_0 \times \Omega, \\ a\nabla y(t, \mathbf{x}, \omega) \cdot \mathbf{n} = \alpha(t, \mathbf{x}, \omega)(u(t, \mathbf{x}, \omega) - y(t, \mathbf{x}, \omega)), & \text{on } [0, T] \times \partial D_1 \times \Omega, \end{cases} \tag{1.1}$$

where Robin coefficients α, diffusion coefficients a, boundary condition u, source term f, and initial condition y^0 are random fields with continuous and bounded covariance functions.

The model (1.1) describes the process of heat as well as mass transfer with random Robin factors and diffusion factors, which can be found in [3, 4] and the references therein. These uncertain factors are involved in thermal environments. The temperature is quantified stochastically. In the review [3], this model can describe random high-cycle temperature fluctuations observed at the upper core structure of fast-breeder reactors or random variations in heat transfer coefficients around the stator vanes of gas turbines. The model (1.1) also appears in robust optimal boundary control problems as state equations, such as [25, 26]. The effective implementation of these applications needs to solve (1.1) efficiently.

To solve PDEs with random coefficients, there are many numerical algorithms that include polynomial chaos method, stochastic collocation method, stochastic finite element method, MC method, and so on (see, e.g., [1, 9, 11, 21, 22, 27, 29, 31]). Where the MC method (see, e.g., [8, 10, 12, 24]) is easy to implement and the convergence property of the MC method is not dependent on the dimension of the parameters in a random model. When the MC method is adopted, we first need independent sampling and then deal with independent numerical simulation per sample. These simulations are affected by independent diffusion factors, Robin factors, forces, initial and boundary conditions. To improve calculating efficiency, the ensemble algorithms are widely applied (see, e.g., [13, 17, 20, 23, 24]). In [23, 24], the authors studied the parabolic problem with random coefficients by using the ensemble method, and obtained an error estimate. But the error estimate therein is not optimal in physical space. The authors of [18] combined the ensemble MC with the HDG method to obtain an optimal error estimate in space. For the heat conduction model with random Robin factors, we have not found other relevant results about ensemble algorithms besides our recent work [30].

Look back at the numerical algorithm in the article [30]. $\bar{\alpha}(t, \mathbf{x})$ and $\bar{a}(t, \mathbf{x})$ stand for the ensemble mean for the Robin factors as well as the diffusion factors, respectively, and we denote the fluctuations, $\alpha'_j := \bar{\alpha} - \alpha_j$, $a'_j := \bar{a} - a_j$. We adopt a uniform partition on the interval $[0, T]$ as well as $t_n = n\Delta t, n = 0, 1, 2, ..., N$. Let $y^n_j, f^n_j, u^n_j, a^n_j, \alpha^n_j, \bar{a}^n$, $\bar{\alpha}^n$, ${a'_j}^n$ and ${\alpha'_j}^n$ be the values of functions $y_j, f_j, u_j, a_j, \alpha_j, \bar{a}$, $\bar{\alpha}$, a'_j and α'_j at $t = t_n$, respectively. $y_j = y(t, \mathbf{x}; \omega_j)$ and the others are similar. In [30], based on the implicit Euler method, we developed a variational average ensemble method for the random transient heat problem (1.1). Conveniently, we called the ***variational average ensemble Monte Carlo (VAEMC)*** method: find y^{n+1}_j $(j = 1, 2, \ldots, J; n = 0, 1, \ldots, N - 1)$ such that for all $v \in H^1(\bar{D})$,

$$\left(\frac{y^{n+1}_j - y^n_j}{\Delta t}, v\right) + \left(\bar{a}^{n+1}\nabla y^{n+1}_j, \nabla v\right) + \left(\left(a^{n+1}_j - \bar{a}^{n+1}\right)\nabla y^n_j, \nabla v\right) + \left(\bar{\alpha}^{n+1} y^{n+1}_j, v\right)_{\partial D_1}$$
$$+ \left(\left(\alpha^{n+1}_j - \bar{\alpha}^{n+1}\right) y^n_j, v\right)_{\partial D_1} = \left(f^{n+1}_j, v\right) + \left(\alpha^{n+1}_j u^{n+1}_j, v\right)_{\partial D_1},$$

After arrangement, that is

$$\left(\frac{1}{\Delta t}y_j^{n+1}, v\right) + \left(\bar{a}^{n+1}\nabla y_j^{n+1}, \nabla v\right) + \left(\bar{\alpha}^{n+1}y_j^{n+1}, v\right)_{\partial D_1}$$
$$= \left(f_j^{n+1}, v\right) + \left(\alpha_j^{n+1}u_j^{n+1}, v\right)_{\partial D_1} + \left(\frac{1}{\Delta t}y_j^n, v\right) + \left(\left(a_j'^{\,n+1}\right)\nabla y_j^n, \nabla v\right) + \left(\left(\alpha_j'^{\,n+1}\right)y_j^n, v\right)_{\partial D_1}. \quad (1.2)$$

Denote $a^* = \max\limits_{1\leqslant j\leqslant J}\left\|\frac{a_j'}{\bar{a}}\right\|_\infty$ and $\alpha^* = \max\limits_{1\leqslant j\leqslant J}\left\|\frac{\alpha_j'}{\bar{\alpha}}\right\|_\infty$. From [30], we base on the following numerical stability condition $a^* < 1$ and $\alpha^* < 1$ for the first-order ensemble time-stepping methods (1.2). In the ensemble algorithm in the literatures [17–20, 23, 24, 30], both stability and convergence were conditionally dependent on the ratio between the fluctuating and average values of the relevant factors.

Now, instead of employing ensemble means of Robin coefficients as well as diffusion coefficients in scheme (1.2), we propose a maximum ensemble of these coefficients at each time step. The process gets a shared coefficient matrix for the ensemble group to better improve computational efficiency. This is a new variational ensemble Monte Carlo numerical method, which we called the ***variational MAX ensemble Monte Carlo (VMEMC)*** method to conveniently distinguish it from other methods. In contrast with related methodologies, the novelty of this algorithm is that the numerical stability and convergence are unconditional.

The following is the outline of this work. Some mathematical preliminaries and notations are listed in Section 2. The full discretization VMEMC scheme for the random transient heat model (1.1) is given in Section 3. In Section 4, the unconditional numerical stability and error estimate are analyzed. Several numerical tests are shown in Section 5. Related discussions of this algorithm and corresponding summaries are presented in Section 6.

2. Mathematical preliminaries

In this section, we will give some notations and mathematical preliminaries. For simplicity, $d\mathbf{x}, d\mathbf{s}$, and dt in some expressions will be omitted when there is no confusion. The boundaries ∂D_0 and ∂D_1 concern the experimentally accessible and inaccessible parts, respectively. Throughout this work C is a positive constant; it has different values in different places and does not rely on time step Δt, sample size J, or mesh size h.

2.1. Basic preliminary.

Let $\|\cdot\|$ and $(\cdot,\cdot)$ be the $L^2(D)$ norm as well as inner product, respectively. Simultaneously, $\|\cdot\|_{\partial D}$ and $(\cdot,\cdot)_{\partial D}$ stand for the $L^2(\partial D)$ norm and the inner product. The Sobolev space $W^{s,q}(D)$ with the norm $\|v\|_{W^{s,q}(D)}$, here $s \in \mathbb{N}$ (natural number set) and $q \geq 1$. We denote $H^s(D) = W^{s,2}(D)$. Particularly, $H^1(D)$ is equipped with the norm $\|\cdot\|_1 = \|\cdot\|_{1,D}$, which is defined by

$$\|y\|_{1,D} = \left(\|y\|^2 + \|\nabla y\|^2\right)^{\frac{1}{2}}.$$

Let $H^{-s}(D)$ be the dual space of the bounded linear functions on $H^s(D)$, with norm $\|f\|_{-s} = \sup_{0\neq v\in H^s(D)}(f, v)/\|v\|_s$. Denote

$$L^\infty(D) = \{v : v \text{ is a measurable function and } |v|_\infty < +\infty\},$$

where $|v|_\infty = \text{ess } \sup_{\mathbf{x}\in D}|v|$.

$(\Omega, \mathcal{F}, P)$ is a complete probability space. $Z \in L^1_P(\Omega)$ is a random variable. Denote

$$\mathbb{E}[Z] = \int_\Omega Z(\omega) dP(\omega).$$

Let $\delta = (\delta_1, \ldots, \delta_d)$ is a d-tuple with the length $|\delta| = \sum_{i=1}^{d} \delta_i, \delta_i \in N^+$. The stochastic Sobolev spaces $\widetilde{W}^{s,q}(D) = L^q_P(\Omega, W^{s,q}(D))$ contains stochastic function $v : \Omega \times D \to R$, which is measurable with respect to the product σ-algebra $\mathcal{F} \otimes B(D)$. The norm of $\widetilde{W}^{s,q}(D)$ is defined by

$$\|v\|_{\widetilde{W}^{s,q}(D)} = \left(\mathbb{E}\left[\|v\|^q_{W^{s,q}(D)}\right]\right)^{1/q} = \left(\mathbb{E}\left[\sum_{|\delta \leq s|} \int_D \left|\partial^\delta v\right|^q\right]\right)^{1/q}, 1 \leq q < +\infty.$$

Let $\widetilde{H}^s(D) = \widetilde{W}^{s,2}(D) \simeq L^2_P(\Omega) \otimes H^s(D)$.

We will use the tensor product Hilbert space

$$X = \widetilde{L}^2\left(0, T; H^1(D)\right) \simeq L^2_P\left(0, T; H^1(D); \Omega\right)$$

with its inner product

$$(v, u)_X \equiv \mathbb{E}\left[\int_0^T \int_D (\nabla v \cdot \nabla u + vu)\right].$$

The induced norm is given by

$$\|v\|_X = \left(\mathbb{E}\left[\int_0^T \int_D \left(|\nabla v|^2 + v^2\right)\right]\right)^{1/2}.$$

Lemma 2.1. *The norm* $\|\cdot\|_{1,\partial D_1}$ *is defined by*

$$\|y\|^2_{1,\partial D_1} = \int_D |\nabla y|^2 + \int_{\partial D_1} |y|^2.$$

This norm and the standard norm $\|\cdot\|_1$ *in* $H^1(D)$ *are equivalent. Particularly, there exist constants* $C_P, C'_P > 0$ *such that* $\forall v \in H^1(D)$

$$C_P\|v\|_1 \leq \|v\|_{1,\partial D_1}, \quad C'_P\|v\|_{1,\partial D_1} \leq \|v\|_1.$$

Proof. cf. [14, 15]. □

The weak solution of the problem (1.1) is defined as follows: a function $y \in X$ is a weak solution of (1.1) if it satisfies the initial condition $y(0, \mathbf{x}, \omega) = y^0(\mathbf{x}, \omega) \in L^2_P\left(H^1(D); \Omega\right)$ and for $T > 0$,

$$\int_0^T (y_t, v)\, dt + \int_0^T \int_\Omega \int_D a\nabla y \cdot \nabla v d\mathbf{x} dP(\omega)dt + \int_0^T \int_\Omega \int_{\partial D_1} \alpha yv d\mathbf{s} dP(\omega)dt$$
$$= \int_0^T \int_\Omega \int_D fv d\mathbf{x} dP(\omega)dt + \int_0^T \int_\Omega \int_{\partial D_1} \alpha uv d\mathbf{s} dP(\omega)dt \tag{2.1}$$

for all $v \in X$.

We emphasize that the assumed regularity only requires the random fields to be square integrable. Assumptions on the finite-dimensional noise, that is, all the involved random input data depend on a finite number of real-valued random variables. Assumptions of input data for coefficients are considered:

(**A1**) $a = a(t, \mathbf{x}, \omega) \in L_P^2\left(0, T; L^2(D); \Omega\right)$ is uniformly bounded and coercive, i.e., $\exists\, a_{\min}$, $a_{\max} > 0$ such that $Pro\{\omega \in \Omega : a(t, \mathbf{x}, \omega) \in [a_{\min}, a_{\max}], \forall (t, \mathbf{x}) \in [0, T] \times D\} = 1$.

(**A2**) $\alpha = \alpha(t, \mathbf{x}, \omega) \in L_P^2\left(0, T; L^2\left(\partial D_1\right); \Omega\right)$ is uniformly bounded and coercive too, i.e., $\exists\, \alpha_{\min}$, $\alpha_{\max} > 0$ such that $Pro\{\omega \in \Omega : \alpha(t, \mathbf{x}, \omega) \in [\alpha_{\min}, \alpha_{\max}], \forall (t, \mathbf{x}) \in [0, T] \times \partial D_1\} = 1$.

