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Research article

Note on prescribed-time stability of impulsive piecewise-smooth differential systems and application in networks

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Abstract: We explored the prescribed-time stability (PTSt) of impulsive piecewise smooth differential systems (IPSDS) based on the Lyapunov theory and set-valued analysis technology, allowing flexibility in selecting the settling time as desired. Furthermore, by developing a feedback controller, we employed the theoretical results to evaluate the synchronization behavior of impulsive piecewise-smooth network systems (IPSNS) within a prescribed time frame and obtained novel criteria to guarantee the synchronization objective. A numerical example was presented to validate the accuracy of the results.

Keywords: feedback controller; impulsive piecewise-smooth differential systems; prescribed-time stability; prescribed-time synchronization

1. Introduction

Stability is a critical metric for assessing system responsiveness and control in the fields of automation, biology, aerospace, and engineering. The traditional stability [1] always studied the asymptotic tendency of a system over an infinite time horizon. However, as more strict stability concepts, finite-time stability (FTS) [2,3] and fixed-time stability (FxTS) [4] gradually attracted the attention of academics due to the growing need for control accuracy and response time. For instance, a novel FTS theory was proposed in [5], where system stability could be achieved within a finite time using a linear time-varying feedback controller. Although FTS has more stringent requirements than traditional stability, the settling time is limited by the system's initial values. FxTS is introduced to get around this restriction. Kong et al. [6] expanded on the FxTS theory and established stability criteria for fuzzy and discontinuous systems. However, the settling time in FxTS is limited due to the system's parameter values. PTSt, a more flexible stability, has gained scholars' interest. PTSt allows the settling time to be selected freely during the design stage and guarantees the system's stability within a prescribed time, which has excellent theoretical value and practical application prospects. For example, Holloway and Krstic [7] presented a novel approach that allows linear systems to be observed in a prescribed time. Zhou and Shi [8] investigated a class of nonlinear systems' prescribed-time stabilization problem and proposed a stabilizing strategy based on a linear time-varying feedback controller, which was extended by [9].

Piecewise-smooth differential systems (PSDS) consist of numerous subsystems, where each subsystem demonstrates smooth dynamical properties inside its state space but may exhibit discontinuous or abrupt behavior at the boundaries. These systems are often used in many real-world applications, including collision and friction phenomena in mechanical systems [10] and switching behavior in electrical circuits [11]. The researches on PSDS stability analysis are challenging, particularly in achieving system stability within a prescribed time. Samadi [12] focused on stability analysis and controller synthesis for a class of PSDS. Chen and Du [13] explored the stability and the response behavior of homoclinic loops in PSDS when subjected to small perturbations and derived the corresponding stability conditions. Samadi and Rodrigues [14] provided a systematic and generalized approach to the stability analysis of PSDS by introducing a unified dissipative framework. Glendinning and Jeffrey [15] provided a comprehensive introduction to piecewise smooth dynamical systems. Li et al. [16] developed two new lemmas related to PTSt in PSDS to achieve prescribed-time synchronization objectives. Although FTS and FxTS have made significant progress compared to traditional stability, they are limited by the system's initial conditions and parameter values. It is worth noting that although some progress has been made for PSDS stability, there is an urgent need for a stability concept that can control the convergence time more flexibly and should not be subject to the initial conditions and parameter limitations. This further emphasizes the need to study PTSt. From the preceding discussion, it is evident that the current research on the stability of PSDS primarily concentrates on the conventional stability theory, while few investigations are reported for PTSt. Filling this theoretical gap is our first motivation.

Many systems experience instantaneous changes, particularly impulse effects, during their dynamic processes. These changes occur in a very short time but significantly impact the systems. Studying such systems helps to simulate complex real-world phenomena accurately and improves the response speed and precision of control systems. Furthermore, the stability of impulsive systems follows into the spotlight. For example, Haddad et al. [17] proposed a mathematical model of impulsive systems and analyzed the stability using Lyapunov method. Nersesov and Haddad [18] developed vector Lyapunov functions theory for impulsive systems with large scale. Xi et al. [19] proposed a new analytical method, using Lyapunov function and characteristics of impulsive time-varying systems, to derive sufficient conditions for the uniform FTS. Xi et al. [20] solved practical FTS problem of nonlinear systems by delayed impulse control approach. Zhao et al. [21] investigated the FTS of linear time-varying singular systems with impulse effects. Jamal et al. [22] studied the FxTS of dynamical systems with impulse effects. Wang and Abdurahman [23] explored the FxTS of general impulsive systems and the synchronization problem of complex networks with hybrid impulses. Li et al. [24] analyzed the influence of impulse series on the stability of the system and derived fixed-time stability for impulsive systems. He et al. [25] addressed the problem of prescribed-time stabilization of nonlinear systems by impulse modulation. IPSDS are more complex and intriguing than traditional ones without piecewise-smooth feature, due to the mixed temporal characteristics. Li et al. [26] proposed a new PTSt theorem for IPSDS and demonstrated its application in network synchronization. In piecewise-smooth network systems with impulse effects, impulsive and piecewise properties complicate the synchronization problem. Traditional PTSy methods may be difficult to be applied directly or do not have the flexibility to deal with these complexities. Based on the above discussions, despite the significant impact of impulse effects on dynamic systems, further comprehensive research, particularly for PTSt, on IPSDS remains insufficient. Studying the PTSt of IPSDS is our further motivation rooted in the first motivation.

In modern engineering, network structures are widely found in various systems, such as social networks, biological networks, power networks, and communication networks. The nodes of networks are connected by edges, which form complex topologies to reflect real systems' complexity and diversity. Investigating the dynamic behavior for these networks, especially synchronization problem, is important for understanding and controlling the network systems. Coraggio et al. [27] proposed a new distributed discontinuous coupling method to synchronize network systems. Dieci and Elia [28] introduced and analyzed the principal stability function applicable to Filippov networks. The synchronization problem of complex networks has become one of the hot research topics, which contains different types of synchronization phenomena such as global synchronization, cluster synchronization. In particular, the prescribed-time synchronization (PTSy), as a special synchronization, has received much attention due to its ability to achieve synchronization within a specified time, see [29–32]. Detailly, Chen et al. [29] achieved PTSy of complex dynamic networks containing directed spanning trees by designing a suitable controller. Xu and Liu [30] proposed a series of sufficient conditions to ensure that the multi-weighted directed complex network can achieve PTSy by constructing an appropriate Lyapunov function. Yang et al. [31] derived the conditions for fixed-time synchronization of bidirectional associative memory memristive neural networks (BAMMNNs); Based on it, the PTSy of BAMMNNs was implemented. Tang et al. [32] proposed a prescribed-time controller that did not require fractional power and sign functions to achieve PTSy for a switched network. Although many synchronization results are addressed for complex networks, they mainly focus on network systems with continuous dynamic properties. For those network systems with impulse effects or piecewise-smooth characteristics, existing synchronization methods for PTSy show limitations in dealing with transient behavior or boundary behavior. The above discussions suggest that PTSy is an important problem to be solved in complex network systems with impulse effects and piecewise-smooth properties. The study on PTSy of IPSNS is our third motivation.

Based on the previous discussions, this work intends to investigate the PTSt of IPSDS, allowing flexibility in choosing the settling time as needed. Furthermore, the theoretical results are used to tackle the synchronization problem in IPSNS by importing a feedback controller. The significant contributions are as follows:

1) Employing the Lyapunov inequality and set-valued analysis technology, we construct a theorem on the PTSt of IPSDS, supplying sufficient criteria for the PTSt. Particularly, the setting time can be flexibly selected as required, regardless of the system's initial values and control parameters. This responds to our first and second motivations.

2) A feedback controller is designed to guarantee that the IPSNS can achieve synchronization within a prescribed time under the established stability theorem. This responds to our third motivation.

