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Research article

H^1 solutions for a modified Korteweg-de Vries-Burgers type equation

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Abstract: This paper modeled the dynamics of microbubbles coated with viscoelastic shells using the modified Korteweg-de Vries-Burgers equation, a nonlinear third-order partial differential equation. This study focused on the well-posedness of the Cauchy problem associated with this equation.

Keywords: existence; uniqueness; stability; a modified Korteweg-de Vries-Burgers type equation; cauchy problem

1. Introduction

The equation:

$$\begin{cases} \partial_t u + \partial_x f(u) - \beta^2 \partial_x^2 u + \delta \partial_x^3 u + \kappa u + \gamma^2 |u| u = 0, & 0 < t < T, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

was originally derived in [14, 17] with $f(u) = au^2$ focusing on microbubbles coated by viscoelastic shells. These structures are crucial in ultrasound diagnosis using contrast agents, and the dynamics of individual coated bubbles are explored, taking into account nonlinear competition and dissipation factors such as dispersion, thermal effects, and drag force.

The coefficients β^2, δ, κ, and γ^2 are related to the dissipation, the dispersion, the thermal conduction dissipation, and to the drag force, repsctively.

If $\kappa = \gamma = 0$, we obtain the Kudryashov-Sinelshchikov [18] Korteweg-de Vries-Burgers [3, 20] equation

$$\partial_t u + a\partial_x u^2 - \beta^2 \partial_x^2 u + \delta \partial_x^3 u = 0, \tag{1.2}$$

that models pressure waves in liquids with gas bubbles, taking into account heat transfer and viscosity. The mathematical results on Eq (1.2) are the following:

- analysis of exact solutions in [13],
- existence of the traveling waves in [2],
- well-posedness and asymptotic behavior in [7, 11].

If $\beta = 0$, we derive the Korteweg-de Vries equation:

$$\partial_t u + a\partial_x u^2 + \delta\partial_x^3 u = 0, \tag{1.3}$$

which describes surface waves of small amplitude and long wavelength in shallow water. Here, $u(t, x)$ represents the wave height above a flat bottom, x corresponds to the distance in the propagation direction, and t denotes the elapsed time. In [4,6,10,12,15,16], the completele integrability of Eq (1.3) and the existence of solitary wave solutions are proved.

Through the manuscript, we will assume

- on the coefficients
$$\beta,\ \delta,\ \kappa,\ \gamma \in \mathbb{R}, \quad \beta,\ \delta,\ \gamma \neq 0; \tag{1.4}$$
- on the flux f, one of the following conditions:
$$f(u) = au^2 + bu^3, \tag{1.5}$$
$$f \in C^1(\mathbb{R}), \quad |f'(u)| \leq C_0\,(1 + |u|)\,, \quad u \in \mathbb{R}, \tag{1.6}$$
  for some positive constant C_0;
- on the initial value
$$u_0 \in H^1(\mathbb{R}). \tag{1.7}$$

The main result of this paper is the following theorem.

Theorem 1.1. *Assume Eqs* (1.5)–(1.7). *For fixed $T > 0$, there exists a unique distributional solution u of Eq* (1.1)*, such that*

$$\begin{aligned} &u \in L^\infty(0, T; H^1(\mathbb{R})) \cap L^4(0,\ T; W^{1,4}(\mathbb{R})) \cap L^6(0,\ T; W^{1,6}(\mathbb{R}))\\ &\partial_x^2 u \in L^2((0,\ T) \times \mathbb{R}). \end{aligned} \tag{1.8}$$

Moreover, if u_1 and u_2 are solutions to Eq (1.1) *corresponding to the initial conditions $u_{1,0}$ and $u_{2,0}$, respectively, it holds that:*

$$\|u_1(t,\cdot) - u_2(t,\cdot)\|_{L^2(\mathbb{R})} \leq e^{C(T)t}\left\|u_{1,0} - u_{2,0}\right\|_{L^2(\mathbb{R})}, \tag{1.9}$$

for some suitable $C(T) > 0$, and every, $0 \leq t \leq T$.

Observe that Theorem 1.1 gives the well-posedness of (1.1), without conditions on the constants. Moreover, the proof of Theorem 1.1 is based on the Aubin-Lions Lemma [5, 21]. The analysis of Eq (1.1) is more delicate than the one of Eq (1.2) due to the presence of the nonlinear sources and the very general assumptions on the coefficients.

The structure of the paper is outlined as follows. Section 2 is dedicated to establishing several a priori estimates for a vanishing viscosity approximation of Eq (1.1). These estimates are crucial for proving our main result, which is presented in Section 3.

2. Vanishing viscosity approximation

To establish existence, we utilize a vanishing viscosity approximation of equation (1.1), as discussed in [19]. Let $0 < \varepsilon < 1$ be a small parameter, and denote by $u_\varepsilon \in C^\infty([0,T) \times \mathbb{R})$ the unique classical solution to the following problem [1,9]:

$$\begin{cases} \partial_t u_\varepsilon + \partial_x f(u_\varepsilon) - \beta^2 \partial_x^2 u_\varepsilon + \delta \partial_x^3 u_\varepsilon + \kappa u \\ \qquad +\gamma^2 |u| u = -\varepsilon \partial_x^4 u_\varepsilon, & 0 < t < T, \quad x \in \mathbb{R}, \\ u_\varepsilon(0,x) = u_{\varepsilon,0}(x), & x \in \mathbb{R}, \end{cases} \tag{2.1}$$

where $u_{\varepsilon,0}$ is a C^∞ approximation of u_0, such that

$$\left\| u_{\varepsilon,0} \right\|_{H^1(\mathbb{R})} \le \|u_0\|_{H^1(\mathbb{R})}. \tag{2.2}$$

Let us prove some a priori estimates on u_ε, denoting with C_0 constants which depend only on the initial data, and with $C(T)$ the constants which depend also on T.

