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Theory article

Multi-cluster flocking of the thermodynamic Cucker-Smale model with a unit-speed constraint under a singular kernel

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Abstract: This paper presents several sufficient frameworks for multi-cluster flocking of the thermodynamic Cucker-Smale model with a unit-speed constraint (in short, TCSUS) under a singular kernel. By providing precise estimates and deriving the dissipative structure of TCSUS, it was proved that under specific well-prepared conditions for particle positions and fully separated initial velocities, multi-cluster flocking occurs in the TCSUS system under a strong singular kernel. Furthermore, the velocities and temperatures converge to the average final data for each cluster group.

Keywords: Cucker-Smale Model; thermodynamic; unit-speed; singular kernel; multi-cluster flocking; convergence value

1. Introduction

Emergent dynamics in interacting multi-agent systems are commonly observed in nature. Natural phenomena, including animal migration [1], bacterial movement [2], and synchronization of coupled cells [3] and fireflies [4], exhibit collective behaviors. For instance, in the field of ecology, collective behaviors can facilitate population reproduction, predator evasion, and the reduction of competition among individuals. Ultimately, these behaviors can enhance the population's safety coefficient. Therefore, studying collective behaviors is of significant importance and meaning.

To investigate aggregation phenomena, biophysicist T. Vicsek et al. conducted numerical experiments to elucidate the mechanisms underlying collective motion [5]. A. Jadbabaie subsequently verified these experiments through analytical methods [6]. Following the pioneering work of T. Vicsek et al. numerous mathematical models have been developed to study emergent behavior. Professors Cucker and Smale proposed the Cucker-Smale model, which characterizes aggregation phenomena [7]. The Cucker-Smale model describes a flocking dynamic system with position and velocity following Newtonian dynamics. For the $i-th$ particle in the Cucker-Smale model, let $x_i \in \mathbb{R}^d$

and $v_i \in \mathbb{R}^d$ denote its position and velocity. Here, $\mathbb{R}^d$ represents d-dimensional Euclidean space. The Cucker-Smale model is governed by

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \quad t > 0, \quad i \in [N] := \{1, 2, \cdots, N\}, \\ \dfrac{dv_i}{dt} = \dfrac{\kappa}{N-1} \displaystyle\sum_{j \neq i} \phi(\|x_i - x_j\|)(v_j - v_i), \\ (x_i(0), v_i(0)) = (x_i^0, v_i^0) \in \mathbb{R}^d \times \mathbb{R}^d, \end{cases} \tag{1.1}$$

where N represents the number of particles, κ denotes the non-negative coupling strength, and $\|\cdot\|$ denotes the standard l^2-norm. ϕ signifies the communication weight. The Cucker-Smale model offers unique advantages in mathematical analysis due to its high degree of symmetry. Additionally, the solution's large-time behavior is determined solely by the initial conditions and the interaction function ϕ. Since its proposal, the Cucker-Smale model has been the subject of extensive research, with scholars exploring diverse communication weights $\phi(r)$ tailored to specific application contexts. For instance, the authors of [8] adopted the communication weight $\phi(r)$ as $\phi(r) = \dfrac{1}{r^\beta}$, known as the singular kernel, to conduct a detailed analysis of its clustering behavior.

However, the Cucker-Smale model cannot describe the aggregative behaviors influenced by external factors, including light and temperature. For instance, Bhaya et al. [9] and Jakob et al. [10] observed that cyanobacteria actively migrate toward light sources under certain conditions. Ha et al. [11] investigated the effect of temperature on aggregative behavior, resulting in the development of a thermodynamic Cucker-Smale (in short, TCS) model. Since then, the TCS model has also been a subject of extensive research. Two sufficient frameworks for the emergence of mono-cluster flocking on a digraph for the continuous and discrete models were presented in [12]. The emergent behaviors of a TCS ensemble confined in a harmonic potential field was studied in [13]. The coupling of a kinetic TCS equation and viscous fluid system was proposed and considered in [14, 15]. Based on system (1.1), we set T_i to denote the temperature of the $i-th$ particle, and then the TCS model is governed by

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \quad t > 0, \quad i \in [N] := \{1, 2, \cdots, N\}, \\ \dfrac{dv_i}{dt} = \dfrac{\kappa_1}{N-1} \displaystyle\sum_{j \neq i} \phi(\|x_i - x_j\|)\left(\dfrac{v_j}{T_j} - \dfrac{v_i}{T_i}\right), \\ \dfrac{d}{dt}\left(T_i + \dfrac{1}{2}\|v_i\|^2\right) = \dfrac{\kappa_2}{N-1} \displaystyle\sum_{j \neq i} \zeta(\|x_i - x_j\|)(\dfrac{1}{T_i} - \dfrac{1}{T_j}), \\ (x_i(0), v_i(0), T_i(0)) = (x_i^0, v_i^0, T_i^0) \in \mathbb{R}^d \times \mathbb{R}^d \times (0, +\infty), \end{cases} \tag{1.2}$$

where N denotes the number of particles, while κ_1 and κ_2 represent strictly positive coupling strengths. Moreover, ϕ, ζ, which are mappings from $(0, +\infty) \to (0, +\infty)$, serve as the communication weights. These functions are non-negative, locally Lipschitz continuous, and monotonically decreasing.

Based on the conceptual framework for the unit-speed Cucker-Smale model presented in [16], Ahn modified the velocity coupling term to guarantee that each velocity possesses a constant positive modulus [17], as follows:

$$
\begin{cases}
\dfrac{dx_i}{dt} = v_i, \quad t > 0, \quad i \in [N] := \{1, 2, \cdots, N\}, \\
\dfrac{dv_i}{dt} = \dfrac{\kappa_1}{N-1} \sum_{j \neq i} \phi(\|x_i - x_j\|) \left(\dfrac{v_j}{T_j} - \dfrac{\langle v_i, v_j \rangle v_i}{T_j \|v_i\|^2} \right), \\
\dfrac{d}{dt} \left(T_i + \dfrac{1}{2}\|v_i\|^2 \right) = \dfrac{\kappa_2}{N-1} \sum_{j \neq i} \zeta(\|x_i - x_j\|)(\dfrac{1}{T_i} - \dfrac{1}{T_j}), \\
(x_i(0), v_i(0), T_i(0)) = (x_i^0, v_i^0, T_i^0) \in \mathbb{R}^d \times \mathbb{S}^{d-1} \times (0, +\infty),
\end{cases}
\tag{1.3}
$$

where N, κ_1, κ_2, and communication weights ϕ, ζ are defined as above. The term $\mathbb{S}^{d-1}$ denotes the unit $(d-1)$-sphere. However, the author of [17] only addressed the scenarios where the communication weights ϕ and ζ are non-negative, bounded, locally Lipschitz continuous, and monotonically decreasing.

Furthermore, the author of [18] employed suitable subdivided configurations $\{Z_\alpha\}_{\alpha=1}^n$ and demonstrated that the velocity and temperature of all agents within each cluster group converge to identical values. In addition, based on the results of [17], the authors of [19] proved that asymptotic flocking occurs when the communication weights ϕ, ζ are singular kernels.

This article considers the multi-cluster flocking dynamics of the thermodynamic Cucker-Smale model with a unit-speed constraint (TCSUS) under a singular kernel. The system (1.3) is reorganized into a multi-cluster framework, which is governed by,

$$
\begin{cases}
\dfrac{dx_{\alpha i}}{dt} = v_{\alpha i}, \quad t > 0, \quad i \in [N_\alpha] := \{1, 2, \cdots, N_\alpha\}, \quad \alpha \in [n] := \{1, 2, \cdots n\}, \quad n \geq 3, \\
\dfrac{dv_{\alpha i}}{dt} = \dfrac{\kappa_1}{N-1} \sum_{\substack{j=1 \\ j \neq i}}^{N_\alpha} \phi(\|x_{\alpha i} - x_{\alpha j}\|) \left(\dfrac{v_{\alpha j}}{T_{\alpha j}} - \dfrac{\langle v_{\alpha i}, v_{\alpha j} \rangle v_{\alpha i}}{T_{\alpha j} \|v_{\alpha i}\|^2} \right) + \dfrac{\kappa_1}{N-1} \sum_{\beta \neq \alpha} \sum_{j=1}^{N_\beta} \phi(\|x_{\alpha i} - x_{\beta j}\|) \left(\dfrac{v_{\beta j}}{T_{\beta j}} - \dfrac{\langle v_{\alpha i}, v_{\beta j} \rangle v_{\alpha i}}{T_{\beta j} \|v_{\alpha i}\|^2} \right), \\
\dfrac{d}{dt} \left(T_{\alpha i} + \dfrac{1}{2}\|v_{\alpha i}\|^2 \right) = \dfrac{\kappa_2}{N-1} \sum_{\substack{j=1 \\ j \neq i}}^{N_\alpha} \zeta(\|x_{\alpha i} - x_{\alpha j}\|)(\dfrac{1}{T_{\alpha i}} - \dfrac{1}{T_{\alpha j}}) + \dfrac{\kappa_2}{N-1} \sum_{\beta \neq \alpha} \sum_{j=1}^{N_\beta} \zeta(\|x_{\alpha i} - x_{\beta j}\|)(\dfrac{1}{T_{\alpha i}} - \dfrac{1}{T_{\beta j}}), \\
(x_{\alpha i}(0), v_{\alpha i}(0), T_{\alpha i}(0)) = (x_{\alpha i}^0, v_{\alpha i}^0, T_{\alpha i}^0) \in \mathbb{R}^d \times \mathbb{S}^{d-1} \times (0, +\infty),
\end{cases}
\tag{1.4}
$$

where t represents time, n represents the number of clusters, N represents the number of particles, and N_α represents the number of particles in the $\alpha - th$ cluster. Additionally, $x_{\alpha i}$, $v_{\alpha i}$, and $T_{\alpha i}$ denote the position, velocity, and temperature of the $i - th$ particle in the $\alpha - th$ cluster, respectively. Furthermore, κ_1 and κ_2 represent strictly positive coupling strengths, and $\mathbb{S}^{d-1}$ is the unit $(d-1)$-sphere. Specifically, we assume that when the communication weights ϕ, ζ are singular kernels, they will take the following explicit assumption: $\phi(r) = \dfrac{1}{r^\lambda}, \zeta(r) = \dfrac{1}{r^\mu} (\lambda, \mu > 0)$.

In fact, system (1.4) corresponds exactly to system (1.3). The formulation of system (1.4) specifically highlights the influence of different clusters on particle dynamics.

Previous studies in [17, 18] thoroughly explored mono-cluster, bi-cluster, and multi-cluster flocking behaviors in system (1.4) under a standard kernel. More recently, studies cited in [19] have concentrated on mechanical flocking and thermal homogenization within the TCSUS model under a singular kernel. However, a comprehensive study about the multi-cluster flocking of TCSUS under a singular kernel remains largely unexplored. In this paper, we mainly focus on studying the multi-cluster flocking under a strong singular kernel and provide relevant conclusions and estimates.

