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Research article

Finite-in-time flocking of the thermodynamic Cucker–Smale model

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Abstract: We illustrate finite-in-time flocking in the thermodynamic Cucker–Smale (TCS) model. First, we extend the original TCS model to allow for a continuous vector field with a locally Lipschitz continuity. Then, within this system, we derive appropriate dissipative inequalities concerning the position-velocity-temperature using several preparatory estimates. Subsequently, based on initial data and system parameters, we formulate sufficient conditions to guarantee the desired finite-time flocking in each case where the communication weight conditions are divided into two scenarios: one with a positive lower bound and another with nonnegativity and monotonicity. Finally, we provide several numerical simulations and compare them with the analytical results.

Keywords: Continuous dynamics; dissipative structure; finite-in-time flocking; multi-agent system; thermodynamic Cucker–Smale

1. Introduction

The term "*flocking*" denotes a collective phenomenon wherein the velocities of all agents controlled by a system converge to a common value. Flocking behaviors are commonly observed in various real-world scenarios, such as the coordinated movements of migratory birds and fish [14], the flow of crowds in a human gathering [26], and the collective motion of bacterial colonies [36]. Many scholars have devoted considerable efforts to mathematically describe the flocking behaviors due to the widespread occurrence of flocking. Over the past few decades, many mathematical models have been proposed and studied within the fields of mathematics and science to capture the flocking phenomena. One particularly successful model is the Cucker–Smale model, which was introduced in a seminal paper [14]. This model was devised within a Newtonian framework and was governed by the following

Cauchy problem in terms of the *position–velocity*, $\{(x_i, v_i)\}_{i=1}^N$:

$$\begin{cases} \dot{x}_i = v_i, \quad t > 0, \quad i \in [N] := \{1, \ldots, N\}, \\ \dot{v}_i = \dfrac{\kappa}{N} \displaystyle\sum_{j=1}^{N} \phi(\|x_i - x_j\|) \left(v_j - v_i\right), \\ (x_i(0), v_i(0)) = (x_i^{\mathrm{in}}, v_i^{\mathrm{in}}) \in \mathbb{R}^d \times \mathbb{R}^d, \end{cases} \tag{1.1}$$

where κ denotes the coupling strength, which is a nonnegative constant, N corresponds to the number of particles, and $\|\cdot\|$ denotes the standard ℓ_2-norm. We set $\mathbb{R}_+ := [0, \infty)$, and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ represents a communication weight. The dissipative velocity structure of the Cucker–Smale model and its variants have garnered significant attention in the mathematics community. Examples include the collision avoidance [5, 13, 30, 33], the unit-speed constraint [9], the inter-particle bonding force [31], fixed and switching topologies [29], bi-cluster flocking [12], stochastic flocking [7], pattern formation [11, 32], the mean-field limit [23], orientation flocking [20], kinetic and hydrodynamic descriptions [6, 17, 25, 27, 28, 34], and random environment [18].

However, the studies mentioned above were conducted without considering the temperature field. In other words, the original Cucker–Smale model (1.1) could not describe the flocking behaviors that arose from differences in the internal energy between particles. The authors of [24] extended the original Cucker–Smale model (1.1) to a thermodynamic system based on multi-temperature Eulerian fluid dynamics and suitable assumptions of spatial homogeneity to investigate a more realistic version of the Cucker–Smale model. Subsequently, a follow-up paper by [22] reduced this system to derive an approximate thermodynamic Cucker–Smale model regarding *position-velocity-temperature* when the diffusion rate was small enough. The approximated thermodynamic Cucker–Smale model (TCS) is governed by the following Cauchy problem for $\{(x_i, v_i, T_i)\}_{i=1}^N$:

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \quad t > 0, \quad i \in [N], \\ \dfrac{dv_i}{dt} = \dfrac{\kappa_1}{N} \displaystyle\sum_{j=1}^{N} \phi(\|x_i - x_j\|) \left(\dfrac{v_j}{T_j} - \dfrac{v_i}{T_i}\right), \\ \dfrac{dT_i}{dt} = \dfrac{\kappa_2}{N} \displaystyle\sum_{j=1}^{N} \zeta(\|x_i - x_j\|) \left(\dfrac{1}{T_i} - \dfrac{1}{T_j}\right), \\ (x_i(0), v_i(0), T_i(0)) = (x_i^0, v_i^0, T_i^0) \in \mathbb{R}^d \times \mathbb{R}^d \times (\mathbb{R}_+ - \{0\}), \end{cases} \tag{1.2}$$

where N denotes the number of particles, κ_1, and κ_2 are coupling strengths, which are positive constants, and the communication weights $\phi, \zeta : \mathbb{R}_+ \to \mathbb{R}_+$ are locally Lipschitz continuous.

The TCS model and its variants have been actively investigated within the mathematics community. The studies encompass a wide range of topics, including particle analyses and derivations [22, 25], collision avoidance [2,10], extension to manifolds [3,4], time delay [8], general digraph structures [16], discrete and continuous models [15], the hydrodynamic limit [19], the unit-speed constraint [1], and the mean-field limit [21].

In this paper, we primarily focus on understanding the finite-in-time flocking of a (1.2)-type model. The impetus for this research arises from the observation that investigations into the TCS model have predominantly centered on analyzing asymptotic behaviors. Indeed, research on finite-in-time flocking holds potential applications in various fields, including national defense science, robotics, spacecrafts, autonomous vehicles, and semiconductors. Particularly, this research is attractive when contrasted with studies on asymptotic behavior because it can contribute to enhancing the control accuracy, efficiency, stability, and speed of multi-agent flocking systems, while reducing several numerical and machine errors. Moreover, such investigations have the potential to enhance the performance and guarantee stability in practical real-world applications.