The rest of this paper is organized as follows: In Section 2, we introduce basic symbols, definitions, and lemmas. In Section 3, we present a PTSt theorem applicable to IPSDS. In Section 4, we design a feedback controller to achieve synchronization in IPSNS within a prescribed time. In Section 5, we provide an example to illustrate the correctness of the theoretical results. In Section 6, we conclude the paper.

2. Preliminaries

Notations. Let $\mathcal{R}$, $\mathcal{R}_+$, $\mathcal{N}$, $\mathcal{N}_+$ denote the sets of real numbers, positive real numbers, natural numbers, and positive integers, respectively. Let $\mathcal{R}^n$ represent the n-dimensional column vector, $\mathcal{R}^{n\times m}$ the set of $n\times m$ real matrices, I_n the identity matrix of dimension n. $A>0(A<0)$ denotes positive (negative) definite matrix. A^T represents the transpose of matrix A. $\|A\|=\sqrt{\lambda_{max}(A^TA)}$, where $\lambda_{max}(A^TA)$ is the maximun eigenvalue of matrix A^TA. $A^\varsigma=\frac{1}{2}(A+A^T)$. For any $x=(x_1,x_2,\cdots,x_n)^T$, $y=(y_1,y_2,\cdots,y_n)^T$, $x\circ y=(x_1y_1,x_2y_2,\cdots,x_ny_n)^T$. $\otimes$ represents the Kronecker produced.

Consider a series of non-empty, open, and disjoint finite sets $\mathcal{P}_1,\mathcal{P}_2,\cdots,\mathcal{P}_\ell$ such that $Q\subseteq\bigcup_{\ell=1}^{\ell}\bar{\mathcal{P}}_{\ell}\subseteq\mathcal{R}^n$ is a finite collection, and $\bar{\mathcal{P}}_i\cap\bar{\mathcal{P}}_j$ is a line. A differential system with impulse effects is concerned,

$$\begin{cases}\dot{x}(t)=g\big(x(t)\big), & t\neq t_k\\ \Delta x(t)=\alpha_k x(t^-), t=t_k,\\ x(0)=x_0,\end{cases} \tag{1}$$

where $g:Q\mapsto\mathcal{R}^n$ is a vector field, α_k is a constant. For $k\in\mathcal{N}$, $\Delta x(t_k)=x(t_k^+)-x(t_k^-)$, in which $x(t_k^+)=\lim_{t\to t_k^+}x(t)$, $x(t_k^-)=\lim_{t\to t_k^-}x(t)$. The impulse time sequence $\{t_k\}_{k\in\mathcal{N}}$ satisfies $0=t_0<t_1<t_2<\cdots<t_k\cdots$, $\lim_{k\to\infty}t_k=+\infty$. If $g(x)$ is a finite set of vector fields, i.e.,

$$g(x)=G_{\ell}(x), x\in\mathcal{P}_{\ell},$$

then system (1) is IPSDS, where $\ell=1,2,\cdots,\ell$, each vector field $G_{\ell}(x)$ is smooth in x for any $x\in\mathcal{P}_{\ell}$ and continuously extends on the boundary $\partial\mathcal{P}_{\ell}$. Additionally, assume that the solution $x(t)$ of system (1) is always right-discontinuous and has left-hand limits. Then, the solution $x(t)$ can be defined as Filippov solutions.

Remark 1. We focus on one-dimensional discontinuity manifolds, which appear at the switching boundaries of PSDSs and play a crucial role in describing the systems' switching behavior and trajectory evolutions. However, except for these low-dimensional discontinuity manifolds, higher-order cases are also crucial in complex systems. Higher-order discontinuity manifolds involve more complex geometric structures and can lead to more intricate dynamic phenomena. Difonzo [33] systematically analyzed these higher-order manifolds using set-valued theory and geometric methods and introduced the "isochronous attainable manifolds", which can help one understand these manifolds in higher-dimensional systems. In the future, we will apply higher-order manifolds to more complex control systems to gain deeper insights into the overall dynamic behavior and switching mechanisms of such systems.

Definition 1. [34] A vector function $x(t): \mathcal{R}^+ \mapsto \mathcal{R}^n$ is a Filippov solution of system (1) if it is absolutely continuous on $[t_k, t_{k+1})$, $k \in \mathcal{N}$, and for almost every $t > 0$,

$$\begin{cases} x(t) \in \mathcal{G}[g](x), & t \neq t_k \\ \Delta x(t) = \alpha_k x(t^-), & t = t_k, \\ x(0) = x_0, \end{cases}$$

where Filippov set-valued function $\mathcal{G}[g]: \mathcal{R}^n \mapsto \mathcal{A}(\mathcal{R}^n)$, with $\mathcal{A}(\mathcal{R}^n)$ is the set of all subsets of $\mathcal{R}^n$, is given by

$$\mathcal{G}[g](x) = \bigcap_{\varepsilon>0} \bigcap_{a(\mathcal{P})=0} \overline{co}\{g(\mathcal{B}(x, \varepsilon))\backslash \mathcal{P}\},$$

$a(\cdot)$ denotes the Lebesgue measure, $\overline{co}(\cdot)$ represents the closed convex hull, $\mathcal{B}(x, \varepsilon) = \{x \in \mathcal{R}^n: \|x\| \leq \varepsilon\}$.

Definition 2. [35] Let $v: \mathcal{R}^n \mapsto \mathcal{R}$ be a locally Lipschitz function. The set-valued Lie derivative $\mathcal{L}_{\mathcal{G}[g]}v(x): \mathcal{R}^n \mapsto \mathcal{A}(\mathcal{R}^n)$ of v concerning $\mathcal{G}[g]$, is given by

$$\mathcal{L}_{\mathcal{G}[g]}v(x) \triangleq \{b \in \mathcal{R}: \exists c \in \mathcal{G}[g](x) \text{ such that } d^T c = b, \forall d \in \partial v(x)\},$$

where $\partial v(x) \triangleq \left\{ \lim_{\hbar \to +\infty} \nabla v(x_\hbar) : x_\hbar \to x, x_\hbar \notin \mathcal{P} \cup \Omega_v \right\}$ is the generalized gradient of v at any $x \in \mathcal{R}^n$, and Ω_v is the set of zero Lebesgue measures.

Definition 3. [2] System (1) is finite-time stable, if for $\forall x_0 \in \mathcal{R}^n$, there exists a function $0 \leq T(x) < +\infty$, such that

$$\begin{cases} \lim_{t \to T_0} \|x(t)\| = 0, \\ \|x(t)\| \equiv 0, & t \geq T_0, \end{cases}$$

where $T_0 = T(x_0)$ is the settling time function.

Definition 4. [36] Given a prescribed time constant $\hat{T} > 0$, system (1) is prescribed-time stable, if it is finite-time stable and the settling time $T_0 \leq \hat{T}$.

Lemma 1. [37] If $x(t)$ be a Filippov solution of system (1), $v: \mathcal{R}^n \mapsto \mathcal{R}$ be a locally Lipschitz regular function, then $\dot{v}(x(t))$ exists and $\dot{v}(x(t)) \in \mathcal{L}_{\mathcal{G}[g]}v(x(t))$ almost everywhere.

3. Main results

We construct a PTSt theorem for IPSDS in which the settling time is freely chosen and unaffected by the system's initial values and control parameters.