For simplicity, we apply the following notation:

Notation 1.1. For the vector $u \in \mathbb{R}^d$, we denote by $\|u\|$ and u^i the Euclidean l^2-norm of u and its $i-th$ component, respectively. The standard inner product of two vectors $u, v \in \mathbb{R}^d$ is denoted by $\langle u, v\rangle$. The distance between two sets A and B is denoted by $d(A, B) := \inf\{d(x, y) : x \in A, y \in B\}$. For simplicity, we define $[N] := \{1, 2, \cdots, N\}$. Given fixed real sequences $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$, we define the family of sets $\{I_\alpha\}_{\alpha=1}^n$ such that $I_\alpha := [a_\alpha, b_\alpha]$. The solution of system (1.3) is denoted by $A := (a_1, a_2, \cdots, a_N)$, where $A \in \{X, V, T\}$, and $a \in \{x, v, T\}$. After we reorganize the system (1.3) into a multi-cluster configuration (1.4), if we denote the solution of the $\alpha - th$ cluster as $A_\alpha := (a_{\alpha1}, a_{\alpha2}, \cdots, a_{\alpha N_\alpha})$, where $A \in \{X, V, T\}$ and $a \in \{x, v, T\}$, it becomes evident that $A = (a_1, a_2, \cdots, a_N) = (A_1, A_2, \cdots, A_n)$. In addition, we define $a^0 := a(0)$. The generic constant C may differ from line to line. We define the centers for position, velocity, and temperature of the $\alpha - th$ cluster as $x_\alpha^{cen} := \frac{1}{N_\alpha}\sum_{i\in[N_\alpha]} x_{\alpha i}$, $v_\alpha^{cen} := \frac{1}{N_\alpha}\sum_{i\in[N_\alpha]} v_{\alpha i}$, and $T_\alpha^{cen} := \frac{1}{N_\alpha}\sum_{i\in[N_\alpha]} T_{\alpha i}$, respectively. Furthermore, we define the minimum temperature of the $\alpha - th$ cluster as $T_{\alpha m}(t) := \min_{i\in[N_\alpha]} T_{\alpha i}(t)$ and the maximum temperature of the $\alpha - th$ cluster as $T_{\alpha M}(t) := \max_{i\in[N_\alpha]} T_{\alpha i}(t)$. We define the minimum temperature of the whole system as $T_m(t) := \min_{\alpha\in[n]} T_{\alpha m}(t)$ and the maximum temperature of the whole system as $T_M(t) := \max_{\alpha\in[n]} T_{\alpha M}(t)$. The minimum inner product throughout the system is denoted as $A(v) := \min_{\substack{\alpha,\beta\in[n] \\ i\in[N_\alpha], j\in[N_\beta]}} \langle v_{\alpha i}, v_{\beta j}\rangle$.

First, we define the $L^\infty-$ diameters for position, velocity, and temperature of each cluster group $Z_\alpha := \{(x_{\alpha i}, v_{\alpha i}, T_{\alpha i})\}_{i=1}^{N_\alpha}$ as follows:

•(position-velocity-temperature diameters for the $\alpha - th$ cluster group)

$D_{X_\alpha} := \max_{i,j\in[N_\alpha]} \|x_{\alpha i} - x_{\alpha j}\|, \qquad D_{V_\alpha} := \max_{i,j\in[N_\alpha]} \|v_{\alpha i} - v_{\alpha j}\|, \qquad D_{T_\alpha} := \max_{i,j\in[N_\alpha]} |T_{\alpha i} - T_{\alpha j}|,$

•(position-velocity-temperature diameters for the whole system)

$D_X := \sum_{\alpha=1}^n D_{X_\alpha}, \qquad D_V := \sum_{\alpha=1}^n D_{V_\alpha}, \qquad D_T := \sum_{\alpha=1}^n D_{T_\alpha}.$

Then, we define the following three configuration vectors for each cluster group:

$X_\alpha := (x_{\alpha1}, x_{\alpha2}, \cdots, x_{\alpha N_\alpha})$, $V_\alpha := (v_{\alpha1}, v_{\alpha2}, \cdots, v_{\alpha N_\alpha})$, $T_\alpha := (T_{\alpha1}, T_{\alpha2}, \cdots, T_{\alpha N_\alpha})$, where $1 \le \alpha \le n$.

Next, we introduce the definition of the multi-cluster flocking behavior of system (1.4):

Definition 1.1. Let $Z = \{(x_i, v_i, T_i)\}_{i=1}^N$ be a solution to system (1.4). Then, the configuration Z exhibits multi-cluster flocking if there exist n cluster groups $Z_\alpha = \{(x_{\alpha i}, v_{\alpha i}, T_{\alpha i})\}_{i=1}^{N_\alpha}$ such that the following assertions hold for $2 \le n \le N$ and $1 \le \alpha \le n$:

•$|Z_\alpha| = N_\alpha \ge 1, \sum_{\alpha=1}^n |Z_\alpha| = \sum_{\alpha=1}^n N_\alpha = N, \bigcup_{\alpha=1}^n Z_\alpha = Z,$

•$\forall \alpha \in [n],\ \sup_{0\le t<\infty} \max_{1\le k,l\le N_\alpha} \|x_{\alpha k} - x_{\alpha l}\| < \infty, \lim_{t\to\infty} \max_{1\le k,l\le N_\alpha} \|v_{\alpha k} - v_{\alpha l}\| = 0, \lim_{t\to\infty} \max_{1\le k,l\le N_\alpha} |T_{\alpha k} - T_{\alpha l}| = 0,$

•$\inf_{0\le t<\infty} \min_{k,l} \|x_{\alpha k} - x_{\beta l}\| = \infty, 1 \le k \le N_\alpha, 1 \le l \le N_\beta, 1 \le \alpha \ne \beta \le n.$

To describe adequate frameworks for multi-cluster flocking estimation, we display the admissible data and conditions $(\mathcal{H})$ as follows:

$$(\mathcal{H}) := \{(X(0), V(0), T(0)) \in \mathbb{R}^{2dN} \times (0, +\infty)^N | (\mathcal{H}_0), (\mathcal{H}_1), (\mathcal{H}_2), \text{ and } (\mathcal{H}_3) \text{ hold.}\}$$

(i) $(\mathcal{H}_0)$ (Notation): For brevity, we have the following notation:

$$\begin{aligned}
&T_m^\infty := T_m(0), T_M^\infty := T_M(0), \delta_0 := \inf_{0\le t\le\infty} \min_{1\le i\ne j\le N} \|x_i(t) - x_j(t)\|,\\
&r_0 := \min_{\alpha<\beta,i,j} (x_{\beta j}^k(0) - x_{\alpha i}^k(0)),\ R_0 := \max_{\alpha<\beta,i,j} (x_{\beta j}^k(0) - x_{\alpha i}^k(0)) \text{ for some fixed } 1 \le k \le d,\\
&\Lambda_0\left(D_X^\infty\right) := \frac{\kappa_1 \min\left(N_1, \cdots, N_\alpha\right) A(v^0)\phi\left(D_X^\infty\right)}{(N-1)T_M^\infty},\ \bar{\Lambda}_0\left(D_X^\infty\right) := \frac{\kappa_2(\min(N_1, \ldots, N_\alpha) - 2)\zeta(D_X^\infty)}{N(T_M^\infty)^2},\\
&\Lambda := \frac{D_V(0)}{\Lambda_0} + \frac{4(n-1)N\kappa_1}{(N-1)T_m^\infty \Lambda_0^2}\phi\left(\frac{r_0}{2}\right) + \frac{4(n-1)N\kappa_1}{(N-1)\min_{1\le\alpha\le n-1} d(I_\alpha, I_{\alpha+1})T_m^\infty \Lambda_0} \int_{\frac{r_0}{2}}^\infty \phi(s)ds,\\
&\Lambda_\alpha := \frac{\kappa_1(N_\alpha - 1)\phi(\delta_0)}{(N-1)T_m^\infty}\Lambda + \frac{\kappa_1\left(N - N_\alpha\right)}{(N-1)T_m^\infty \left(\min_{1\le\alpha\le n-1} d\left(I_\alpha, I_{\alpha+1}\right)\right)} \int_{\frac{r_0}{2}}^\infty \phi(s)ds.
\end{aligned}$$

(ii) $(\mathcal{H}_1)$ (Well prepared conditions): There exists a strictly positive number $D_X^\infty > 0$ such that

$$D_X^\infty \ge D_X(0) + \max\left(\Lambda, \frac{D_V(0)T_M^\infty}{\kappa_1 A(v^0)\phi(D_X^\infty)}\right), \quad \text{and} \quad \lambda,\ \mu > 1 \text{ (strong kernel).}$$

(iii) $(\mathcal{H}_2)$ (Separated initial data): For fixed $1 \le k \le d$ *in* $(\mathcal{H}_0)$, there exist real sequences $\{a_i\}_{i=1}^n$ *and* $\{b_i\}_{i=1}^n$ such that the initial data and system parameters are appropriately split as follows:

$$\begin{aligned}
&r_0 > 0, \quad a_1 < b_1 < a_2 < b_2 \cdots < a_n < b_n, \quad I_\alpha = [a_\alpha, b_\alpha] \subset [-1, 1], \quad I_\alpha \cap I_\beta = \emptyset (\beta \ne \alpha),\\
&[v_{\alpha i}^k(0) - \Lambda_\alpha, v_{\alpha i}^k(0) + \Lambda_\alpha] \subset I_\alpha = [a_\alpha, b_\alpha] \subset [-1, 1], \quad \alpha, \beta = 1, \ldots, n, \quad i = 1, \ldots, N_\alpha.
\end{aligned}$$

(iv) $(\mathcal{H}_3)$ (Small fluctuations): The local velocity perturbation for all cluster groups is sufficiently small as follows:

$$D_V(0) \le \frac{\kappa_1 A(v^0)}{T_M^\infty} \int_{D_X(0)}^{D_X^\infty} \phi(s)ds. \tag{1.5}$$

Finally, we present the main theorems of this article.

Theorem 1.1. Assume that $Z_\alpha = \{(x_{\alpha i}, v_{\alpha i}, T_{\alpha i})\}_{i=1}^{N_\alpha}$ is a solution to system (1.4). Suppose that $(\mathcal{H})$ holds. It follows that

$$\min_{\alpha\ne\beta,i,j} \left\|x_{\alpha i}(t) - x_{\beta j}(t)\right\| \ge \left(\min_{1\le\alpha\le n-1} d\left(I_\alpha, I_{\alpha+1}\right)\right)t + \frac{r_0}{2}, \quad t \in (0, +\infty). \tag{1.6}$$

Theorem 1.2. Assume that $Z_\alpha = \{(x_{\alpha i}, v_{\alpha i}, T_{\alpha i})\}_{i=1}^{N_\alpha}$ is a solution to system (1.4). Suppose that $(\mathcal{H})$ holds. Then, it follows that for $t \in (0, +\infty)$:

1. (Velocity alignment for each cluster group)

$$D_V(t) \le D_V(0) \exp\left(-\Lambda_0 t\right) + \frac{2\kappa_1(n-1)N}{T_m^\infty \Lambda_0(N-1)} \exp\left(-\frac{\Lambda_0}{2}t\right)\phi\left(\frac{r_0}{2}\right)$$

$$+\frac{2\kappa_1(n-1)N}{T_m^\infty \Lambda_0(N-1)}\phi\left(\frac{(\min_{1\le\alpha\le n-1} d(I_\alpha, I_{\alpha+1}))\,t + r_0}{2}\right). \tag{1.7}$$

2. (Temperature equilibrium for each cluster group)

$$\begin{aligned} D_T(t) \le & D_T(0)\exp\left(-\bar{\Lambda}_0 t\right) + \frac{2\kappa_2(n-1)N}{N-1}\left(\frac{1}{T_m^\infty}-\frac{1}{T_M^\infty}\right)\exp\left(-\frac{\bar{\Lambda}_0}{2}t\right)\zeta\left(\frac{r_0}{2}\right) \\ & + \frac{2\kappa_2(n-1)N}{N-1}\left(\frac{1}{T_m^\infty}-\frac{1}{T_M^\infty}\right)\zeta\left(\frac{(\min_{1\le\alpha\le n-1} d(I_\alpha, I_{\alpha+1}))\,t + r_0}{2}\right). \end{aligned} \tag{1.8}$$

Remark 1.1. It is evident that Theorems 1.1 and 1.2 demonstrate that system (1.4) exhibits the phenomenon of multi-cluster flocking.