To set up the main stages, we first let $\overrightarrow{\mathrm{sig}}_\alpha(\cdot)$ for $\alpha \in (0,1)$ be

$$\overrightarrow{\mathrm{sig}}_\alpha(u) := (\mathrm{sgn}(u_1)|u_1|^\alpha, \dots, \mathrm{sgn}(u_d)|u_d|^\alpha), \quad u := (u_1, \dots, u_d) \in \mathbb{R}^d,$$

where $\mathrm{sgn}(\cdot)$ is given by

$$\mathrm{sgn}(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$$

In what follows, we present a motivating example for the finite-time flocking modeling of (1.2). Consider the following Cauchy problem for an ODE (ordinary differential equation) with a non-Lipschitz continuous vector field as follows: For $\alpha \in (0,1)$,

$$\begin{cases} \dot{x} = -\mathrm{sgn}(x)|x|^\alpha, \quad t > 0, \\ x(0) = x^0 \neq 0. \end{cases}$$

Then, it is straightforward to check that the explicit solution is given by the following:

$$x(t) = \begin{cases} \mathrm{sgn}(x^0)\Big(|x^0|^{1-\alpha} - t(1-\alpha)\Big)^{\frac{1}{1-\alpha}}, & 0 < t \le \tau^*, \\ 0, & t \ge \tau^*, \end{cases}$$

where $\tau^* = \frac{|x^0|^{1-\alpha}}{1-\alpha}$. Thus, the solution $x = x(t)$ reaches zero at the finite time τ^*.

$$\lim_{t \to \tau^*-} x(t) = 0.$$

This example serves as the guiding illustration for the modeling approach throughout this paper.

Then, motivated from the above example, by combining $\overrightarrow{\mathrm{sig}}_\alpha$ and $\overrightarrow{\mathrm{sig}}_\beta$ with the locally Lipschitz continuous vector field of (1.2), we propose an autonomous dynamical system with a non-locally Lipschitz continuous vector field. This system is described by the following Cauchy problem regarding $\{(x_i, v_i, T_i)\}_{i=1}^N$:

$$\frac{dx_i}{dt} = v_i, \quad t > 0, \quad i \in [N], \tag{1.3a}$$

$$\frac{dv_i}{dt} = \frac{\kappa_1}{N} \sum_{j=1}^N \overrightarrow{\mathrm{sig}}_\alpha\left(\phi(\|x_i - x_j\|)\left(\frac{v_j}{T_j} - \frac{v_i}{T_i}\right)\right), \tag{1.3b}$$

$$\frac{dT_i}{dt} = \frac{\kappa_2}{N} \sum_{j=1}^N \overrightarrow{\mathrm{sig}}_\beta\left(\zeta(\|x_i - x_j\|)\left(\frac{1}{T_i} - \frac{1}{T_j}\right)\right), \tag{1.3c}$$

$$(x_i(0), v_i(0), T_i(0)) = (x_i^0, v_i^0, T_i^0) \in \mathbb{R}^d \times \mathbb{R}^d \times (\mathbb{R}_+ - \{0\}), \tag{1.3d}$$

where N denotes the number of particles, κ_1, and κ_2 are coupling strengths, which are positive constants, $\alpha, \beta \in (0, 1)$ are constant parameters, and we assume that the communication weights $\phi, \zeta : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous.

Here, we mention that only the finite-time flocking Cucker-Smale model and its variants designed based on the original Cucker–Smale model (1.1) have been primarily addressed within the mathematical community, to name a few, the standard analysis [37] and unknown intrinsic dynamics [35]. In other words, only finite-in-time flocking without considering thermodynamics has been studied. Accordingly, we are primarily interested in the following issue:

- (Main issue): In terms of initial data and system parameters, under what sufficient conditions can the finite-in-time flocking of (1.3) be achieved?

Subsequently, we provide a basic notion for the finite-in-time flocking of (1.3) as follows:

Definition 1.1. (Finite-in-time flocking) *Suppose that $Z = \{(x_i, v_i, T_i)\}_{i=1}^N$ is a solution to* (3.1). *Then, the configuration Z exhibits finite-in-time flocking if there exists a finite time $t^* \in \mathbb{R}_+$ such that the following assertions hold:*

$$\begin{aligned}
&(i) \text{ (Group formation)} \iff \sup_{t\in\mathbb{R}_+} \max_{i,j\in[N]} \|x_i(t) - x_j(t)\| < \infty,\\
&(ii) \text{ (Finite-in-time velocity alignment)} \iff \sup_{t\in[t^*,\infty)} \max_{i,j\in[N]} \|v_i(t) - v_j(t)\| = 0,\\
&(iii) \text{ (Finite-in-time temperature equilibrium)} \iff \sup_{t\in[t^*,\infty)} \max_{i,j\in[N]} |T_i(t) - T_j(t)| = 0.
\end{aligned}$$

Notation. We set the following notation for concision:

$$\begin{aligned}
&\|\cdot\| = \text{standard Euclidean } l_2\text{-norm}, \quad \langle\cdot,\cdot\rangle = \text{standard inner product}, \quad T^\infty := \frac{\sum_{i=1}^N T_i^0}{N},\\
&X =: (x_1, \dots, x_N), \quad V =: (v_1, \dots, v_N), \quad T =: (T_1, \dots, T_N), \quad \bar{T} := (T_1 - T^\infty, \dots, T_N - T^\infty),\\
&\mathbb{R}_+ := [0, \infty), \quad x^i = \text{the } i\text{-th component of } x \in \mathbb{R}^d, \quad [N] := \{1, \dots, N\}.
\end{aligned}$$

The main results of this paper can be summarized as follows: Assume that $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $\inf_{s\in\mathbb{R}_+} \zeta(s) > 0$, and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies

$$\inf_{s\in\mathbb{R}_+} \phi(s) > 0 \quad \text{or} \quad \phi(r_2) \ge \phi(r_1) \text{ for } r_1 \ge r_2 \ge 0.$$

⋄ Result A (Dissipative structure for the temperature and a finite-in-time temperature equilibrium): There exists a positive constant $C_1 > 0$ depending on initial data and system parameters such that for a.e. $t \in \mathbb{R}_+ - \{0\}$,

$$\frac{d\|\bar{T}\|}{dt} \le -C_1\|\bar{T}\|^\beta.$$

Therefore, there are two positive constants $t_1 > 0$ and $T^\infty > 0$ satisfying the following:

$$T_i(t) \equiv T^\infty \quad \text{for} \quad i \in [N], \quad t \ge t_1.$$

$\diamond$ Result B (Dissipative structure for velocity): There exists a positive constant $C_2 > 0$ depending on the initial data and system parameters such that for a.e. $t > t_1$,

$$\frac{d\|V\|}{dt} \le -C_2\|V\|^{\alpha}.$$

$\diamond$ Result C (Global existence of solutions and finite-in-time flocking): We demonstrate the global existence of the solutions and the desired finite-in-time flocking in (1.3) as follows: There is a finite time $t_0 \ge 0$ such that for $t \in [t_0, \infty)$,

$$\sup_{t\in\mathbb{R}_+}\max_{i,j\in[N]}\|x_i(t) - x_j(t)\| < \infty, \quad \max_{i,j\in[N]}\|v_i(t) - v_j(t)\| \equiv 0, \quad \max_{i,j\in[N]}|T_i(t) - T_j(t)| \equiv 0.$$

The rest of this paper is organized as follows. In Section 2, we provide basic estimates for the temperatures and the propagation of the conserved quantities in the system (1.3). In Section 3, by dividing the communication weight into two cases, we verify the global existence of solutions and finite-in-time flocking of (1.3) under suitable admissible data, respectively. In Section 4, we present several numerical simulations and compare them with the analytical results in Section 3. Finally, Section 5 summarizes the primary outcomes of this study and discuss future work.