Denote the ensemble group of solution variables for the equation (1.1) by $y_j(t, \mathbf{x}) = y\left(t, \mathbf{x}; \omega_j\right)$, $j = 1, 2, ..., J$, with corresponding boundary conditions. $[0, T] = \bigcup_{n=0}^{N-1} [t_n, t_{n+1}]$ is an isometric partition of the interval. These norms are utilized in the numerical error discussion: $\forall -1 \leq s < \infty$,

$$|||v|||_{\infty,s} := \max_{0\leq n\leq N} \|v^n\|_s\,, \quad |||v|||_{p,s} := \left(\Delta t \sum_{n=0}^{N} \|v^n\|_s^p\right)^{1/p}.$$

We use the MC method for random sampling because it is non-intrusive, easy to implement, and its convergence is independent of the dimension of the uncertain model parameters. For $j \in \{1, 2, ..., J\}$, we let $a_{\max}^n = \max\limits_{1\leq j\leq J} \sup\limits_{x\in\Omega} a_j\left(t_n, \mathbf{x}\right)$ and $\alpha_{\max}^n = \max\limits_{1\leq j\leq J} \sup\limits_{x\in\Omega} \alpha_j\left(t_n, \mathbf{x}\right)$, where $a_j\left(t, \mathbf{x}\right) = a\left(t, \mathbf{x}, \omega_j\right)$, $\alpha_j\left(t, \mathbf{x}\right) = \alpha\left(t, \mathbf{x}, \omega_j\right)$. Without causing confusion, we abuse the fluctuating component symbols $a'^{\,n}_j := a_{\max}^n - a_j\left(t_n, \mathbf{x}\right)$ and $\alpha'^{\,n}_j := \alpha_{\max}^n - a_j\left(t_n, \mathbf{x}\right)$. Suppressing the spatial discretization, utilizing an implicit-explicit for time-direct to the model (1.1), we propose the ***VMEMC scheme***: find $y_j^{n+1} \in H^1(\bar{D})$ such that

$$\begin{aligned}&\left(\frac{y_j^{n+1} - y_j^n}{\Delta t}, v\right) + \left(a_{\max}^{n+1} \nabla y_j^{n+1}, \nabla v\right) - \left(a'^{\,n+1}_j \nabla y_j^n, \nabla v\right) + \left(\alpha_{\max}^{n+1} y_j^{n+1}, v\right)_{\partial D_1} \\ &- \left(\alpha'^{\,n+1}_j y_j^n, v\right)_{\partial D_1} = \left(f_j^{n+1}, v\right) + \left(\alpha_j^{n+1} u_j^{n+1}, v\right)_{\partial D_1}, j = 1, \ldots, J, \quad \forall v \in H^1(\bar{D}). \end{aligned} \tag{2.2}$$

Using a standard finite element (FE) method to discretize the space, we obtain the following block linear system for each ensemble member j :

$$\begin{aligned}&\left(\frac{1}{\Delta t} M + a_{\max}^{n+1} D + \alpha_{\max}^{n+1} M_{\partial D_1}\right) y_j^{n+1} \\ &= \left(f_j^{n+1} + \alpha_j^{n+1} u_j^{n+1} + \frac{1}{\Delta t} M + a'^{n+1}_j D + \alpha'^{n+1}_j M_{\partial D_1}\right) y_j^n, \end{aligned} \tag{2.3}$$

where M is the mass matrix, $M_{\partial D_1}$ is the boundary mass matrix, and D is the diffusion matrix.

The above linear system is equivalent to the following:

$$[A] \left[y_1 \left| y_2 \right| \ldots \mid y_J\right] = \left[b_1 \left| b_2 \right| \ldots \mid b_J\right], \tag{2.4}$$

where A is the resulting coefficient matrix (independent of j). The matrix A is symmetric positive definite (SPD) since both $\frac{1}{\Delta t} M$, $a_{\max}^{n+1} D$, and $\alpha_{\max}^{n+1} M_{\partial D_1}$ are SPD. The system (2.4) can be solved with efficient block solvers [7,16,28]. Further, since only one coefficient matrix is required for computation each time step, the storage requirement is thereby reduced.

2.2. Finite elements preliminary.

Suppose $\mathcal{T}_h$ is a quasi-uniform triangulation of the domain $\bar{D}$, such that $\bar{D} = \bigcup_{K\in\mathcal{T}_h} \bar{K}$. Let h_K be the diameter of the element K. Define $h = \max_{K\in\mathcal{T}_h} h_K$. Let $\mathbf{P}_k$ be the set of polynomials of degree k. Denote the finite elements (FEs) space V_h of the domain $\bar{D}$ by

$$V_h = span\{\varphi_k\}_1^{n_{\bar{D}}} \subset \left\{v_h \in H^1(D) \cap H^{k+1}(\bar{D}) : v_h|_{\bar{K}} \in \mathbf{P}_k, \forall K \in \mathcal{T}_h\right\},$$

the FEs space S_h of the trace space corresponding to V_h by $S_h = span\{\phi_i\}_1^{n_{\partial D_1}} \subset V_{h,\partial D_1}$, here the space $V_{h,\partial D_1}$ is defined as the restriction of V_h on ∂D_1, $n_{\bar{D}}$ is the number of nodes on $\bar{D}$, $n_{\partial D_1}$ is the number of nodes on ∂D_1. The spaces above satisfy the following approximation properties [2, 5] : $\forall 1 \le l \le k$,

$$\inf_{v_h\in H^1(\bar{D})} \{\|y - v_h\| + h\,\|\nabla(y - v_h)\|\} \le Ch^{l+1}|y|_{l+1}, \quad y \in H^1(\bar{D}) \cap H^{l+1}(\bar{D}). \tag{2.5}$$

$I_h : C(\bar{D}) \to V_h$ or $I_h : C(\partial D_1) \to S_h$ denotes the interpolation operator related to the spaces V_h or S_h.

3. variational MAX ensemble Monte Carlo (VMEMC) algorithm

We first present a full discretization VMEMC method with finite elements for the VMEMC scheme (2.2) of the random transient heat model (1.1). And then the VMEMC algorithm is proposed. The ***full-discrete VMEMC scheme***: find $y_{j,h}^{n+1} \in V_h$ such that

$$\begin{aligned}&\left(\frac{y_{j,h}^{n+1} - y_{j,h}^{n}}{\Delta t}, v_h\right) + \left(a_{\max}^{n+1}\nabla y_{j,h}^{n+1}, \nabla v_h\right) - \left(a_j'^{\,n+1}\nabla y_{j,h}^{n}, \nabla v_h\right) + \left(\alpha_{\max}^{n+1} y_{j,h}^{n+1}, v_h\right)_{\partial D_1}\\ &- \left(\alpha_j'^{\,n+1} y_{j,h}^{n}, v_h\right)_{\partial D_1} = \left(f_j^{n+1}, v_h\right) + \left(\alpha_j^{n+1} u_j^{n+1}, v_h\right)_{\partial D_1}, \quad \forall v_h \in V_h,\end{aligned} \tag{3.1}$$

here the initial value $y_{j,h}^0 \in V_h$ which $\left(y_{j,h}^0, v_h\right) = \left(y_j^0, v_h\right), \forall v_h \in V_h$.

As a comparison, here let's list the ***full-discrete VAEMC scheme*** [30] : find $y_{j,h}^{n+1} \in V_h$ such that,

$$\begin{aligned}&\left(\frac{y_{j,h}^{n+1} - y_{j,h}^{n}}{\Delta t}, v_h\right) + \left(\bar{a}^{n+1}\nabla y_{j,h}^{n+1}, \nabla v_h\right) - \left(a_j'^{\,n+1}\nabla y_{j,h}^{n}, \nabla v_h\right) + \left(\bar{\alpha}^{n+1} y_{j,h}^{n+1}, v_h\right)_{\partial D_1}\\ &- \left(\alpha_j'^{\,n+1} y_{j,h}^{n}, v_h\right)_{\partial D_1} = \left(f_j^{n+1}, v_h\right) + \left(\alpha_j^{n+1} u_j^{n+1}, v_h\right)_{\partial D_1}, \forall v_h \in V_h,\end{aligned} \tag{3.2}$$

here the initial value $y_{j,h}^0 \in V_h$ which $\left(y_{j,h}^0, v_h\right) = \left(y_j^0, v_h\right), \forall v_h \in V_h$.

By the way, the ***full-discrete variational Monte Carlo (VMC) scheme***: find $y_{j,h}^{n+1} \in V_h$ such that,

$$\begin{aligned}&\left(\frac{y_{j,h}^{n+1} - y_{j,h}^{n}}{\Delta t}, v_h\right) + \left(a_j^{n+1}\nabla y_{j,h}^{n+1}, \nabla v_h\right) + \left(\alpha_j^{n+1} y_{j,h}^{n+1}, v_h\right)_{\partial D_1}\\ &= \left(f_j^{n+1}, v_h\right) + \left(\alpha_j^{n+1} u_j^{n+1}, v_h\right)_{\partial D_1}, \forall v_h \in V_h,\end{aligned} \tag{3.3}$$

here the initial value $y_{j,h}^0 \in V_h$ which $\left(y_{j,h}^0, v_h\right) = \left(y_j^0, v_h\right), \forall v_h \in V_h$.

Remark 3.1. *(1) The symbols α'^{n+1}_j and a'^{n+1}_j both appear in VMEMC and VAEMC schemes. Although they all represent fluctuations in corresponding random coefficients, they all mean different things.*
(2) Although the VMEMC and VAEMC schemes are formally similar, the novelty of the VMEMC scheme is its unconditional numerical stability and convergence. And the VMEMC scheme is more efficient in the numerical simulation.
(3) If the random thermal conductivity and convective coefficient are independent of time, such as, $\alpha := \alpha(x, \omega)$, $a := a(x, \omega)$, the accordingly analyzed analysis is analogous to that presented below. It (VMEMC) is unconditionally stable as well as the first-order accurate approach.
(4) Assume that the random thermal conductivity and convective coefficient are provided with uncertain temperature-dependent values as [6], e.g., $a := a(y)$, $\alpha := \alpha(y)$. Let y^n_h be the fully discrete approximate solution at time t_n, $a^n_h = a\left(y^n_h\right)$, $a'^{\,n}_h = a_{\max} - a^n_h$, $\alpha^n_h = \alpha\left(y^n_h\right)$, $\alpha'^{\,n}_h = \alpha_{\max} - \alpha^n_h$. Then we can replace them accordingly in the VMEMC scheme. Similarly, unconditionally stable and first-order accurate methods under assumptions that two coefficients are continuously differentiable can also be obtained.

Applying the variational MAX ensemble scheme to the stochastic model (1.1), we first take the MC approach for random sampling. After sampling with independent identically distributed (i.i.d.), we solve the corresponding deterministic PDE for each sample. Furthermore, the solution for model (1.1) is made from the expectation of the solutions of the deterministic PDEs. A variational MAX ensemble-based Monte-Carlo algorithm (i.e., VMEMC algorithm) is proposed to quantifies uncertainty and improve its computational efficiency. The VMEMC algorithm includes the following three steps (S):

VMEMC algorithm:

S1. Choose a group of random samples for the stochastic Robin coefficient, diffusion coefficient, source term, boundary, and initial conditions, $\alpha_j \equiv \alpha(\cdot,\cdot,\omega_j)$, $a_j \equiv a(\cdot,\cdot,\omega_j)$, $f_j \equiv f(\cdot,\cdot,\omega_j)$, $u_j \equiv u(\cdot,\cdot,\omega_j)$, and $y^0_j \equiv y^0(\cdot,\omega_j)$ for per $j \in \{1, 2, ..., J\}$. Therefore, the solutions $y(\cdot,\cdot,\omega_j)$ are i.i.d..

S2. Denote $y^n_j = y(t_n, \mathbf{x}, \omega_j)$, $a^n_{\max} = \max\limits_{1\le j\le J} \sup\limits_{x\in\Omega} a_j(t_n, \mathbf{x})$ and $\alpha^n_{\max} = \max\limits_{1\le j\le J} \sup\limits_{x\in\Omega} \alpha_j(t_n, \mathbf{x})$. For $n = 0, \ldots, N-1$ and the j-th member, we find y^{n+1}_j to satisfy the VMEMC scheme. One finds the FE solution $y_h(\cdot,\cdot,\omega_j)$ after choosing an approximate FE space.

S3. Take the VMEMC sample average $\frac{1}{J}\sum_{j=1}^{J} y_h(\cdot,\cdot,\omega_j)$ as an approximation to the expectation $\mathbb{E}[y]$. Given an object of interest $G(y)$, we are going to analyze these outputs for the variational ensemble systems, $G(y_h(\cdot,\cdot,\omega_1)), \ldots, G\left(y_h(\cdot,\cdot,\omega_J)\right)$, to obtain the stochastic information.

4. Stability and error analysis

The unconditional stability and error estimates are presented in this section.

4.1. Stability analysis

The unconditional stability for the VMEMC method is below.