Theorem 1.3. Assume that $Z_\alpha := \{(x_{\alpha i}, v_{\alpha i}, T_{\alpha i})\}_{i=1}^{N_\alpha}$ is a solution to system (1.4). Then, under the sufficient frameworks $(\mathcal{H})$, there exist some strictly positive convergence constants V_1, V_2 and T_1, T_2 that satisfy the subsequent criteria for $t \in (0, +\infty)$.
1. (Velocity convergence value for each cluster group) If we define $v_\alpha^\infty := \lim_{t\to\infty} v_\alpha^{cen}$, then the existence of v_α^∞ is guaranteed, and the two values V_1 and V_2 satisfy the following inequality for all $\alpha \in [n]$ and $i_\alpha \in [N_\alpha]$.

$$\frac{V_1}{t^{\lambda-1}} \le \sum_{\alpha=1}^{n}\left\|v_{\alpha i_\alpha}(t) - v_\alpha^\infty\right\| \le \frac{V_2}{t^{\lambda-1}},\ t \to \infty. \tag{1.9}$$

2. (Temperature convergence value for each cluster group) If we define $T_\alpha^\infty := \lim_{t\to\infty} T_\alpha^{cen}$, then the existence of T_α^∞ is guaranteed, and the two values T_1 and T_2 satisfy the following inequality for all $\alpha \in [n]$ and $i_\alpha \in [N_\alpha]$.

$$\frac{T_1}{t^{\mu-1}} \le \sum_{\alpha=1}^{n}\left\|T_{\alpha i_\alpha}(t) - T_\alpha^\infty\right\| \le \frac{T_2}{t^{\mu-1}},\ t \to \infty. \tag{1.10}$$

Remark 1.2. In [18], Ahn proved that when the communication weights ϕ and ζ are standard kernels, which are non-negative, bounded, locally Lipschitz continuous, monotonically decreasing, and integrable, the multi-cluster flocking of the system (1.4) is exhibited under some sufficient framework. However, when the communication weights ϕ and ζ are singular kernels, they will blow up as r approaches 0, meaning they will not be bounded and integrable over the interval $(0, +\infty)$. In [19], Ahn et al. proved the mono-cluster flocking of system (1.4) under the singular kernel. However, the multi-cluster flocking of system (1.4) under a singular kernel remains unexplored. In this article, by employing a new sufficient framework, we address the non-regularity of the singular kernel at $r = 0$ and obtain the multi-cluster flocking of the system (1.4) under the singular kernels based on the work of [19].

This article is organized as follows. In Section 2, several basic results of the TCSUS model are briefly reviewed initially. Subsequently, some previous results related to the TCSUS model under a singular kernel in [19] are reviewed to prepare for the proof of multi-cluster flocking. In Section 3, several fundamental frameworks for achieving multi-cluster flocking in system (1.4) with a strongly singular kernel are provided, and appropriate dissipative differential inequalities for position, velocity, and temperature are derived. Then, by using self-consistent parameters for these inequalities, we derive sufficient conditions to ensure multi-cluster flocking of system (1.4) based on initial data and system parameters.

2. Preliminaries

This section reviews several basic results for the TCSUS to guarantee its multi-cluster flocking. These estimates will be crucial throughout this paper.

2.1. Basic estimates

Proposition 2.1. For $\tau \in (0, +\infty)$, let (X, V, T) be a solution to system (1.4) in the time-interval $(0, \tau)$. Then, the following assertions hold:

(1) (Conservation laws): The modulus of velocities and the total sum of temperatures are conserved.

$$\frac{d}{dt}\sum_{\alpha=1}^{n}\sum_{\alpha i=1}^{N_\alpha} T_{\alpha i}(t) = 0, \quad \|v_{\alpha i}(t)\| = 1, \quad t \in (0, \tau).$$

(2) (Monotonicity of temperature): External temperatures $T_m(t)$ and $T_M(t)$ are monotonically increasing and decreasing, respectively, and one has positivity and uniform boundedness.

$$0 < T_m^\infty \le T_{\alpha i}(t) \le T_M^\infty, \quad \alpha \in [n], \quad i \in [N_\alpha], \quad t \in [0, \tau).$$

(3) (Monotonicity of $A(v)$): If the initial data satisfies that $A(v^0) := \min\langle v^0_{\alpha i}, v^0_{\beta j}\rangle > 0$, then $A(v)$ is monotonically increasing: If $0 \le s \le t < \tau$, then $0 < A(v^0) \le A(v)(s) \le A(v)(t) \le 1$.

Proof. (1) To demonstrate speed conservation, we take the inner product of the second equation in system (1.4) with $2v_{\alpha i}$ to obtain

$$\begin{aligned}\left\langle 2v_{\alpha i}, \frac{dv_{\alpha i}}{dt}\right\rangle &= \frac{2\kappa_1}{N-1}\sum_{\substack{j=1\\ j\neq i}}^{N_\alpha}\phi(\|x_{\alpha i}-x_{\alpha j}\|)\left(\frac{\langle v_{\alpha i}, v_{\alpha j}\rangle}{T_{\alpha j}} - \frac{\langle v_{\alpha i}, v_{\alpha j}\rangle\langle v_{\alpha i}, v_{\alpha i}\rangle}{T_{\alpha j}\|v_{\alpha i}\|^2}\right)\\ &\quad + \frac{2\kappa_1}{N-1}\sum_{\beta\neq\alpha}\sum_{j=1}^{N_\beta}\phi(\|x_{\alpha i}-x_{\beta j}\|)\left(\frac{\langle v_{\beta j}, v_{\alpha i}\rangle}{T_{\beta j}} - \frac{\langle v_{\alpha i}, v_{\beta j}\rangle\langle v_{\alpha i}, v_{\alpha i}\rangle}{T_{\beta j}\|v_{\alpha i}\|^2}\right) = 0.\end{aligned}$$

This implies that $\frac{d\|v_{\alpha i}\|^2}{dt} = 0$, *i.e.*, $\|v_{\alpha i}(t)\| = \|v_{\alpha i}(0)\| = 1$. Then, we employ $\zeta(\|x_i - x_j\|) = \zeta(\|x_j - x_i\|)$ and exchange $\alpha i \leftrightarrow \alpha j$ and $\alpha i \leftrightarrow \beta j$ respectively to get

$$\begin{aligned}\frac{d}{dt}\sum_{\alpha=1}^{n}\sum_{\alpha i=1}^{N_\alpha} T_{\alpha i}(t) &= \frac{\kappa_2}{N-1}\sum_{\alpha=1}^{n}\sum_{\alpha i=1}^{N_\alpha}\sum_{\substack{j=1\\ j\neq i}}^{N_\alpha}\zeta(\|x_{\alpha i}-x_{\alpha j}\|)\left(\frac{1}{T_{\alpha i}} - \frac{1}{T_{\alpha j}}\right)\\ &\quad + \frac{\kappa_2}{N-1}\sum_{\alpha=1}^{n}\sum_{\alpha i=1}^{N_\alpha}\sum_{\beta\neq\alpha}\sum_{j=1}^{N_\beta}\zeta(\|x_{\alpha i}-x_{\beta j}\|)\left(\frac{1}{T_{\alpha i}} - \frac{1}{T_{\beta j}}\right),\\ &= \frac{\kappa_2}{N-1}\sum_{\alpha=1}^{n}\sum_{\alpha j=1}^{N_\alpha}\sum_{\substack{i=1\\ i\neq j}}^{N_\alpha}\zeta(\|x_{\alpha j}-x_{\alpha i}\|)\left(\frac{1}{T_{\alpha j}} - \frac{1}{T_{\alpha i}}\right)\\ &\quad + \frac{\kappa_2}{N-1}\sum_{\beta=1}^{n}\sum_{\beta j=1}^{N_\beta}\sum_{\alpha\neq\beta}\sum_{i=1}^{N_\alpha}\zeta(\|x_{\alpha j}-x_{\beta i}\|)\left(\frac{1}{T_{\beta j}} - \frac{1}{T_{\alpha i}}\right) = 0. \qquad (2.1)\end{aligned}$$

(2) We induce that $\alpha_t i_t$ depends on time $t \in [0, \tau)$ satisfying $T_m(t) = T_{\alpha_t i_t}(t)$, and then we have

$$\frac{dT_{\alpha_t i_t}}{dt} = \frac{\kappa_2}{N-1}\sum_{\substack{j=1\\ j\neq i}}^{N_{\alpha t}} \underbrace{\zeta(\|x_{\alpha_t i_t} - x_{\alpha_t j}\|)}_{>0} \underbrace{(\frac{1}{T_{\alpha_t i_t}} - \frac{1}{T_{\alpha_t j}})}_{\geq 0} + \frac{\kappa_2}{N-1}\sum_{\beta\neq\alpha_t}\sum_{j=1}^{N_\beta} \underbrace{\zeta(\|x_{\alpha_t i_t} - x_{\beta j}\|)}_{>0} \underbrace{(\frac{1}{T_{\alpha_t i_t}} - \frac{1}{T_{\beta j}})}_{\geq 0}.$$

Therefore, $\frac{dT_{\alpha_t i_t}}{dt} \geq 0$, *i.e.*, $T_m(t)$ is increasing. By the same token, we get that $T_M(t)$ is decreasing. This implies that $0 < T_m^\infty \leq T_{\alpha i}(t) \leq T_M^\infty, \quad \alpha \in [n], \quad i \in [N_\alpha], \quad t \in [0, \tau)$.

(3) We split the proof into two steps:

• First, we show that the functional $A(v)$ is strictly positive in the time interval $(0, \tau), A(v) > 0$.

• Second, we verify that in the time interval $(0, \tau)$, $\frac{d}{dt}A(v) \geq 0$.

Step A: For fixed $t \in (0, \tau)$, we choose two indices $\alpha_t i_t, \beta_t j_t$ $(\alpha,\beta \in [n], i \in [N_\alpha], j \in [N_\beta])$ such that $\langle v_{\alpha_t i_t}, v_{\beta_t j_t}\rangle = A(v)(t)$. Then, we define a temporal set S_1 and its supremum τ_1^* as

$$S_1 := \{t \in (0, \tau) | A(v)(t) > 0\}, \quad \tau_1^* := \sup S_1.$$

Since $A(v^0) > 0$, and $A(v)$ is continuous, the set S_1 is an open set, and $0 < \tau_1^* \leq \tau$. Next, we claim $\tau_1^* = \tau$.