We begin by proving the following lemma:

Lemma 2.1. *Let $T > 0$ be fixed. There exists a constant $C(T) > 0$, which does not depend on ε, such that*

$$\begin{aligned} \|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} &+ 2\gamma^2 e^{|\kappa| t} \int_0^t \int_{\mathbb{R}} e^{-|\kappa| s} u_\varepsilon^2 |u_\varepsilon| ds dx \\ &+ 2\beta^2 e^{|\kappa| t} \int_0^t e^{-|\kappa| s} \|\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})} ds \\ &+ 2\varepsilon e^{|\kappa| t} \int_0^t e^{-|\kappa| s} \left\| \partial_x^2 u_\varepsilon(s,\cdot) \right\|^2_{L^2(R)} \le C(T), \end{aligned} \tag{2.3}$$

for every $0 \le t \le T$.

Proof. For $0 \le t \le T$. Multiplying equations (2.1) by $2u_\varepsilon$, and integrating over $\mathbb{R}$ yields

$$\begin{aligned} \frac{d}{dt} \|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} &= 2 \int_{\mathbb{R}} u_\varepsilon \partial_t u_\varepsilon dx \\ &= \underbrace{-2 \int_{\mathbb{R}} u_\varepsilon f'(u_\varepsilon) \partial_x u_\varepsilon dx}_{=0} + 2\beta^2 \int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon dx - 2\delta \int_{\mathbb{R}} u_\varepsilon \partial_x^3 u_\varepsilon dx \\ &\quad - \kappa \|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} - 2\gamma^2 \int_{\mathbb{R}} |u_\varepsilon| u_\varepsilon^2 dx - 2\varepsilon \int_{\mathbb{R}} u_\varepsilon \partial_x^4 u_\varepsilon dx \\ &= -2\beta^2 \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\delta \int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx \\ &\quad - \kappa \|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} - 2\gamma^2 \int_{\mathbb{R}} |u_\varepsilon| u_\varepsilon^2 dx + 2\varepsilon \int_{\mathbb{R}} \partial_x u_\varepsilon \partial_x^3 u_\varepsilon dx \\ &= -2\beta^2 \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} - \kappa \|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \\ &\quad - 2\gamma^2 \int_{\mathbb{R}} |u_\varepsilon| u_\varepsilon^2 dx - 2\varepsilon \left\| \partial_x^2 u_\varepsilon(t,\cdot) \right\|^2_{L^2(\mathbb{R})}. \end{aligned}$$

Thus, it follows that

$$\begin{aligned}\frac{d}{dt}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} &+2\beta^2\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &+2\gamma^2\int_{\mathbb{R}}|u_\varepsilon|u_\varepsilon^2dx+2\varepsilon\left\|\partial_x^2u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &=\kappa\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\le|\kappa|\,\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Therefore, applying the Gronwall's lemma and using Eq (2.2), we obtain

$$\begin{aligned}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}&+2\beta^2e^{|\kappa|t}\int_0^t e^{-|\kappa|s}\|\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\\ &+2\gamma^2e^{|\kappa|t}\int_0^t\int_{\mathbb{R}}e^{-|\kappa|t}|u_\varepsilon|u_\varepsilon^2dsdx+2\varepsilon\left\|\partial_x^2u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\ &+2\varepsilon e^{|\kappa|t}\int_0^t e^{-|\kappa|s}\left\|\partial_x^2u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\ &\le C_0e^{|\kappa|t}\le C(T),\end{aligned}$$

which gives Eq (2.3). □

Lemma 2.2. *Fix $T>0$ and assume* (1.5). *There exists a constant $C(T)>0$, independent of ε, such that*

$$\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\le C(T), \tag{2.4}$$

$$\begin{aligned}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}&+\beta^2\int_0^t\left\|\partial_x^2u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\\ &+2\varepsilon\int_0^t\left\|\partial_x^3u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})}ds\le C(T),\end{aligned} \tag{2.5}$$

$$\int_0^t\|\partial_x u_\varepsilon(s,\cdot)\|^4_{L^4(\mathbb{R})}ds\le C(T), \tag{2.6}$$

holds for every $0\le t\le T$.

Proof. Let $0\le t\le T$. Consider A, B as two real constants, which will be specified later. Thanks to Eq (1.5), multiplying Eq (2.1) by

$$-2\partial_x^2u_\varepsilon+Au_\varepsilon^2+Bu_\varepsilon^3,$$

we have that

$$\begin{aligned}&\left(-2\partial_x^2u_\varepsilon+Au_\varepsilon^2+Bu_\varepsilon^3\right)\partial_tu_\varepsilon+2a\left(-2\partial_x^2u_\varepsilon+Au_\varepsilon^2+Bu_\varepsilon^3\right)u_\varepsilon\partial_xu_\varepsilon\\ &\quad+3b\left(-2\partial_x^2u_\varepsilon+Au_\varepsilon^2+Bu_\varepsilon^3\right)u_\varepsilon^2\partial_xu_\varepsilon-\beta^2\left(-2\partial_x^2u_\varepsilon+Au_\varepsilon^2+Bu_\varepsilon^3\right)\partial_x^2u_\varepsilon\\ &\quad+\delta\left(-2\partial_x^2u_\varepsilon+Au_\varepsilon^2+Bu_\varepsilon^3\right)\partial_x^3u_\varepsilon+\kappa\left(-2\partial_x^2u_\varepsilon+Au_\varepsilon^2+Bu_\varepsilon^3\right)u_\varepsilon\\ &\quad+\gamma^2\left(-2\partial_x^2u_\varepsilon+Au_\varepsilon^2+Bu_\varepsilon^3\right)|u_\varepsilon|u_\varepsilon=-\varepsilon\left(-2\partial_x^2u_\varepsilon+Au_\varepsilon^2+Bu_\varepsilon^3\right)\partial_x^4u_\varepsilon.\end{aligned} \tag{2.7}$$