Theorem 4.1. *Assume $f_j \in L^2\left(0, T; H^{-1}(D)\right)$, $u_j \in L^2\left(0, T; L^2(\partial D_1)\right)$, and hypothesises* (**A1**) *and* (**A2**) *are satisfied. Then the VMEMC method is unconditionally stable. Furthermore, the numerical solution* *of* (3.1) *holds*

$$
\begin{aligned}
&\mathbb{E}\left[\left\|y_{j,h}^{N}\right\|^2\right] + \Delta t\mathbb{E}\left[\left\|\sqrt{a_{\max}^{n+1}}\nabla y_{j,h}^{N}\right\|^2\right] + \Delta t\mathbb{E}\left[\left\|\sqrt{\alpha_{\max}^{n+1}}y_{j,h}^{N}\right\|_{\partial D_1}^2\right]\\
&+\sum_{n=0}^{N-1}\left(\mathbb{E}\left[\left\|y_{j,h}^{n+1}-y_{j,h}^{n}\right\|^2\right] + \Delta t\mathbb{E}\left[\left\|\sqrt{a_j'^{\,n+1}}\nabla(y_{j,h}^{n+1}-y_{j,h}^{n})\right\|^2\right]\right.\\
&\left.+\ \Delta t\mathbb{E}\left[\left\|\sqrt{\alpha_j'^{\,n+1}}(y_{j,h}^{n+1}-y_{j,h}^{n})\right\|_{\partial D_1}^2\right]\right) + \frac{C_P^2\Delta t}{4}\sum_{n=0}^{N-1}\mathbb{E}\left[\left\|\sqrt{a_j^{n+1}}y_{j,h}^{n+1}\right\|_1^2\right]\\
\leq&\mathbb{E}\left[\left\|y_{j,h}^{0}\right\|^2\right] + \Delta t\mathbb{E}\left[\left\|\sqrt{a_{\max}^{n+1}}\nabla y_{j,h}^{0}\right\|^2\right] + \Delta t\mathbb{E}\left[\left\|\sqrt{\alpha_{\max}^{n+1}}y_{j,h}^{0}\right\|_{\partial D_1}^2\right]\\
&+\frac{8\Delta t}{C_P^2 a_{\min}}\sum_{n=0}^{N-1}\mathbb{E}\left[\left\|f^{n+1}\right\|_{-1}^2\right] + \frac{8\alpha_{\max}^2\Delta t}{C_P^2 a_{\min}}\sum_{n=0}^{N-1}\mathbb{E}\left[\|u_j^{n+1}\|_{\partial D_1}^2\right].
\end{aligned}\tag{4.1}
$$

Proof. Taking $v_h = 2\Delta t y_{j,h}^{n+1}$ in (3.1), we obtain

$$
\begin{aligned}
2\left(y_{j,h}^{n+1}-y_{j,h}^{n},y_{j,h}^{n+1}\right) &+ 2\Delta t\left(a_{\max}^{n+1}\nabla y_{j,h}^{n+1},\nabla y_{j,h}^{n+1}\right) - 2\Delta t\left(a_j'^{\,n+1}\nabla y_{j,h}^{n},\nabla y_{j,h}^{n+1}\right)\\
&+ 2\Delta t\left(\alpha_{\max}^{n+1}y_{j,h}^{n+1},y_{j,h}^{n+1}\right)_{\partial D_1} - 2\Delta t\left(\alpha_j'^{\,n+1}y_{j,h}^{n},y_{j,h}^{n+1}\right)_{\partial D_1}\\
&=\left(f_j^{n+1},2\Delta t y_{j,h}^{n+1}\right) + \left(\alpha_j^{n+1}u_j^{n+1},2\Delta t y_{j,h}^{n+1}\right)_{\partial D_1}.
\end{aligned}\tag{4.2}
$$

Noting that

$$
\begin{aligned}
&2\Delta t\left(a_{\max}^{n+1}\nabla y_{j,h}^{n+1},\nabla y_{j,h}^{n+1}\right) - 2\Delta t\left(a_j'^{\,n+1}\nabla y_{j,h}^{n},\nabla y_{j,h}^{n+1}\right) = 2\Delta t\left(a_{\max}^{n+1}\nabla y_{j,h}^{n+1},\nabla\left(y_{j,h}^{n+1}-y_{j,h}^{n}\right)\right)\\
&+ 2\Delta t\left(a_j^{n+1}\nabla y_{j,h}^{n},\nabla\left(y_{j,h}^{n+1}-y_{j,h}^{n}\right)\right) + 2\Delta t\left(a_j^{n+1}\nabla y_{j,h}^{n},\nabla y_{j,h}^{n}\right)
\end{aligned}\tag{4.3}
$$

and

$$
\begin{aligned}
&2\Delta t\left(\alpha_{\max}^{n+1}y_{j,h}^{n+1},y_{j,h}^{n+1}\right)_{\partial D_1} - 2\Delta t\left(\alpha_j'^{\,n+1}y_{j,h}^{n},y_{j,h}^{n+1}\right)_{\partial D_1} = 2\Delta t\left(\alpha_{\max}^{n+1}y_{j,h}^{n+1},y_{j,h}^{n+1}-y_{j,h}^{n}\right)_{\partial D_1}\\
&+ 2\Delta t\left(\alpha_j^{n+1}y_{j,h}^{n},y_{j,h}^{n+1}-y_{j,h}^{n}\right)_{\partial D_1} + 2\Delta t\left(\alpha_j^{n+1}y_{j,h}^{n},y_{j,h}^{n}\right)_{\partial D_1},
\end{aligned}\tag{4.4}
$$

applying the polarization identity, integrating over the probability space, and rearranging terms in equation (4.2), we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\left\|y_{j,h}^{n+1}\right\|^2\right] - \mathbb{E}\left[\left\|y_{j,h}^{n}\right\|^2\right] + \mathbb{E}\left[\left\|y_{j,h}^{n+1}-y_{j,h}^{n}\right\|^2\right] + \Delta t\mathbb{E}\left[\left\|\sqrt{a_{\max}^{n+1}}\nabla y_{j,h}^{n+1}\right\|^2\right]\\
&-\Delta t\mathbb{E}\left[\left\|\sqrt{a_{\max}^{n+1}}\nabla y_{j,h}^{n}\right\|^2\right] + \Delta t\mathbb{E}\left[\left\|\sqrt{a_j'^{\,n+1}}\nabla(y_{j,h}^{n+1}-y_{j,h}^{n})\right\|^2\right] + \Delta t\mathbb{E}\left[\left\|\sqrt{a_j^{n+1}}\nabla y_{j,h}^{n+1}\right\|^2\right]\\
&+\Delta t\mathbb{E}\left[\left\|\sqrt{a_j^{n+1}}\nabla y_{j,h}^{n}\right\|^2\right] + \Delta t\mathbb{E}\left[\left\|\sqrt{\alpha_{\max}^{n+1}}y_{j,h}^{n+1}\right\|_{\partial D_1}^2\right] - \Delta t\mathbb{E}\left[\left\|\sqrt{\alpha_{\max}^{n+1}}y_{j,h}^{n}\right\|_{\partial D_1}^2\right]
\end{aligned}
$$

$$+\Delta t\mathbb{E}\left[\left\|\sqrt{\alpha_j'^{\,n+1}}(y_{j,h}^{n+1}-y_{j,h}^{n})\right\|_{\partial D_1}^2\right]+\Delta t\mathbb{E}\left[\left\|\sqrt{\alpha_j^{n+1}}y_{j,h}^{n+1}\right\|_{\partial D_1}^2\right]+\Delta t\mathbb{E}\left[\left\|\sqrt{\alpha_j^{n+1}}y_{j,h}^{n}\right\|_{\partial D_1}^2\right]$$
$$=2\Delta t\mathbb{E}\left[\left(f_j^{n+1},y_{j,h}^{n+1}\right)\right]+2\Delta t\mathbb{E}\left[\left(\alpha_j^{n+1}u_j^{n+1},y_{j,h}^{n+1}\right)_{\partial D_1}\right]. \tag{4.5}$$

Combing Cauchy-Schwarz, Young's inequalities with the trace theorem and the Sobolev embedding theorem to the right-hand side (RHS) of (4.5), we have

$$2\Delta t\mathbb{E}\left[\left(f_j^{n+1},y_{j,h}^{n+1}\right)\right]\le\frac{\Delta t}{\epsilon_1 a_{\min}}\mathbb{E}\left[\left\|f^{n+1}\right\|_{-1}^2\right]+\epsilon_1\Delta t\mathbb{E}\left[\left\|\sqrt{a_j^{n+1}}y_{j,h}^{n+1}\right\|_1^2\right], \tag{4.6}$$

$$2\Delta t\mathbb{E}\left[\left(\alpha_j^{n+1}u_j^{n+1},y_{j,h}^{n+1}\right)_{\partial D_1}\right]\le\frac{\Delta t\alpha_{\max}^2}{\epsilon_2 a_{\min}}\mathbb{E}\left[\|u_j^{n+1}\|_{\partial D_1}^2\right]+\epsilon_2\Delta t\mathbb{E}\left[\left\|\sqrt{a_j^{n+1}}y_{j,h}^{n+1}\right\|_1^2\right]. \tag{4.7}$$

Under Lemma 2.1, we obtain $\left\|\sqrt{a_j^{n+1}}\nabla y_{j,h}^{n+1}\right\|^2+\left\|\sqrt{\alpha_j^{n+1}}y_{j,h}^{n+1}\right\|_{\partial D_1}^2\ge\frac{C_P^2}{2}\left\|\sqrt{a_j^{n+1}}y_{j,h}^{n+1}\right\|_1^2$. Multiplying by Δt and integrating over the probability space, we can yield

$$\frac{C_P^2\Delta t}{2}\mathbb{E}\left[\left\|\sqrt{a_j^{n+1}}y_{j,h}^{n+1}\right\|_1^2\right]\le\Delta t\mathbb{E}\left[\left\|\sqrt{a_j^{n+1}}\nabla y_{j,h}^{n+1}\right\|^2\right]+\Delta t\mathbb{E}\left[\left\|\sqrt{\alpha_j^{n+1}}y_{j,h}^{n+1}\right\|_{\partial D_1}^2\right]. \tag{4.8}$$

For Eq (4.5), dropping two positive terms $\Delta t\mathbb{E}\left[\left\|\sqrt{a_j^{n+1}}\nabla y_{j,h}^{n}\right\|^2\right]$ and $\Delta t\mathbb{E}\left[\left\|\sqrt{\alpha_j^{n+1}}y_{j,h}^{n}\right\|_{\partial D_1}^2\right]$, useing estimate Eqs (4.6) and (4.7) with $\epsilon_1=\epsilon_2=\frac{C_P^2}{8}$, we can obtain

$$\mathbb{E}\left[\|y_{j,h}^{n+1}\|^2\right]-\mathbb{E}\left[\|y_{j,h}^{n}\|^2\right]+\mathbb{E}\left[\|y_{j,h}^{n+1}-y_{j,h}^{n}\|^2\right]+\Delta t\mathbb{E}\left[\left\|\sqrt{a_{\max}^{n+1}}\nabla y_{j,h}^{n+1}\right\|^2\right]$$
$$-\Delta t\mathbb{E}\left[\left\|\sqrt{a_{\max}^{n+1}}\nabla y_{j,h}^{n}\right\|^2\right]+\Delta t\mathbb{E}\left[\left\|\sqrt{a_j'^{\,n+1}}\nabla(y_{j,h}^{n+1}-y_{j,h}^{n})\right\|^2\right]+\Delta t\mathbb{E}\left[\left\|\sqrt{\alpha_{\max}^{n+1}}y_{j,h}^{n+1}\right\|_{\partial D_1}^2\right]$$
$$-\Delta t\mathbb{E}\left[\left\|\sqrt{\alpha_{\max}^{n+1}}y_{j,h}^{n}\right\|_{\partial D_1}^2\right]+\Delta t\mathbb{E}\left[\left\|\sqrt{\alpha_j'^{\,n+1}}(y_{j,h}^{n+1}-y_{j,h}^{n})\right\|_{\partial D_1}^2\right]+\frac{C_P^2\Delta t}{4}\mathbb{E}\left[\left\|\sqrt{a_j^{n+1}}y_{j,h}^{n+1}\right\|_1^2\right]$$
$$\le\frac{8\Delta t}{C_P^2 a_{\min}}\mathbb{E}\left[\left\|f^{n+1}\right\|_{-1}^2\right]+\frac{8\alpha_{\max}^2\Delta t}{C_P^2 a_{\min}}\mathbb{E}\left[\|u_j^{n+1}\|_{\partial D_1}^2\right]. \tag{4.9}$$

Summing from $n=0$ to $n=N-1$, one obtains the stability result Eq (4.1) . □

Remark 4.1. *(1) The stability result above shows that we have controlled over the temperature approximation in both $L^\infty\left(0,T;L^2(\Omega)\right)$ and $L^2\left(0,T;H^1(\Omega)\right)$, unconditionally. Specifically, this numerical stability is independent of both the time step size and the grid size constraints, as well as the fluctuations in the random coefficients. This feature is different from the VAEMC approach ((see, e.g., [23, 30]).*
(2)Compared to the standard Backward Euler method, we see that the numerical dissipation is enhanced with the additional term

$$\Delta t\mathbb{E}\left[\left\|\sqrt{a_j'^{\,n+1}}\nabla(y_{j,h}^{n+1}-y_{j,h}^{n})\right\|^2\right]+\Delta t\mathbb{E}\left[\left\|\sqrt{\alpha_j'^{\,n+1}}(y_{j,h}^{n+1}-y_{j,h}^{n})\right\|_{\partial D_1}^2\right].$$

(3) If we consider the assumption of minimum regularity, that is,

$$f \in \widetilde{L}^2\left(0, T; H^{-1}(D)\right), u \in \widetilde{L}^2\left(0, T; H^{-1}(\partial D_1)\right),$$

hypotheses (**A1**) *and* (**A2**) *are satisfied, then the VMEMC method also possesses unconditional stability.*

4.2. Error analysis

In this subsection, we first decompose the overall error of the VMEMC algorithm into the statistical error and the discretization error. Next, we sequentially estimate the statistical error (Theorem 4.2) and the discretization error (Theorem 4.3). Finally, we combine these two error estimates to obtain the overall error estimate of the VMEMC algorithm (Theorem 4.4).