Suppose the contrary holds, *i.e.*, $\tau_1^* < \tau$, which implies $A(v)(\tau_1^* - 0) = 0$. We differentiate $A(v)$ with respect to time $t \in (0, \tau_1^*)$ to find

$$\begin{aligned}
\frac{d}{dt}A(v) &= \langle \dot v_{\alpha_t i_t}, v_{\beta_t j_t}\rangle + \langle v_{\alpha_t i_t}, \dot v_{\beta_t j_t}\rangle \\
&= \frac{\kappa_1}{N-1}\sum_{\substack{j=1\\ j\neq i_t}}^{N_{\alpha t}} \phi(\|x_{\alpha_t i_t} - x_{\alpha j}\|)\left(\frac{\langle v_{\alpha j}, v_{\beta_t j_t}\rangle - \langle v_{\alpha_t i_t}, v_{\alpha j}\rangle\langle v_{\alpha_t i_t}, v_{\beta_t j_t}\rangle}{T_{\alpha_t j}}\right) \\
&\quad + \frac{\kappa_1}{N-1}\sum_{\beta\neq\alpha_t}\sum_{j=1}^{N_\beta} \phi(\|x_{\alpha_t i_t} - x_{\beta j}\|)\left(\frac{\langle v_{\beta_t j_t}, v_{\beta j}\rangle - \langle v_{\alpha_t i_t}, v_{\beta j}\rangle\langle v_{\alpha_t i_t}, v_{\beta_t j_t}\rangle}{T_{\beta j}}\right) \\
&\quad + \frac{\kappa_1}{N-1}\sum_{\substack{j=1\\ j\neq j_t}}^{N_{\beta t}} \phi(\|x_{\beta_t j_t} - x_{\beta_t j}\|)\left(\frac{\langle v_{\alpha_t i_t} v_{\beta_t j}\rangle - \langle v_{\beta_t j_t}, v_{\beta_t j}\rangle\langle v_{\alpha_t i_t} v_{\beta_t j_t}\rangle}{T_{\beta_t j}}\right) \\
&\quad + \frac{\kappa_1}{N-1}\sum_{\beta\neq\beta_t}\sum_{j=1}^{N_\beta} \phi(\|x_{\beta_t j_t} - x_{\beta j}\|)\left(\frac{\langle v_{\alpha_t i_t} v_{\beta j}\rangle - \langle v_{\beta_t j_t}, v_{\beta j}\rangle\langle v_{\alpha_t i_t} v_{\beta_t j_t}\rangle}{T_{\beta j}}\right).
\end{aligned} \tag{2.2}$$

For any $v_{\alpha i}, v_{\beta j}$ which are the components of V, the unit-speed constraint yields

$$|\langle v_{\alpha i}, v_{\beta j}\rangle| \leq \|v_{\alpha i}\| \cdot \|v_{\beta j}\| \leq 1.$$

Thus, the positivity and minimality of $A(v)$ lead to

$$\langle v_{\gamma k}, v_{\beta_t j_t}\rangle \geq A(v) := \langle v_{\alpha_t i_t}, v_{\beta_t j_t}\rangle \geq \langle v_{\alpha_t i_t}, v_{\beta_t j_t}\rangle\langle v_{\alpha_t i_t}, v_{\gamma k}\rangle$$

and

$$\langle v_{\gamma k}, v_{\alpha_t i_t}\rangle \geq A(v) := \langle v_{\alpha_t i_t}, v_{\beta_t j_t}\rangle \geq \langle v_{\alpha_t i_t}, v_{\beta_t j_t}\rangle\langle v_{\beta_t j_t}, v_{\gamma k}\rangle, \quad \forall \gamma \in [n].$$

Since each temperature is bounded below by a positive constant, each summand of Eq (2.2) is non-negative, and one has $\frac{d}{dt}A(v) \geq 0,\ \ t \in (0, \tau_1^*)$. Therefore, $A(v)(\tau_1^* - 0) \geq A(v)(0) > 0$, which is contradictory to $A(v)(\tau_1^* - 0) = 0$. Finally, we have $\tau_1^* = \tau$ and $A(v) > 0,\ t \in (0, \tau)$.

Step B: It follows from Eq (2.2) that $\frac{d}{dt}A(v) \geq 0,\ t \in (0, \tau)$. □

Remark 2.1. Based on Proposition 2.1, it can be immediately inferred that the system (1.4) can be simplified into the following system:

$$\begin{cases} \dfrac{dx_{\alpha i}}{dt} = v_{\alpha i}, \quad t > 0, \quad i \in [N_\alpha] := \{1, 2, \cdots, N_\alpha\}, \quad \alpha \in [n] := \{1, 2, \cdots n\}, \quad n \geq 3, \\ \dfrac{dv_{\alpha i}}{dt} = \dfrac{\kappa_1}{N-1} \displaystyle\sum_{\substack{j=1 \\ j \neq i}}^{N_\alpha} \phi(\|x_{\alpha i} - x_{\alpha j}\|) \left(\dfrac{v_{\alpha j} - \langle v_{\alpha i}, v_{\alpha j} \rangle v_{\alpha i}}{T_{\alpha j}} \right) + \dfrac{\kappa_1}{N-1} \displaystyle\sum_{\beta \neq \alpha} \sum_{j=1}^{N_\beta} \phi(\|x_{\alpha i} - x_{\beta j}\|) \left(\dfrac{v_{\beta j} - \langle v_{\alpha i}, v_{\beta j} \rangle v_{\alpha i}}{T_{\beta j}} \right) \\ \dfrac{dT_{\alpha i}}{dt} = \dfrac{\kappa_2}{N-1} \displaystyle\sum_{\substack{j=1 \\ j \neq i}}^{N_\alpha} \zeta(\|x_{\alpha i} - x_{\alpha j}\|)(\dfrac{1}{T_{\alpha i}} - \dfrac{1}{T_{\alpha j}}) + \dfrac{\kappa_2}{N-1} \displaystyle\sum_{\beta \neq \alpha} \sum_{j=1}^{N_\beta} \zeta(\|x_{\alpha i} - x_{\beta j}\|)(\dfrac{1}{T_{\alpha i}} - \dfrac{1}{T_{\beta j}}), \\ (x_{\alpha i}(0), v_{\alpha i}(0), T_{\alpha i}(0)) = (x_{\alpha i}^0, v_{\alpha i}^0, T_{\alpha i}^0) \in \mathbb{R}^d \times \mathbb{S}^{d-1} \times (0, +\infty). \end{cases}$$

2.2. *Previous results*

In this subsection, we will give some previous results about the mono-cluster flocking of system (1.3) under a singular kernel. These results are necessary for later sections.

Definition 2.1. We suppose that $t_0 \in (0, +\infty)$ is the first collision time of the system (1.3) ensemble, and the $l-th$ particle is one of the such colliding particles at time t_0. Then, we denote by $[l]$ the collection of all particles colliding with the $l-th$ particle at time t_0,

$$[l] := \{i \in [N] | \lim_{t \to t_0 -} \|x_i(t) - x_l(t)\| = 0\}.$$

For $\forall t \in [0, t_0)$ and $\forall i \notin [l]$, we define the constant δ such that δ is a strictly positive real number satisfying $\|x_l(t) - x_i(t)\| \geq \delta > 0$. Then, we define the following $L^\infty -$ diameters as follows:

$$D_{X,[l]} := \max_{i,j \in [l]} \|x_i - x_j\|, \quad D_{V,[l]} := \max_{i,j \in [l]} \|v_i - v_j\|, \quad A_{[l]}(v) := \min_{i,j \in [l]} \langle v_i, v_j \rangle.$$

Proposition 2.2. [19] Let (X, V, T) be a solution to system (1.3). If $A(v_0) > 0$, then sub-ensemble diameters satisfy the following system of dissipative differential inequalities: For *a.e.* $t \in (0, t_0)$,

$$\begin{cases} \left| \dfrac{d}{dt} D_{X,[l]} \right| \leq D_{V,[l]}, \quad t > 0, \\ \dfrac{d}{dt} D_{V,[l]} \leq -\dfrac{\kappa_1 |[l]| A_{[l]}(v^0)}{(N-1) T_M^\infty} \phi(D_{X,[l]}) D_{V,[l]} + \dfrac{4\kappa_1 (N - |[l]|) \phi(\delta)}{(N-1) T_m^\infty}, \quad t > 0, \\ \dfrac{dD_T}{dt} \leq \dfrac{\kappa_2 (N-2) \zeta(D_X)}{(N-1)(T_M^\infty)^2} D_T. \end{cases} \tag{2.3}$$

Proposition 2.3 (Collision avoidance)**.** [19] We suppose that communication weight and initial data satisfy $\lambda > 1, A(v_0) > 0, \min_{1 \leq i \neq j \leq N} \|x_i(0) - x_j(0)\| > 0$, and let (X, V, T) be a solution to the system (1.3). Then, collision avoidance occurs, *i.e.*, $x_i(t) \neq x_j(t)\ \ (i \neq j, i, j \in [N], t \geq 0)$.

Proposition 2.4. [19] We suppose that communication weight and initial data satisfy the following conditions:

(1) The parameter λ and initial configuration satisfy $\lambda > 1, A(v^0) > 0, \min_{1\le i\neq j\le N} \|x_i(0) - x_j(0)\| > 0$.

(2) If there exists a positive constant D_X^∞ such that $D_X(0) + \dfrac{D_V(0)T_M^\infty}{\kappa_1 A(v^0)\phi(D_X^\infty)} < D_X^\infty$ and (X, V, T) is a solution to system (1.3), then thermodynamic flocking emerges asymptotically:

(i) $\sup_{0\le t<+\infty} D_X(t) \le D_X^\infty, \quad D_V(t) \le D_V(0)\exp\left(-\dfrac{\kappa_1 A(v^0)\phi(D_X^\infty)}{T_M^\infty}\right)$.

(ii) $D_T(t) \le D_T(0)\exp\left(-\dfrac{\kappa_2\zeta(D_X^\infty)(N-2)}{|T_M^\infty|^2(N-1)}\right)$.

(iii) (A uniform positive lower bound for relative distances) We suppose that $D_V(0) \le \dfrac{\kappa_1 A(v^0)}{T_M^\infty}\displaystyle\int_{D_X(0)}^{D_X^\infty}\phi(s)ds$ holds, and (X, V, T) is a global solution of the system (1.3). Then, there exists a strictly positive lower bound for the relative spatial distances, *i.e.*, $\delta_0 := \inf_{0\le t\le\infty}\min_{1\le i\neq j\le N}\|x_i(t) - x_j(t)\| > 0$.

3. Multi-cluster flocking of TCSUS

In this section, we derive suitable dissipative differential inequalities initially with respect to position, velocity, and temperature. By employing a bootstrapping technique with these inequalities, we apply appropriate sufficient conditions based on the initial conditions and system parameters to ensure multi-cluster flocking within system (1.4).

3.1. Dissipative structure

In the following, we derive several dissipative differential inequalities with respect to position-velocity-temperature to establish adequate frameworks based on system parameters and initial conditions.

Lemma 3.1 (Dissipative structure)**.** Assume that $Z_\alpha = \{(x_{\alpha i}, v_{\alpha i}, T_{\alpha i})\}_{i=1}^{N_\alpha}$ is a solution to the system (1.4). We define $\phi_M(t) := \max_{\alpha\neq\beta,i,j}\phi(\|x_{\beta j} - x_{\alpha i}\|)$ and $\zeta_M(t) := \max_{\alpha\neq\beta,i,j}\zeta(\|x_{\beta j} - x_{\alpha i}\|)$.

Then, we have the following three differential inequalities for *a.e.* $t \in (0, +\infty)$:

1. $\left|\dfrac{dD_X}{dt}\right| \le D_V$,
2. $\dfrac{dD_T}{dt} \le -\dfrac{\kappa_2(\min(N_1,\cdots,N_\alpha) - 2)\zeta(D_X)}{(N-1)(T_M^\infty)^2}D_T + \dfrac{2\kappa_2(n-1)N}{N-1}\zeta_M(\dfrac{1}{T_m^\infty} - \dfrac{1}{T_M^\infty})$,
3. $\dfrac{dD_V}{dt} \le -\dfrac{\kappa_1\min(N_1,\ldots,N_\alpha)A(v^0)\phi(D_X)}{(N-1)T_M^\infty}D_V + \dfrac{2\kappa_1(n-1)N\phi_M}{(N-1)T_m^\infty}$.