2. Preliminaries

In this section, we present the propagation of conserved quantities for the *velocity–temperature* and the entropy principle with the monotonicity of the max–min temperatures and a uniform boundedness of temperatures in the proposed system (1.3). These results will be crucially employed to demonstrate the finite-in-time flocking and global existence of solutions of (1.3). Now, we begin by dealing with several conserved quantities and the entropy principle in (1.3). Here, we define the entropy $\mathcal{S}$ of (1.3) as follows:

$$\mathcal{S}(t) := \sum_{i=1}^{N} \ln(T_i(t)).$$

Proposition 2.1. (Basic facts) *Assume that* $\{(x_i, v_i, T_i)\}_{i=1}^N$ *is a solution to the system* (1.3) *and* $T_i(t) > 0$ *for* $t \ge 0$ *and* $i \in [N]$. *Then, the following assertions hold for* $t \ge 0$*:*

1. *(Conserved velocity and temperature sums): The total sums* $\sum_{i=1}^N v_i(t)$ *and* $\sum_{i=1}^N T_i(t)$ *are conserved:*
$$\sum_{i=1}^{N} v_i(t) = \sum_{i=1}^{N} v_i^0, \qquad \sum_{i=1}^{N} T_i(t) = \sum_{i=1}^{N} T_i^0 =: NT^{\infty}.$$
2. *(Entropy principle): The total entropy* $\mathcal{S}(t)$ *is monotonically increasing:*
$$\frac{d\mathcal{S}(t)}{dt} = \frac{\kappa_2}{2N}\sum_{i,j=1}^{N} \zeta(\|x_j - x_i\|)^{\beta}\left|\frac{1}{T_i} - \frac{1}{T_j}\right|^{\beta+1} \ge 0.$$

Proof. To verify the first assertion, we use the definition of $\overrightarrow{\mathrm{sig}}_\beta$ and the standard trick of interchanging i and j and dividing by 2 to have the following:

$$\frac{d}{dt}\left(\sum_{i=1}^{N} T_i\right) = \frac{\kappa_2}{N}\sum_{i,j=1}^{N} \overrightarrow{\mathrm{sig}}_\beta\left(\zeta(\|x_i - x_j\|)\left(\frac{1}{T_i} - \frac{1}{T_j}\right)\right)$$

$$
\begin{aligned}
&= \frac{\kappa_2}{N}\sum_{i,j=1}^{N}\overrightarrow{\mathrm{sig}}_\beta\left(\zeta(\|x_i-x_j\|)\left(\frac{1}{T_j}-\frac{1}{T_i}\right)\right)\\
&= \frac{\kappa_2}{2N}\sum_{i,j=1}^{N}\overrightarrow{\mathrm{sig}}_\beta\left(\zeta(\|x_i-x_j\|)\left(\frac{1}{T_j}-\frac{1}{T_i}\right)+\zeta(\|x_i-x_j\|)\left(\frac{1}{T_i}-\frac{1}{T_j}\right)\right)\\
&= 0.
\end{aligned}
$$

As in the same method as above, we check the following:

$$
\begin{aligned}
\frac{d}{dt}\left(\sum_{i=1}^{N}v_i\right) &= \frac{\kappa_1}{N}\sum_{i,j=1}^{N}\overrightarrow{\mathrm{sig}}_\alpha\left(\phi(\|x_i-x_j\|)\left(\frac{v_j}{T_j}-\frac{v_i}{T_i}\right)\right)\\
&= \frac{\kappa_1}{N}\sum_{i,j=1}^{N}\overrightarrow{\mathrm{sig}}_\alpha\left(\phi(\|x_i-x_j\|)\left(\frac{v_i}{T_i}-\frac{v_j}{T_j}\right)\right)\\
&= \frac{\kappa_1}{2N}\sum_{i,j=1}^{N}\overrightarrow{\mathrm{sig}}_\alpha\left(\phi(\|x_i-x_j\|)\left(\frac{v_j}{T_j}-\frac{v_i}{T_i}\right)+\phi(\|x_i-x_j\|)\left(\frac{v_i}{T_i}-\frac{v_j}{T_j}\right)\right)=0.
\end{aligned}
$$

Next, to get the second assertion, it follows by a direct calculation with the standard trick that

$$
\begin{aligned}
\frac{d\mathcal{S}}{dt} &= \sum_{i=1}^{N}\frac{\dot{T}_i}{T_i}\\
&= \frac{\kappa_2}{N}\sum_{i,j=1}^{N}\frac{1}{T_i}\cdot\overrightarrow{\mathrm{sig}}_\beta\left(\zeta(\|x_i-x_j\|)\left(\frac{1}{T_i}-\frac{1}{T_j}\right)\right)\\
&= \frac{\kappa_2}{2N}\sum_{i,j=1}^{N}\left(\frac{1}{T_i}-\frac{1}{T_j}\right)\cdot\overrightarrow{\mathrm{sig}}_\beta\left(\zeta(\|x_i-x_j\|)\left(\frac{1}{T_i}-\frac{1}{T_j}\right)\right)\\
&= \frac{\kappa_2}{2N}\sum_{i,j=1}^{N}\zeta(\|x_i-x_j\|)^\beta\left(\frac{1}{T_i}-\frac{1}{T_j}\right)\cdot\overrightarrow{\mathrm{sig}}_\beta\left(\frac{1}{T_i}-\frac{1}{T_j}\right)\\
&\geq \frac{\kappa_2}{2N}\sum_{i,j=1}^{N}\zeta(\|x_j-x_i\|)^\beta\left|\frac{1}{T_i}-\frac{1}{T_j}\right|^{\beta+1}\geq 0,
\end{aligned}
$$

where we applied the following relation:

$$
\left\langle\overrightarrow{\mathrm{sig}}_\beta(u),u\right\rangle\geq\|u\|^{\beta+1},\quad u:=(u^1,\dots,u^d)\in\mathbb{R}^d. \tag{2.1}
$$

Finally, we obtain the desired proposition. □

Remark 2.1. *By the virtue of velocity conservation, we may consider the case* $\sum_{i=1}^{N}v_i^0=0$ *only.*

In what follows, we show the monotonicity of $\max_{i\in[N]}T_i$ and $\min_{i\in[N]}T_i$ with a uniform boundedness of $\{T_i\}_{i=1}^N$ in the system (1.3).