Theorem 1. Given a prescribed time $\hat{T}$ and $x(0) \in \mathcal{B}(x, \varepsilon)$, system (1) is prescribed-time stable, provided there exist a positive definite and regular locally Lipschitz function $\mathcal{V}: \mathcal{R}^n \mapsto \mathcal{R}^n$, K_∞-class functions λ, μ, and constants $0 < \gamma \le 1,\ p, q > 0,\ 0 < \delta < 1,\ 0 < \epsilon < 1$, such that

$$(C_1)\ \psi(t) \le -\frac{\beta}{\hat{T}}\left(p\mathcal{V}^{\delta}\big(x(t)\big) - q\mathcal{V}\big(x(t)\big)\right),\ \forall x(t) \in \mathcal{R}^n \backslash \{0\},\ t \ne t_k,$$

$$(C_2)\ \mathcal{V}\big(x(t)\big) \le \gamma \mathcal{V}\big(x(t^-)\big),\ \forall x \in \mathcal{R}^n \backslash \{0\},\ t = t_k,$$

$$(C_3)\ \lambda(\|x(t)\|) \le \mathcal{V}\big(x(t)\big) \le \mu(\|x(t)\|),$$

$$(C_4)\ max_{k \in \{1,\,2,\,\cdots,\,M_0\}}\{t_k - t_{k-1}\} \le \frac{\epsilon \hat{T}}{M_0},$$

where $\psi(t) \in \mathcal{L}_{\mathcal{G}[g]}\mathcal{V}\big(x(t)\big)$, $\beta = \frac{\epsilon}{(1-\epsilon)^2(1-\delta)q}$, $M_0 = min\left\{\varrho \in \mathcal{N}_+ | \varrho \ge \frac{-\ln \pi - q\beta\epsilon}{ln\,\gamma}\right\}$, $\pi = \mu(\varepsilon)\left(\frac{q}{\epsilon p}\right)^{\frac{1}{1-\delta}}$.

Proof. Based on Lemma 1, for almost every $t > 0$, there exists

$$\psi(t) = \dot{\mathcal{V}}\big(x(t)\big) \in \mathcal{L}_{\mathcal{G}[g]}\mathcal{V}\big(x(t)\big).$$

From condition (C_3), we can obtain that

$$\|x(t)\| \le \lambda^{-1}\left(\mathcal{V}\big(x(t)\big)\right),$$

$$\mathcal{V}\big(x(0)\big) \le \mu(\|x(0)\|).$$

From $x(0) \in \mathcal{B}(x, \varepsilon)$, then $\|x(0)\| \le \varepsilon$. Combined with the definition of π, yields

$$\mathcal{V}\big(x(0)\big) \le \mu(\|x(0)\|) \le \mu(\varepsilon) = \pi\left(\frac{\epsilon p}{q}\right)^{\frac{1}{1-\delta}}.$$

Due to the uncertainty in π, we cannot determine the relationship between $\mathcal{V}\big(x(0)\big)$ and $\left(\frac{\epsilon p}{q}\right)^{\frac{1}{1-\delta}}$. Therefore, we can analyze the PTSt of IPSDS (1) under the following two scenarios:

Case I: $\mathcal{V}\big(x(0)\big) \le \left(\frac{\epsilon p}{q}\right)^{\frac{1}{1-\delta}}$. In this case, there is for $\forall t \ge 0$, $\mathcal{V}\big(x(t)\big) \le \left(\frac{\epsilon p}{q}\right)^{\frac{1}{1-\delta}}$. There must be at least $t \ge 0$ such that $\mathcal{V}\big(x(t)\big) > \left(\frac{\epsilon p}{q}\right)^{\frac{1}{1-\delta}}$, if this conclusion is false. Let $t_s = inf\left\{t \ge 0 | \mathcal{V}\big(x(t)\big) > \left(\frac{\epsilon p}{q}\right)^{\frac{1}{1-\delta}}\right\}$, it follows that $\mathcal{V}\big(x(t_s)\big) = \left(\frac{\epsilon p}{q}\right)^{\frac{1}{1-\delta}}$. Thus, for $t \in [0, t_s]$, $\mathcal{V}\big(x(t)\big) \le \mathcal{V}\big(x(t_s)\big)$. According to condition (C_2), t_s is not an impulse moment. Then we can get

$$\dot{\mathcal{V}}\big(x(t_s)\big) \geq 0.$$

It follows from condition (C_1) that

$$\begin{aligned}\dot{\mathcal{V}}\big(x(t_s)\big) &\leq -\frac{\beta}{\hat{T}}\Big(p\mathcal{V}^{\delta}\big(x(t_s)\big) - q\mathcal{V}\big(x(t_s)\big)\Big)\\ &= -\frac{\beta}{\hat{T}}\big[p - q\mathcal{V}^{1-\delta}\big(x(t_s)\big)\big]\mathcal{V}^{\delta}\big(x(t_s)\big)\\ &= -\frac{(1-\epsilon)p\beta}{\hat{T}}\mathcal{V}^{\delta}\big(x(t_s)\big).\end{aligned}$$

Letting $\varpi = \frac{(1-\epsilon)p\beta}{\hat{T}}$, it can be inferred that $\varpi > 0$, namely

$$\dot{\mathcal{V}}\big(x(t_s)\big) < 0.$$

This contradicts the above conclusion and, therefore, for $\forall t \geq 0$, $\mathcal{V}\big(x(t)\big) \leq \left(\frac{\epsilon p}{q}\right)^{\frac{1}{1-\delta}}$.

When $t \neq t_k$, it follows from condition (C_1) that

$$\begin{aligned}\dot{\mathcal{V}}\big(x(t)\big) &\leq -\frac{\beta}{\hat{T}}\Big(p\mathcal{V}^{\delta}\big(x(t)\big) - q\mathcal{V}\big(x(t)\big)\Big)\\ &= -\frac{\beta}{\hat{T}}\big[p - q\mathcal{V}^{1-\delta}\big(x(t)\big)\big]\mathcal{V}^{\delta}\big(x(t)\big)\\ &= -\frac{(1-\epsilon)p\beta}{\hat{T}}\mathcal{V}^{\delta}\big(x(t)\big),\end{aligned}$$

from which we can get when $t \neq t_k$,

$$\dot{\mathcal{V}}\big(x(t)\big) \leq -\varpi\mathcal{V}^{\delta}\big(x(t)\big). \tag{2}$$

Considering the impulse effect, divide the interval $[0, t_\infty)$ into $[0, t_1)$, $[t_1, t_2)$, $\cdots$, $[t_{M-1}, t_M)\cdots$. Then, it can be deduced from (2) that

$$\mathcal{V}^{1-\delta}\big(x(t)\big) \leq \mathcal{V}^{1-\delta}\big(x(0)\big) - \varpi(1-\delta)t, t \neq t_k. \tag{3}$$

Integrating on both sides of (2) from 0 to t, where $t \in [0, t_1)$, one has

$$\mathcal{V}^{1-\delta}\big(x(t)\big) \leq \mathcal{V}^{1-\delta}\big(x(0)\big) - \varpi(1-\delta)t.$$

Hence, (3) holds for $t \in [0, t_1)$. Assume that (3) is correct for $t \in [t_{M-2}, t_{M-1})$, where $M \geq 2$. Subsequently, for $t \in [t_{M-2}, t_{M-1})$, there is

$$\mathcal{V}^{1-\delta}\big(x(t)\big) \leq \mathcal{V}^{1-\delta}\big(x(0)\big) - \varpi(1-\delta)t.$$

From this, it can be easily inferred that $\mathcal{V}^{1-\delta}(x(t_{M-1}^{-})) \leq \mathcal{V}^{1-\delta}(x(0)) - \varpi(1-\delta)t\_\{M-1\}$. Because $\gamma^{1-\delta} \leq 1$, we can deduce from condition (C_2) that

$$\mathcal{V}^{1-\delta}(x(t_{M-1})) \leq \gamma^{1-\delta}\mathcal{V}^{1-\delta}(x(t_{M-1}^{-})) \leq \mathcal{V}^{1-\delta}(x(0)) - \varpi(1-\delta)t_{M-1}.$$