Proof. The first assertion follows directly from the Cauchy-Schwarz inequality. To prove the second assertion, we select two indices, M_t and m_t depending on t, such that

$$D_{T_\alpha}(t) = T_{\alpha M_t}(t) - T_{\alpha m_t}(t), \quad 1 \le m_t, M_t \le N_\alpha.$$

Then, for *a.e.* $t \in (0, +\infty)$, one can show that by using the definitions of M_t and m_t,

$$\frac{dD_{T_\alpha}}{dt} = \frac{\kappa_2}{N-1}\sum_{\substack{j=1\\ j\neq i}}^{N_\alpha}\zeta(\|x_{\alpha M_t}-x_{\alpha j}\|)(\frac{1}{T_{\alpha M_t}}-\frac{1}{T_{\alpha j}}) - \frac{\kappa_2}{N-1}\sum_{\substack{j=1\\ j\neq i}}^{N_\alpha}\zeta(\|x_{\alpha m_t}-x_{\alpha j}\|)(\frac{1}{T_{\alpha m_t}}-\frac{1}{T_{\alpha j}})$$
$$+\frac{\kappa_2}{N-1}\sum_{\beta\neq\alpha}\sum_{j=1}^{N_\beta}\zeta(\|x_{\alpha M_t}-x_{\beta j}\|)(\frac{1}{T_{\alpha M_t}}-\frac{1}{T_{\beta j}}) - \frac{\kappa_2}{N-1}\sum_{\beta\neq\alpha}\sum_{j=1}^{N_\beta}\zeta(\|x_{\alpha m_t}-x_{\beta j}\|)(\frac{1}{T_{\alpha m_t}}-\frac{1}{T_{\beta j}})$$
$$=: \mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4.$$

(i) (Estimate of $\mathcal{I}_1+\mathcal{I}_2$) In the same method as the proof of Proposition 2.2, the following inequality holds for *a.e.* $t\in(0,+\infty)$:

$$\mathcal{I}_1+\mathcal{I}_2 \le -\frac{\kappa_2(N_\alpha-2)\zeta(D_{X_\alpha})}{(N-1)(T_M^\infty)^2}D_{T_\alpha}.$$

(ii) (Estimate of $\mathcal{I}_3+\mathcal{I}_4$) From Proposition 2.1 and the definitions of ζ_M and ζ_m, we derive the following inequality for *a.e.* $t\in(0,+\infty)$:

$$\mathcal{I}_3+\mathcal{I}_4 \le \frac{\kappa_2}{N-1}\left|\sum_{\beta\neq\alpha}\sum_{j=1}^{N_\beta}\zeta(\|x_{\alpha M_t}-x_{\beta j}\|)(\frac{1}{T_{\alpha M_t}}-\frac{1}{T_{\beta j}})\right| + \frac{\kappa_2}{N-1}\left|\sum_{\beta\neq\alpha}\sum_{j=1}^{N_\beta}\zeta(\|x_{\alpha m_t}-x_{\beta j}\|)(\frac{1}{T_{\alpha m_t}}-\frac{1}{T_{\beta j}})\right|$$
$$\le \frac{2\kappa_2(N-N_\alpha)\zeta_M}{N-1}(\frac{1}{T_m^\infty}-\frac{1}{T_M^\infty}).$$

Thus, combining $\mathcal{I}_1+\mathcal{I}_2$ and $\mathcal{I}_3+\mathcal{I}_4$ yields that, for *a.e.* $t\in(0,+\infty)$,

$$\frac{dD_{T_\alpha}}{dt} \le -\frac{\kappa_2(N_\alpha-2)\zeta(D_{X_\alpha})}{(N-1)(T_M^\infty)^2}D_{T_\alpha} + \frac{2\kappa_2(N-N_\alpha)\zeta_M}{N-1}(\frac{1}{T_m^\infty}-\frac{1}{T_M^\infty}).$$

Therefore, summing for α from 1 to n to the above inequality, we obtain that for *a.e.* $t\in(0,+\infty)$,

$$\frac{dD_T}{dt} \le -\frac{\kappa_2(\min(N_1,\cdots,N_\alpha)-2)\zeta(D_X)}{(N-1)(T_M^\infty)^2}D_T + \frac{2\kappa_2(n-1)N}{N-1}\zeta_M(\frac{1}{T_m^\infty}-\frac{1}{T_M^\infty}).$$

To verify the third assertion, we select two indices M_t and m_t depending on t which satisfy

$$D_{V_\alpha}(t) = \|v_{\alpha M_t}(t)-v_{\alpha m_t}(t)\|, \quad 1\le m_t, M_t\le N_\alpha.$$

Hence, we attain that for *a.e.* $t\in(0,+\infty)$,

$$\frac{1}{2}\frac{dD_{V_\alpha}^2}{dt} = \langle v_{\alpha M_t}-v_{\alpha m_t}, \dot v_{\alpha M_t}-\dot{v_{\alpha m_t}}\rangle$$
$$= \Big\langle v_{\alpha M_t}-v_{\alpha m_t}, \frac{\kappa_1}{N-1}\sum_{j=1}^{N_\alpha}\phi_{\alpha M_t j}\Big(\frac{(v_{\alpha j}-\langle v_{\alpha M_t}, v_{\alpha j}\rangle v_{\alpha M_t})}{T_{\alpha j}}\Big)$$
$$- \frac{\kappa_1}{N-1}\sum_{j=1}^{N_\alpha}\phi_{\alpha m_t j}\Big(\frac{(v_{\alpha j}-\langle v_{\alpha m_t}, v_{\alpha j}\rangle v_{\alpha m_t})}{T_{\alpha j}}\Big)\Big\rangle$$
$$+ \Big\langle v_{\alpha M_t}-v_{\alpha m_t}, \frac{\kappa_1}{N-1}\sum_{\beta\neq\alpha}\sum_{j=1}^{N_\alpha}\phi(\|x_{\alpha M_t - x_{\beta j}}\|)\Big(\frac{(v_{\beta j}-\langle v_{\alpha M_t}, v_{\beta j}\rangle v_{\alpha M_t})}{T_{\beta j}}\Big)$$
$$- \frac{\kappa_1}{N-1}\sum_{\beta\neq\alpha}\sum_{j=1}^{N_\alpha}\phi(\|x_{\alpha m_t - x_{\beta j}}\|)\Big(\frac{(v_{\beta j}-\langle v_{\alpha m_t}, v_{\beta j}\rangle v_{\alpha m_t})}{T_{\beta j}}\Big)\Big\rangle =: J_1+J_2.$$

(iii) (Estimate of J_1) Replacing $[l]$ with $[N_\alpha]$ and in the same method as the proof of Proposition 2.2, for *a.e.* $t \in (0, +\infty)$, we have

$$J_1 \le -\frac{\kappa_1 N_\alpha A(v^0)}{(N-1)T_M^\infty}\phi(D_{X_\alpha})D_{V_\alpha}^2.$$

(iv) (Estimate of J_2) We employ the identities

$$\|v_{\beta j} - \langle v_{\alpha M_t}, v_{\beta j}\rangle v_{\alpha M_t}\| \le 1, \quad \|v_{\beta j} - \langle v_{\alpha m_t}, v_{\beta j}\rangle v_{\alpha m_t}\| \le 1$$

with the Cauchy-Schwarz inequality and Proposition 2.3 to estimate that for *a.e.* $t \in (0, +\infty)$,

$$\begin{aligned} J_2 \le & \frac{\kappa_1 D_{V_\alpha}}{N-1}\left\| \sum_{\beta\neq\alpha}\sum_{j=1}^{N_\beta}\phi(\|x_{\alpha M_t} - x_{\beta j}\|)\left(\frac{v_{\beta j} - \langle v_{\alpha M_t}, v_{\beta j}\rangle v_{\alpha M_t}}{T_{\beta j}}\right)\right\| \\ & + \frac{\kappa_1 D_{V_\alpha}}{N-1}\left\| \sum_{\beta\neq\alpha}\sum_{j=1}^{N_\beta}\phi(\|x_{\alpha m_t} - x_{\beta j}\|)\left(\frac{v_{\beta j} - \langle v_{\alpha m_t}, v_{\beta j}\rangle v_{\alpha m_t}}{T_{\beta j}}\right)\right\| \le \frac{2\kappa_1(N-N_\alpha)\phi_M D_{V_\alpha}}{(N-1)T_m^\infty}. \end{aligned}$$

Then, we combine J_1 and J_2 to derive that for *a.e.* $t \in (0, +\infty)$,

$$\frac{dD_{V_\alpha}}{dt} \le -\frac{\kappa_1 N_\alpha A(v^0)}{(N-1)T_M^\infty}\phi(D_{X_\alpha})D_{V_\alpha} + \frac{2\kappa_1(N-N_\alpha)\phi_M}{(N-1)T_m^\infty}.$$

Summing the above inequality from $\alpha = 1$ to n, we obtain that

$$\frac{dD_V}{dt} \le -\frac{\kappa_1 \min(N_1, \ldots, N_\alpha)A(v^0)\phi(D_X)}{(N-1)T_M^\infty}D_V + \frac{2\kappa_1(n-1)N\phi_M}{(N-1)T_m^\infty},$$

since the monotonicity of ϕ implies that $\min(\phi(D_{X_1}), \ldots, \phi(D_{X_\alpha})) \ge \phi(D_X)$. Finally, we demonstrate the third assertion. □

3.2. *Multi-cluster flocking*

The proofs of Theorems 1.1–1.3 are presented in this subsection. First of all, we give a brief comment regarding $(\mathcal{H})$.

The assumption $(\mathcal{H}_1)$ is the sufficient condition which guarantees group formation within each cluster. The assumption $(\mathcal{H}_2)$ implies that the initial positions for each cluster group should be sufficiently separated from each other to achieve a multi-cluster flocking result. In fact, if $v_{\alpha i}^k(0)$ is covered by $I_\alpha := [a_\alpha, b_\alpha]$, then we can take sufficiently small κ_1 such that $[v_{\alpha i}^k(0) - \Lambda_\alpha, v_{\alpha i}^k(0) + \Lambda_\alpha] \subset I_\alpha$ because Λ_α is linearly proportional to κ_1.

The assumption $(\mathcal{H}_3)$ ensures that a uniformly strictly positive lower bound exists for relative distances. Here, we can find the admissible data meeting the assumption $(\mathcal{H}_3)$ requirements when κ_1 is sufficiently small. Moreover, under sufficiently large r_0, suitable temperature initial data and small coupling strength regime, we can check that the sufficient framework $(\mathcal{H})$ is admissible data.

Lemma 3.2. Assume that $Z_\alpha = \{(x_{\alpha i}, v_{\alpha i}, T_{\alpha i})\}_{i=1}^{N_\alpha}$ is a solution to the system (1.4) and suppose that $(\mathcal{H})$ holds. We define the following set:

$$S_2 := \left\{ s > 0 \,\middle|\, \min_{\alpha\neq\beta,i,j}\|x_{\alpha i}(t) - x_{\beta j}(t)\| \ge \left(\min_{1\le\alpha\le n-1} d(I_\alpha, I_{\alpha+1})\right)t + \frac{r_0}{2}, \quad t \in [0, s)\right\}.$$

Then, S_2 is nonempty, and it follows that $D_X(t) \le D_X^\infty, t \in [0, T^*]$, where $T^* := \sup S_2$.