Proposition 2.2. (Essential estimates for temperature dynamics) *Suppose that* $\{(x_i, v_i, T_i)\}_{i=1}^N$ *is a solution to* (1.3). *Then,* $\min_{i\in[N]} T_i(t)$ *is monotonically increasing and* $\max_{i\in[N]} T_i(t)$ *is monotonically decreasing in time. In addition, all temperatures satisfy the following uniform boundedness for* $t \geq 0$ *and* $i \in [N]$*:*

$$0 < \min_{i\in[N]} T_i^0 =: T_m^\infty \leq T_i(t) \leq \max_{i\in[N]} T_i^0 =: T_M^\infty.$$

Proof. We take two indices $M_t, m_t \in [N]$ such that

$$T_{M_t} := \max_{i\in[N]} T_i(t), \qquad T_{m_t} := \min_{i\in[N]} T_i(t).$$

For a sufficiently small real number $\epsilon > 0$, we set

$$S =: \left\{ t > 0 \,\middle|\, \min_{i\in[N]} T_i(s) > T_m^\infty - \epsilon, \quad \forall s \in [0, t) \right\},$$

and let $\bar{S} := \sup S$. Then, S is nonempty due to the continuity of $\min_{i\in[N]} T_i$ and $T_i^0 > 0$ for each $i \in [N]$. Moreover, one has

$$\min_{i\in[N]} T_i(\bar{S}-) = T_m^\infty - \epsilon.$$

Now, we claim that $\bar{S} = \infty$. For the proof by contradiction, suppose that

$$\bar{S} < \infty.$$

Then, from Eq (1.3c), for a.e. $t \in (0, \bar{S})$, we derive the following:

$$\frac{dT_{m_t}}{dt} = \frac{\kappa_2}{N} \sum_{j=1}^N \overrightarrow{\mathrm{sig}}_\beta \left(\zeta(\|x_i - x_j\|) \left(\frac{1}{T_{m_t}} - \frac{1}{T_j} \right) \right) \geq 0.$$

Thus, T_{m_t} is monotonically increasing and

$$\min_{i\in[N]} T_i(\bar{S}-) > T_m^\infty - \epsilon.$$

This outcome is contradictory. Therefore, because ϵ is arbitrary, we immediately attain

$$\inf_{t\in\mathbb{R}_+} \min_{i\in[N]} T_i(t) \geq T_m^\infty$$

and T_{m_t} is monotonically increasing in time. Subsequently, in the same way as the proof of the above assertion, we can also obtain

$$\sup_{t\in\mathbb{R}_+} \max_{i\in[N]} T_i(t) \leq T_M^\infty$$

and T_{M_t} is monotonically decreasing in time. In conclusion, we reach the following desired results:

$$0 < T_m^\infty \leq T_i(t) \leq T_M^\infty.$$

□

3. Finite-in-time flocking of (1.3)

In this section, we verify the finite-in-time flocking of the proposed system (1.3), which is governed by the following Cauchy Problem:

$$\frac{dx_i}{dt} = v_i, \quad t > 0, \quad i \in [N], \tag{3.1a}$$

$$\frac{dv_i}{dt} = \frac{\kappa_1}{N}\sum_{j=1}^{N} \overrightarrow{\mathrm{sig}}_\alpha\left(\phi(\|x_i - x_j\|)\left(\frac{v_j}{T_j} - \frac{v_i}{T_i}\right)\right), \tag{3.1b}$$

$$\frac{dT_i}{dt} = \frac{\kappa_2}{N}\sum_{j=1}^{N} \overrightarrow{\mathrm{sig}}_\beta\left(\zeta(\|x_i - x_j\|)\left(\frac{1}{T_i} - \frac{1}{T_j}\right)\right), \tag{3.1c}$$

$$(x_i(0), v_i(0), T_i(0)) = (x_i^0, v_i^0, T_i^0) \in \mathbb{R}^d \times \mathbb{R}^d \times (\mathbb{R}_+ - \{0\}), \tag{3.1d}$$

where we assume that ζ has a positive lower bound, and ϕ either has a positive lower bound or satisfies the nonnegativity and monotonicity.

Before we move on further, we present a basic concept for asymptotic flocking and recall the definition of finite-in-time flocking in the system (3.1):

Definition 3.1. (Asymptotic flocking) *Let* $Z = \{(x_i, v_i, T_i)\}_{i=1}^N$ *be a solution to the system* (3.1). *Then, the configuration Z exhibits asymptotic flocking if the following assertions hold:*

$$(i) \text{ (Group formation) } \iff \sup_{t\in\mathbb{R}_+} \max_{i,j\in[N]} \|x_i(t) - x_j(t)\| < \infty,$$

$$(ii) \text{ (Asymptotic velocity alignment) } \iff \lim_{t\to\infty} \max_{i,j\in[N]} \|v_i(t) - v_j(t)\| = 0,$$

$$(iii) \text{ (Asymptotic temperature equilibrium) } \iff \lim_{t\to\infty} \max_{i,j\in[N]} |T_i(t) - T_j(t)| = 0.$$

Definition 3.2. (Finite-in-time flocking) *Suppose that* $Z = \{(x_i, v_i, T_i)\}_{i=1}^N$ *is a solution to* (3.1). *Then, the configuration Z exhibits finite-in-time flocking if there exists a finite time* $t^* \in \mathbb{R}_+$ *such that the following assertions hold:*

$$(i) \text{ (Group formation) } \iff \sup_{t\in\mathbb{R}_+} \max_{i,j\in[N]} \|x_i(t) - x_j(t)\| < \infty,$$

$$(ii) \text{ (Finite-in-time velocity alignment) } \iff \sup_{t\in[t^*,\infty)} \max_{i,j\in[N]} \|v_i(t) - v_j(t)\| = 0,$$

$$(iii) \text{ (Finite-in-time temperature equilibrium) } \iff \sup_{t\in[t^*,\infty)} \max_{i,j\in[N]} |T_i(t) - T_j(t)| = 0.$$

Our main goal in this section is to demonstrate several sufficient frameworks regarding initial data and system parameters for the finite-in-time flocking of (3.1) to emerge.