Observe that

$$
\begin{aligned}
&\int_{\mathbb{R}} \left(-2\partial_x^2 u_\varepsilon + Au_\varepsilon^2 + Bu_\varepsilon^3\right)\partial_t u_\varepsilon dx\\
&\qquad = \frac{d}{dt}\left(\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{A}{3}\int_{\mathbb{R}} u_\varepsilon^3 dx + \frac{B}{4}\int_{\mathbb{R}} u_\varepsilon^4 dx\right),\\
&2a\int_{\mathbb{R}} \left(-2\partial_x^2 u_\varepsilon + Au_\varepsilon^2 + Bu_\varepsilon^3\right) u_\varepsilon \partial_x u_\varepsilon dx = -4a\int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx,\\
&3b\int_{\mathbb{R}} \left(-2\partial_x^2 u_\varepsilon + Au_\varepsilon^2 + Bu_\varepsilon^3\right) u_\varepsilon^2 \partial_x u_\varepsilon dx = -6b\int_{\mathbb{R}} u_\varepsilon^2 \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx,\\
&-\beta^2\int_{\mathbb{R}} \left(-2\partial_x^2 u_\varepsilon + Au_\varepsilon^2 + Bu_\varepsilon^3\right)\partial_x^2 u_\varepsilon dx\\
&\qquad = 2\beta^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2A\beta^2\int_{\mathbb{R}} u_\varepsilon(\partial_x u_\varepsilon)^2 dx + 3B\beta^2\int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^2 dx,\\
&\delta\int_{\mathbb{R}} \left(-2\partial_x^2 u_\varepsilon + Au_\varepsilon^2 + Bu_\varepsilon^3\right)\partial_x^3 u_\varepsilon dx = -2A\delta\int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx - 3B\delta\int_{\mathbb{R}} u_\varepsilon^2 \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx,\\
&\kappa\int_{\mathbb{R}} \left(-2\partial_x^2 u_\varepsilon + Au_\varepsilon^2 + Bu_\varepsilon^3\right) u_\varepsilon dx\\
&\qquad = 2\kappa\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + A\kappa\int_{\mathbb{R}} u_\varepsilon^3 dx + B\kappa\int_{\mathbb{R}} u_\varepsilon^4 dx,\\
&\gamma^2\int_{\mathbb{R}} \left(-2\partial_x^2 u_\varepsilon + Au_\varepsilon^2 + Bu_\varepsilon^3\right)|u_\varepsilon| u_\varepsilon dx\\
&\qquad = -2\gamma^2\int_{\mathbb{R}} |u_\varepsilon| u_\varepsilon \partial_x^2 u_\varepsilon dx + A\gamma^2\int_{\mathbb{R}} |u| u_\varepsilon^3 dx + B\gamma^2\int_{\mathbb{R}} |u_\varepsilon| u^4 dx,\\
&-\varepsilon\int_{\mathbb{R}} \left(-2\partial_x^2 u_\varepsilon + Au_\varepsilon^2 + Bu_\varepsilon^3\right)\partial_x^4 u_\varepsilon dx\\
&\qquad = -2\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2A\varepsilon\int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^3 u_\varepsilon dx + 3B\varepsilon\int_{\mathbb{R}} u_\varepsilon^2 \partial_x u_\varepsilon \partial_x^3 u_\varepsilon dx\\
&\qquad = -2\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - A\varepsilon\int_{\mathbb{R}} (\partial_x u_\varepsilon)^3 dx - 6B\varepsilon\int_{\mathbb{R}} u_\varepsilon(\partial_x u_\varepsilon)^2\partial_x^2 u_\varepsilon dx\\
&\qquad\quad - 3B\varepsilon\left\|u_\varepsilon(t,\cdot)\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad = -2\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - A\varepsilon\int_{\mathbb{R}} (\partial_x u_\varepsilon)^3 dx + 2B\varepsilon\int_{\mathbb{R}} (\partial_x u_\varepsilon)^4 dx\\
&\qquad\quad - 3B\varepsilon\left\|u_\varepsilon(t,\cdot)\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Therefore, an integration on $\mathbb{R}$ gives

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{A}{3}\int_{\mathbb{R}} u_\varepsilon^3 dx + \frac{B}{4}\int_{\mathbb{R}} u_\varepsilon^4 dx\right)\\
&\qquad + \beta^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\qquad = -(4a + A\delta)\int_{\mathbb{R}} u_\varepsilon \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx - 3(2b + B\delta)\int_{\mathbb{R}} u_\varepsilon^2 \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx
\end{aligned}
$$

$$
\begin{aligned}
&- 2A\beta^2 \int_{\mathbb{R}} u_\varepsilon(\partial_x u_\varepsilon)^2 dx - 3B\beta^2 \int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^2 dx \\
&- \kappa \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} - \frac{A\kappa}{3}\int_{\mathbb{R}} u_\varepsilon^3 dx - \frac{B\kappa}{4}\int_{\mathbb{R}} u_\varepsilon^4 dx \\
&+ 2\gamma^2 \int_{\mathbb{R}} |u_\varepsilon| u_\varepsilon \partial_x^2 u_\varepsilon dx - A\gamma^2 \int_{\mathbb{R}} |u_\varepsilon| u_\varepsilon^3 dx - B\gamma^2 \int_{\mathbb{R}} |u_\varepsilon| u_\varepsilon^4 dx \\
&- A\varepsilon \int_{\mathbb{R}} (\partial_x u_\varepsilon)^3 dx + 2B\varepsilon \int_{\mathbb{R}} (\partial_x u_\varepsilon)^4 dx - 3B\varepsilon \left\| u_\varepsilon(t,\cdot)\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Taking

$$
(A,\ B) = \left(-\frac{4a}{\delta},\ -\frac{2b}{\delta}\right),
$$

we get

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} - \frac{4a}{3\delta}\int_{\mathbb{R}} u_\varepsilon^3 dx - \frac{b}{\delta}\int_{\mathbb{R}} u_\varepsilon^4 dx\right) \\
&\quad + 2\beta^2 \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon \left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&= \frac{8a\beta^2}{\delta}\int_{\mathbb{R}} u_\varepsilon(\partial_x u_\varepsilon)^2 dx + \frac{6b\beta^2}{\delta}\int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^2 dx \\
&\quad - \kappa \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{4a\kappa}{3\delta}\int_{\mathbb{R}} u_\varepsilon^3 dx + \frac{b\kappa}{2}\int_{\mathbb{R}} u_\varepsilon^4 dx \\
&\quad + 2\gamma^2 \int_{\mathbb{R}} |u_\varepsilon| u_\varepsilon \partial_x^2 u_\varepsilon dx + \frac{4a\gamma^2}{\delta}\int_{\mathbb{R}} |u_\varepsilon| u_\varepsilon^3 dx + \frac{2b\gamma^2}{\delta}\int_{\mathbb{R}} |u_\varepsilon| u_\varepsilon^4 dx \\
&\quad + \frac{4a\varepsilon}{\delta}\int_{\mathbb{R}} (\partial_x u_\varepsilon)^3 dx - \frac{4b\varepsilon}{\delta}\int_{\mathbb{R}} (\partial_x u_\varepsilon)^4 dx + \frac{6b\varepsilon}{\delta}\left\| u_\varepsilon(t,\cdot)\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
\tag{2.8}
$$