Proof. We observe that S_2 is nonempty due to the assumption $(\mathcal{H}_2)$ and the continuity of $\|x_{\alpha i}(t) - x_{\beta j}(t)\|$. Then, we just need to prove $D_X(t) \le D_X^\infty, t \in [0, T^*]$. First of all, we consider the following set:

$$S_3 := \{s > 0 \mid \forall t \in [0, s], D_X(t) \le D_X^\infty, \quad s \le T^*\}. \tag{3.1}$$

We set $\sup S_3 =: T^{**}$. Then, we have $D_X(T^{**}) = D_X^\infty$ and suppose that $T^{**} < T^*$ for the proof by contradiction. Then, for $\forall t \in [0, T^{**}]$, one has

$$-\frac{\kappa_1 \min(N_1, \cdots, N_\alpha)\, \phi(D_X)}{(N-1)T_M^\infty} \le -\frac{\kappa_1 \min(N_1, \cdots, N_\alpha)\, \phi\left(D_X^\infty\right)}{(N-1)T_M^\infty}. \tag{3.2}$$

Thus, for *a.e.* $t \in (0, T^{**})$, the second assertion of Lemma 3.1 and the above estimates lead to the following inequalities:

$$\begin{aligned} \frac{dD_V}{dt} &\le -\frac{\kappa_1 \min(N_1, \cdots, N_\alpha)\, A(v^0)\phi(D_X)}{(N-1)T_M^\infty} D_V + \frac{2\kappa_1(n-1)N\phi_M}{(N-1)T_m^\infty} \\ &\le -\frac{\kappa_1 \min(N_1, \cdots, N_\alpha)\, A(v^0)\phi\left(D_X^\infty\right)}{(N-1)T_M^\infty} D_V + \frac{2\kappa_1(n-1)N\phi_M}{(N-1)T_m^\infty} \\ &= -\Lambda_0 D_V + \frac{2\kappa_1(n-1)N\phi_M}{(N-1)T_m^\infty}. \end{aligned} \tag{3.3}$$

For $t \in [0, T^{**}]$, we integrate inequality (3.3) from time 0 to t through multiplying both sides of the inequality by the integral factor $\exp(\Lambda_0 t)$.

$$\begin{aligned} D_V(t) &\le D_V(0)\exp(-\Lambda_0 t) + \int_0^t \frac{2\kappa_1(n-1)N\phi_M}{(N-1)T_m^\infty} \exp\Big(\Lambda_0(s-t)\Big)ds \\ &= D_V(0)\exp(-\Lambda_0 t) + \left(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t\right) \frac{2\kappa_1(n-1)N\phi_M}{(N-1)T_m^\infty} \exp\Big(\Lambda_0(s-t)\Big)ds \\ &\le D_V(0)\exp(-\Lambda_0 t) + \frac{2\kappa_1(n-1)N}{(N-1)\Lambda_0 T_m^\infty}\phi\left(\frac{r_0}{2}\right)\left[\exp\left(-\frac{\Lambda_0}{2}t\right) - \exp(-\Lambda_0 t)\right] \\ &\qquad + \frac{2\kappa_1(n-1)N}{(N-1)\Lambda_0 T_m^\infty}\phi\left(\frac{(\min_{1\le\alpha\le n-1} d(I_\alpha, I_{\alpha+1}))\,t + r_0}{2}\right)\left[1 - \exp\left(-\frac{\Lambda_0}{2}t\right)\right] \\ &\le D_V(0)\exp(-\Lambda_0 t) + \frac{2\kappa_1(n-1)N}{(N-1)\Lambda_0 T_m^\infty}\phi\left(\frac{r_0}{2}\right)\exp\left(-\frac{\Lambda_0}{2}t\right) \\ &\qquad + \frac{2\kappa_1(n-1)N}{(N-1)\Lambda_0 T_m^\infty}\phi\left(\frac{(\min_{1\le\alpha\le n-1} d(I_\alpha, I_{\alpha+1}))\,t + r_0}{2}\right), \end{aligned} \tag{3.4}$$

where we used the definition of S_3 and the fact that $\phi_M \le \phi\left(\left(\min_{1\le\alpha\le n-1} d(I_\alpha, I_{\alpha+1})\right)t + \frac{r_0}{2}\right)$.

In the latter case, we estimate from inequality (3.4) that for $t \in [0, T^{**}]$,

$$D_X(t) \le D_X(0) + \int_0^t D_V(s)ds$$

$$
\begin{aligned}
&\le D_X(0) + \int_0^t \Bigg[D_V(0)\exp(-\Lambda_0 s) + \frac{2\kappa_1 N(n-1)}{(N-1)T_m^\infty \Lambda_0}\exp\left(-\frac{\Lambda_0}{2}s\right)\phi\left(\frac{r_0}{2}\right) \\
&\quad + \frac{2\kappa_1 N(n-1)}{(N-1)T_m^\infty \Lambda_0}\phi\left(\frac{(\min_{1\le\alpha\le n-1} d(I_\alpha, I_{\alpha+1}))\, s + r_0)}{2}\right)\Bigg] ds \\
&< D_X(0) + \Lambda \le D_X^\infty. \qquad (3.5)
\end{aligned}
$$

Accordingly, $D_X(T^{**}) < D_X^\infty$, which is contradictory to $D_X(T^{**}) = D_X^\infty$. Finally, $\sup S_3 = T^{**} = T^*$. We have reached the desired lemma. □

Proof of Theorem 1.1. Following Lemma 3.2, we just need to prove that $T^* = \infty$, which is equivalent to

$$
\min_{\alpha\neq\beta,i,j} \left\| x_{\alpha i}(t) - x_{\beta j}(t) \right\| \ge \left(\min_{1\le\alpha\le n-1} d(I_\alpha, I_{\alpha+1}) \right) t + \frac{r_0}{2}, \quad t \in (0, +\infty). \qquad (3.6)
$$

For the proof by contradiction, we suppose that $T^* < \infty$. From the definition of S_2, we select four indices that satisfy

$$
1 \le \alpha^* < \beta^* \le n, \quad i^* \in \{1, \cdots, N_{\alpha^*}\} \quad \text{and} \quad j^* \in \left\{1, \cdots, N_{\beta^*}\right\} \qquad (3.7)
$$

such that

$$
\left\| x_{\alpha^* i^*}(T^*) - x_{\beta^* j^*}(T^*) \right\| = \left(\min_{1\le\alpha\le n-1} d(I_\alpha, I_{\alpha+1}) \right) T^* + \frac{r_0}{2}. \qquad (3.8)
$$

Then, we show that for the $k \in \{1, \cdots, d\}$ chosen *in* $(\mathcal{H}_0)$,

$$
\begin{aligned}
\left\| x_{\alpha^* i^*}(T^*) - x_{\beta^* j^*}(T^*) \right\| &\ge x^k_{\beta^* j^*}(T^*) - x^k_{\alpha i^*}(T^*) \\
&= x^k_{\beta^* j^*}(0) - x^k_{\alpha^* i^*}(0) + \int_0^{T^*} \left(v^k_{\beta^* j^*}(t) - v^k_{\alpha^* i^*}(t) \right) dt \\
&\ge r_0 + \int_0^{T^*} \left(v^k_{\beta^* j^*}(t) - v^k_{\alpha^* i^*}(t) \right) dt.
\end{aligned}
$$

From the third assertion of Proposition 2.3, we set a positive number δ_0 such that $\delta_0 := \inf_{0\le t\le\infty} \min_{1\le i\neq j\le N} \|x_i(t) - x_j(t)\| > 0$.

Next, we integrate the second equation of system (1.4) and employ the following relation:

$$
1 - \langle v_{\alpha i}, v_{\alpha j} \rangle = \frac{\|v_{\alpha i} - v_{\alpha j}\|^2}{2}
$$

and

$$
\left\| v_{\alpha j} - \left\langle v_{\alpha i}, v_{\alpha j} \right\rangle v_{\alpha i} \right\|^2 = 1 - \left\langle v_{\alpha i}, v_{\alpha j} \right\rangle^2 = \left(1 - \left\langle v_{\alpha i}, v_{\alpha j} \right\rangle\right)\left(1 + \left\langle v_{\alpha i}, v_{\alpha j} \right\rangle\right) \le D^2_{V_\alpha}
$$

to attain that for $t \in [0, T^*]$,

$$
\begin{aligned}
\left| v^k_{\alpha i}(t) - v^k_{\alpha i}(0) \right| &\le \|v_{\alpha i}(t) - v_{\alpha i}(0)\| \le \int_0^t \|\dot{v}_{\alpha i}\| ds \\
&\le \frac{\kappa_1 (N_\alpha - 1)\phi(\delta_0)}{(N-1)T_m^\infty} \int_0^t D_{V_\alpha}(s) ds + \frac{\kappa_1 (N - N_\alpha)}{(N-1)T_m^\infty} \int_0^t \phi_M(s) ds \\
&\le \frac{\kappa_1 (N_\alpha - 1)\phi(\delta_0)}{(N-1)T_m^\infty} D_{V_\alpha}(s) ds + \frac{\kappa_1 (N - N_\alpha)}{(N-1)T_m^\infty} \int_0^\infty \phi\left(\min_{1\le\alpha\le n-1} d(I_\alpha, I_{\alpha+1})\, s + \frac{r_0}{2} \right) ds
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{\kappa_1(N_\alpha-1)\phi(\delta_0)}{(N-1)T_m^\infty}D_V(s)ds+\frac{\kappa_1(N-N_\alpha)}{(N-1)T_m^\infty}\int_0^\infty \phi\left(\min_{1\le\alpha\le n-1} d(I_\alpha,I_{\alpha+1})\,s+\frac{r_0}{2}\right)ds\\
&\le \frac{\kappa_1(N_\alpha-1)\phi(\delta_0)}{(N-1)T_m^\infty}\Lambda+\frac{\kappa_1(N-N_\alpha)}{(N-1)T_m^\infty}\int_0^\infty \phi\left(\min_{1\le\alpha\le n-1} d(I_\alpha,I_{\alpha+1})\,s+\frac{r_0}{2}\right)ds\\
&= \frac{\kappa_1(N_\alpha-1)\phi(\delta_0)}{(N-1)T_m^\infty}\Lambda+\frac{\kappa_1(N-N_\alpha)}{(N-1)T_m^\infty(\min_{1\le\alpha\le n-1} d(I_\alpha,I_{\alpha+1}))}\int_{\frac{r_0}{2}}^\infty \phi(s)ds=:\Lambda_\alpha,
\end{aligned}
$$

where we used $\phi \le \phi(\delta_0)$, $\left\|v_{\beta j}-\left\langle v_{\alpha i},v_{\beta j}\right\rangle v_{\alpha i}\right\|\le 1$, and Λ was estimated in inequality (3.5). Therefore, it follows by $(\mathcal{H}_2)$ that for $\alpha=1,\cdots,n$,

$$
\begin{aligned}
v_{\alpha i}^k(0)+\Lambda_\alpha &\ge v_{\alpha i}^k(0)+\left|v_{\alpha i}^k(t)-v_{\alpha i}^k(0)\right|\ge v_{\alpha i}^k(t)=v_{\alpha i}^k(0)+v_{\alpha i}^k(t)-v_{\alpha i}^k(0)\\
&\ge v_{\alpha i}^k(0)-\left|v_{\alpha i}^k(t)-v_{\alpha i}^k(0)\right|\ge v_{\alpha i}^k(0)-\Lambda_\alpha \Longrightarrow v_{\alpha i}^k(t)\in I_\alpha.
\end{aligned}
$$

By using the assumption $(\mathcal{H}_2)$, we derive that

$$
\begin{aligned}
\left\|x_{\alpha^* i^*}(T^*)-x_{\beta^* j^*}(T^*)\right\| &\ge r_0+\int_0^{T^*}\left(v_{\beta^* j^*}^k(t)-v_{\alpha^* i^*}^k(t)\right)dt\\
&> \frac{r_0}{2}+\min_{1\le\alpha\le n-1} d(I_\alpha,I_{\alpha+1})\,T_*,
\end{aligned}
$$

which gives a contradiction to $T^*<\infty$. Consequently, we conclude that $T^*=\infty$. Subsequently, we claim that $T^*=\infty$, which is crucial to derive the multi-cluster flocking estimate of the system (1.4). □

Proof of Theorem 1.2. We apply the second assertion of Lemma 3.1, the definition of the set S_2, and Theorem 1.1 to have that for *a.e.* $t\in(0,+\infty)$,

$$
\begin{aligned}
\frac{dD_V}{dt} &\le -\Lambda_0 D_V+\frac{2\kappa_1(n-1)N}{T_m^\infty(N-1)}\phi_M\\
&\le -\Lambda_0 D_V+\frac{2\kappa_1(n-1)N}{T_m^\infty(N-1)}\phi\left(\frac{r_0}{2}+\min_{1\le\alpha\le n-1} d(I_\alpha,I_{\alpha+1})\,t\right).
\end{aligned}
\tag{3.9}
$$

Similar to inequality (3.4), we recall that for $t\in(0,+\infty)$,

$$
\begin{aligned}
D_V(t)\le & D_V(0)\exp(-\Lambda_0 t)+\frac{2\kappa_1(n-1)N}{T_m^\infty\Lambda_0(N-1)}\exp\left(-\frac{\Lambda_0}{2}t\right)\phi\left(\frac{r_0}{2}\right)\\
&+\frac{2\kappa_1(n-1)N}{T_m^\infty\Lambda_0(N-1)}\phi\left(\frac{(\min_{1\le\alpha\le n-1} d(I_\alpha,I_{\alpha+1}))\,t+r_0}{2}\right).
\end{aligned}
\tag{3.10}
$$

Hence, we reach the desired first assertion.