The full-discrete VMEMC approximate solution of (1.1) is defined as $\Psi_h^n \equiv \frac{1}{J}\sum_{j=1}^{J} y_{j,h}^n$. Noting that $\mathbb{E}[y^n] = \mathbb{E}\left[y_j^n\right]$, we can divide the overall error $\mathbb{E}[y^n] - \Psi_h^n$ as

$$\mathbb{E}[y^n] - \Psi_h^n = \left(\mathbb{E}\left[y_{j,h}^n\right] - \Psi_h^n\right) + \left(\mathbb{E}\left[y_j^n\right] - \mathbb{E}\left[y_{j,h}^n\right]\right) := \mathcal{E}_S^n + \mathcal{E}_h^n.$$

Here the statistical error, $\mathcal{E}_S^n = \mathbb{E}\left[y_{j,h}^n\right] - \Psi_h^n$, is estimated using the sample number J. The second term, $\mathcal{E}_h^n = \mathbb{E}\left[y_j^n - y_{j,h}^n\right]$, that is corresponding to the total error of the time discretization and the FE discretization, is controlled by the time step as well as mesh size. Next, we examine the bounds of $\mathcal{E}_S^n$ and $\mathcal{E}_h^n$.

Theorem 4.2. *Assume $f_j \in L^2\left(0, T; H^{-1}(D)\right)$, $u_j \in L^2\left(0, T; L^2(\partial D_1)\right)$, and hypotheses* (**A1**) *and* (**A2**) *are held, for the numerical solution to the VMEMC* (3.1)*, then*

$$\begin{aligned}
&\mathbb{E}\left[\left\|\mathcal{E}_S^N\right\|^2\right] + \Delta t\mathbb{E}\left[\left\|\sqrt{a_{\max}^{n+1}}\nabla\mathcal{E}_S^N\right\|^2\right] + \Delta t\mathbb{E}\left[\left\|\sqrt{\alpha_{\max}^{n+1}}\mathcal{E}_S^N\right\|_{\partial D_1}^2\right] \\
&+ \sum_{n=0}^{N-1}\left(\mathbb{E}\left[\left\|\mathcal{E}_S^{n+1} - \mathcal{E}_S^n\right\|^2\right] + \Delta t\mathbb{E}\left[\left\|\sqrt{a_j'^{\,n+1}}\nabla(\mathcal{E}_S^{n+1} - \mathcal{E}_S^n)\right\|^2\right]\right. \\
&\left. + \Delta t\mathbb{E}\left[\left\|\sqrt{\alpha_j'^{\,n+1}}(\mathcal{E}_S^{n+1} - \mathcal{E}_S^n)\right\|_{\partial D_1}^2\right]\right) + \frac{C_P^2\Delta t}{4}\sum_{n=0}^{N-1}\mathbb{E}\left[\left\|\sqrt{a_j^{n+1}}\mathcal{E}_S^{n+1}\right\|_1^2\right] \\
\le&\frac{C}{J}\left(\mathbb{E}\left[\left\|y_{j,h}^0\right\|^2\right] + \Delta t\mathbb{E}\left[\left\|\sqrt{a_{\max}^{n+1}}\nabla y_{j,h}^0\right\|^2\right] + \Delta t\mathbb{E}\left[\left\|\sqrt{\alpha_{\max}^{n+1}}y_{j,h}^0\right\|_{\partial D_1}^2\right]\right. \\
&\left. + \frac{8\Delta t}{C_P^2 a_{\min}}\sum_{n=0}^{N-1}\mathbb{E}\left[\left\|f^{n+1}\right\|_{-1}^2\right] + \frac{8\alpha_{\max}^2\Delta t}{C_P^2 a_{\min}}\sum_{n=0}^{N-1}\mathbb{E}\left[\|u_j^{n+1}\|_{\partial D_1}^2\right]\right).
\end{aligned}$$

Proof. The process of this proof is similar to the Theorem 5 in [30]. □

The $\mathcal{E}_h^n$ is examined in the following. Recall, the true solution satisfies

$$\left(\frac{y_j^{n+1} - y_j^n}{\Delta t}, v\right) + \left(a_j^{n+1}\nabla y_j^{n+1}, \nabla v\right) + \left(\alpha_j^{n+1} y_j^{n+1}, v\right)_{\partial D_1}$$

$$= \left(f_j^{n+1}, v\right) + \left(\frac{y_j^{n+1} - y_j^n}{\Delta t} - y_{j,t}^{n+1}, v\right) + (\alpha_j^{n+1} u_j^{n+1}, v)_{\partial D_1} \quad , \forall v \in H^1(D). \tag{4.10}$$

The error at the time $t = t_n$ is denoted by $e_j^n = \left(y_j^n - I_h y_j^n\right) - \left(y_{j,h}^n - I_h y_j^n\right) = \phi_j^n - \psi_{j,h}^n$, where $I_h y_j^n$ is a Lagrange interpolation [2, 5] of y_j^n. Taking $v = v_h \in V_h$ and subtracting Eq (3.1) from (4.10), respectively, we have the error equation:

$$\begin{aligned}\left(\frac{e_j^{n+1} - e_j^n}{\Delta t}, v_h\right) + \left(a_{\max}^{n+1} \nabla e_j^{n+1}, \nabla v_h\right) - \left({a'_j}^{n+1} \nabla e_j^n, \nabla v_h\right) \\ + \left(\alpha_{\max}^{n+1} e_j^{n+1}, v_h\right)_{\partial D_1} - \left({\alpha'_j}^{n+1} e_j^n, v_h\right)_{\partial D_1} = \xi\left(y_j^{n+1}, v_h\right),\end{aligned} \tag{4.11}$$

here $\xi\left(y_j^{n+1}, v_h\right)$ is defined as

$$\begin{aligned}\xi\left(y_j^{n+1}, v_h\right) := \left(\frac{y_j^{n+1} - y_j^n}{\Delta t} - y_{j,t}^{n+1}, v_h\right) + \left(\left(a_{\max}^{n+1} - a_j^{n+1}\right) \nabla y_j^{n+1}, \nabla v_h\right) - \left({a'_j}^{n+1} \nabla y_j^n, \nabla v_h\right) \\ + \left(\left(\alpha_{\max}^{n+1} - \alpha_j^{n+1}\right) y_j^{n+1}, v_h\right)_{\partial D_1} - \left({\alpha'_j}^{n+1} y_j^n, v_h\right)_{\partial D_1}.\end{aligned} \tag{4.12}$$

The following regularity assumptions are needed:

$$\begin{aligned}&y_j \in L^\infty\left(0, T; H^1(D) \cap H^{k+1}(D)\right), \nabla y_j \in L^\infty\left(0, T; L^\infty(D)\right), \\ &y_{j,t} \in L^2\left(0, T; H^1(D)\right), y_{j,tt} \in L^2\left(0, T; H^{-1}(D)\right).\end{aligned} \tag{4.13}$$

As a preparatory step for estimating the discretization error, we first estimate each term of the $\xi\left(y_j^{n+1}, v_h\right)$ in the following lemma.

Lemma 4.1. *Suppose every y_j fulfills the equation* (1.1) *and the regularity assumptions* (4.13)*, then,*

$$\begin{aligned}\mathbb{E}\left[\xi\left(y_j^{n+1}, v_h\right)\right] \le \left(\sum_{i=3}^{7} \frac{\epsilon_i}{2}\right) \mathbb{E}\left[\left\| \sqrt{a_j^{n+1}} v_h \right\|_1^2\right] + \frac{\Delta t}{2 a_{\min} \epsilon_3} \mathbb{E}\left[\|y_{j,tt}\|^2_{L^2\left(t_n, t_{n+1}; H^{-1}(D)\right)}\right] \\ + \left(\epsilon_4^{-1} + \epsilon_5^{-1}\right) \Delta t \frac{a_{\max}^2}{2 a_{\min}} \mathbb{E}\left[\|\nabla y_{j,t}\|^2_{L^2\left(t_n, t_{n+1}; L^2(D)\right)}\right] \\ + \left(\epsilon_6^{-1} + \epsilon_7^{-1}\right) \Delta t \frac{\alpha_{\max}^2}{2 \alpha_{\min}} \mathbb{E}\left[\|y_{j,t}\|^2_{L^2\left(t_n, t_{n+1}; L^2(\partial D_1)\right)}\right].\end{aligned} \tag{4.14}$$

Proof. Applying Lemma 2.1, the Cauchy-Schwarz and Young's inequalities, the Taylor's Theorem with integral remainder, we obtain the following results for $\xi\left(y_j^{n+1}, v_h\right)$ is defined by Eq (4.12). The first term of $\xi\left(y_j^{n+1}, v_h\right)$ is estimated as

$$\left(\frac{y_j^{n+1} - y_j^n}{\Delta t} - y_{j,t}^{n+1}, v_h\right) \le \frac{\Delta t}{2 a_{\min} \epsilon_3} \|y_{j,tt}\|^2_{L^2\left(t_n, t_{n+1}; H^{-1}(D)\right)} + \frac{\epsilon_3}{2} \left\| \sqrt{a_j^{n+1}} v_h \right\|_1^2. \tag{4.15}$$

The second and third terms of $\xi\left(y_j^{n+1}, v_h\right)$ are estimated as

$$\left(\left(a_{\max}^{n+1} - a_j^{n+1}\right) \nabla y_j^{n+1}, \nabla v_h\right) - \left({a'_j}^{n+1} \nabla y_j^n, \nabla v_h\right)$$

$$=\left(a_{\max}^{n+1}\nabla\left(y_j^{n+1}-y_j^n\right),\nabla v_h\right)-\left(a_j^{n+1}\nabla\left(y_j^{n+1}-y_j^n\right),\nabla v_h\right), \tag{4.16}$$

$$\left(a_{\max}^{n+1}\nabla\left(y_j^{n+1}-y_j^n\right),\nabla v_h\right)\le\frac{a_{\max}^2\Delta t}{2a_{\min}\epsilon_4}\left\|\nabla y_{j,t}\right\|^2_{L^2\left(t_n,t_{n+1};L^2(D)\right)}+\frac{\epsilon_4}{2}\left\|\sqrt{a_j^{n+1}}\nabla v_h\right\|^2, \tag{4.17}$$

$$-\left(a_j^{n+1}\nabla\left(y_j^{n+1}-y_j^n\right),\nabla v_h\right)\le\frac{a_{\max}^2\Delta t}{2a_{\min}\epsilon_5}\left\|\nabla y_{j,t}\right\|^2_{L^2\left(t_n,t_{n+1};L^2(D)\right)}+\frac{\epsilon_5}{2}\left\|\sqrt{a_j^{n+1}}\nabla v_h\right\|^2. \tag{4.18}$$

The last two terms of $\xi\left(y_j^{n+1},v_h\right)$ are estimated as

$$\begin{aligned}&\left(\left(\alpha_{\max}^{n+1}-\alpha_j^{n+1}\right)y_j^{n+1},v_h\right)_{\partial D_1}-\left(\alpha_j'^{\,n+1}y_j^n,v_h\right)_{\partial D_1}\\&=\left(\alpha_{\max}^{n+1}\left(y_j^{n+1}-y_j^n\right),v_h\right)_{\partial D_1}-\left(\alpha_j^{n+1}\left(y_j^{n+1}-y_j^n\right),v_h\right)_{\partial D_1},\end{aligned} \tag{4.19}$$

$$\left(\alpha_{\max}^{n+1}\left(y_j^{n+1}-y_j^n\right),v_h\right)_{\partial D_1}\le\frac{\alpha_{\max}^2\Delta t}{2\alpha_{\min}\epsilon_6}\left\|y_{j,t}\right\|^2_{L^2\left(t_n,t_{n+1};L^2(\partial D_1)\right)}+\frac{\epsilon_6}{2}\left\|\sqrt{\alpha_j^{n+1}}v_h\right\|^2_{\partial D_1}, \tag{4.20}$$

$$-\left(\alpha_j^{n+1}\left(y_j^{n+1}-y_j^n\right),v_h\right)_{\partial D_1}\le\frac{\alpha_{\max}^2\Delta t}{2\alpha_{\min}\epsilon_7}\left\|y_{j,t}\right\|^2_{L^2\left(t_n,t_{n+1};L^2(\partial D_1)\right)}+\frac{\epsilon_7}{2}\left\|\sqrt{\alpha_j^{n+1}}v_h\right\|^2_{\partial D_1}. \tag{4.21}$$

Combining the above estimates Eqs (4.15)–(4.21) , using $\|\nabla v_h\|\le\|v_h\|_1$ and $\|v_h\|_{\partial D_1}\le\|v_h\|_1$, and integrating over the probability space, we can yield Eq (4.13). □

With the results of Lemma 4.1, the major convergence result of this work can be proved next.