Since $0 < \varepsilon < 1$, due to the Young inequality and (2.3),

$$
\begin{aligned}
&\frac{8a\beta^2}{\delta}\int_{\mathbb{R}} |u_\varepsilon|(\partial_x u_\varepsilon)^2 dx \\
&\qquad \le 4\int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^2 dx + \frac{4a^2\beta^4}{\delta^2}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\qquad \le 4\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{4a^2\beta^4}{\delta^2}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\qquad \le C_0\left(1 + \|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}, \\
&\left|\frac{6b\beta^2}{\delta}\right|\int_{\mathbb{R}} u_\varepsilon^2(\partial_x u_\varepsilon)^2 dx \le \left|\frac{6b\beta^2}{\delta}\right|\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}, \\
&\left|\frac{4a\kappa}{3\delta}\right|\int_{\mathbb{R}} |u_\varepsilon|^3 dx \le \left|\frac{4a\kappa}{3\delta}\right|\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\qquad \le C(T)\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}, \\
&\left|\frac{b\kappa}{2}\right|\int_{\mathbb{R}} u_\varepsilon^4 dx \le \left|\frac{b\kappa}{2}\right|\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}
\end{aligned}
$$

$$
\begin{aligned}
&\le C(T)\,\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})},\\
2\gamma^2\int_{\mathbb{R}}|u_\varepsilon|u_\varepsilon\partial_x^2u_\varepsilon dx&\le 2\int_{\mathbb{R}}\left|\frac{\gamma^2|u_\varepsilon|u_\varepsilon}{\beta}\right|\left|\beta\partial_x^2u_\varepsilon\right|dx\\
&\le\frac{\gamma^4}{\beta^2}\int_{\mathbb{R}}u_\varepsilon{}^4dx+\beta^2\left\|\partial_x^2u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le\frac{\gamma^4}{\beta^2}\|u_\varepsilon\|^2_{L^\infty((0.T)\times\mathbb{R})}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta^2\left\|\partial_x^2u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le C(T)\,\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}+\beta^2\left\|\partial_x^2u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},\\
\left|\frac{4a\gamma^2}{\delta}\right|\int_{\mathbb{R}}|u_\varepsilon||u_\varepsilon|^3dx&=\left|\frac{4a\gamma^2}{\delta}\right|\int_{\mathbb{R}}u_\varepsilon^4dx\\
&\le\left|\frac{4a\gamma^2}{\delta}\right|\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\le C(T)\,\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})},\\
\left|\frac{2b\gamma^2}{\delta}\right|\int_{\mathbb{R}}|u_\varepsilon|u_\varepsilon{}^4dx&\le\left|\frac{2b\gamma^2}{\delta}\right|\|u_\varepsilon\|^3_{L^\infty((0,T)\times\mathbb{R})}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\le C(T)\,\|u_\varepsilon\|^3_{L^\infty((0,T)\times\mathbb{R})},\\
\left|\frac{4a\varepsilon}{\delta}\right|\int_{\mathbb{R}}|\partial_xu_\varepsilon|^3dx&\le\left|\frac{4a\varepsilon}{\delta}\right|\|\partial_xu_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\left|\frac{4a\varepsilon}{\delta}\right|\int_{\mathbb{R}}(\partial_xu_\varepsilon)^4dx\\
&\le\left|\frac{4a}{\delta}\right|\|\partial_xu_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\left|\frac{4a\varepsilon}{\delta}\right|\int_{\mathbb{R}}(\partial_xu_\varepsilon)^4dx.
\end{aligned}
$$

It follows from Eq (2.8) that

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\partial_xu_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}-\frac{4a}{3\delta}\int_{\mathbb{R}}u_\varepsilon^3dx-\frac{b}{\delta}\int_{\mathbb{R}}u_\varepsilon^4dx\right)\\
&\quad+\beta^2\left\|\partial_x^2u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}+2\varepsilon\left\|\partial_x^3u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\le C_0\left(1+\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)\|\partial_xu_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+C(T)\,\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\\
&\quad+C(T)\,\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}+C(T)\,\|u_\varepsilon\|^3_{L^\infty((0,T)\times\mathbb{R})}\\
&\quad+C_0\varepsilon\int_{\mathbb{R}}(\partial_xu_\varepsilon)^4dx+C_0\varepsilon\left\|u_\varepsilon(t,\cdot)\partial_x^2u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
&\quad+C_0\,\|\partial_xu_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned}\tag{2.9}
$$

[8, Lemma 2.3] says that

$$
\int_{\mathbb{R}}(\partial_xu_\varepsilon)^4dx\le 9\int_{\mathbb{R}}u_\varepsilon^2(\partial_x^2u_\varepsilon)^2dx\le 9\,\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\tag{2.10}
$$

Moreover, we have that

$$
\left\|u_\varepsilon(t,\cdot)\partial_x^2u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}=\int_{\mathbb{R}}u_\varepsilon^2(\partial_x^2u_\varepsilon)^2dx\le\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\tag{2.11}
$$