3.1. First methodology

In this subsection, we reduce the proposed system (3.1) to a suitable dissipative structure in terms of the *position–velocity–temperature* when each of ϕ and ζ has a positive lower bound. Afterward, we obtain the desired finite-in-time flocking from this structure.

Lemma 3.1. (Dissipative structure for temperature) *Assume that* $\{(x_i, v_i, T_i)\}_{i=1}^N$ *is a solution to* (3.1) *satisfying the following:*

$$\zeta_m := \inf_{s\in\mathbb{R}_+} \zeta(s) > 0. \tag{3.2}$$

Then, for a.e. $t \in \mathbb{R}_+ - \{0\}$, *we have*

$$\left|\frac{d\|X\|}{dt}\right| \le \|V\| \qquad \textit{and} \qquad \frac{d\|\bar{T}\|}{dt} \le -\frac{\kappa_2 \zeta_m^\beta}{(2N)^{\frac{1-\beta}{2}}(T_M^\infty)^{2\beta}}\|\bar{T}\|^\beta.$$

Proof. If we differentiate $\|X\|^2$ with respect to t, for a.e. $t \in \mathbb{R}_+ - \{0\}$, one has

$$2\|X\|\left|\frac{d\|X\|}{dt}\right| = \left|\frac{d\|X\|^2}{dt}\right| = 2\,|\langle X, V\rangle| \le 2\|X\|\|V\|.$$

Thus, for a.e. $t \in \mathbb{R}_+ - \{0\}$,

$$\left|\frac{d\|X\|}{dt}\right| \le \|V\|.$$

Next, it follows from Eq (3.1c), Proposition 2.2, and the standard trick of interchanging i and j and dividing 2 that for $t \in \mathbb{R}_+ - \{0\}$,

$$\begin{aligned}
\frac{1}{2}\frac{d\|\bar{T}\|^2}{dt} &= \frac{\kappa_2}{N}\sum_{i,j=1}^N \bar{T}_i \overrightarrow{\mathrm{sig}}_\beta\left(\zeta(\|x_i - x_j\|)\left(\frac{1}{T_i} - \frac{1}{T_j}\right)\right) \\
&= \frac{\kappa_2}{2N}\sum_{i,j=1}^N (\bar{T}_i - \bar{T}_j)\overrightarrow{\mathrm{sig}}_\beta\left(\zeta(\|x_i - x_j\|)\left(\frac{1}{T_i} - \frac{1}{T_j}\right)\right) \\
&= -\frac{\kappa_2}{2N}\sum_{i,j=1}^N (\bar{T}_i - \bar{T}_j)\overrightarrow{\mathrm{sig}}_\beta\left(\zeta(\|x_i - x_j\|)\left(\frac{T_i - T_j}{T_i T_j}\right)\right) \\
&= -\frac{\kappa_2}{2N}\sum_{i,j=1}^N \zeta(\|x_i - x_j\|)^\beta(\bar{T}_i - \bar{T}_j)\overrightarrow{\mathrm{sig}}_\beta\left(\frac{T_i - T_j}{T_i T_j}\right) \\
&\le -\frac{\kappa_2 \zeta_m^\beta}{2N}\sum_{i,j=1}^N \frac{1}{(T_i T_j)^\beta}(\bar{T}_i - \bar{T}_j)\overrightarrow{\mathrm{sig}}_\beta\left(T_i - T_j\right) \\
&\le -\frac{\kappa_2 \zeta_m^\beta}{2N}\sum_{i,j=1}^N \frac{1}{(T_i T_j)^\beta}|\bar{T}_i - \bar{T}_j|^{\beta+1} \\
&\le -\frac{\kappa_2 \zeta_m^\beta}{2N(T_M^\infty)^{2\beta}}\sum_{i,j=1}^N |\bar{T}_i - \bar{T}_j|^{\beta+1},
\end{aligned}$$

where we used (2.1). Then, employing the following relations:

$$\left(\sum_{i=1}^N |a_i|\right)^\gamma \le \sum_{i=1}^N |a_i|^\gamma \quad \text{for} \quad a_i \in \mathbb{R} \text{ and } \gamma \in (0,1), \qquad \sum_{i=1}^N |\bar{T}_i - \bar{T}_j|^2 = 2N\|\bar{T}\|^2,$$

for $t \in \mathbb{R}_+ - \{0\}$, we show

$$\begin{aligned}\frac{1}{2}\frac{d\|\bar{T}\|^2}{dt} &\leq -\frac{\kappa_2\zeta_m^\beta}{2N(T_M^\infty)^{2\beta}}\sum_{i,j=1}^N |\bar{T}_i - \bar{T}_j|^{\beta+1} \\ &\leq -\frac{\kappa_2\zeta_m^\beta}{2N(T_M^\infty)^{2\beta}}\left(\sum_{i,j=1}^N |\bar{T}_i - \bar{T}_j|^2\right)^{\frac{\beta+1}{2}} \\ &= -\frac{\kappa_2\zeta_m^\beta}{2N(T_M^\infty)^{2\beta}}\left(2N\|\bar{T}\|^2\right)^{\frac{\beta+1}{2}} \\ &= -\frac{\kappa_2\zeta_m^\beta}{(2N)^{\frac{1-\beta}{2}}(T_M^\infty)^{2\beta}}\|\bar{T}\|^{\beta+1}.\end{aligned}$$

This implies that for a.e. $t \in \mathbb{R}_+ - \{0\}$,

$$\frac{d\|\bar{T}\|}{dt} \leq -\frac{\kappa_2\zeta_m^\beta}{(2N)^{\frac{1-\beta}{2}}(T_M^\infty)^{2\beta}}\|\bar{T}\|^\beta.$$

In conclusion, we verify the desired lemma. □

Now, we provide the finite-in-time temperature equilibrium of the system (3.1) under (3.2).