Theorem 4.3. *Suppose y_j satisfies the assumptions of Lemma 4.1. Moreover, suppose $y_{j,h}^0\in V_h$ is an approximation of y_j^0 with the accuracy of the interpolation. Then, the VMEMC solution of* (3.1) *satisfies*

$$\begin{aligned}&\mathbb{E}\left[\left\|e_j^N\right\|^2\right]+\sum_{n=0}^{N-1}\mathbb{E}\left[\left\|e_j^{n+1}-e_j^n\right\|^2\right]+\frac{C_P^2\Delta t}{4}\sum_{n=0}^{N-1}\mathbb{E}\left[\left\|\sqrt{a_j^{n+1}}e_j^{n+1}\right\|_1^2\right]\\&+\Delta t\mathbb{E}\left[\left\|\sqrt{a_{\max}^N}\nabla e_j^N\right\|^2\right]+\Delta t\sum_{n=0}^{N-1}\mathbb{E}\left[\left\|\sqrt{a_j'^{\,n+1}}\nabla\left(e_j^{n+1}-e_j^n\right)\right\|^2\right]\\&+\Delta t\mathbb{E}\left[\left\|\sqrt{\alpha_{\max}^N}e_{j,h}^N\right\|^2_{\partial D_1}\right]+\Delta t\sum_{n=0}^{N-1}\mathbb{E}\left[\left\|\sqrt{\alpha_j'^{\,n+1}}\left(e_{j,h}^{n+1}-e_{j,h}^n\right)\right\|^2_{\partial D_1}\right]\\&\le C\left(\frac{1}{a_{\min}}+\frac{a_{\max}^2}{a_{\min}}+\frac{\alpha_{\max}^2}{\alpha_{\min}}\right)\Delta t^2+C\left(\frac{a_{\max}^2}{a_{\min}}+\frac{a_{\max}^2}{a_{\min}}\Delta t^2+a_{\max}\Delta t+a_{\max}\Delta t^2\right)h^{2k}\\&\quad+C\left(\frac{\alpha_{\max}^2}{a_{\min}}+\frac{1}{a_{\min}}+1+\Delta t+a_{\max}\Delta t+\alpha_{\max}\Delta t+\alpha_{\max}\Delta t^2\right)h^{2k+2}.\end{aligned} \tag{4.22}$$

Proof. Noting $e_j^n=\phi_j^n-\psi_{j,h}^n$, rearranging the equation (4.11) with $v_h=2\Delta t\psi_{j,h}^{n+1}$, we have

$$\begin{aligned}&\left(\psi_{j,h}^{n+1}-\psi_{j,h}^n,2\psi_{j,h}^{n+1}\right)+2\Delta t\left(a_{\max}^{n+1}\nabla\psi_{j,h}^{n+1},\nabla\psi_{j,h}^{n+1}\right)-2\Delta t\left(a_j'^{\,n+1}\nabla\psi_{j,h}^n,\nabla\psi_{j,h}^{n+1}\right)\\&\quad+2\Delta t\left(\alpha_{\max}^{n+1}\psi_{j,h}^{n+1},\psi_{j,h}^{n+1}\right)_{\partial D_1}-2\Delta t\left(\alpha_j'^{\,n+1}\psi_{j,h}^n,\psi_{j,h}^{n+1}\right)_{\partial D_1}\end{aligned}$$

$$
\begin{aligned}
&= \left(\phi_j^{n+1} - \phi_j^n, 2\psi_{j,h}^{n+1}\right) - \xi\left(y_j^{n+1}, 2\Delta t \psi_{j,h}^{n+1}\right) \\
&\quad + 2\Delta t\left(a_{\max}^{n+1}\nabla\phi_j^{n+1}, \nabla\psi_{j,h}^{n+1}\right) - 2\Delta t\left(a_j'^{\,n+1}\nabla\phi_j^n, \nabla\psi_{j,h}^{n+1}\right) \\
&\quad + 2\Delta t\left(\alpha_{\max}^{n+1}\phi_j^{n+1}, \psi_{j,h}^{n+1}\right)_{\partial D_1} - 2\Delta t\left(\alpha_j'^{\,n+1}\phi_j^n, \psi_{j,h}^{n+1}\right)_{\partial D_1}. \qquad (4.23)
\end{aligned}
$$

Integrating over the probability space and applying the polarization identity, we arrive at

$$
\begin{aligned}
&\mathbb{E}\left[\left\|\psi_{j,h}^{n+1}\right\|^2\right] - \mathbb{E}\left[\left\|\psi_{j,h}^{n}\right\|^2\right] + \mathbb{E}\left[\left\|\psi_{j,h}^{n+1} - \psi_{j,h}^n\right\|^2\right] \\
&+ \Delta t\mathbb{E}\left[\left\|\sqrt{a_{\max}^{n+1}}\nabla\psi_{j,h}^{n+1}\right\|^2\right] - \Delta t\mathbb{E}\left[\left\|\sqrt{a_{\max}^{n+1}}\nabla\psi_{j,h}^{n}\right\|^2\right] \\
&+ \Delta t\mathbb{E}\left[\left\|\sqrt{a_j'^{\,n+1}}\nabla\left(\psi_{j,h}^{n+1} - \psi_{j,h}^n\right)\right\|^2\right] + \Delta t\mathbb{E}\left[\left\|\sqrt{a_j^{n+1}}\nabla\psi_{j,h}^{n+1}\right\|^2\right] \\
&+ \Delta t\mathbb{E}\left[\left\|\sqrt{a_j^{n+1}}\nabla\psi_{j,h}^{n}\right\|^2\right] + \Delta t\mathbb{E}\left[\left\|\sqrt{\alpha_{\max}^{n+1}}\psi_{j,h}^{n+1}\right\|_{\partial D_1}^2\right] \\
&- \Delta t\mathbb{E}\left[\left\|\sqrt{\alpha_{\max}^{n+1}}\psi_{j,h}^{n}\right\|_{\partial D_1}^2\right] + \Delta t\mathbb{E}\left[\left\|\sqrt{\alpha_j'^{\,n+1}}\left(\psi_{j,h}^{n+1} - \psi_{j,h}^n\right)\right\|_{\partial D_1}^2\right] \\
&+ \Delta t\mathbb{E}\left[\left\|\sqrt{\alpha_j^{n+1}}\psi_{j,h}^{n+1}\right\|_{\partial D_1}^2\right] + \Delta t\mathbb{E}\left[\left\|\sqrt{\alpha_j^{n+1}}\psi_{j,h}^{n}\right\|_{\partial D_1}^2\right] \\
=& - 2\Delta t\mathbb{E}\left[\xi\left(y_j^{n+1}, \psi_{j,h}^{n+1}\right)\right] + \mathbb{E}\left[\left(\phi_j^{n+1} - \phi_j^n, 2\psi_{j,h}^{n+1}\right)\right] \\
&+ 2\Delta t\mathbb{E}\left[\left(a_{\max}^{n+1}\nabla\phi_j^{n+1}, \nabla\psi_{j,h}^{n+1}\right)\right] - 2\Delta t\mathbb{E}\left[\left(a_j'^{\,n+1}\nabla\phi_j^n, \nabla\psi_{j,h}^{n+1}\right)\right] \\
&+ 2\Delta t\mathbb{E}\left[\left(\alpha_{\max}^{n+1}\phi_j^{n+1}, \psi_{j,h}^{n+1}\right)_{\partial D_1}\right] - 2\Delta t\mathbb{E}\left[\left(\alpha_j'^{\,n+1}\phi_j^n, \psi_{j,h}^{n+1}\right)_{\partial D_1}\right]. \qquad (4.24)
\end{aligned}
$$

From Lemma 4.1 with $v_h = \psi_{j,h}^{n+1}$, the first term on the RHS of Eq (4.24) can be expressed as

$$
\begin{aligned}
&- 2\Delta t\mathbb{E}\left[\xi\left(y_j^{n+1}, \psi_{j,h}^{n+1}\right)\right] \leq \left(\sum_{i=3}^{7}\epsilon_i\right)\Delta t\mathbb{E}\left[\left\|\sqrt{a_j^{n+1}}\psi_{j,h}^{n+1}\right\|_1^2\right] \\
&+ \frac{\Delta t^2}{a_{\min}\epsilon_3}\mathbb{E}\left[\left\|y_{j,tt}\right\|_{L^2\left(t_n,t_{n+1};H^{-1}(D)\right)}^2\right] + \left(\epsilon_4^{-1} + \epsilon_5^{-1}\right)\Delta t^2\frac{a_{\max}^2}{a_{\min}}\mathbb{E}\left[\left\|\nabla y_{j,t}\right\|_{L^2\left(t_n,t_{n+1};L^2(D)\right)}^2\right] \\
&+ \left(\epsilon_6^{-1} + \epsilon_7^{-1}\right)\Delta t^2\frac{\alpha_{\max}^2}{\alpha_{\min}}\mathbb{E}\left[\left\|y_{j,t}\right\|_{L^2\left(t_n,t_{n+1};L^2(\partial D_1)\right)}^2\right]. \qquad (4.25)
\end{aligned}
$$

Applying Lemma 2.1, the Taylor's theorem with integral remainder, the Young's inequality, and the Cauchy-Schwarz inequality on the second term of RHS of (4.24), we have

$$
\mathbb{E}\left[\left(\phi_j^{n+1} - \phi_j^n, 2\psi_{j,h}^{n+1}\right)\right] \leq \frac{1}{\epsilon_8 a_{\min}}\mathbb{E}\left[\|\phi_t\|_{L^2\left(t_n,t_{n+1};L^2(D)\right)}^2\right] + \epsilon_8\Delta t\mathbb{E}\left[\left\|\sqrt{a_j^{n+1}}\psi_{j,h}^{n+1}\right\|^2\right]. \qquad (4.26)
$$

Using the Young's inequality and the Cauchy-Schwarz inequality to the third and fourth terms, we obtain

$$
2\Delta t\mathbb{E}\left[\left(a_{\max}^{n+1}\nabla\phi_j^{n+1}, \nabla\psi_{j,h}^{n+1}\right)\right] \leq \frac{a_{\max}^2\Delta t}{\epsilon_9 a_{\min}}\mathbb{E}\left[\left\|\nabla\phi_j^{n+1}\right\|^2\right] + \epsilon_9\Delta t\mathbb{E}\left[\left\|\sqrt{a_j^{n+1}}\nabla\psi_{j,h}^{n+1}\right\|^2\right], \qquad (4.27)
$$

$$-2\Delta t\mathbb{E}\left[\left(a_j'^{\,n+1}\nabla\phi_j^n,\nabla\psi_{j,h}^{n+1}\right)\right]\le\frac{a_{\max}^2\Delta t}{\epsilon_{10}a_{\min}}\mathbb{E}\left[\left\|\nabla\phi_j^n\right\|^2\right]+\epsilon_{10}\Delta t\mathbb{E}\left[\left\|\sqrt{a_j^{n+1}}\nabla\psi_{j,h}^{n+1}\right\|^2\right].\tag{4.28}$$

Using the Young's inequality and the Cauchy-Schwarz inequality as well as the trace theorem to the two boundary terms, we have

$$2\Delta t\mathbb{E}\left[\left(\alpha_{\max}^{n+1}\phi_j^{n+1},\psi_{j,h}^{n+1}\right)_{\partial D_1}\right]\le\frac{\alpha_{\max}^2\Delta t}{\epsilon_{11}a_{\min}}\mathbb{E}\left[\left\|\phi_j^{n+1}\right\|_{\partial D_1}^2\right]+\epsilon_{11}\Delta t\mathbb{E}\left[\left\|\sqrt{a_j^{n+1}}\psi_{j,h}^{n+1}\right\|^2\right],\tag{4.29}$$

$$-2\Delta t\mathbb{E}\left[\left(\alpha_j'^{\,n+1}\phi_j^n,\psi_{j,h}^{n+1}\right)_{\partial D_1}\right]\le\frac{\alpha_{\max}^2\Delta t}{\epsilon_{12}a_{\min}}\mathbb{E}\left[\left\|\phi_j^n\right\|_{\partial D_1}^2\right]+\epsilon_{12}\Delta t\mathbb{E}\left[\left\|\sqrt{a_j^{n+1}}\psi_{j,h}^{n+1}\right\|^2\right].\tag{4.30}$$

Under Lemma 2.1, proceeding in similar fashion with equation (4.8), we obtain

$$\frac{C_P^2\Delta t}{2}\mathbb{E}\left[\left\|\sqrt{a_j^{n+1}}\psi_{j,h}^{n+1}\right\|_1^2\right]\le\Delta t\mathbb{E}\left[\left\|\sqrt{a_j^{n+1}}\nabla\psi_{j,h}^{n+1}\right\|^2\right]+\Delta t\mathbb{E}\left[\left\|\sqrt{\alpha_j^{n+1}}\psi_{j,h}^{n+1}\right\|_{\partial D_1}^2\right].\tag{4.31}$$