Consequentially, by Eqs (2.9)–(2.11), we have that

$$
\frac{d}{dt}\left(\|\partial_xu_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}-\frac{4a}{3\delta}\int_{\mathbb{R}}u_\varepsilon^3dx-\frac{b}{\delta}\int_{\mathbb{R}}u_\varepsilon^4dx\right)
$$

$$
\begin{aligned}
&+\beta^2 \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon \left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}\\
\le\, & C_0\left(1+\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + C(T)\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\\
&+ C(T)\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})} + C(T)\|u_\varepsilon\|^3_{L^\infty((0,T)\times\mathbb{R})}\\
&+ C_0\varepsilon\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + C_0\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

An integration on $(0,t)$ and Eqs (2.2) and (2.3) give

$$
\begin{aligned}
&\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} - \frac{4a}{3\delta}\int_{\mathbb{R}} u_\varepsilon^3 dx - \frac{b}{\delta}\int_{\mathbb{R}} u_\varepsilon^4 dx\\
&\quad + \beta^2\int_0^t \left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds + 2\varepsilon\int_0^t \left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds\\
&\le C_0\left(1+\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\right)\int_0^t \|\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})} ds\\
&\quad + C(T)\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} t + C(T)\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})} t + C(T)\|u_\varepsilon\|^3_{L^\infty((0,T)\times\mathbb{R})} t\\
&\quad + C_0\varepsilon\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_0^t \left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds + C_0\int_0^t \|\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})} ds\\
&\le C(T)\left(1+\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} + \|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})} + \|u_\varepsilon\|^3_{L^\infty((0,T)\times\mathbb{R})}\right).
\end{aligned}
$$

Therefore, by Eq (2.3),

$$
\begin{aligned}
&\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\int_0^t \left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds + 2\varepsilon\int_0^t \left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds\\
&\le C(T)\left(1+\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} + \|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})} + \|u_\varepsilon\|^3_{L^\infty((0,T)\times\mathbb{R})}\right)\\
&\quad + \frac{4a}{3\delta}\int_{\mathbb{R}} u_\varepsilon^3 dx + \frac{b}{\delta}\int_{\mathbb{R}} u_\varepsilon^4 dx\\
&\le C(T)\left(1+\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} + \|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})} + \|u_\varepsilon\|^3_{L^\infty((0,T)\times\mathbb{R})}\right)\\
&\quad + \left|\frac{4a}{3\delta}\right|\int_{\mathbb{R}} |u_\varepsilon|^3 dx + \left|\frac{b}{\delta}\right|\int_{\mathbb{R}} u_\varepsilon^4 dx\\
&\le C(T)\left(1+\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} + \|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})} + \|u_\varepsilon\|^3_{L^\infty((0,T)\times\mathbb{R})}\right)\\
&\quad + \left|\frac{4a}{3\delta}\right|\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \left|\frac{b}{\delta}\right|\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\le C(T)\left(1+\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} + \|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})} + \|u_\varepsilon\|^3_{L^\infty((0,T)\times\mathbb{R})}\right).
\end{aligned}
\tag{2.12}
$$

We prove Eq (2.4). Thanks to the Hölder inequality,

$$
u_\varepsilon^2(t,x) = 2\int_{-\infty}^x u_\varepsilon\partial_x u_\varepsilon dx \le 2\int_{\mathbb{R}} |u_\varepsilon||\partial_x u_\varepsilon| dx \le 2\|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}.
$$

Hence, we have that

$$
\|u_\varepsilon(t,\cdot)\|^4_{L^\infty(\mathbb{R})} \le 4\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}. \tag{2.13}
$$

Thanks to Eqs (2.3) and (2.12), we have that

$$
\begin{aligned}
&\|u_\varepsilon\|^4_{L^\infty((0,T)\times\mathbb{R})}\\
&\quad \le C(T)\left(1+\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} + \|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})} + \|u_\varepsilon\|^3_{L^\infty((0,T)\times\mathbb{R})}\right).
\end{aligned}
\tag{2.14}
$$

Due to the Young inequality,

$$\begin{aligned} C(T)\|u_\varepsilon\|^3_{L^\infty((0,T)\times\mathbb{R})} \leq& \frac{1}{2}\|u_\varepsilon\|^4_{L^\infty((0,T)\times\mathbb{R})} + C(T)\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}, \\ C(T)\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} \leq& C(T)\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})} + C(T). \end{aligned}$$

By Eq (2.14), we have that

$$\frac{1}{2}\|u_\varepsilon\|^4_{L^\infty((0,T)\times\mathbb{R})} - C(T)\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})} - C(T) \leq 0,$$

which gives Eq (2.4).

Equation (2.5) follows from Eqs (2.4) and (2.12).

Finally, we prove Eq (2.6). We begin by observing that, from Eqs (2.4) and (2.10), we have

$$\|\partial_x u_\varepsilon(t,\cdot)\|^4_{L^4(\mathbb{R})} \leq C(T)\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

An integration on $(0,t)$ and Eqs (2.5) give Eq (2.6). □

Lemma 2.3. *Fix $T > 0$ and assume* (1.6). *There exists a constant $C(T) > 0$, independent of ε, such that Eq* (2.4) *holds. Moreover, we have Eqs* (2.5) *and* (2.6).

Proof. Let $0 \leq t \leq T$. Multiplying Eq (2.1) by $-2\partial_x^2 u_\varepsilon$, an integration on $\mathbb{R}$ gives

$$\begin{aligned} \frac{d}{dt}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} =& -2\int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_t u_\varepsilon dx \\ =& -2\int_{\mathbb{R}} f'(u_\varepsilon)\partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx - 2\beta^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} - 2\delta\int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^3 u_\varepsilon dx \\ & - 2\kappa\int_{\mathbb{R}} u_\varepsilon \partial_x^2 u_\varepsilon dx - 2\gamma^2\int_{\mathbb{R}} |u_\varepsilon| u_\varepsilon \partial_x^2 u_\varepsilon dx + 2\varepsilon\int_{\mathbb{R}} \partial_x^2 u_\varepsilon \partial_x^4 u_\varepsilon dx \\ =& -2\int_{\mathbb{R}} f'(u_\varepsilon)\partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx - 2\beta^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\ & + 2\kappa\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\gamma^2\int_{\mathbb{R}} |u_\varepsilon| u_\varepsilon \partial_x^2 u_\varepsilon dx - 2\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}. \end{aligned}$$