Proposition 3.1. (Finite-in-time temperature equilibrium) *Let* $\{(x_i, v_i, T_i)\}_{i=1}^N$ *be a solution to the system* (3.1) *such that* (3.2) *holds. Then, there exists a finite time* $t_1 \in \mathbb{R}_+$ *satisfying*

$$T_i(t) = T^\infty, \quad t \geq t_1 \quad \text{and} \quad i \in [N].$$

Proof. Applying the comparison principle to the dissipative inequality for $\bar{T}$ studied in Lemma 3.1 yields that for $t \in \mathbb{R}_+$,

$$\|\bar{T}(t)\| \leq \left(\|\bar{T}(0)\|^{1-\beta} - \frac{\kappa_2\zeta_m^\beta(1-\beta)}{(2N)^{\frac{1-\beta}{2}}(T_M^\infty)^{2\beta}} \cdot t\right)^{\frac{1}{1-\beta}}.$$

Therefore, if we define t_1 as

$$t_1 := \frac{(2N)^{\frac{1-\beta}{2}}(T_M^\infty)^{2\beta}}{\kappa_2\zeta_m^\beta(1-\beta)}\|\bar{T}(0)\|^{1-\beta},$$

then we can prove

$$\|\bar{T}(t)\| = 0, \quad t \geq t_1.$$

Finally, we reach the desired proposition. □

Due to Proposition 3.1, if we assume that the condition (3.2) holds, then the system (3.1) can be converted into the following system when $t \geq t_1$:

$$\begin{cases}\dfrac{dx_i}{dt} = v_i, \quad t > t_1, \quad i \in [N], \\ \dfrac{dv_i}{dt} = \dfrac{\kappa_1}{N(T^\infty)^\alpha}\displaystyle\sum_{j=1}^N \overrightarrow{\text{sig}}_\alpha\left(\phi(\|x_i - x_j\|)\left(v_j - v_i\right)\right), \\ (x_i(t_1), v_i(t_1)) = (x_i^{t_1}, v_i^{t_1}) \in \mathbb{R}^d \times \mathbb{R}^d, \quad \displaystyle\sum_{i=1}^N v_i^{t_1} = 0.\end{cases} \tag{3.3}$$

Next, we reduce the system (3.3) to a dissipative structure concerning velocity to obtain its finite-in-time velocity alignment.

Lemma 3.2. (Dissipative structure for velocity) *Suppose that* $\{(x_i, v_i)\}_{i=1}^N$ *is a solution to* (3.3) *such that*

$$\phi_m := \inf_{s\in\mathbb{R}_+} \phi(s) > 0. \tag{3.4}$$

Then, for a.e. $t > t_1$*, we derive the following:*

$$\frac{d\|V\|}{dt} \le -\frac{\kappa_1 \phi_m^\alpha}{(2N)^{\frac{1-\alpha}{2}}(T^\infty)^\alpha}\|V\|^\alpha.$$

Proof. We employ Eqs (3.1b) and (3.4), and the standard trick of interchanging i and j and dividing 2 to estimate

$$\begin{aligned}
\frac{1}{2}\frac{d\|V\|^2}{dt} &= \sum_{i=1}^N \langle \dot{v}_i, v_i\rangle \\
&= \frac{\kappa_1}{N(T^\infty)^\alpha}\sum_{i,j=1}^N \left\langle \overrightarrow{\mathrm{sig}}_\alpha\left(\phi(\|x_i - x_j\|)\left(v_j - v_i\right)\right), v_i\right\rangle \\
&= -\frac{\kappa_1}{2N(T^\infty)^\alpha}\sum_{i,j=1}^N \left\langle \overrightarrow{\mathrm{sig}}_\alpha\left(\phi(\|x_i - x_j\|)\left(v_i - v_j\right)\right), (v_i - v_j)\right\rangle \\
&= -\frac{\kappa_1}{2N(T^\infty)^\alpha}\sum_{i,j=1}^N \phi(\|x_i - x_j\|)^\alpha \left\langle \overrightarrow{\mathrm{sig}}_\alpha\left(v_i - v_j\right), (v_i - v_j)\right\rangle \\
&\le -\frac{\kappa_1\phi_m^\alpha}{2N(T^\infty)^\alpha}\sum_{i,j=1}^N \left\langle \overrightarrow{\mathrm{sig}}_\alpha\left(v_i - v_j\right), (v_i - v_j)\right\rangle \\
&\le -\frac{\kappa_1\phi_m^\alpha}{2N(T^\infty)^\alpha}\sum_{i,j=1}^N \|v_i - v_j\|^{\alpha+1},
\end{aligned}$$

where we used Eq (2.1) to the last estimate. Then, if we apply the following relations:

$$\left(\sum_{i=1}^N |a_i|\right)^\gamma \le \sum_{i=1}^N |a_i|^\gamma \quad \text{for} \quad a_i \in \mathbb{R} \text{ and } \gamma \in (0,1), \qquad \sum_{i=1}^N \|v_i - v_j\|^2 = 2N\|V\|^2, \tag{3.5}$$

then we see that for $t \in \mathbb{R}_+ - \{0\}$,

$$\begin{aligned}
\frac{1}{2}\frac{d\|V\|^2}{dt} &\le -\frac{\kappa_1\phi_m^\alpha}{2N(T^\infty)^\alpha}\sum_{i,j=1}^N \|v_i - v_j\|^{\alpha+1} \\
&\le -\frac{\kappa_1\phi_m^\alpha}{2N(T^\infty)^\alpha}\left(\sum_{i,j=1}^N \|v_i - v_j\|^2\right)^{\frac{\alpha+1}{2}}
\end{aligned}$$

$$= -\frac{\kappa_1 \phi_m^\alpha}{(2N)^{\frac{1-\alpha}{2}} (T^\infty)^\alpha} \|V\|^{\alpha+1}.$$

Accordingly, for a.e. $t \in \mathbb{R}_+ - \{0\}$,

$$\frac{d\|V\|}{dt} \le -\frac{\kappa_1 \phi_m^\alpha}{(2N)^{\frac{1-\alpha}{2}} (T^\infty)^\alpha} \|V\|^{\alpha}.$$

We attain the desired lemma. □

Subsequently, we derive the finite-in-time velocity alignment of (3.3) when (3.4) is assumed.

Proposition 3.2. (Finite-in-time velocity alignment) *Let* $\{(x_i, v_i)\}_{i=1}^N$ *be a solution to the system* (3.3) *satisfying* (3.4). *Then, there exists a finite time* $t_2 \ge t_1$ *such that*

$$v_i(t) = 0, \quad t \ge t_2 \quad \textit{and} \quad i \in [N].$$

Proof. We apply the comparison principle to Lemma 3.2 to get that for $t \ge t_1$,

$$\|V(t)\| \le \left(\|V(t_1)\|^{1-\alpha} - \frac{\kappa_1 \phi_m^\alpha (1-\alpha)}{(2N)^{\frac{1-\alpha}{2}} (T^\infty)^\alpha} \cdot (t - t_1) \right)^{\frac{1}{1-\alpha}}.$$

Herein, if we set t_2 as

$$t_2 := t_1 + \frac{(2N)^{\frac{1-\alpha}{2}} (T^\infty)^\alpha}{\kappa_1 \phi_m^\alpha (1-\alpha)} \|V(t_1)\|^{1-\alpha},$$

then we find

$$\|V(t)\| = 0, \quad t \ge t_2.$$

We finish the desired proposition. □

As a direct consequence of the finite-in-time velocity alignment and finite-in-time temperature equilibrium results, we can deduce the group formation of the system (3.1).