Dropping the positive terms $\Delta t\mathbb{E}\left[\left\|\sqrt{a_j^{n+1}}\nabla\psi_{j,h}^n\right\|^2\right]$ and $\Delta t\mathbb{E}\left[\left\|\sqrt{\alpha_j^{n+1}}\psi_{j,h}^n\right\|_{\partial D_1}^2\right]$, using the estimate (4.25)-(4.31), noting that $\|\nabla v\|\le\|v\|_1$, $\|v\|\le\|v\|_1$, selecting $\epsilon_i=C_P^2/40$ for $i=3,4,...,12$, in equation (4.24), then we can obtain,

$$\begin{aligned}
&\mathbb{E}\left[\left\|\psi_{j,h}^{n+1}\right\|^2\right]-\mathbb{E}\left[\left\|\psi_{j,h}^n\right\|^2\right]+\mathbb{E}\left[\left\|\psi_{j,h}^{n+1}-\psi_{j,h}^n\right\|^2\right]+\frac{C_P^2\Delta t}{4}\mathbb{E}\left[\left\|\sqrt{a_j^{n+1}}\psi_{j,h}^{n+1}\right\|_1^2\right]\\
&+\Delta t\left(\mathbb{E}\left[\left\|\sqrt{a_{\max}^{n+1}}\nabla\psi_{j,h}^{n+1}\right\|^2\right]-\mathbb{E}\left[\left\|\sqrt{a_{\max}^{n+1}}\nabla\psi_{j,h}^n\right\|^2\right]\right)\\
&+\Delta t\mathbb{E}\left[\left\|\sqrt{a_j'^{\,n+1}}\nabla\left(\psi_{j,h}^{n+1}-\psi_{j,h}^n\right)\right\|^2\right]+\Delta t\mathbb{E}\left[\left\|\sqrt{\alpha_j'^{\,n+1}}\left(\psi_{j,h}^{n+1}-\psi_{j,h}^n\right)\right\|_{\partial D_1}^2\right]\\
&+\Delta t\left(\mathbb{E}\left[\left\|\sqrt{\alpha_{\max}^{n+1}}\psi_{j,h}^{n+1}\right\|_{\partial D_1}^2\right]-\mathbb{E}\left[\left\|\sqrt{\alpha_{\max}^{n+1}}\psi_{j,h}^n\right\|_{\partial D_1}^2\right]\right)\\
\le&\frac{40\Delta t^2}{a_{\min}C_P^2}\mathbb{E}\left[\left\|y_{j,tt}\right\|_{L^2(t_n,t_{n+1};H^{-1}(D))}^2\right]+\frac{40}{C_P^2a_{\min}}\mathbb{E}\left[\|\phi_t\|_{L^2(t_n,t_{n+1};L^2(D))}^2\right]\\
&+\frac{80\Delta t^2}{C_P^2}\frac{a_{\max}^2}{a_{\min}}\mathbb{E}\left[\left\|\nabla y_{j,t}\right\|_{L^2(t_n,t_{n+1};L^2(D))}^2\right]+\frac{80\Delta t^2}{C_P^2}\frac{\alpha_{\max}^2}{\alpha_{\min}}\mathbb{E}\left[\left\|y_{j,t}\right\|_{L^2(t_n,t_{n+1};L^2(\partial D_1))}^2\right]\\
&+\frac{40a_{\max}^2\Delta t}{a_{\min}C_P^2}\left(\mathbb{E}\left[\left\|\nabla\phi_j^{n+1}\right\|^2\right]+\mathbb{E}\left[\left\|\nabla\phi_j^n\right\|^2\right]\right)\\
&+\frac{40\alpha_{\max}^2\Delta t}{a_{\min}C_P^2}\left(\mathbb{E}\left[\left\|\phi_j^{n+1}\right\|_{\partial D_1}^2\right]+\mathbb{E}\left[\left\|\phi_j^n\right\|_{\partial D_1}^2\right]\right).
\end{aligned}\tag{4.32}$$

Summing $n=0$ to $n=N-1$, collecting the constants, and reorganizing the items, we obtain

$$\mathbb{E}\left[\left\|\psi_{j,h}^N\right\|^2\right]+\sum_{n=0}^{N-1}\mathbb{E}\left[\left\|\psi_{j,h}^{n+1}-\psi_{j,h}^n\right\|^2\right]+\frac{C_P^2\Delta t}{4}\sum_{n=0}^{N-1}\mathbb{E}\left[\left\|\sqrt{a_j^{n+1}}\psi_{j,h}^{n+1}\right\|_1^2\right]$$

$$+\Delta t\mathbb{E}\left[\left\|\sqrt{a_{\max}^N}\nabla\psi_{j,h}^N\right\|^2\right]+\Delta t\sum_{n=0}^{N-1}\mathbb{E}\left[\left\|\sqrt{a_j'^{\,n+1}}\nabla\left(\psi_{j,h}^{n+1}-\psi_{j,h}^n\right)\right\|^2\right]$$

$$+\Delta t\mathbb{E}\left[\left\|\sqrt{\alpha_{\max}^N}\psi_{j,h}^N\right\|_{\partial D_1}^2\right]+\Delta t\sum_{n=0}^{N-1}\mathbb{E}\left[\left\|\sqrt{\alpha_j'^{\,n+1}}\left(\psi_{j,h}^{n+1}-\psi_{j,h}^n\right)\right\|_{\partial D_1}^2\right]$$

$$\leq \mathbb{E}\left[\left\|\psi_{j,h}^0\right\|^2\right]+\Delta t\mathbb{E}\left[\left\|\sqrt{a_{\max}^0}\nabla\psi_{j,h}^0\right\|^2\right]+\Delta t\mathbb{E}\left[\left\|\sqrt{\alpha_{\max}^0}\psi_{j,h}^0\right\|_{\partial D_1}^2\right]$$

$$+\frac{C\Delta t^2}{a_{\min}}\mathbb{E}\left[\left\|y_{j,tt}\right\|_{L^2\left(0,T;H^{-1}(D)\right)}^2\right]+\frac{C}{a_{\min}}\mathbb{E}\left[\|\phi_t\|_{L^2\left(0,T;L^2(D)\right)}^2\right]$$

$$+\frac{C\Delta t^2a_{\max}^2}{a_{\min}}\mathbb{E}\left[\left\|\nabla y_{j,t}\right\|_{L^2\left(0,T;L^2(D)\right)}^2\right]+\frac{C\Delta t^2\alpha_{\max}^2}{\alpha_{\min}}\mathbb{E}\left[\left\|y_{j,t}\right\|_{L^2\left(0,T;L^2(\partial D_1)\right)}^2\right]$$

$$+\frac{Ca_{\max}^2\Delta t}{a_{\min}}\sum_{n=0}^{N-1}\left(\mathbb{E}\left[\left\|\nabla\phi_j^{n+1}\right\|^2\right]+\mathbb{E}\left[\left\|\nabla\phi_j^n\right\|^2\right]\right)$$

$$+\frac{C\alpha_{\max}^2\Delta t}{a_{\min}}\sum_{n=0}^{N-1}\left(\mathbb{E}\left[\left\|\phi_j^{n+1}\right\|_{\partial D_1}^2\right]+\mathbb{E}\left[\left\|\phi_j^n\right\|_{\partial D_1}^2\right]\right). \tag{4.33}$$

Then,

$$\mathbb{E}\left[\left\|\psi_{j,h}^N\right\|^2\right]+\sum_{n=0}^{N-1}\mathbb{E}\left[\left\|\psi_{j,h}^{n+1}-\psi_{j,h}^n\right\|^2\right]+\frac{C_P^2\Delta t}{4}\sum_{n=0}^{N-1}\mathbb{E}\left[\left\|\sqrt{a_j^{n+1}}\psi_{j,h}^{n+1}\right\|_1^2\right]$$

$$+\Delta t\mathbb{E}\left[\left\|\sqrt{a_{\max}^N}\nabla\psi_{j,h}^N\right\|^2\right]+\Delta t\sum_{n=0}^{N-1}\mathbb{E}\left[\left\|\sqrt{a_j'^{\,n+1}}\nabla\left(\psi_{j,h}^{n+1}-\psi_{j,h}^n\right)\right\|^2\right]$$

$$+\Delta t\mathbb{E}\left[\left\|\sqrt{\alpha_{\max}^N}\psi_{j,h}^N\right\|_{\partial D_1}^2\right]+\Delta t\sum_{n=0}^{N-1}\mathbb{E}\left[\left\|\sqrt{\alpha_j'^{\,n+1}}\left(\psi_{j,h}^{n+1}-\psi_{j,h}^n\right)\right\|_{\partial D_1}^2\right]$$

$$\leq C\left(\frac{1}{a_{\min}}+\frac{a_{\max}^2}{a_{\min}}+\frac{\alpha_{\max}^2}{\alpha_{\min}}\right)\Delta t^2+C\frac{a_{\max}^2}{a_{\min}}h^{2k}+C\frac{\alpha_{\max}^2}{a_{\min}}h^{2k+2}+\frac{C}{a_{\min}}h^{2k+2}. \tag{4.34}$$

Inequalities $\left\|e_j^n\right\|^2\leq 2\left\|\phi_j^n\right\|^2+2\left\|\psi_{j,h}^n\right\|^2$ and $\left\|\phi_j^{n+1}-\phi_j^n\right\|^2\leq C\Delta t\int_{t_n}^{t_{n+1}}\left\|\phi_{j,t}(s)\right\|^2ds$ imply the result (4.22). □

Combining the MC sampling error and the discretization error, one can acquire the total error of the VMEMC FE approximation.

Theorem 4.4. *Assume that $f_j\in L^2\left(0,T;H^{-1}(D)\right)$, $u_j\in L^2\left(0,T;L^2(\partial D_1)\right)$, y_j satisfies the assumptions of Lemma 4.1. Moreover, suppose $y_{j,h}^0\in V_h$ is an approximation of y_j^0 with the accuracy of the interpolation. Then,*

$$\mathbb{E}\left[\left\|\mathbb{E}\left[y^N\right]-\Psi_h^N\right\|^2\right]+\Delta t\mathbb{E}\left[\left\|\sqrt{a_{\max}^{n+1}}\nabla\left(\mathbb{E}\left[y^N\right]-\Psi_h^N\right)\right\|^2\right]$$

$$
\begin{aligned}
&+\Delta t\mathbb{E}\left[\left\|\sqrt{\alpha_{\max}^{n+1}}\left(\mathbb{E}\left[y^N\right]-\Psi_h^N\right)\right\|_{\partial D_1}^2\right]+\sum_{n=0}^{N-1}\mathbb{E}\left[\left\|\left(\mathbb{E}\left[y^{n+1}\right]-\Psi_h^{n+1}\right)-\left(\mathbb{E}\left[y^n\right]-\Psi_h^n\right)\right\|^2\right]\\
&+\Delta t\sum_{n=0}^{N-1}\mathbb{E}\left[\left\|\sqrt{a_j'^{\,n+1}}\nabla(\left(\mathbb{E}\left[y^{n+1}\right]-\Psi_h^{n+1}\right)-\left(\mathbb{E}\left[y^n\right]-\Psi_h^n\right))\right\|^2\right]\\
&+\Delta t\sum_{n=0}^{N-1}\mathbb{E}\left[\left\|\sqrt{\alpha_j'^{\,n+1}}(\left(\mathbb{E}\left[y^{n+1}\right]-\Psi_h^{n+1}\right)-\left(\mathbb{E}\left[y^n\right]-\Psi_h^n\right))\right\|_{\partial D_1}^2\right]\\
&+\frac{C_P^2\Delta t}{4}\sum_{n=0}^{N-1}\mathbb{E}\left[\left\|\sqrt{a_j^{n+1}}\left(\mathbb{E}\left[y^{n+1}\right]-\Psi_h^{n+1}\right)\right\|_1^2\right]\\
&\leq\frac{C}{J}\left(\mathbb{E}\left[\|y_{j,h}^0\|^2\right]+\Delta t\mathbb{E}\left[\left\|\sqrt{a_{\max}^{n+1}}\nabla y_{j,h}^0\right\|^2\right]+\Delta t\mathbb{E}\left[\left\|\sqrt{\alpha_{\max}^{n+1}}y_{j,h}^0\right\|_{\partial D_1}^2\right]\right.\\
&\left.+\frac{\Delta t}{a_{\min}}\sum_{n=0}^{N-1}\mathbb{E}\left[\|f_j^{n+1}\|_{-1}^2\right]+\frac{\alpha_{\max}^2\Delta t}{a_{\min}}\sum_{n=0}^{N-1}\mathbb{E}\left[\|u_j^{n+1}\|_{\partial D_1}^2\right]\right)\\
&+C\left(\frac{1}{a_{\min}}+\frac{a_{\max}^2}{a_{\min}}+\frac{\alpha_{\max}^2}{\alpha_{\min}}\right)\Delta t^2+C\left(\frac{a_{\max}^2}{a_{\min}}+\frac{a_{\max}^2}{a_{\min}}\Delta t^2+a_{\max}\Delta t+a_{\max}\Delta t^2\right)h^{2k}\\
&+C\left(\frac{\alpha_{\max}^2}{a_{\min}}+\frac{1}{a_{\min}}+1+\Delta t+a_{\max}\Delta t+\alpha_{\max}\Delta t+\alpha_{\max}\Delta t^2\right)h^{2k+2}. \qquad (4.35)
\end{aligned}
$$

Proof. In view of the first term on the left-hand side (LHS) of (4.35), the triangle inequality and Young inequality suggest

$$\mathbb{E}\left[\left\|\mathbb{E}\left[y^N\right]-\Psi_h^N\right\|^2\right]\leq 2\left(\mathbb{E}\left[\left\|\mathbb{E}\left[y^N\right]-\mathbb{E}\left[y_h^N\right]\right\|^2\right]+\mathbb{E}\left[\left\|\mathbb{E}\left[y_h^N\right]-\Psi_h^N\right\|^2\right]\right).$$

Using Jensen inequality to the first term on the RHS of the above inequality, one gets

$$\mathbb{E}\left[\left\|\mathbb{E}\left[y^N\right]-\mathbb{E}\left[y_h^N\right]\right\|^2\right]\leq\mathbb{E}\left[\mathbb{E}\left[\|y^N-y_h^N\|^2\right]\right]=\mathbb{E}\left[\|y^N-y_h^N\|^2\right].$$

Thus, the result can be obtained by Theorem 4.2 and Theorem 4.3. The remaining terms of the LHS of (4.35) can be handled similarly. □

Remark 4.2. *Collecting these constants, the RHS of* (4.35) *can be expressed as*

$$\frac{C}{J}+C\Delta t^2+Ch^{2k}.$$

5. Numerical tests

For comparison purposes, here we use the numerical examples in our recent work [30] to test the various performances of the VMEMC algorithm. Example One is a deterministic heat transfer model with a known exact solution, which is intended to show the convergence rate of time and space, respectively. A random transient heat equation is in Example Two, which is used to test Theorem 4.4 and reveal the effectiveness of the VMEMC algorithm.