Therefore, we have that

$$\begin{aligned} \frac{d}{dt}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^2\left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon\left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\ = -2\int_{\mathbb{R}} f'(u_\varepsilon)\partial_x u_\varepsilon \partial_x^2 u_\varepsilon dx + 2\kappa\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\gamma^2\int_{\mathbb{R}} |u_\varepsilon| u_\varepsilon \partial_x^2 u_\varepsilon dx. \end{aligned} \tag{2.15}$$

Due Eqs (1.6) and (2.3) and the Young inequality,

$$\begin{aligned} 2\int_{\mathbb{R}} |f'(u_\varepsilon)||\partial_x u_\varepsilon||\partial_x^2 u_\varepsilon| dx \leq& C_0\int_{\mathbb{R}} |\partial_x u_\varepsilon \partial_x^2 u_\varepsilon| dx + C_0\int_{\mathbb{R}} |u_\varepsilon \partial_x u_\varepsilon||\partial_x^2 u_\varepsilon| dx \\ =& 2\int_{\mathbb{R}} \left|\frac{C_0\sqrt{3}\partial_x u_\varepsilon}{2\beta}\right|\left|\frac{\beta\partial_x^2 u_\varepsilon}{\sqrt{3}}\right| dx + 2\int_{\mathbb{R}} \left|\frac{C_0\sqrt{3}u_\varepsilon \partial_x u_\varepsilon}{2\beta}\right|\left|\sqrt{3}\beta\partial_x^2 u_\varepsilon\right| dx \end{aligned}$$

$$
\begin{aligned}
&\le C_0 \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + C_0 \int_{\mathbb{R}} u_\varepsilon^2 (\partial_x u_\varepsilon)^2 dx + \frac{2\beta^2}{3} \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\le C_0 \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + C_0 \|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})} \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{2\beta^2}{3} \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\le C_0 \left(1 + \|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\right) \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{2\beta^2}{3} \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})},
\end{aligned}
$$

$$
\begin{aligned}
2\gamma^2 \int_{\mathbb{R}} |u_\varepsilon| u_\varepsilon \partial_x^2 u_\varepsilon dx &\le 2\gamma^2 \int_{\mathbb{R}} u_\varepsilon^2 |\partial_x^2 u_\varepsilon| dx = 2 \int_{\mathbb{R}} \left| \frac{\sqrt{3}\gamma^2 u_\varepsilon^2}{\beta} \right| \left| \frac{\beta \partial_x^2 u_\varepsilon}{\sqrt{3}} \right| dx \\
&\le \frac{3\gamma^4}{\beta^2} \int_{\mathbb{R}} u_\varepsilon^4 dx + \frac{\beta^2}{3} \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\le \frac{3\gamma^4}{\beta^2} \|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})} \|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{3} \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\le C(T) \|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})} + \frac{\beta^2}{3} \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

It follows from Eq (2.15) that

$$
\begin{aligned}
&\frac{d}{dt} \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2 \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} + 2\varepsilon \left\|\partial_x^3 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\qquad \le C_0 \left(1 + \|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\right) \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + C(T) \|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}.
\end{aligned}
$$

Integrating on $(0, t)$, by Eq (2.3), we have that

$$
\begin{aligned}
&\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2 \int_0^t \left\|\partial_x^2 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} ds + 2\varepsilon \int_0^t \left\|\partial_x^3 u_\varepsilon(s,\cdot)\right\|^2_{L^2(\mathbb{R})} \\
&\quad \le C_0 + C_0 \left(1 + \|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\right) \int_0^t \|\partial_x u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})} ds + C(T) \|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})} t \qquad (2.16) \\
&\quad \le C(T) \left(1 + \|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\right).
\end{aligned}
$$

Thanks to Eqs (2.3), (2.13), and (2.16), we have that

$$
\|u_\varepsilon\|^4_{L^\infty((0,T)\times\mathbb{R})} \le C(T) \left(1 + \|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\right).
$$

Therefore,

$$
\|u_\varepsilon\|^4_{L^\infty((0,T)\times\mathbb{R})} - C(T) \|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})} - C(T) \le 0,
$$

which gives (2.4).

Equation (2.5) follows from (2.4) and (2.16), while, arguing as in Lemma 2.2, we have Eq (2.6). □

Lemma 2.4. *Fix $T > 0$. There exists a constant $C(T) > 0$, independent of ε, such that*

$$
\int_0^t \|\partial_x u_\varepsilon(s,\cdot)\|^6_{L^6(\mathbb{R})} ds \le C(T), \qquad (2.17)
$$

for every $0 \le t \le T$.

Proof. Let $0 \leq t \leq T$. We begin by observing that,

$$\int_{\mathbb{R}} (\partial_x u_\varepsilon)^6 dx \leq \|\partial_x u_\varepsilon(t,\cdot)\|^4_{L^\infty(\mathbb{R})} \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}. \tag{2.18}$$

Thanks to the Hölder inequality,

$$\begin{aligned}(\partial_x u_\varepsilon(t,x))^2 =& 2\int_{-\infty}^{x} \partial_x u_\varepsilon \partial_x^2 u_\varepsilon dy \leq 2\int_{\mathbb{R}} |\partial_x u_\varepsilon||\partial_x^2 u_\varepsilon| dx \\ \leq& 2\,\|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Hence,

$$\|u(t,\cdot)\|^4_{L^\infty(\mathbb{R})} \leq 4\,\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

It follows from Eq (2.18) that

$$\int_{\mathbb{R}} (\partial_x u_\varepsilon)^6 dx \leq 4\,\|\partial_x u_\varepsilon(t,\cdot)\|^4_{L^2(\mathbb{R})} \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

Therefore, by Eq (2.5),

$$\int_{\mathbb{R}} (\partial_x u_\varepsilon)^6 dx \leq C(T) \left\|\partial_x^2 u_\varepsilon(t,\cdot)\right\|^2_{L^2(\mathbb{R})}.$$

An integration on $(0,t)$ and Eq (2.5) gives (2.17). □

3. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1.