Integrating on both sides of (2) from t_{M-1} to t, where $t \in [t_{M-1}, t_M)$, we have

$$\begin{aligned}\mathcal{V}^{1-\delta}(x(t)) &\leq \mathcal{V}^{1-\delta}(x(t_{M-1})) - \varpi(1-\delta)(t - t_{M-1}) \\ &\leq \mathcal{V}^{1-\delta}(x(0)) - \varpi(1-\delta)t_{M-1} - \varpi(1-\delta)(t - t_{M-1}) \\ &= \mathcal{V}^{1-\delta}(x(0)) - \varpi(1-\delta)t.\end{aligned}$$

Hence, (3) holds for $t \in [t_{M-1}, t_M)$. In summary, (3) is deemed to be correct when $t \neq t_k$. Considering condition (C_2), $\mathcal{V}^{1-\delta}(x(t)) \leq \mathcal{V}^{1-\delta}(x(0)) - \varpi(1-\delta)t$ for $\forall t \geq 0$. Then, T_0 satisfies

$$\begin{aligned}T_0 &= \frac{\mathcal{V}^{1-\delta}(x(0))}{\varpi(1-\delta)} \\ &\leq \left(\left(\frac{\epsilon p}{q}\right)^{\frac{1}{1-\delta}}\right)^{1-\delta} \cdot \frac{1}{1-\delta} \cdot \frac{\hat{T}}{(1-\epsilon)p\beta} \\ &= \frac{\epsilon p}{q} \cdot \frac{1}{1-\delta} \cdot \frac{\hat{T}}{(1-\epsilon)p} \cdot \frac{(1-\epsilon)^2(1-\delta)q}{\epsilon} \\ &= (1-\epsilon)\hat{T}.\end{aligned}$$

We can then deduce that as $t \to T_0$, $\mathcal{V}(x(t)) \to 0$. By combining (2) and condition (C_2), we have for $\forall t \geq 0$, $\dot{\mathcal{V}}(x(t)) \leq 0$. Therefore, as $t \geq T_0$, $0 \leq \mathcal{V}(x(t)) \leq \mathcal{V}(x(T_0)) \to 0$. Consequently, as $t \geq T_0$, $V(x(t)) \equiv 0$, Since $\|x(t)\| \leq \lambda^{-1}\left(\mathcal{V}(x(t))\right)$, $T_0 \leq (1-\epsilon)\hat{T}$ and $\dot{\mathcal{V}}(x(t)) \leq 0$, for $\forall t \geq 0$, it can be inferred that $\|x(t)\| \to 0$, $t \to \hat{T}$; $\|x(t)\| \equiv 0$, $t \geq \hat{T}$.

Thus, the IPSDS (1) is deemed to be prescribed-time stable when $\mathcal{V}(x(0)) \leq \left(\frac{\epsilon p}{q}\right)^{\frac{1}{1-\delta}}$.

Case II: $\mathcal{V}(x(0)) > \left(\frac{\epsilon p}{q}\right)^{\frac{1}{1-\delta}}$. From condition (C_1), we can obtain that when $t \neq t_k$,

$$\dot{\mathcal{V}}(x(t)) \leq \tfrac{q\beta}{\hat{T}}\mathcal{V}(x(t)). \tag{4}$$

Let $M(t)$ represent the count of impulses within $(0, t]$. Then, it can be deduced from (4) that, when $t \neq t_k$,

$$\mathcal{V}(x(t)) \leq exp\left\{M(t)\, ln\,\gamma + \tfrac{q\beta}{\hat{T}}t\right\}\mathcal{V}(x(0)), \tag{5}$$

Integrating on both sides of (4) from 0 to t, where $t \in [0, t_1)$, one has

$$\mathcal{V}(x(t)) \leq exp\left\{\frac{q\beta}{\hat{T}}t\right\}\mathcal{V}(x(0)).$$

It is evident that when $t \in [0, t_1)$, $M(t) = 0$. Hence, (5) holds for $t \in [0, t_1)$. Assume that (5) is correct for $t \in [t_{M-2}, t_{M-1})$, where $M \geq 2$. As $t \in [t_{M-2}, t_{M-1})$, $M(t) = M - 2$, then we can get

$$\mathcal{V}(x(t)) \leq exp\left\{(M-2)\ln\gamma + \frac{q\beta}{\hat{T}}t\right\}\mathcal{V}(x(0)).$$

From this, it can be easily inferred that $\mathcal{V}(x(t_{M-1}^-)) \leq exp\left\{(M-2)\ln\gamma + \frac{q\beta}{\hat{T}}t\_\{M-1\}\right\}\mathcal{V}(x(0))$. We can deduce from condition (C_2) that

$$\begin{aligned}\mathcal{V}(x(t_{M-1})) &\leq \gamma\mathcal{V}(x(t_{M-1}^-))\\ &\leq \gamma exp\left\{(M-2)\ln\gamma + \frac{q\beta}{\hat{T}}t_{M-1}\right\}\mathcal{V}(x(0))\\ &= exp\left\{(M-1)\ln\gamma + \frac{q\beta}{\hat{T}}t_{M-1}\right\}\mathcal{V}(x(0)).\end{aligned}$$

Integrating on both sides of (4) from t_{M-1} to t, where $t \in [t_{M-1}, t_M)$, we have

$$\begin{aligned}\mathcal{V}(x(t)) &\leq exp\left\{\frac{q\beta}{\hat{T}}(t - t_{M-1})\right\}\mathcal{V}(x(t_{M-1}))\\ &\leq exp\left\{\frac{q\beta}{\hat{T}}(t - t_{M-1})\right\}exp\left\{(M-1)\ln\gamma + \frac{q\beta}{\hat{T}}t_{M-1}\right\}\mathcal{V}(x(0))\\ &= exp\left\{(M-1)\ln\gamma + \frac{q\beta}{\hat{T}}t\right\}\mathcal{V}(x(0)).\end{aligned}$$

As $t \in [t_{M-2}, t_{M-1})$, $M(t) = M - 1$, hence, (5) holds for $t \in [t_{M-1}, t_M)$. In summary, (5) is deemed to be correct when $t \neq t_k$. Considering condition (C_2), $\mathcal{V}(x(t)) \leq exp\left\{M(t)\ln\gamma + \frac{q\beta}{\hat{T}}t\right\}\mathcal{V}(x(0))$ for $\forall t \geq 0$. Based on (5) and condition (C_4), we can conclude that

$$\begin{aligned}\mathcal{V}\left(x\left(t_{M_0}\right)\right) &\leq exp\left\{M_0\ln\gamma + \frac{q\beta}{\hat{T}}t_{M_0}\right\}\mathcal{V}(x(0))\\ &\leq exp\left\{\frac{-\ln\pi - q\beta\epsilon}{\ln\gamma}\ln\gamma + \frac{q\beta}{\hat{T}}\epsilon\hat{T}\right\}\pi\left(\frac{\epsilon p}{q}\right)^{\frac{1}{1-\delta}}\\ &= exp\{-\ln\pi\}\pi\left(\frac{\epsilon p}{q}\right)^{\frac{1}{1-\delta}}\\ &= \left(\frac{\epsilon p}{q}\right)^{\frac{1}{1-\delta}}.\end{aligned}$$

From this, we can obtain $\mathcal{V}\left(x\left(t_{M_0}\right)\right) \leq \left(\frac{\epsilon p}{q}\right)^{\frac{1}{1-\delta}}$. According to Case I, T_0 satisfies

$$
\begin{aligned}
T_0 &= \frac{\mathcal{V}^{1-\delta}\left(x\left(t_{M_0}\right)\right)}{\varpi(1-\delta)} \\
&\le \left(\left(\frac{\epsilon p}{q}\right)^{\frac{1}{1-\delta}}\right)^{1-\delta} \cdot \frac{1}{1-\delta} \cdot \frac{\hat{T}}{(1-\epsilon)p\beta} \\
&\le \frac{\epsilon p}{q} \cdot \frac{1}{1-\delta} \cdot \frac{\hat{T}}{(1-\epsilon)p} \cdot \frac{(1-\epsilon)^2(1-\delta)q}{\epsilon} \\
&= (1-\epsilon)\hat{T}.
\end{aligned}
$$