5.1. Example one

We take $y_j = (1 + \omega_j)\exp(t + 1)\cos(2\pi x_1)\cos(2\pi x_2)$ as the manufactured solution; here $j \in \{1, 2, 3\}$, $\omega_j \in [0, 1]$ is a random variable, $(x_1, x_2) \in [0, 1]^2$, and $t \in [0, 1]$. The upper and lower boundary of the domain are zero Neumann boundary; the left and right boundary are Robin boundaries. The Robin coefficient and thermal conductivity diffusion factor are specified by $\alpha_j = a_j = 2 + (1 + \omega_j)\sin(x_1 x_2)\sin(t)$. The source term, initial condition, and Robin boundary function should be adjusted appropriately.

Table 1. Errors and convergence orders on temporal step.

Δt	$1/2$	$1/2^2$	$1/2^3$	$1/2^4$
$\omega_1 = 0.2$, $h = 1/2^9$				
$\mathcal{E}_{L^2}^{E,1}$ *of VAEMC*	0.04729	0.02254	0.01091	0.00537
Rate		1.068	1.046	1.022
$\mathcal{E}_{L^2}^{E,1}$ *of VMEMC*	0.07742	0.03724	0.01812	0.00893
Rate		1.055	1.039	1.020
$\omega_1 = 0.7$, $h = 1/2^9$				
$\mathcal{E}_{L^2}^{E,2}$ *of VAEMC*	0.02952	0.01384	0.00659	0.00317
Rate		1.093	1.069	1.055
$\mathcal{E}_{L^2}^{E,2}$ *of VMEMC*	0.01504	0.00711	0.00344	0.00171
Rate		1.079	1.048	1.009
$\omega_1 = 0.8$, $h = 1/2^9$				
$\mathcal{E}_{L^2}^{E,3}$ *of VAEMC*	0.05090	0.02377	0.01132	0.00546
Rate		1.098	1.070	1.049
$\mathcal{E}_{L^2}^{E,3}$ *of VMEMC*	0.00600	0.00286	0.00134	0.00060
Rate		1.068	1.093	1.150

In this experiment, the ensemble schemes (3.1) and (3.2) are applied to simulate the ensemble set that includes three samples where $\omega_1 = 0.2$, $\omega_2 = 0.7$, $\omega_3 = 0.8$. We consider $\mathcal{E}_{L^2}^{E,j} = |||y_j^n - y_{j,h}^n|||_{2,0}$, $j = 1, 2, 3$. We use isometric time partition and linear FEs ($k = 1$) on uniform space partition. Moreover, we take Δt from $1/2$ to $1/2^4$ with $h = 1/2^9$ to verify the convergence order in the temporal direction. The numerical results on the VMEMC and the VAEMC schemes are shown in Table 1. Meanwhile, we select h from $1/2^4$ to $1/2^7$ with $\Delta t = 1/2^9$ to consider the convergence order in space. The resulted numerical errors on the VMEMC and the VAEMC schemes are listed in Table 2.

Both the VMEMC and the VAEMC algorithms, the convergence orders on temporal direction are about one from Table 1, and coincide with theoretical findings. From Table 2, the rates of convergence on the VMEMC scheme in spatial direction are close to 2, which is higher than the theoretical result (Theorem 4.4). This may be the cause of the higher regularity of source term f and boundary function u in this numerical test.

Table 2. Errors and convergence orders on spatial step.

h	$1/2^4$	$1/2^5$	$1/2^6$	$1/2^7$
	$\omega_1 = 0.2$, $\Delta t = 1/2^9$			
$\mathcal{E}^{E,1}_{L^2}$ *of VAEMC*	0.14645	0.03808	0.00965	0.00246
Rate		1.943	1.979	1.969
$\mathcal{E}^{E,1}_{L^2}$ *of VMEMC*	0.14648	0.03812	0.00970	0.00251
Rate		1.942	1.974	1.946
	$\omega_1 = 0.7$, $\Delta t = 1/2^9$			
$\mathcal{E}^{E,2}_{L^2}$ *of VAEMC*	0.20886	0.05419	0.01364	0.00338
Rate		1.946	1.989	2.011
$\mathcal{E}^{E,2}_{L^2}$ *of VMEMC*	0.20891	0.05424	0.01370	0.00344
Rate		1.945	1.985	1.993
	$\omega_1 = 0.8$, $\Delta t = 1/2^9$			
$\mathcal{E}^{E,3}_{L^2}$ *of VAEMC*	0.22145	0.05743	0.01444	0.00356
Rate		1.946	1.991	2.018
$\mathcal{E}^{E,3}_{L^2}$ *of VMEMC*	0.22150	0.05749	0.01450	0.00362
Rate		1.945	1.987	2.001

5.2. Example two

The transient heat model (1.1) with random factors is considered on $[0,1]^2$. The upper and lower boundary of the domain are zero Neumann boundary; the left and right boundary are Robin boundaries. Inspired by numerical examples in literature [25, 30], we choose the exact solution, the diffusion factor and Robin factor are as follow:

$$y(t,x,\omega) = (1+\omega_1+\sum_{i=2}^{5}\omega_i)\exp(t+1)\cos(2\pi x_1)\cos(2\pi x_2),$$

$$a(x,\omega_1) = 2 + (1+\omega_1)\sin(x_1x_2),$$

$$\begin{aligned}\alpha(x,\omega_2,\cdots,\omega_5) =& 10 + \exp((\omega_2\cos(\pi x_2)+\omega_3\sin(\pi x_2))\exp(-\frac{3}{4})\\ &+ (\omega_4\cos(\pi x_1)+\omega_5\sin(\pi x_1))\exp(-\frac{3}{4})).\end{aligned}$$

The random variables $\omega_1,\cdots,\omega_5$ are independent. ω_1 is uniformly distributed on $[0,1]$. $\omega_2,\cdots,\omega_5$ are uniformly distributed in the interval $[-0.5, 0.5]$. The Robin boundary condition, the source term, and initial conditions, are chosen to match the exact solution. Let $\mathcal{E}_{L^2} = \sqrt{\frac{1}{J}\sum_{j=1}^{J}|||y_j^n - y_{j,h}^n|||^2_{2,0}}$, $j = 1,2,3$.

(1) Convergence orders for time and physic space. Analogy to Example One, we use linear FEs ($k = 1$) on uniform partition physical space and isometric partition in time direction. The corresponding average numerical errors on VAEMC and VMEMC schemes are shown in Table 3 ($h = \frac{1}{2^8}$, $J = 50$) and Table 4 ($\Delta t = \frac{1}{2^9}$, $J = 50$).

Table 3. Average errors and convergence orders on temporal step.

Δt	$1/2$	$1/2^2$	$1/2^3$	$1/2^4$
$\mathcal{E}_{L^2}$ $of\ VAEMC$	0.01472	0.00727	0.00358	0.00180
Rate		1.017	1.020	0.990
$\mathcal{E}_{L^2}$ $of\ VMEMC$	0.02233	0.01134	0.00574	0.00294
Rate		0.977	0.982	0.962

From Table 3, the convergence order on temporal direction is according to Theorem 4.4. Looking at Table 4, the orders of convergence on the VMEMC scheme on spatial direction are close to 2, which is higher than the theoretical result (Theorem 4.4). This may be the cause of the higher regularity of source term f and boundary function u.

Table 4. Average errors and convergence orders on spatial step.

h	$1/2^4$	$1/2^5$	$1/2^6$	$1/2^7$
$\mathcal{E}_{L^2}$ $of\ VAEMC$	0.16484	0.04296	0.01084	0.00270
Rate		1.939	1.985	2.001
$\mathcal{E}_{L^2}$ $of\ VMEMC$	0.18079	0.04713	0.01191	0.00299
Rate		1.939	1.984	1.993

Table 5. One group of values of $\mathcal{E}^*_{L^2}(h = \Delta t = \frac{1}{2^6})$.

J	10	20	40	80	160
$\mathcal{E}^*_{L^2}$	0.03431	0.01763	0.02839	0.01077	0.00661

(2) Convergence order for MC sampling. We take the VMEMC solution choosing $J_0 = 5000$ samples as our benchmark, vary the values of J in the VMEMC simulations, and then evaluate the approximation errors based on the benchmark. We repeat such error analysis for $R = 12$ independent replicas and compute the average of the output errors. Denote the VMEMC solution at time t_n in the $r-$th independent replica by $\Psi^{n,r}_{J,h} \equiv \frac{1}{J}\sum_{j=1}^{J} y^{n,r}_{j,h}$, where $y^{n,r}_{j,h}$ is result of the VMEMC (3.1) in the $r-$th experiment. Denote

$$\mathcal{E}^*_{L^2} = \sqrt{\frac{1}{R}\sum_{r=1}^{R} |||\Psi^{n,r}_{J_0,h} - \Psi^{n,r}_{J,h}|||^2_{2,0}}.$$

We run $R = 12$ times and take the logarithm for the numerical results of $\mathcal{E}^*_{L^2}$ at $J = 10, 20, 40, 80, 160$ are depicted in the left subgraph of Figure 1. One group of values of $\mathcal{E}^*_{L^2}$ can be seen in Table 5 and the corresponding linear regression model is shown in the right subgraph of Figure 1.

We apply linear regression analysis on the numerical errors, which show

$$\mathcal{E}^*_{L^2} \approx 0.1237 J^{-0.546}.$$

These results shows that the rate of convergence with respect to J is close to -0.5, which coincides with our theoretical results in Theorem 4.4.

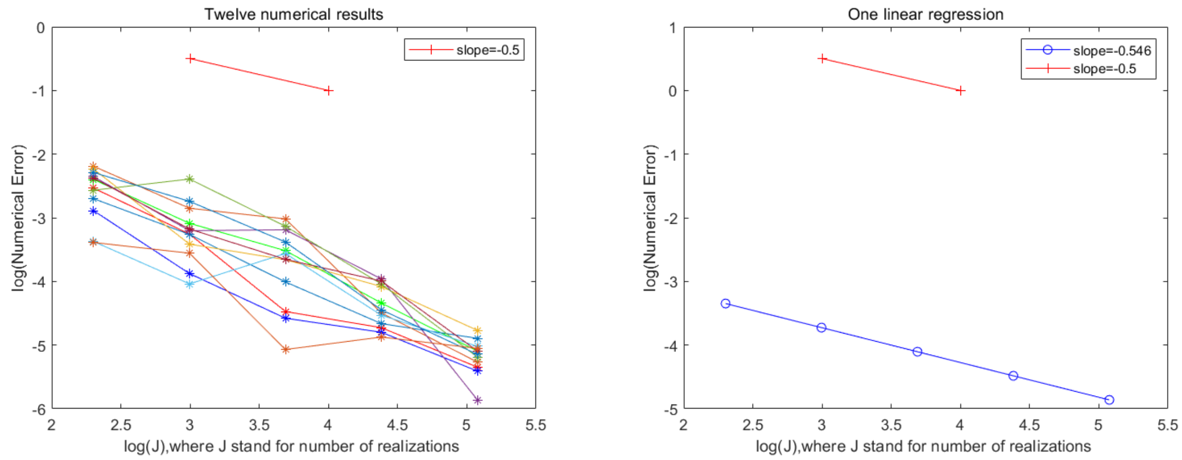

Figure 1. Convergence order of the VMEMC algorithm for sampling number J. Left: the 12 numerical results of $\mathcal{E}^*_{L^2}$. Right: the linear regression for one group of numerical results of $\mathcal{E}^*_{L^2}$.

(3) Comparison. Using the mesh size $h = \Delta t = 1/2^6$ and the sample size $J = 100$, we obtain the mean and the variance of the solutions at $t = 0.2$. The results are plotted in Figure 2 and Figure 3 for different algorithms.

From Figure 2 and Figure 3, it is seen that both VMEMC and VAEMC approximations are excellent.

To check out the performance of the VMEMC algorithm, we compare the result with that of individual finite element MC (IFE-MC) simulations applying the same groups of sample values. The difference from the mean to the IFE-MC solution is plotted in Figure 4.

The accurate approximation to the IFE-MC implemented for the VMEMC scheme is very nice. From Figure 4, the difference on the order of 10^{-3} signifies that the VMEMC approximation can also achieve basically accurate approximation as the IFE-MC implements.

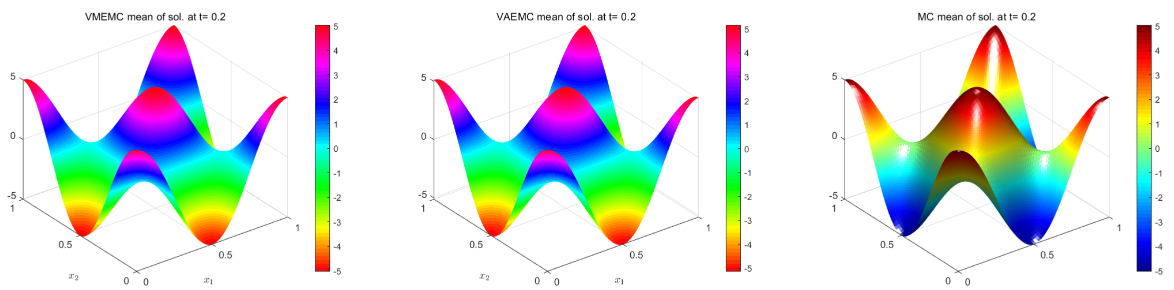

Figure 2. Three means. Left: mean of the VMEMC solution. Middle: mean of the VAEMC solution. Right: mean of the IFE-MC solution.