We begin by proving the following result.

Lemma 3.1. *Fix $T > 0$. Then,*

$$\textit{the family } \{u_\varepsilon\}_{\varepsilon>0} \textit{ is compact in } L^2_{loc}((0,T)\times\mathbb{R}). \tag{3.1}$$

Consequently, there exist a subsequence $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ and $u \in L^2_{loc}((0,T)\times\mathbb{R})$ such that

$$u_{\varepsilon_k} \to u \textit{ in } L^2_{loc}((0,T)\times\mathbb{R}) \textit{ and a.e. in } (0,T)\times\mathbb{R}. \tag{3.2}$$

Moreover, u is a solution of Eq (1.1)*, satisfying Eq* (1.8)*.*

Proof. We begin by proving Eq (3.1). To prove Eq (3.1), we rely on the Aubin-Lions Lemma (see [5,21]). We recall that

$$H^1_{loc}(\mathbb{R}) \hookrightarrow\hookrightarrow L^2_{loc}(\mathbb{R}) \hookrightarrow H^{-1}_{loc}(\mathbb{R}),$$

where the first inclusion is compact and the second one is continuous. Owing to the Aubin-Lions Lemma [21], to prove Eq (3.1), it suffices to show that

$$\{u_\varepsilon\}_{\varepsilon>0} \text{ is uniformly bounded in } L^2(0,T;H^1_{loc}(\mathbb{R})), \tag{3.3}$$

$$\{\partial_t u_\varepsilon\}_{\varepsilon>0} \text{ is uniformly bounded in } L^2(0,T;H^{-1}_{loc}(\mathbb{R})). \tag{3.4}$$

We prove Eq (3.3). Thanks to Lemmas 2.1–2.3,

$$\|u_\varepsilon(t,\cdot)\|^2_{H^1(\mathbb{R})} = \|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \le C(T).$$

Therefore,

$$\{u_\varepsilon\}_{\varepsilon>0} \text{ is uniformly bounded in } L^\infty(0,T;H^1(\mathbb{R})),$$

which gives Eq (3.3).

We prove Eq (3.4). Observe that, by Eq (2.1),

$$\partial_t u_\varepsilon = -\partial_x\left(G(u_\varepsilon)\right) - f'(u_\varepsilon)\partial_x u_\varepsilon - \kappa u_\varepsilon - \gamma^2|u_\varepsilon|u_\varepsilon,$$

where

$$G(u_\varepsilon) = \beta^2\partial_x u_\varepsilon - \delta\partial_x^2 u_\varepsilon - \varepsilon\partial_x^3 u_\varepsilon. \tag{3.5}$$

Since $0<\varepsilon<1$, thanks to Eq (2.5), we have that

$$\begin{aligned} &\beta^2\|\partial_x u_\varepsilon\|^2_{L^2((0,T)\times\mathbb{R})},\ \delta^2\left\|\partial_x^2 u_\varepsilon\right\|^2_{L^2((0,T)\times\mathbb{R})} \le C(T),\\ &\varepsilon^2\left\|\partial_x^3 u_\varepsilon\right\|^2_{L^2((0,T)\times\mathbb{R})} \le C(T). \end{aligned} \tag{3.6}$$

Therefore, by Eqs (3.5) and (3.6), we have that

$$\{\partial_x\left(G(u_\varepsilon)\right)\}_{\varepsilon>0} \quad \text{is bounded in } L^2(0,T;H^{-1}(\mathbb{R})). \tag{3.7}$$

We claim that

$$\int_0^T\int_{\mathbb{R}}(f'(u_\varepsilon))^2(\partial_x u_\varepsilon)^2 dtdx \le C(T). \tag{3.8}$$

Thanks to Eqs (2.4) and (2.5),

$$\int_0^T\int_{\mathbb{R}}(f'(u_\varepsilon))^2(\partial_x u_\varepsilon)^2 dtdx \le \|f'\|^2_{L^\infty(-C(T),C(T))}\int_0^T \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}dt \le C(T).$$

Moreover, thanks to Eq (2.3),

$$|\kappa|\int_0^T\int_{\mathbb{R}}(u_\varepsilon)^2 dx \le C(T). \tag{3.9}$$

We have that

$$\gamma^2\int_0^T\int_{\mathbb{R}}(|u_\varepsilon|u_\varepsilon)^2 dsdx \le C(T). \tag{3.10}$$

In fact, thanks to Eqs (2.3) and (2.4),

$$\begin{aligned}\gamma^2\int_0^T\int_{\mathbb{R}}(|u_\varepsilon|u_\varepsilon)^2 dsdx \le&\gamma^2\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})}\int_0^T\int_{\mathbb{R}}(u_\varepsilon)^2 dsdx\\ \le&C(T)\int_0^T\int_{\mathbb{R}}(u_\varepsilon)^2 dsdx \le C(T).\end{aligned}$$

Therefore, Eq (3.4) follows from Eqs (3.7)–(3.10).

Thanks to the Aubin-Lions Lemma, Eqs (3.1) and (3.2) hold.

Consequently, arguing as in [5, Theorem 1.1], u is solution of Eq (1.1) and, thanks to Lemmas 2.1–2.3 and Eqs (2.4), (1.8) holds. □

Proof of Theorem 1.1. Lemma 3.1 gives the existence of a solution of Eq (1.1).