Proposition 3.3. (Group formation) *Assume that* $\{(x_i, v_i)\}_{i=1}^N$ *is a solution to the system* (3.1) *such that* (3.2) *and* (3.4) *hold. Then, one has the following:*

$$\sup_{t \in \mathbb{R}_+} \|X(t)\| \le \max \left(\sup_{s \in [0, t_1)} \|X(s)\|, \|X(t_1)\| + \int_{t_1}^{t_2} \|V(s)\| ds \right) < \infty.$$

Proof. For $t \ge t_1$, we use the first assertion of Lemma 3.1 and Proposition 3.2 to lead to the following:

$$\|X(t)\| \le \|X(t_1)\| + \int_{t_1}^{t} \|V(s)\| ds \le \|X(t_1)\| + \int_{t_1}^{\infty} \|V(s)\| ds = \|X(t_1)\| + \int_{t_1}^{t_2} \|V(s)\| ds.$$

Consequently, we induce the below relation:

$$\sup_{t \in \mathbb{R}_+} \|X(t)\| \le \max \left(\sup_{s \in [0, t_1)} \|X(s)\|, \|X(t_1)\| + \int_{t_1}^{t_2} \|V(s)\| ds \right) < \infty.$$

Hence, we obtain the desired proposition. □

Therefore, by combining the above lemmas and propositions, we demonstrate the global existence of the solutions and finite-in-time flocking of (3.1) as follows:

Theorem 3.1. (Global existence of solutions and finite-in-time flocking) *Let two communication weights satisfy*

$$\phi_m := \inf_{s\in\mathbb{R}_+} \phi(s) > 0 \qquad \textit{and} \qquad \zeta_m := \inf_{s\in\mathbb{R}_+} \zeta(s) > 0,$$

and let $\{(x_i, v_i, T_i)\}_{i=1}^N$ *be a solution to the system* (3.1). *Then, the global existence of the solutions of* (3.1) *can be guaranteed and moreover, the following finite-in-time flocking holds: there exists a finite time* $t_2 \in \mathbb{R}_+$ *such that for* $i \in [N]$ *and* $t \geq t_2$,

$$\sup_{t\in\mathbb{R}_+} \|X(t)\| < \infty, \qquad v_i(t) = 0, \qquad T_i(t) = T^\infty.$$

Proof. From Propositions 3.1–3.3, it remains to prove the global existence of the solutions to the system (3.1). Indeed, this is obvious because $\{(x_i, v_i, T_i)\}_{i=1}^N$ is uniformly bounded in time by the arguments studied in Propositions 3.1–3.3. Finally, we reach the desired theorem. □

Before we end this subsection, we note the following crucial remark:

Remark 3.1. *Although we obtain the global existence of solutions, we can not guarantee the uniqueness (in particular, backward uniqueness) of the solution to the system* (3.1) *because its vector field is not locally Lipschitz continuous and furthermore, not one-sided locally Lipschitz continuous. Indeed, it is also unknown whether the forward uniqueness of solution can be established in* (3.1).

3.2. Second methodology

In this subsection, we consider the following continuous communication weight ϕ satisfying the nonnegativity and monotonicity in the system (3.1):

$$\phi : \mathbb{R}_+ \to \mathbb{R}_+, \qquad \phi(r_2) \geq \phi(r_1) \quad \text{for} \quad r_1 \geq r_2 \geq 0.$$

In this situation, we can also deduce suitable frameworks for the finite-in-time flocking of the system (3.1) in a different way than Section 3.1.

Theorem 3.2. (Global existence of solutions and finite-in-time flocking) *Let* $\{(x_i, v_i, T_i)\}_{i=1}^N$ *be a solution to* (3.1) *with the following assumptions for* ϕ *and* ζ*:*

$$\phi, \zeta : \mathbb{R}_+ \to \mathbb{R}_+, \qquad \phi(r_2) \geq \phi(r_1) \quad \textit{for} \quad r_1 \geq r_2 \geq 0, \qquad \zeta_m := \inf_{s\in\mathbb{R}_+} \zeta(s) > 0.$$

Further assume that there exists a positive number $X^\infty \in \mathbb{R}_+ - \{0\}$ *satisfying* $\phi(\sqrt{2}X^\infty) > 0$ *and*

$$\|X(t_1)\| + \frac{(2N)^{\frac{1-\alpha}{2}}(T^\infty)^\alpha}{\kappa_1(2-\alpha)\phi(\sqrt{2}X^\infty)^\alpha}\|V(t_1)\|^{2-\alpha} < X^\infty, \tag{3.6}$$

where t_1 *is defined by*

$$t_1 := \frac{(2N)^{\frac{1-\beta}{2}}(T_M^\infty)^{2\beta}}{\kappa_2\zeta_m^\beta(1-\beta)}\|\bar{T}(0)\|^{1-\beta}.$$

Then, the global existence of the solutions is guaranteed and there exists a finite time $t_3 \in \mathbb{R}_+$ satisfying for $i \in [N]$ and $t \geq t_3$,

$$\sup_{t \in \mathbb{R}_+} \|X(t)\| < \max\left(\sup_{t \in [0,t_1)} \|X(t)\|, X^\infty \right) < \infty, \qquad v_i(t) = 0, \qquad T_i(t) = T^\infty.$$

Proof. Due to Proposition 3.1, recall that the following temperature equilibrium holds; there exists a finite time $t_1 \in \mathbb{R}_+$ such that for $i \in [N]$ and $t \geq t_1$,

$$T_i(t) = T^\infty.$$

Then, the system (3.1) can be written as follows for $t \geq t_1$:

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \quad t > t_1, \quad i \in [N], \\ \dfrac{dv_i}{dt} = \dfrac{\kappa_1}{N(T^\infty)^\alpha} \displaystyle\sum_{j=1}^N \overrightarrow{\mathrm{sig}}_\alpha \left(\phi(\|x_i - x_j\|) \left(v_j - v_i \right) \right), \\ (x_i(t_1), v_i(t_1)) = (x_i^{t_1}, v_i^{t_1}) \in \mathbb{R}^d \times \mathbb{R}^d, \quad \displaystyle\sum_{i=1}^N v_i^{t_1} = 0. \end{cases} \tag{3.7}$$