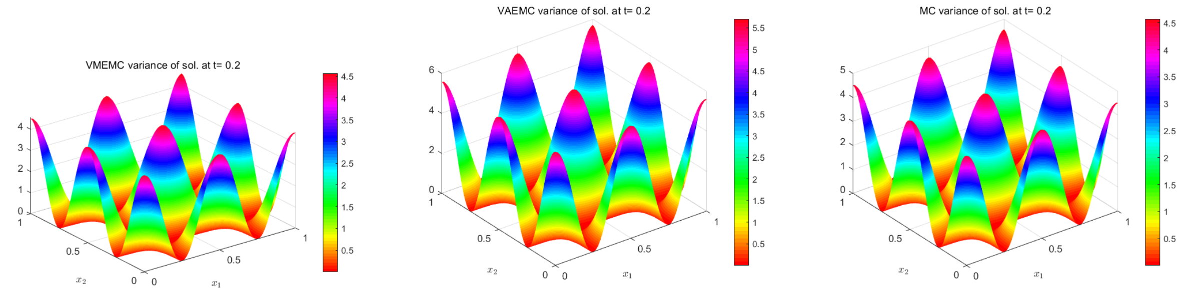

Figure 3. Variances. Left: variance of the VMEMC solution. Middle: variance of the VAEMC solution. Right: variance of the IFE-MC solution.

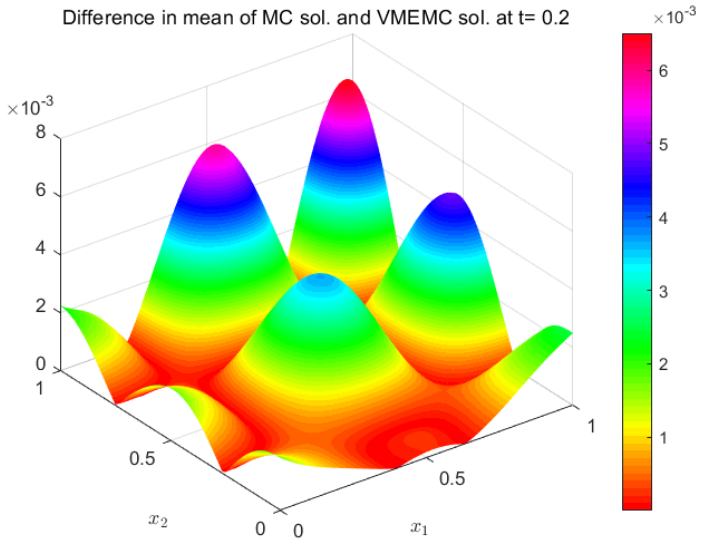

Figure 4. Difference of the VMEMC mean and the IFE-MC solution.

(4) Efficiency. To test the efficiency, setting the same time as well as space size, we record the CPU running times of the VMEMC, VAEMC, and MC algorithms in Table 6. Here we apply LU factorization because the discrete system's size is small.

From Table 6, when $J > 1$, we can show that both the VMEMC and the VAEMC algorithms take fewer CPU times than the MC algorithm. In contrast with the non-ensemble scheme, these ensemble methods improve the calculational efficiency by about 70%–80%. As the size of the ensemble increases, the efficiency of these ensemble algorithms improves, particularly for the VMEMC algorithm.

Table 6. CPU time (s) ($\Delta t = \frac{1}{2^4}, h = \frac{1}{2^8}$).

J	10	50	100	200
CPU time for VMEMC	37.65	150.21	295.59	569.66
CPU time for VAEMC	38.37	152.47	285.91	607.01
CPU time for MC	65.27	288.10	559.29	1100.71

6. Discussion and conclusion

The core idea of the ensemble algorithm is to develop a semi-implicit and semi-explicit temporal discretization scheme for PDEs. It will select a representative $\hat{a}$ of the random coefficient as the implicit term, and take the fluctuation $||a_j - \hat{a}||$ between random coefficients and this representation as the explicit term. This results in a shared coefficient matrix for each ensemble member at every time step. Consequently, storage requirements are reduced, and solution speed is increased. In general, as long as the relative fluctuation $\frac{||a_j-\hat{a}||}{\hat{a}}$ is not very large, such as $\frac{||a_j-\hat{a}||}{\hat{a}} < 1$, the ensemble numerical scheme can be stable. A large amount of literature [17–20, 23, 24, 30] discussed the numerical stability conditions and error analysis of the ensemble algorithm based on taking the mean as representation $\hat{a}$.

Based on taking the maximum value of the random coefficient as representation, an unconditionally stable ensemble scheme, VMEMC, is established in this work. First, we apply the VMEMC algorithm to deal with the transient heat equation with the stochastic diffusion factors and the Robin factors. Without any stability conditions, we prove that the algorithm is numerically stable. Additionally, we also provide the overall error estimate for the algorithm. Moreover, numerical experiments are conducted to demonstrate the application of the ensemble schemes and to validate the established properties of the algorithm.

Important next steps include accommodating phase changes in the solid material (e.g., transitions to a liquid phase) and incorporating additional physical processes into the boundary conditions (e.g., surface-to-ambient radiation). The stochastic Robin boundary control problem is worth investigating. The VMEMC method can also be developed for nonlinear stochastic parabolic equations.

Author contributions

Tingfu Yao: Methodology, writing original draft, software; Changlun Ye: Software and supervision; Xianbing Luo: Review, editing, and supervision; Shuwen Xiang: Supervision and validation.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (Granted No. 11961008, 12361033) and The Natural Science Research Foundation of Education Department of Guizhou Province (Granted NO. QJJ[2024]190).

Conflict of interest

The authors declare there is no conflict of interest.

References

1. I. Babuška, F. Nobile, R. Tempone, A stochastic collocation method for elliptic partial differential equations with random input data, *SIAM J Numer Anal*, **52** (2010), 317–355. https://doi.org/10.1137/050645142
2. S. C. Brenner, L. R. Scott, *The mathematical theory of finite element methods, 3rd edn.*, Springer, New York, 2008. https://doi.org/10.1007/978-0-387-75934-0
3. R. Chiba, *Stochastic analysis of heat conduction and thermal stresses in solids: a review, Chapter 9 in Heat Transfer Phenomena and Applications*, IntechOpen, London, 2012. https://doi.org/10.5772/50994
4. R. Chiba, Stochastic heat conduction analysis of a functionally graded annular disc with spatially random heat transfer coefficients, *Appl. Math. Model.*, **33** (2009), 507–523. https://doi.org/10.1016/j.apm.2007.11.014
5. A. Ern, J. L. Guermond, *Theory and Practice of Finite Elements*, Springer-Verlag, New York, 2004. https://doi.org/10.1007/978-1-4757-4355-5
6. J. A. Fiordilino, M. Winger, Unconditionally energy stable and first-order accurate numerical schemes for the heat equation with uncertain temperature-dependent conductivity, *Int. J. Num. Ana. Model.*, **20** (2023), 805–831. https://doi.org/10.4208/ijnam2023-1035
7. X. Feng, Y. Luo, L. Vo, Z. Wang, An efficient iterative method for solving parameter-dependent and random convection-diffusion problems, *J. Sci. Comput.*, **90** (2022), 72. https://doi.org/10.1007/s10915-021-01737-z
8. G. Fishman., *Monte Carlo: Concepts, Algorithms, and Applications*, Springer, New York ,1996. https://doi.org/10.1007/978-1-4757-2553-7
9. B. Ganapathysubramanian, N. Zabaras, Sparse grid collocation schemes for stochastic natural convection problems, *J. Comput. Phys.*, **225** (2007), 652–685. https://doi.org/10.1016/j.jcp.2006.12.014
10. M. B. Giles, Multilevel Monte Carlo methods, *Acta Numer.*, **24** (2015), 259–328. https://doi.org/10.1017/S09624929
11. M. D. Gunzburger, C. G. Webster, G. Zhang, Stochastic finite element methods for partial differential equations with random input data, *Acta Numer.*, **23** (2014), 521–650. https://doi.org/10.1017/S0962492914000075
12. J. C. Helton, F. J. Davis, Latin hypercube sampling and the propagation of uncertainty in analyses of complex systems, *Reliab. Eng. Syst. Safe.*, **81** (2003), 23–69. https://doi.org/10.1016/S0951-8320(03)00058-9
13. N. Jiang, W. Layton, An algorithm for fast calculation of flow ensembles, *Int. J. Uncertain. Quan.,* **4** (2014), 273–301. https://doi.org/10.1615/Int.J.UncertaintyQuantification.2014007691
14. B. Jin, J. Zou, Numerical estimation of the Robin coefficient in a stationary diffusion equation, *IMA J. Numer. Anal.* , **30** (2010), 677–701. https://doi.org/10.1093/imanum/drn066
15. B. Jin, J. Zou, Numerical identification of a Robin coefficient in parabolic problems, *Math. Comp.*, **81** (2012), 1369–1398. https://doi.org/10.1090/S0025-5718-2012-02559-2

16. L. Ju, W. Leng, Z. Wang, S. Yuan, Numerical investigation of ensemble methods with block iterative solvers for evolution problems, *Discrete Contin. Dyn. Syst. Ser. B.*, **25** (2020), 4905–4923. https://doi.org/10.3934/dcdsb.2020132

17. M. Li, X. Luo, An EMC-HDG scheme for the convection-diffusion equation with random diffusivity, *Numer Algor*, **90** (2022), 1755–1776. https://doi.org/10.1007/s11075-021-01250-2

18. M. Li, X. Luo, An MLMCE-HDG method for the convection diffusion equation with random diffusivity, *Comput. Math. Appl.*, **127** (2022), 127–143. https://doi.org/10.1007/s11075-021-01250-2

19. M. Li, X. Luo, An ensemble Monte Carlo HDG method for parabolic PDEs with random coefficients, *Int. J. Comput. Math.*, **100** (2022), 405–421. https://doi.org/10.1016/j.camwa.2022.10.002

20. M. Li, X. Luo, A multilevel Monte Carlo ensemble and hybridizable discontinuous Galerkin method for a stochastic parabolic problem, *Numer Methods Partial Differ Equ*, **39** (2023), 2840–2864. https://doi.org/10.1002/num.22990

21. K. Liu, B. M. Riviere, Discontinuous galerkin methods for elliptic partial differential equations with random coefficients, *Int. J. Comput. Math.*, **90** (2013), 2477–2490. https://doi.org/10.1080/00207160.2013.784280

22. G. J. Lord, C. E. Powell, T. Shardlow, *An introduction to computational stochastic PDEs*, Cambridge University Press, New York, 2014. https://doi.org/10.1017/CBO9781139017329

23. Y. Luo, Z. Wang, An ensemble algorithm for numerical solutions to deterministic and random parabolic PDEs, *SIAM J. Numer. Anal.*, **56** (2018), 859–876. https://doi.org/10.1137/17M1131489

24. Y. Luo, Z. Wang, A multilevel Monte Carlo ensemble scheme for solving random parabolic PDEs, *SIAM J. Sci. Comput.*, **41** (2019), A622–A642. https://doi.org/10.1137/18M1174635

25. J. Martínez-Frutos, M. Kessler, A. Münch, F. Periago, Robust optimal Robin boundary control for the transient heat equation with random input data, *Int. J. Numer. Methods Eng.*, **108** (2016), 116–135. https://doi.org/10.1002/nme.5210

26. J. Martínez-Frutos, F. P. Esparza, *Optimal control of PDEs under uncertainty. An introduction with application to optimal shape design of structures*, Springer, Cham, 2018. https://doi.org/10.1007/978-3-319-98210-6

27. L. Mathelin, M. Y. Hussaini, T. A. Zang, Stochastic approaches to uncertainty quantification in CFD simulations, *Numer Algor*, **38** (2005), 209–236. https://doi.org/10.1007/BF02810624

28. J. Meng, P. Y. Zhu, H. B. Li, A block method for linear systems with multiple right-hand sides, *J. Comput. Appl. Math.*, **255** (2014), 544–554. https://doi.org/10.1016/j.cam.2013.06.014

29. D. Xiu, J. S. Hesthaven, High-order collocation methods for differential equations with random inputs, *SIAM J. Sci. Comput.*, **27** (2005), 1118–1139. https://doi.org/10.1137/040615201

30. T. F. Yao, C. L. Ye, X. B. Luo, S. W. Xiang, An ensemble scheme for the numerical solution of a random transient heat equation with uncertain inputs, *Numer Algor*, **94** (2023), 643–668. https://doi.org/10.1007/s11075-023-01514-z

31. X. Zhu, E. M. Linebarger, D. Xiu, Multi-fidelity stochastic collocation method for computation of statistical moments, *J. Comput. Phys.*, **341** (2017), 386–396. https://doi.org/10.1016/j.jcp.2017.04.022