To prove the second assertion, we employ the third assertion of Lemma 3.1 and Theorem 1.1 to get that for $t\in(0,+\infty)$,

$$
\begin{aligned}
\frac{dD_T}{dt} &\le -\frac{\kappa_2(\min(N_1,\cdots,N_\alpha)-2)\zeta(D_X)}{(N-1)\left(T_M^\infty\right)^2}D_T+\frac{2\kappa_2(n-1)N}{N-1}\zeta_M\left(\frac{1}{T_m^\infty}-\frac{1}{T_M^\infty}\right)\\
&\le -\bar{\Lambda}_0 D_T+\frac{2\kappa_2(n-1)N}{N-1}\left(\frac{1}{T_m^\infty}-\frac{1}{T_M^\infty}\right)\zeta\left(\left(\min_{1\le\alpha\le n-1} d(I_\alpha,I_{\alpha+1})\right)t+\frac{r_0}{2}\right).
\end{aligned}
\tag{3.11}
$$

We use Gronwall's lemma to yield that for $t \in (0, +\infty)$,

$$\begin{aligned}
D_T(t) &\le D_T(0)\exp(-\bar{\Lambda}_0 t) + \int_0^t \frac{2\kappa_2(n-1)N\zeta_M}{N-1}\left(\frac{1}{T_m^\infty} - \frac{1}{T_M^\infty}\right)\exp\Big(\bar{\Lambda}_0(s-t)\Big)ds \\
&= D_T(0)\exp(-\bar{\Lambda}_0 t) + \left(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t\right)\frac{2\kappa_2(n-1)N\zeta_M}{N-1}\left(\frac{1}{T_m^\infty} - \frac{1}{T_M^\infty}\right)\exp\Big(\bar{\Lambda}_0(s-t)\Big)ds \\
&\le D_T(0)\exp(-\bar{\Lambda}_0 t) + \frac{2\kappa_2(n-1)N}{N-1}\left(\frac{1}{T_m^\infty} - \frac{1}{T_M^\infty}\right)\zeta\left(\frac{r_0}{2}\right)\left[\exp\left(-\frac{\bar{\Lambda}_0}{2}t\right) - \exp(-\bar{\Lambda}_0 t)\right] \\
&\quad + \frac{2\kappa_2(n-1)N}{N-1}\left(\frac{1}{T_m^\infty} - \frac{1}{T_M^\infty}\right)\zeta\left(\frac{(\min_{1\le\alpha\le n-1} d(I_\alpha, I_{\alpha+1}))\,t + r_0}{2}\right)\left[1 - \exp\left(-\frac{\bar{\Lambda}_0}{2}t\right)\right] \\
&\le D_T(0)\exp\Big(-\bar{\Lambda}_0 t\Big) + \frac{2\kappa_2(n-1)N}{N-1}\left(\frac{1}{T_m^\infty} - \frac{1}{T_M^\infty}\right)\exp\left(-\frac{\bar{\Lambda}_0}{2}t\right)\zeta\left(\frac{r_0}{2}\right) \\
&\quad + \frac{2\kappa_2(n-1)N}{N-1}\left(\frac{1}{T_m^\infty} - \frac{1}{T_M^\infty}\right)\zeta\left(\frac{(\min_{1\le\alpha\le n-1} d(I_\alpha, I_{\alpha+1}))\,t + r_0}{2}\right).
\end{aligned} \tag{3.12}$$

We conclude the desired second assertion. □

As a direct consequence, we present the following result that the velocity and temperature of each agent in each cluster group converge to some same non-negative value, respectively. We prove the following lemma at first.

Lemma 3.3. Assume that $Z_\alpha = \{x_{\alpha i}, v_{\alpha i}, T_{\alpha i}\}_{i=1}^{N_\alpha}$ is a solution to the system (1.4). Each local average $(x_\alpha^{cen}, v_\alpha^{cen}, T_\alpha^{cen})$ then satisfies the following relations:

$$\begin{cases}
\dfrac{dx_\alpha^{cen}}{dt} = v_\alpha^{cen}, \quad t > 0, \quad \alpha \in \{1, \cdots, n\}, \quad n \ge 3, \\
N_\alpha \dot{v}_\alpha^{cen} = \dfrac{\kappa_1}{N-1}\displaystyle\sum_{1\le i\ne j\le N_\alpha}\phi(\|x_{\alpha i} - x_{\alpha j}\|)\frac{v_{\alpha i}\|v_{\alpha j} - v_{\alpha i}\|^2}{2T_{\alpha j}} \\
\qquad + \dfrac{\kappa_1}{N-1}\displaystyle\sum_{\beta\ne\alpha}\sum_{i=1}^{N_\alpha}\sum_{j=1}^{N_\beta}\phi(\|x_{\alpha i} - x_{\beta j}\|)\left(v_{\beta j} - v_{\alpha i} + \frac{v_{\alpha i}\|v_{\alpha j} - v_{\alpha i}\|^2}{2}\right)\frac{1}{T_{\beta j}}, \\
N_\alpha \dot{T}_\alpha^{cen} = \dfrac{\kappa_2}{N-1}\displaystyle\sum_{\beta\ne\alpha}\sum_{i=1}^{N_\alpha}\sum_{j=1}^{N_\beta}\zeta(\|x_{\alpha i} - x_{\beta j}\|)(\frac{1}{T_{\alpha i}} - \frac{1}{T_{\beta j}}).
\end{cases} \tag{3.13}$$

Proof. The first assertion is trivial. For the second assertion, we take $\sum_{i=1}^{N_\alpha}$ to $\dot{v}_{\alpha i}$ and use the standard trick of interchanging i and j to obtain that

$$\begin{aligned}
\frac{\|v_{\alpha i} - v_{\alpha j}\|^2}{2} &= \frac{1}{2}\langle v_{\alpha i} - v_{\alpha j}, v_{\alpha i} - v_{\alpha j}\rangle \\
&= \frac{1}{2}\Big(\langle v_{\alpha i}, v_{\alpha i}\rangle + \langle v_{\alpha j}, v_{\alpha j}\rangle - 2\langle v_{\alpha i}, v_{\alpha j}\rangle\Big) \\
&= 1 - \langle v_{\alpha i}, v_{\alpha j}\rangle.
\end{aligned} \tag{3.14}$$

Therefore, we have

$$N_\alpha \dot{v}_\alpha^{cen} = \frac{\kappa_1}{N-1}\sum_{1\le i\ne j\le N_\alpha}\phi(\|x_{\alpha i} - x_{\alpha j}\|)\left(\frac{v_{\alpha j} - v_{\alpha i} + v_{\alpha i} - \langle v_{\alpha i}, v_{\alpha j}\rangle v_{\alpha i}}{T_{\alpha j}}\right)$$

$$
\begin{aligned}
&+\frac{\kappa_1}{N-1}\sum_{\beta\neq\alpha}\sum_{i=1}^{N_\alpha}\sum_{j=1}^{N_\beta}\phi(\|x_{\alpha i}-x_{\beta j}\|)\left(\frac{v_{\beta j}-v_{\alpha i}+v_{\alpha i}-\langle v_{\alpha i},v_{\beta j}\rangle v_{\alpha i}}{T_{\beta j}}\right)\\
=&\frac{\kappa_1}{N-1}\sum_{1\le i\neq j\le N_\alpha}\phi(\|x_{\alpha i}-x_{\alpha j}\|)\left(v_{\alpha j}-v_{\alpha i}+\frac{v_{\alpha i}\|v_{\alpha j}-v_{\alpha i}\|^2}{2}\right)\frac{1}{T_{\alpha j}}\\
&+\frac{\kappa_1}{N-1}\sum_{\beta\neq\alpha}\sum_{i=1}^{N_\alpha}\sum_{j=1}^{N_\beta}\phi(\|x_{\alpha i}-x_{\beta j}\|)\left(v_{\beta j}-v_{\alpha i}+\frac{v_{\alpha i}\|v_{\alpha j}-v_{\alpha i}\|^2}{2}\right)\frac{1}{T_{\beta j}}\\
=&\frac{\kappa_1}{N-1}\sum_{1\le i\neq j\le N_\alpha}\phi(\|x_{\alpha i}-x_{\alpha j}\|)\frac{v_{\alpha i}\|v_{\alpha j}-v_{\alpha i}\|^2}{2T_{\alpha j}}\\
&+\frac{\kappa_1}{N-1}\sum_{\beta\neq\alpha}\sum_{i=1}^{N_\alpha}\sum_{j=1}^{N_\beta}\phi(\|x_{\alpha i}-x_{\beta j}\|)\left(v_{\beta j}-v_{\alpha i}+\frac{v_{\alpha i}\|v_{\alpha j}-v_{\alpha i}\|^2}{2}\right)\frac{1}{T_{\beta j}}.
\end{aligned}\tag{3.15}
$$

For the third assertion, we take $\sum_{i=1}^{N_\alpha}$ to $\dot{T}_{\alpha i}$ and again use the standard trick as above. Finally, we prove the lemma. □

Proof of Theorem 1.3. According to Lemma 3.3,

$$
\begin{aligned}
N_\alpha\dot{v}_\alpha^{cen}=&\frac{\kappa_1}{N-1}\sum_{1\le i\neq j\le N_\alpha}\phi\left(\left\|x_{\alpha i}-x_{\alpha j}\right\|\right)\frac{v_{\alpha i}\left\|v_{\alpha j}-v_{\alpha i}\right\|^2}{2T_{\alpha j}}\\
&+\frac{\kappa_1}{N-1}\sum_{\beta\neq\alpha}\sum_{i=1}^{N_\alpha}\sum_{j=1}^{N_\beta}\phi\left(\left\|x_{\alpha i}-x_{\beta j}\right\|\right)\left(v_{\beta j}-v_{\alpha i}+\frac{v_{\alpha i}\left\|v_{\beta j}-v_{\alpha i}\right\|^2}{2}\right)\frac{1}{T_{\beta j}},
\end{aligned}\tag{3.16}
$$

and thus we have

$$
\begin{aligned}
v_\alpha^{cen}(t)=v_\alpha^{cen}(0)&+\frac{\kappa_1}{(N-1)N_\alpha}\sum_{1\le i\neq j\le N_\alpha}\int_0^t\phi\left(\left\|x_{\alpha i}-x_{\alpha j}\right\|\right)\frac{v_{\alpha i}\left\|v_{\alpha j}-v_{\alpha i}\right\|^2}{2T_{\alpha j}}ds\\
&+\frac{\kappa_1}{(N-1)N_\alpha}\sum_{\beta\neq\alpha}\sum_{i=1}^{N_\alpha}\sum_{j=1}^{N_\beta}\int_0^t\phi\left(\left\|x_{\alpha i}-x_{\beta j}\right\|\right)\left(v_{\beta j}-v_{\alpha i}+\frac{v_{\alpha i}\left\|v_{\beta j}-v_{\alpha i}\right\|^2}{2}\right)\frac{1}{T_{\beta j}}ds.
\end{aligned}\tag{3.17}
$$