We can then deduce that as $t \to t_{M_0} + T_0$, $\mathcal{V}(x(t)) \to 0$. By combining (2) and condition (C_2), we have for $\forall t \ge t_{M_0}$, $\dot{\mathcal{V}}(x(t)) \le 0$. Therefore, as $t \ge t_{M_0} + T_0$, $0 \le \mathcal{V}(x(t)) \le \mathcal{V}\left(x\left(t_{M_0} + T_0\right)\right) \to 0$. Consequently, $V(x(t)) \equiv 0$. From $t_{M_0} \le \epsilon\hat{T}$ and $T_0 \le (1-\epsilon)\hat{T}$, $t_{M_0} + T_0 \le \hat{T}$. Similarly to Case I, it can be inferred that $\|x(t)\| \to 0,\ t \to \hat{T};\ \|x(t)\| \equiv 0,\ t \ge \hat{T}$.

Thus, the IPSDS (1) is deemed to be prescribed-time stable when $\mathcal{V}(x(0)) > \left(\frac{\epsilon p}{q}\right)^{\frac{1}{1-\delta}}$. The proof of Theorem 1 is completed.

Remark 2. We introduce a constant gain $\hat{T}$ in the nonlinear function of condition (C_1) to accommodate PTSt. Although [16] also used a constant gain approach, it failed to consider the impact of the impulse effect.

Remark 3. The set-valued Lie derivative can handle discontinuities and piecewise smoothness in systems, whereas the conventional derivatives can be used only for smooth systems. By introducing the set-valued Lie derivative, we can well describe and analyze discontinuous changes in the systems. Therefore, for condition (C_1) in Theorem 1, unlike [25,38], we employ the set-valued Lie derivative. This approach enables us to handle a wider range of functions and more complex systems than conventional ones.

Remark 4. The nonlinear Lyapunov inequality in [26] incorporated time-varying functions, which increased the difficulty in practical applications. In comparison, our nonlinear functions are easier to calculate and analyze.

Remark 5. Our nonlinear function of condition (C_1) includes a positive linear term, enabling our nonlinear Lyapunov inequality to have a broader range of applicability compared to [39].

Remark 6. Research on systems with impulse effects [19–24] primarily focuses on analyzing finite-time/fixed-time stability. In contrast, our attention shifts to the PTSt of such systems. This paper broadens the understanding of stability and provides a detailed analytical framework and explicit time bounds for PTSt. For the PTSt problem of general systems, as discussed in [7,8,16,40], the analysis is typically straightforward since impulse effects are not considered. In this paper, we divide the system's time into series of impulsive intervals and analyze the system's stability within each interval. We also consider the impact of impulsive moments on the system to ensure that impulsive effects do not undermine stability. This piecewise analysis method guarantees that the system can achieve stability within the prescribed time, even with impulse effects.

Remark 7. Our method provides a framework for proving PTSt and effectively handles systems with impulse effects. The approach allows for the analysis of system behavior from different initial states based on the initial value $\mathcal{V}(x(0))$ of the Lyapunov function. This flexibility makes the method applicable to a broader range of systems. Additionally, Theorem 1 gives a clear time bound within which the system state must converge to the equilibrium point, which is crucial for determining the maximum convergence time in practical applications. However, its complexity, potential conservatism, and dependence on specific conditions may limit practical applications of Theorem 1 in certain scenarios. For example, conditions (C_1)–(C_4) are complex, involving multiple parameters and inequalities, which may make it difficult to validate them in practical applications.

Remark 8. In this paper, we study the PTSt of IPSDS without considering the significant factors of stochasticity and fractional order. Work [41–44] have studied the stability of stochastic system with fractional order/impulse effects. These works motivate us to study the stability, particularly PTSt, of IPSDS with stochasticity/fractional order.

4. Application in network

Consider the following IPSNS with N nodes

$$\begin{cases} \dot{x}_i(t) = g(x_i(t)) - m\sum_{j=1}^{N} \iota_{ij}\Pi x_j(t) + \mathcal{u}_i(t), & t \neq t_k \\ \Delta x_i(t) = \alpha_k x_i(t^-), & t = t_k, \end{cases} \tag{6}$$

where $i = 1, 2, \cdots, N$, $x_i(t) = (x_{i1}(t), x_{i2}(t), \cdots, x_{iN}(t))^T$ denotes the i-th node's state vector, piecewise function $g(x_i(t)) = \left(g_1(x_{i1}(t)), g_2(x_{i2}(t)), \cdots, g_N(x_{iN}(t))\right)^T$ represents the i-th node's vector field, m stands for the coupling gain, $\Pi = diag\{\tau_1, \tau_2, \cdots, \tau_N\}$ denotes the internal coupling matrix, $\mathcal{u}_i(t)$ indicates the i-th node's control input, $\bar{L} = (\iota_{ij})_{N\times N}$ is its Laplace matrix.

Remark 9. $|1 + \alpha_k| \leq 1$ holds for this section.

The synchronization target system of the IPSNS (6) is

$$\dot{z}(t) = g(z(t)), \tag{7}$$

where $z(0) = z_0$, $z(t) = (z_1(t), z_2(t), \cdots, z_N(t))^T$.

Let $\omega_i(t) = x_i(t) - z(t)$ denote the synchronization error of the i-th node, $i = 1, 2, \cdots, N$. Subtracting the synchronization target system (7) from the IPSNS (6), give the synchronization error system (SES),

$$\begin{cases} \dot{\omega}_i(t) = \tilde{g}(\omega_i(t)) - m\sum_{j=1}^{N} \iota_{ij}\Pi \omega_j(t) + \mathcal{u}_i(t), & t \neq t_k \\ \Delta \omega_i(t) = \alpha_k \omega_i(t^-), & t = t_k, \end{cases} \tag{8}$$

where $\tilde{g}(\omega_i(t)) = g(x_i(t)) - g(z(t))$, $\omega_i(t) = (\omega_{i1}(t), \omega_{i2}(t), \cdots, \omega_{iN}(t))^T$, $i = 1, 2, \cdots, N$.

Definition 5. Given a prescribed time constant $\hat{T}$, the IPSNS (6) synchronizes with target system (7) within the prescribed time $\hat{T}$, if

$$\begin{cases}\lim_{t\to\hat{T}}\|x_i(t)-z(t)\|=0,\\ \|x_i(t)-z(t)\|\equiv 0, \qquad t>\hat{T},\end{cases}$$

where $i=1,2,\cdots,N$, namely

$$\begin{cases}\lim_{t\to\hat{T}}\|\omega_i(t)\|=0,\\ \|\omega_i(t)\|\equiv 0, \qquad t>\hat{T}.\end{cases}$$

This indicates that only if the IPSNS (6) and target system (7) synchronize within the prescribed time, the SES (8) will be prescribed-time stable.

We designed a feedback controller to attain the synchronization objective,

$$u_i(t)=-\rho_i\omega_i(t)-\vartheta_i sign(\omega_i(t))-\frac{\beta}{\hat{T}}\{sign(\omega_i(t))\circ[\omega_i(t)]^{2\delta-1}-sign(\omega_i(t))\circ[\omega_i(t)]\}, \quad (9)$$

where ρ_i, ϑ_i, $i=1,2,\cdots,N$ are positive constants to be chosen, δ, β and $\hat{T}$ are defined as in Theorem 1.