We prove Eq (1.9). Let u_1 and u_2 be two solutions of Eq (1.1), which verify Eq (1.8), that is,

$$\begin{cases} \partial_t u_i + \partial_x f(u_i) - \beta^2 \partial_x^2 u_i + \delta \partial_x^3 u_i + \kappa u_i + \gamma^2 |u_i| u_i = 0, & 0 < t < T, \quad x \in \mathbb{R}, \\ u_i(0, x) = u_{i,0}(x), & x \in \mathbb{R}, \end{cases} \quad i = 1, 2.$$

Then, the function

$$\omega(t, x) = u_1(t, x) - u_2(t, x), \tag{3.11}$$

is the solution of the following Cauchy problem:

$$\begin{cases} \partial_t \omega + \partial_x \left(f(u_1) - f(u_2) \right) - \beta^2 \partial_x^2 \omega + \delta \partial_x^2 \omega \\ \qquad + \kappa \omega + \gamma^2 \left(|u_1| u_1 - |u_2| u_2 \right) = 0, & 0 < t < T, \quad x \in \mathbb{R}, \\ \omega(0, x) = u_{1,0}(x) - u_{2,0}(x), & x \in \mathbb{R}. \end{cases} \tag{3.12}$$

Fixed $T > 0$, since $u_1,\ u_2 \in H^1(\mathbb{R})$, for every $0 \le t \le T$, we have that

$$\|u_1\|_{L^\infty((0,T)\times\mathbb{R})}, \ \|u_2\|_{L^\infty((0,T)\times\mathbb{R})} \le C(T). \tag{3.13}$$

We define

$$g = \frac{f(u_1) - f(u_2)}{\omega} \tag{3.14}$$

and observe that, by Eq (3.13), we have that

$$|g| \le \|f'\|_{L^\infty(-C(T), C(T))} \le C(T). \tag{3.15}$$

Moreover, by Eq (3.11) we have that

$$\left| |u_1| - |u_2| \right| \le |u_1 - u_2| = |\omega|. \tag{3.16}$$

Observe that thanks to Eq (3.11),

$$\begin{aligned} |u_1| u_1 - |u_2| u_2 =& |u_1| u_1 - |u_1| u_2 + |u_1| u_2 - |u_2| u_2 \\ =& |u_1| \omega + u_2 \left(|u_1| - |u_2| \right). \end{aligned} \tag{3.17}$$

Thanks to Eqs (3.14) and (3.17), Equation (3.12) is equivalent to the following one:

$$\partial_t \omega + \partial_x (g\omega) - \beta^2 \partial_x^2 \omega + \delta \partial_x^3 \omega + \kappa \omega + \gamma^2 |u_1| \omega + \gamma^2 u_2 \left(|u_1| - |u_2| \right) = 0. \tag{3.18}$$

Multiplying Eq (3.18) by 2ω, an integration on $\mathbb{R}$ gives

$$\begin{aligned} &\frac{dt}{dt} \|\omega(t, \cdot)\|_{L^2(\mathbb{R})}^2 = 2 \int_{\mathbb{R}} \omega \partial_t \omega \\ &= -2 \int_{\mathbb{R}} \omega \partial_x (g\omega) dx + 2\beta^2 \int_{\mathbb{R}} \omega \partial_x^2 \omega dx - 2\delta \int_{\mathbb{R}} \omega \partial_x^3 \omega dx \\ &\quad - 2\kappa \|\omega(t, \cdot)\|_{L^2(\mathbb{R})}^2 - 2\gamma^2 \int_{\mathbb{R}} |u_1| \omega^2 dx - 2\gamma^2 \int_{\mathbb{R}} u_2 \left(|u_1| - |u_2| \right) \omega dx \end{aligned}$$

$$
\begin{aligned}
&= 2\int_{\mathbb{R}} g\omega\partial_x\omega dx - 2\beta^2 \|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\delta\int_{\mathbb{R}} \partial_x\omega\partial_x^2\omega dx \\
&\quad - 2\kappa \|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} - 2\gamma^2\int_{\mathbb{R}} |u_1|\omega^2 dx - 2\gamma^2\int_{\mathbb{R}} u_2\left(|u_1|-|u_2|\right)\omega dx \\
&=2\int_{\mathbb{R}} g\omega\partial_x\omega dx - 2\beta^2 \|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} \\
&\quad - 2\kappa \|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} - 2\gamma^2\int_{\mathbb{R}} |u_1|\omega^2 dx - 2\gamma^2\int_{\mathbb{R}} u_2\left(|u_1|-|u_2|\right)\omega dx.
\end{aligned}
$$

Therefore, we have that

$$
\begin{aligned}
&\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^2 \|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\gamma^2\int_{\mathbb{R}} |u_1|\omega^2 dx \\
&\qquad =2\int_{\mathbb{R}} g\omega\partial_x\omega dx - \kappa \|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} - 2\gamma^2\int_{\mathbb{R}} u_2\left(|u_1|-|u_2|\right)\omega dx.
\end{aligned} \tag{3.19}
$$

Due to Eqs (3.13), (3.15) and (3.16) and the Young inequality,

$$
\begin{aligned}
&2\int_{\mathbb{R}} |g||\omega||\partial_x\omega| dx \le 2C(T)\int_{\mathbb{R}} |\omega||\partial_x\omega| dx \\
&\qquad = 2\int_{\mathbb{R}} \left|\frac{C(T)\omega}{\beta}\right| |\beta\partial_x\omega|\, dx \le C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2 \|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}, \\
&2\gamma^2\int_{\mathbb{R}} |u_2|\left|(|u_1|-|u_2|)\right||\omega| dx \le 2\gamma^2\|u_2\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}} \left|(|u_1|-|u_2|)\right||\omega| dx \\
&\qquad \le C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

It follows from Eq (3.19) that

$$
\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2 \|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\gamma^2\int_{\mathbb{R}} |u_1|\omega^2 dx \le C(T)\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}.
$$

The Gronwall Lemma and Eq (3.12) give

$$
\begin{aligned}
&\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2 e^{C(T)t}\int_0^t e^{-C(T)s}\|\partial_x\omega(s,\cdot)\|^2_{L^2(\mathbb{R})} ds \\
&\qquad + 2\gamma^2 e^{C(T)t}\int_0^t\!\!\int_{\mathbb{R}} e^{-C(T)s}|u_1|\omega^2 ds dx \le e^{C(T)t}\|\omega_0\|^2_{L^2(\mathbb{R})}.
\end{aligned} \tag{3.20}
$$

Equation (1.9) follows from Eqs (3.11) and (3.20). □

Author contributions

Giuseppe Maria Coclite and Lorenzo Di Ruvo equally contributed to the methodologies, typesetting, and the development of the paper.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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