For $\forall t_1,t_2\in(0,+\infty)$, we have

$$
\begin{aligned}
&\left\|v_\alpha^{cen}(t_2)-v_\alpha^{cen}(t_1)\right\|\\
\le&\frac{\kappa_1}{(N-1)N_\alpha}\sum_{1\le i\neq j\le N_\alpha}\int_{t_1}^{t_2}\phi\left(\left\|x_{\alpha i}-x_{\alpha j}\right\|\right)\frac{\|v_{\alpha i}\|\cdot\left\|v_{\alpha j}-v_{\alpha i}\right\|^2}{2T_{\alpha j}}ds\\
&+\frac{\kappa_1}{(N-1)N_\alpha}\sum_{\beta\neq\alpha}\sum_{i=1}^{N_\alpha}\sum_{j=1}^{N_\beta}\int_{t_1}^{t_2}\phi\left(\left\|x_{\alpha i}-x_{\beta j}\right\|\right)\left(\left\|v_{\beta j}-v_{\alpha i}\right\|+\frac{\|v_{\alpha i}\|\cdot\left\|v_{\beta j}-v_{\alpha i}\right\|^2}{2}\right)\frac{1}{T_{\beta j}}ds\\
\le&\frac{\kappa_1}{(N-1)N_\alpha}\sum_{1\le i\neq j\le N_\alpha}\int_{t_1}^{t_2}\phi\left(\delta_0\right)\frac{D_V^2}{2T_m^\infty}ds+\frac{\kappa_1}{(N-1)N_\alpha}\sum_{\beta\neq\alpha}\sum_{i=1}^{N_\alpha}\sum_{j=1}^{N_\beta}\int_{t_1}^{t_2}\phi\left(\left\|x_{\alpha i}-x_{\beta j}\right\|\right)\frac{\sqrt{2}+1}{T_m^\infty}ds
\end{aligned}
$$

$$\le \frac{C\kappa_1\phi(\delta_0)}{2T_m^\infty(N-1)N_\alpha}\sum_{1\le i\neq j\le N_\alpha}\int_{t_1}^{t_2}\phi^2\left(\frac{(\min_{1\le\alpha\le n-1} d(I_\alpha,I_{\alpha+1}))\,s+r_0}{2}\right)ds$$
$$+\frac{(\sqrt{2}+1)\kappa_1}{T_m^\infty(N-1)N_\alpha}\sum_{\beta\neq\alpha}\sum_{i=1}^{N_\alpha}\sum_{j=1}^{N_\beta}\int_{t_1}^{t_2}\phi\left(\left(\min_{1\le\alpha\le n-1} d(I_\alpha,I_{\alpha+1})\right)s+\frac{r_0}{2}\right)ds. \tag{3.18}$$

By employing the Cauchy convergence criterion and the existence of $\int_0^\infty \phi\left(\left(\min_{1\le\alpha\le n-1} d(I_\alpha,I_{\alpha+1})\right)s+\frac{r_0}{2}\right)ds$ and $\int_0^\infty \phi^2\left(\frac{(\min_{1\le\alpha\le n-1} d(I_\alpha,I_{\alpha+1}))\,s+r_0}{2}\right)ds$, it is straightforward to observe that $\left\|v_\alpha^{cen}(t_2)-v_\alpha^{cen}(t_1)\right\|$ can be arbitrarily small when both t_1 and t_2 are sufficiently large. Therefore, the existence of $\lim_{t\to\infty} v_\alpha^{cen}(t)$ is guaranteed.

By employing $v_\alpha^\infty := \lim_{t\to\infty} v_\alpha^{cen}(t)$ and

$$v_\alpha^{cen} = v_\alpha^{cen}(0)+\frac{\kappa_1}{(N-1)N_\alpha}\sum_{1\le i\neq j\le N_\alpha}\int_0^t \phi\left(\left\|x_{\alpha i}-x_{\alpha j}\right\|\right)\frac{v_{\alpha i}\left\|v_{\alpha j}-v_{\alpha i}\right\|^2}{2T_{\alpha j}}ds$$
$$+\frac{\kappa_1}{(N-1)N_\alpha}\sum_{\beta\neq\alpha}\sum_{i=1}^{N_\alpha}\sum_{j=1}^{N_\beta}\int_0^t \phi\left(\left\|x_{\alpha i}-x_{\beta j}\right\|\right)\left(v_{\beta j}-v_{\alpha i}+\frac{v_{\alpha i}\left\|v_{\beta j}-v_{\alpha i}\right\|^2}{2}\right)\frac{1}{T_{\beta j}}ds,$$

we have that

$$\left\|v_\alpha^{cen}(t)-v_\alpha^\infty\right\| \le \frac{\kappa_1}{(N-1)N_\alpha}\sum_{1\le i\neq j\le N_\alpha}\int_t^\infty \phi\left(\left\|x_{\alpha i}-x_{\alpha j}\right\|\right)\frac{\|v_{\alpha i}\|\cdot\left\|v_{\alpha j}-v_{\alpha i}\right\|^2}{2T_{\alpha j}}ds$$
$$+\frac{\kappa_1}{(N-1)N_\alpha}\sum_{\beta\neq\alpha}\sum_{i=1}^{N_\alpha}\sum_{j=1}^{N_\beta}\int_t^\infty \phi\left(\left\|x_{\alpha i}-x_{\beta j}\right\|\right)\left(\|v_{\beta j}-v_{\alpha i}\|+\frac{\|v_{\alpha i}\|\cdot\left\|v_{\beta j}-v_{\alpha i}\right\|^2}{2}\right)\frac{1}{T_{\beta j}}ds. \tag{3.19}$$

Then, the multi-flocking estimate studied in Theorem 1.1 and Theorem 1.2 and the monotonicity and non-negativity of ϕ imply that

$$\left\|v_\alpha^{cen}(t)-v_\alpha^\infty\right\| \le O\left(\int_t^\infty \phi\left(\frac{(\min_{1\le\alpha\le n-1} d(I_\alpha,I_{\alpha+1}))\,s+r_0}{2}\right)ds\right)$$
$$\le O(1)\frac{1}{t^{\lambda-1}} \to 0,\ t\to\infty. \tag{3.20}$$

Drawing from from Theorem 1.1 and Theorem 1.2, we observe that

$$\left\|v_{\alpha i}(t)-v_\alpha^{cen}(t)\right\| = O\left(\exp\left(-\frac{\Lambda_0}{2}t\right)+\phi\left(\frac{(\min_{1\le\alpha\le n-1} d(I_\alpha,I_{\alpha+1}))\,s+r_0}{2}\right)\right) \le O(1)\frac{1}{t^\lambda},\ t\to\infty. \tag{3.21}$$

We combine the above estimates to derive that for all $\alpha\in[n]$ and $i\in[N_\alpha]$,

$$\left\|v_{\alpha i}(t)-v_\alpha^\infty\right\| \le \left\|v_\alpha^{cen}(t)-v_\alpha^\infty\right\|+\left\|v_{\alpha i}(t)-v_\alpha^{cen}(t)\right\|$$
$$= O(1)\frac{1}{t^{\lambda-1}}+O(1)\frac{1}{t^\lambda} \le O(1)\frac{1}{t^{\lambda-1}},\ t\to\infty. \tag{3.22}$$

Conversely, it is evident that for $\forall\alpha,\ \beta \in [n]$, and $\forall i \in [N_\alpha],\ j \in [N_\beta]$,

$$
\begin{aligned}
\left\|x_{\alpha i} - x_{\beta j}\right\| &\le \left\|x_{\alpha i}(0) - x_{\beta j}(0)\right\| + \int_0^t \left\|v_{\alpha i} - v_{\beta j}\right\| dt \\
&\le R_0 + \int_0^t D_V(s)ds \le R_0 + (D_V(0) + C_0)\, t,
\end{aligned}
\tag{3.23}
$$

where $C_0 := \dfrac{4\kappa_1(n-1)N}{T_m^\infty \Lambda_0(N-1)}\phi\left(\dfrac{r_0}{2}\right)$. Therefore, the multi-flocking estimate studied in Theorem 1.2 and the monotonicity and non-negativity of ϕ imply that for $\forall\alpha \in [N_\alpha]$

$$
\begin{aligned}
\left\|v_\alpha^{cen}(t) - v_\alpha^\infty\right\| &\ge O\left(\int_t^\infty \phi\left(R_0 + (D_V(0) + C_0)t\right) ds - \exp\left(-\Lambda_0 t\right)\right) \\
&\ge O(1)\frac{1}{t^{\lambda-1}} \to 0,\ t \to \infty.
\end{aligned}
\tag{3.24}
$$

Then, we combine the above estimates to derive that for all $\alpha \in [n]$ and $i \in [N_\alpha]$,

$$
\begin{aligned}
\left\|v_{\alpha i}(t) - v_\alpha^\infty\right\| &\ge \left\|v_\alpha^{cen}(t) - v_\alpha^\infty\right\| - \left\|v_{\alpha i}(t) - v_\alpha^{cen}(t)\right\| \\
&= O(1)\frac{1}{t^{\lambda-1}} - O(1)\frac{1}{t^\lambda} \ge O(1)\frac{1}{t^{\lambda-1}},\ t \to \infty.
\end{aligned}
\tag{3.25}
$$

Finally, there exist $2n$ strictly positive values $V_{\alpha 1},\ V_{\alpha 2}$ such that

$$
\frac{V_{\alpha 1}}{t^{\lambda-1}} \le \left\|v_{\alpha i}(t) - v_\alpha^\infty\right\| \le \frac{V_{\alpha 2}}{t^{\lambda-1}},\ \alpha \in [n],\ i \in [N_\alpha]. \tag{3.26}
$$

Therefore, there exist two strictly positive values $V_1,\ V_2$ such that for all $\alpha \in [n]$ and $i_\alpha \in [N_\alpha]$,

$$
\frac{V_1}{t^{\lambda-1}} \le \sum_{\alpha=1}^n \left\|v_{\alpha i_\alpha}(t) - v_\alpha^\infty\right\| \le \frac{V_2}{t^{\lambda-1}},\ t \to \infty. \tag{3.27}
$$

Similar to the previous proof, the existence of T_α^∞ can be demonstrated, and there exist two positive values T_1 and T_2 such that for all $\alpha \in [n]$ and $i_\alpha \in [N_\alpha]$,

$$
\frac{T_1}{t^{\mu-1}} \le \sum_{\alpha=1}^n \left\|T_{\alpha i_\alpha}(t) - T_\alpha^\infty\right\| \le \frac{T_2}{t^{\mu-1}},\ t \to \infty. \tag{3.28}
$$

We conclude the desired results. □

4. Conclusion

This study provides proof for the fundamental properties and multi-cluster flocking behaviors of the TCSUS system (1.4) under a singular kernel.

Specifically, Propositions 2.1–2.4 establish the foundational characteristics of the TCSUS model and present essential findings that facilitate the investigation of multi-cluster flocking within the TCSUS framework. Lemma 3.1 establishes the dissipative structure of the TCSUS system as derived from its configuration.

Subsequently, the bootstrapping technique is utilized to derive the multi-cluster flocking outcome within a finite time interval. Furthermore, in Theorem 1.1, by enforcing particular initial velocity conditions and applying bootstrapping methods, we ascertain that the divergence rate of distinct clusters is bounded below by a linear function of time.

Theorem 1.2 provides estimates of the position-velocity-temperature L^∞-diameters for all cluster groups by using Gronwall inequalities. Consequently, it is also demonstrated that the velocities and temperatures of all clusters converge to common values, respectively.

Lemma 3.3 establishes the differential equalities for the central velocity and temperature of a cluster, derived by summing the velocities and temperatures of its constituent particles. Finally, Theorem 1.3 provides the convergence values for velocity and temperature within each cluster group by asserting Lemma 3.3 and Theorem 1.2.

Author contributions

Shenglun Yan: Methodology, analysis, calculation, and writing original draft; Wanqian Zhang: Discussion, review and editing; Weiyuan Zou: Supervision, validation, and revision.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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