We incorporate a stacked vector to rewrite the SES (8):

$$\begin{cases}\dot{\omega}(t)=\Lambda(\omega(t))+\Sigma(\omega(t))+U(\omega(t)), & t\neq t_k\\ \Delta\omega_i(t)=\alpha_k\omega_i(t^-), & t=t_k,\end{cases} \quad (10)$$

where $\Sigma(\omega(t))=-m(\bar{L}\otimes\Pi)\omega(t)$, $\omega(t)=(\omega_1^T(t),\omega_2^T(t),\cdots,\omega_N^T(t))^T$, $\Lambda(\omega(t))=\left(\tilde{g}^T(\omega_1(t)),\tilde{g}^T(\omega_2(t)),\cdots,\tilde{g}^T(\omega_N(t))\right)^T$, $U(\omega(t))=(u_1^T(t),u_2^T(t),\cdots,u_N^T(t))^T$.

Two rules are proposed to simplify the computation of the Filippov set-valued function,

(i) If $g:\mathcal{R}^n\mapsto\mathcal{R}^n$ is continuous at $x\in\mathcal{R}$, then $\mathcal{G}[g](x)=g(x)$.

(ii) If $g_1,g_2:\mathcal{R}^n\mapsto\mathcal{R}^n$ are locally bounded at $x\in\mathcal{R}$, then $\mathcal{G}[g_1+g_2](x)\subseteq\mathcal{G}[g_1](x)+\mathcal{G}[g_2](x)$.

Assumption 1. For $\forall y_1,y_2\in\mathcal{R}^n$, there exist $C,D\in\mathcal{R}^{n\times n}$ and a piecewise-smooth function $g:\mathcal{R}^n\mapsto\mathcal{R}^n$ such that

$$(y_1-y_2)^T(\bar{g}(y_1)-\bar{g}(y_2))\leq(y_1-y_2)^TC(y_1-y_2)+(y_1-y_2)^TDsign(y_1-y_2),$$

holds, where $\bar{g}(y_i)\in\mathcal{G}[g](y_i)$, $i=1,2$.

In this article, assume that $x(t_k)=x(t_k^+)$. Next, we propose the synchronization assertion under controller (9).

Theorem 2. Under Assumption 1, the IPSNS (6) synchronizes with the target system (7) within the prescribed time under controller $u_i(t)$, provided there exist matrices $C,D\in\mathcal{R}^{N\times N}$ such that

(i) $I_N\otimes C^{\varsigma}-m\bar{L}^{\varsigma}\otimes\Pi-\rho\otimes I_N\leq 0$,

(ii) $I_N \otimes D - \vartheta \otimes I_N \leq 0$,

(iii) $max_{k\in\{1,\,2,\,\cdots,\,M_0'\}}\{t_k - t_{k-1}\} \leq \frac{\epsilon\hat{T}}{M_0'}$,

where $\rho = diag(\rho_1, \rho_2, \cdots, \rho_N)$, $\vartheta = diag(\vartheta_1, \vartheta_2, \cdots, \vartheta_N)$, $M_0' = min\left\{\varrho' \in \mathcal{N}_+ | \varrho' \geq \frac{-\ln\left(\varepsilon^2 \epsilon^{\frac{1}{1-\delta}}\right) - \frac{\epsilon}{(1-\delta)(1-\epsilon)^2}}{ln\,\gamma}\right\}$, ϵ, ε, and γ are defined in Theorem 1.

Proof. Consider

$$\mathcal{V}(\omega(t)) = \frac{1}{2}\omega^T(t)\omega(t) = \frac{1}{2}\sum_{i=1}^{N}\omega_i^T(t)\omega_i(t).$$

Letting $\lambda(\varepsilon) = \mu(\varepsilon) = \frac{1}{2}\varepsilon^2$, we can confirm that λ and μ meet condition (C_3) in Theorem 1. Now we will deduce the synchronization criteria from two scenarios: $t \neq t_k$ and $t = t_k$.

Case I: $t \neq t_k$. By Lemma 1, we can get

$$\dot{\mathcal{V}}(\omega(t)) = \omega^T(t)\dot{\omega}(t) \in \mathcal{L}_{\mathcal{G}[\Xi]}\mathcal{V}(\omega(t)),$$

where $\Xi = \Lambda(\omega(t)) + \Sigma(\omega(t)) + U(\omega(t))$. From the criteria for calculating the set-valued function $\mathcal{L}_{\mathcal{G}[\Xi]}\mathcal{V}(\omega(t)) \subseteq \mathcal{L}_{\mathcal{G}[\Lambda]+\mathcal{G}[\Sigma]+\mathcal{G}[U]}\mathcal{V}(\omega(t))$, and

$$\begin{aligned}
\mathcal{L}_{\mathcal{G}[\Lambda]+\mathcal{G}[\Sigma]+\mathcal{G}[U]}\mathcal{V}(\omega(t)) &= \omega^T(t)\{\mathcal{G}[\Lambda](\omega(t)) + \mathcal{G}[\Sigma](\omega(t)) + \mathcal{G}[U](\omega(t))\} \\
&= \sum_{i=1}^{N}\omega_i^T(t)\Bigg\{\mathcal{G}[\tilde{g}](\omega_i(t)) - m\sum_{j=1}^{N}\iota_{ij}\Pi\omega_j(t) - \rho_i\omega_i(t) - \vartheta_i\mathcal{G}[sign](\omega_i(t)) \\
&\quad - \frac{\beta}{\hat{T}}\{\mathcal{G}[sign](\omega_i(t)) \circ [\omega_i(t)]^{2\delta-1} - \mathcal{G}[sign](\omega_i(t)) \circ [\omega_i(t)]\}\Bigg\} \\
&= \sum_{i=1}^{N}\omega_i^T(t)\,\mathcal{G}[\tilde{g}](\omega_i(t)) - m\sum_{i=1}^{N}\sum_{j=1}^{N}\omega_i^T(t)\,\iota_{ij}\Pi\omega_j(t) - \sum_{i=1}^{N}\rho_i\omega_i^T(t)\,\omega_i(t) \\
&\quad - \sum_{i=1}^{N}\vartheta_i\omega_i^T(t)\mathcal{G}[sign](\omega_i(t)) \\
&\quad - \frac{\beta}{\hat{T}}\sum_{i=1}^{N}\omega_i^T(t)\{\mathcal{G}[sign](\omega_i(t)) \circ [\omega_i(t)]^{2\delta-1} - \mathcal{G}[sign](\omega_i(t)) \circ [\omega_i(t)]\},
\end{aligned}$$

where $\mathcal{G}[\tilde{g}](\omega_i(t)) = \mathcal{G}[g](x_i(t)) - \mathcal{G}[g](z(t))$, $\mathcal{G}[sign](\omega_i(t)) = \begin{cases} 1 & , \omega_i(t) > 0 \\ -1 & , \omega_i(t) < 0 \\ [-1,1], & \omega_i(t) = 0 \end{cases}$. From Assumption 1, we obtain that

$$\sum_{i=1}^{N} \omega_i^T(t)\, \mathcal{G}[\tilde{g}](\omega_i(t)) = \sum_{i=1}^{N} \omega_i^T(t) \{\mathcal{G}[g](x_i(t)) - \mathcal{G}[g](z(t))\}$$
$$\leq \sum_{i=1}^{N} \omega_i^T(t) C \omega_i(t) + \sum_{i=1}^{N} \omega_i^T(t) D \mathcal{G}[sign](\omega_i(t)).$$

Let

$$\Gamma_1 = \sum_{i=1}^{N} \omega_i^T(t) C \omega_i(t) - \sum_{i=1}^{N}\sum_{j=1}^{N} \omega_i^T(t) \iota_{ij} \Pi \omega_j(t) - \sum_{i=1}^{N} \rho_i \omega_i^T(t)\, \omega_i(t),$$

$$\Gamma_2 = \sum_{i=1}^{N} \omega_i^T(t) D \mathcal{G}[sign](\omega_i(t)) - \sum_{i=1}^{N} \vartheta_i \omega_i^T(t) \mathcal{G}[sign](\omega_i(t)),$$

$$\Gamma_3 = -\frac{\beta}{\hat{T}} \sum_{i=1}^{N} \omega_i^T(t) \{\mathcal{G}[sign](\omega_i(t)) \circ [\omega_i(t)]^{2\delta-1} - \mathcal{G}[sign](\omega_i(t)) \circ [\omega_i(t)]\},$$

Thus,

$$\mathcal{L}_{\mathcal{G}[\Lambda]+\mathcal{G}[\Sigma]+\mathcal{G}[U]} \mathcal{V}(\omega(t)) \leq \Gamma_1 + \Gamma_2 + \Gamma_3.$$

From $A^{\varsigma} = \frac{1}{2}(A + A^T)$, it follows that

$$\omega_i^T(t) C \omega_i(t) = \frac{1}{2}\{\omega_i^T(t) C \omega_i(t) + \omega_i^T(t) C^T \omega_i(t)\}$$
$$= \frac{1}{2}\{\omega_i^T(t)(C + C^T)\omega_i(t)\}$$
$$= \omega_i^T(t) \left\{\frac{1}{2}(C + C^T)\right\} \omega_i(t)$$
$$= \omega_i^T(t) C^{\varsigma} \omega_i(t).$$

Then, Γ_1, Γ_2, Γ_3 can be rewritten as

$$\Gamma_1 = \omega^T(t)(I_N \otimes C^{\varsigma} - m\bar{L}^{\varsigma} \otimes \Pi - \rho \otimes I_N)\omega(t)$$
$$\leq \lambda_{max}(I_N \otimes C^{\varsigma} - m\bar{L}^{\varsigma} \otimes \Pi - \rho \otimes I_N)\omega^T(t)\omega(t),$$

$$\Gamma_2 = \omega^T(t)(I_N \otimes D - \vartheta \otimes I_N)\mathcal{G}[sign](\omega(t))$$
$$\leq \lambda_{max}(I_N \otimes D - \vartheta \otimes I_N)\|\omega(t)\|_1,$$

$$\begin{aligned}\Gamma_3 &= -\frac{\beta}{\hat{T}}\left(\sum_{i=1}^{N}|\omega_i(t)|^{2\delta} - \sum_{i=1}^{N}|\omega_i(t)|^2\right)\\ &\leq -\frac{\beta}{\hat{T}}\left\{\left(\sum_{i=1}^{N}|\omega_i(t)|^2\right)^{\delta} - \sum_{i=1}^{N}|\omega_i(t)|^2\right\}\\ &= -\frac{\beta}{\hat{T}}\left\{2^{\delta}\mathcal{V}^{\delta}\big(\omega(t)\big) - 2\mathcal{V}\big(\omega(t)\big)\right\}.\end{aligned}$$

From $\dot{\mathcal{V}}\big(\omega(t)\big) \in \mathcal{L}_{\mathcal{G}[\Xi]}\mathcal{V}\big(\omega(t)\big) \subseteq \mathcal{L}_{\mathcal{G}[\Lambda]+\mathcal{G}[\Sigma]+\mathcal{G}[U]}\mathcal{V}\big(\omega(t)\big)$, we have

$$\begin{aligned}\dot{\mathcal{V}}\big(\omega(t)\big) &\leq \Gamma_1 + \Gamma_2 + \Gamma_3\\ &\leq \lambda_{max}(I_N \otimes C^{\varsigma} - m\bar{L}^{\varsigma} \otimes \Pi - \rho \otimes I_N)\omega^T(t)\omega(t) + \lambda_{max}(I_N \otimes D - \vartheta \otimes I_N)\|\omega(t)\|_1\\ &\quad -\frac{\beta}{\hat{T}}\left(2^{\delta}\mathcal{V}^{\delta}\big(\omega(t)\big) - 2\mathcal{V}\big(\omega(t)\big)\right).\end{aligned}$$

Based on conditions (i) and (ii), we can obtain that $\lambda_{max}(I_N \otimes C^{\varsigma} - m\bar{L}^{\varsigma} \otimes \Pi - \rho \otimes I_N) \leq 0$, $\lambda_{max}(I_N \otimes D - \vartheta \otimes I_N) \leq 0$. Therefore, the above equation can be reduced to

$$\dot{\mathcal{V}}\big(\omega(t)\big) \leq -\frac{\beta}{\hat{T}}\left(2^{\delta}\mathcal{V}^{\delta}\big(\omega(t)\big) - 2\mathcal{V}\big(\omega(t)\big)\right).$$

Case II: $t = t_k$. From $\omega(t_k) = (1 + \alpha_k)\omega(t_k^-)$, and $|1 + \alpha_k| \leq 1$, we have

$$\begin{aligned}\mathcal{V}\big(\omega(t_k)\big) &= \frac{1}{2}\omega^T(t_k)\omega(t_k)\\ &= \frac{1}{2}(1 + \alpha_k)^2\omega^T(t_k^-)\omega(t_k^-)\\ &\leq \frac{1}{2}\omega^T(t_k^-)\omega(t_k^-)\\ &= \mathcal{V}\big(\omega(t_k^-)\big).\end{aligned}$$

Consequently, when $t = t_k$, $\mathcal{V}\big(\omega(t_k)\big) \leq \mathcal{V}\big(\omega(t_k^-)\big)$.

Therefore, Theorem 2 is proven according to Theorem 1. The proof has been finished.

Remark 10. In contrast to [16], our assumption incorporates the sign function, enabling Theorem 2 to handle functions with discontinuities. Unlike [16], consider impulse effects and divide the proof of Theorem 2 into two distinct cases: $t \neq t_k$ and $t = t_k$.

Remark 11. Theorem 2 is based on Assumption 1. Different from the assumption in [26], our assumption does not include the matrix P.

5. Examples

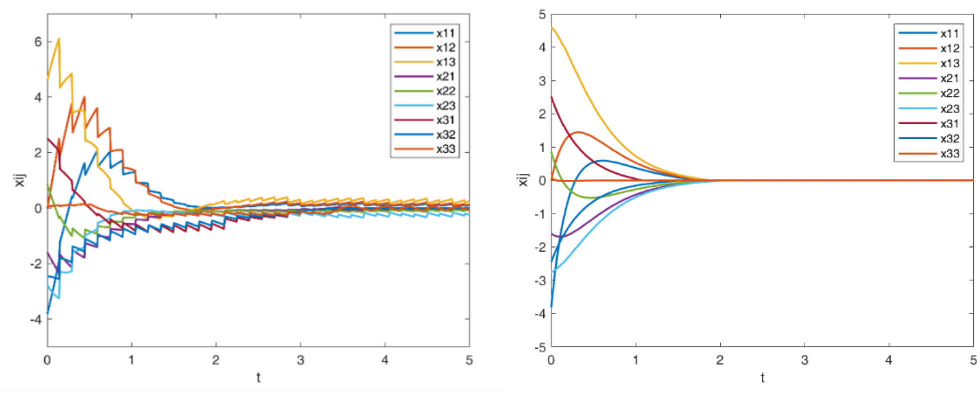
Consider the following three-dimensional chaotic Sprott circuit system,

$$\begin{cases} \dot{x}_i(t) = g(x_i(t)) - m\sum_{j=1}^{3} \iota_{ij}\Pi x_j(t) + u_i(t), t \neq t_k, \\ \Delta x_i(t) = \alpha_k x_i(t^-), \qquad\qquad\qquad\qquad t = t_k, \end{cases}$$

where $x_i(t) = \left(x_{i1}(t), x_{i2}(t), x_{i3}(t)\right)^T, i = 1, 2, 3,$

$$g(x_i(t)) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -0.5 \end{bmatrix} x_i(t) + \begin{bmatrix} 0 \\ 0 \\ sign(x_i(t)) \end{bmatrix}.$$

Let $\hat{T} = 3$, $m = 1$, $\bar{L} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$, $\Pi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\alpha_k = -0.3$. Let $\rho_1 = 3.27$, $\rho_2 = 2.64$, $\rho_3 = 2.74$, $\vartheta_1 = 0.03$, $\vartheta_2 = 0.01$, $\vartheta_3 = 0.04$, $p = 1.26$, $q = 2$, $\delta = 0.33$, $\epsilon = 0.3$, $t_k = 0.15k, k = 0, 1, 2, \cdots$. Let $C = \begin{bmatrix} -0.30 & 0.89 & -0.81 \\ 0.29 & -1.15 & -2.94 \\ -0.79 & -1.07 & 1.44 \end{bmatrix}$, $D = \begin{bmatrix} -0.83 & -0.53 & 0.52 \\ -0.98 & -2.00 & -0.02 \\ -1.16 & 0.96 & -0.03 \end{bmatrix}$, it is easy to verify that C and D satisfy the conditions of Theorem 2. Choose the initial value $x_1(0) = (-3.81, -0.02, 4.60)^T$, $x_2(0) = (-1.60, 0.85, -2.76)^T$, $x_3(0) = (2.51, -2.45, 0.06)^T$. It is clear that Theorem 2's conditions are satisfied. Figure 1(a),(b) shows the trajectories of IPSNS (6) without and with controller (9), respectively. Figure 2 shows the evolution of the synchronization error of the IPSNS under the controller (9), indicating that the SES (10) converges to 0 within $\hat{T} = 3$, i.e., the IPSNS (6) is synchronized with the target system (7) within $\hat{T} = 3$.



(a) (b)

Figure 1. (a) The evolution of the IPSNS (6) without controller (9). (b) The evolution of the IPSNS (6) with controller (9).

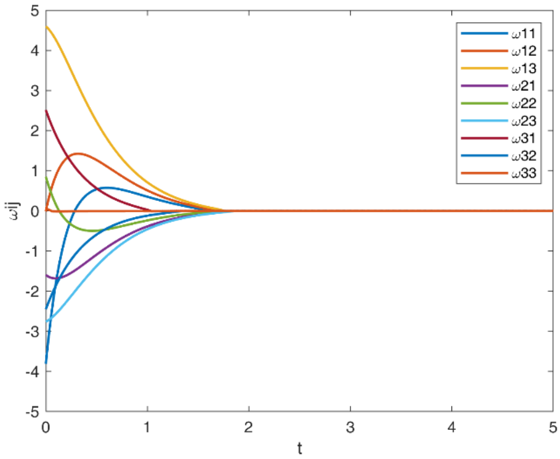
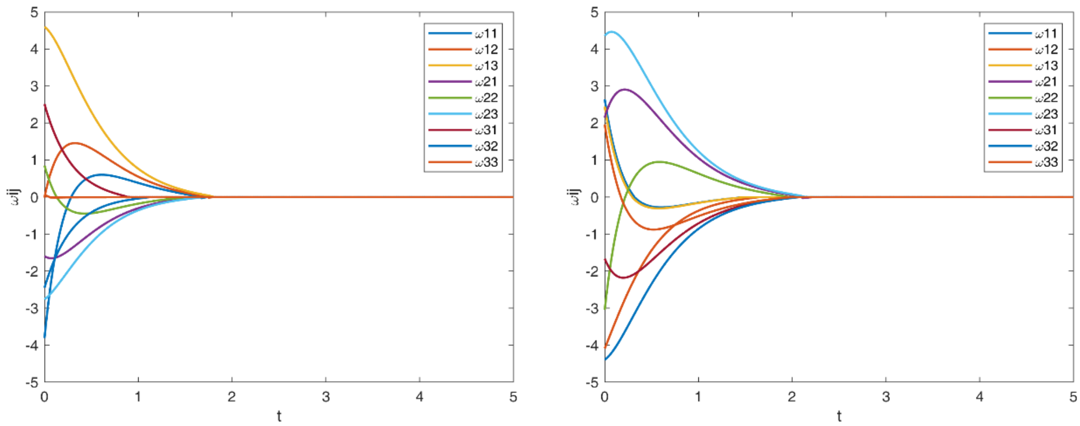


Figure 2. The evolution of the SES (10) under controller (9).

We select the same initial values as previously mentioned and modify the control parameters to see if the control parameters would affect the settling time: $\rho_1' = 3.27$, $\rho_2' = 2.64$, $\rho_3' = 2.74$, $\vartheta_1' = 0.03$, $\vartheta_2' = 0.01$, $\vartheta_3' = 0.04$. Figure 3(a) illustrates the results. We choose the same control parameters as previously mentioned and modify the initial values to see if the initial values would affect the settling time: $x_1'(0) = (2.63, -4.09, 2.44)^T$, $x_2'(0) = (2.14, -3.04, 4.35)^T$, $x_3'(0) = (-1.66, -4.39, 1.96)^T$. Figure 3(b) illustrates the results. Figure 3(a),(b) shows that the SES (10) converges to 0 within $\hat{T} = 3$, i.e., the IPSNS (6) and the target system (7) are synchronized within $\hat{T} = 3$, and the settling time is not affected by the control parameters and the initial values, although the control parameters and the initial values are changed.



(a) (b)

Figure 3. (a) The evolution of the SES (10) with different control parameters under controller (9). (b) The evolution of the SES (10) with different initial value under controller (9).

To test whether the settling time can be set freely, we modify only the settling time to $\hat{T} = 5$, keeping the previous parameter values unchanged. The results can be seen in Figure 4, which demonstrates that even with a change in the settling time, the IPSNS (6) and the target system (7) are synchronized within $\hat{T} = 5$. This implies that the settling time can be adjusted as needed.

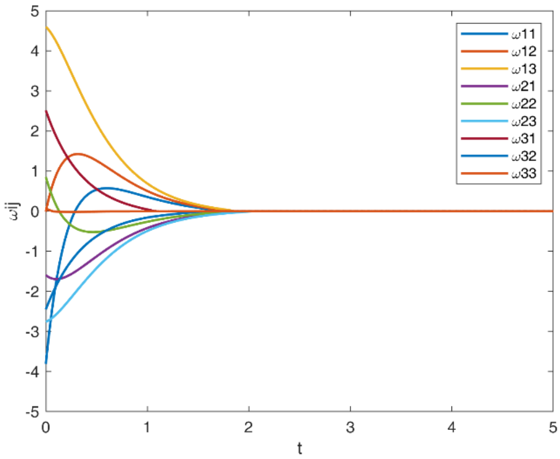


Figure 4. The evolution of the SES (10) with different prescribed time under controller (9).

6. Conclusions

In this paper, the PTSt of IPSDS is investigated, where the settling time can be set arbitrarily, independent of the initial values and control parameters of the system. Furthermore, a feedback controller is developed and the theoretical results are used to derive a new criterion for ensuring the synchronization of the IPSNS within a prescribed time.

In the future, we can research the PTSt of IPSDS with the factors of time-varying/higher-order/fractional order/stochasticity, and we can investigate the PTSt of IPSDS through the event-triggered mechanism. By studying the PTSt of IPSDS with these factors, we can realize more flexible and efficient time control in a wider range of practical applications. Compared with the traditional control methods, these methods can better adapt to dynamically changing environments and limited resource conditions and have a wide range of application prospects.

Author contributions

Chenchen Li: Investigation, Methodology, Writing, Validation; Chunyan Zhang: Investigation, Validation; Lichao Feng: Conceptualization, Investigation, Supervision, Funding acquisition; Zhihui Wu: Writing and Software.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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