Next, in the same way as the proof of Lemma 3.2, we attain from (3.7) that for $t > t_1$,

$$\begin{aligned} \frac{1}{2} \frac{d\|V\|^2}{dt} &= \sum_{i=1}^N \langle \dot{v}_i, v_i \rangle \\ &= \sum_{i=1}^N \left\langle \frac{\kappa_1}{N(T^\infty)^\alpha} \sum_{j=1}^N \overrightarrow{\mathrm{sig}}_\alpha \left(\phi(\|x_i - x_j\|) \left(v_j - v_i \right) \right), v_i \right\rangle \\ &= \frac{\kappa_1}{N(T^\infty)^\alpha} \sum_{i,j=1}^N \left\langle \overrightarrow{\mathrm{sig}}_\alpha \left(\phi(\|x_i - x_j\|) \left(v_j - v_i \right) \right), v_i \right\rangle \\ &= -\frac{\kappa_1}{2N(T^\infty)^\alpha} \sum_{i,j=1}^N \left\langle \overrightarrow{\mathrm{sig}}_\alpha \left(\phi(\|x_i - x_j\|) \left(v_i - v_j \right) \right), (v_i - v_j) \right\rangle \\ &= -\frac{\kappa_1}{2N(T^\infty)^\alpha} \sum_{i,j=1}^N \phi(\|x_i - x_j\|)^\alpha \left\langle \overrightarrow{\mathrm{sig}}_\alpha \left(v_i - v_j \right), (v_i - v_j) \right\rangle \\ &\leq -\frac{\kappa_1 \phi(\sqrt{2}\|X\|)^\alpha}{2N(T^\infty)^\alpha} \sum_{i,j=1}^N \left\langle \overrightarrow{\mathrm{sig}}_\alpha \left(v_i - v_j \right), (v_i - v_j) \right\rangle \\ &\leq -\frac{\kappa_1 \phi(\sqrt{2}\|X\|)^\alpha}{2N(T^\infty)^\alpha} \sum_{i,j=1}^N \|v_i - v_j\|^{\alpha+1}, \end{aligned}$$

because for $i, j \in [N]$, $\|x_i - x_j\| \leq \sqrt{2}\|X\|$ holds and the monotonicity of ϕ leads to

$$\phi(\|x_i - x_j\|) \geq \phi(\sqrt{2}\|X\|).$$

Subsequently, we set

$$S := \left\{ t > 0 \,\middle|\, \|X(s)\| < X^\infty, \quad \forall s \in [t_1, t) \right\},$$

and let $\bar{S} := \sup S$. Here, we notice that S is nonempty from the continuity of $\|X\|$ and the fact that (3.6) implies $\|X(t_1)\| < X^\infty$. In addition, we observe

$$\|X(\bar{S}-)\| = X^\infty. \tag{3.8}$$

Now, using the construction of S and (3.5), for $t \in (t_1, \bar{S})$, we estimate

$$\begin{aligned}
\frac{1}{2}\frac{d\|V\|^2}{dt} &\le -\frac{\kappa_1 \phi(\sqrt{2}X^\infty)^\alpha}{2N(T^\infty)^\alpha} \sum_{i,j=1}^N \|v_i - v_j\|^{\alpha+1} \\
&\le -\frac{\kappa_1 \phi(\sqrt{2}X^\infty)^\alpha}{2N(T^\infty)^\alpha} \left(\sum_{i,j=1}^N \|v_i - v_j\|^2 \right)^{\frac{\alpha+1}{2}} \\
&= -\frac{\kappa_1 \phi(\sqrt{2}X^\infty)^\alpha}{2N(T^\infty)^\alpha} \left(2N\|V\|^2 \right)^{\frac{\alpha+1}{2}} \\
&= -\frac{\kappa_1 \phi(\sqrt{2}X^\infty)^\alpha}{(2N)^{\frac{1-\alpha}{2}}(T^\infty)^\alpha} \|V\|^{\alpha+1}.
\end{aligned}$$

For a.e. $t \in (t_1, \bar{S})$, the above relation yields

$$\frac{d\|V\|}{dt} \le -\frac{\kappa_1 \phi(\sqrt{2}X^\infty)^\alpha}{(2N)^{\frac{1-\alpha}{2}}(T^\infty)^\alpha} \|V\|^\alpha.$$

Thus, it follows from the comparison principle that for $t \in [t_1, \bar{S}]$,

$$\|V(t)\| \le \left(\|V(t_1)\|^{1-\alpha} - \frac{\kappa_1 \phi(\sqrt{2}X^\infty)^\alpha (1-\alpha)}{(2N)^{\frac{1-\alpha}{2}}(T^\infty)^\alpha} \cdot (t - t_1) \right)^{\frac{1}{1-\alpha}}.$$

To prove $\bar{S} = \infty$, we first suppose that $\bar{S} \le t_3$ for the proof by contradiction, where t_3 is defined as follows:

$$t_3 := t_1 + \frac{(2N)^{\frac{1-\alpha}{2}}(T^\infty)^\alpha}{\kappa_1 \phi(\sqrt{2}X^\infty)^\alpha (1-\alpha)} \|V(t_1)\|^{1-\alpha}.$$

Then, for $t \in [t_1, \bar{S}]$, we have

$$\begin{aligned}
\|X(t)\| &\le \|X(t_1)\| + \int_{t_1}^t \|V(s)\| ds \\
&\le \|X(t_1)\| + \int_{t_1}^t \left(\|V(t_1)\|^{1-\alpha} - \frac{\kappa_1 \phi(\sqrt{2}X^\infty)^\alpha (1-\alpha)}{(2N)^{\frac{1-\alpha}{2}}(T^\infty)^\alpha} \cdot (s - t_1) \right)^{\frac{1}{1-\alpha}} ds \\
&\le \|X(t_1)\| + \int_{t_1}^{t_3} \left(\|V(t_1)\|^{1-\alpha} - \frac{\kappa_1 \phi(\sqrt{2}X^\infty)^\alpha (1-\alpha)}{(2N)^{\frac{1-\alpha}{2}}(T^\infty)^\alpha} \cdot (s - t_1) \right)^{\frac{1}{1-\alpha}} ds \\
&= \|X(t_1)\| + \frac{(2N)^{\frac{1-\alpha}{2}}(T^\infty)^\alpha}{\kappa_1 (2-\alpha) \phi(\sqrt{2}X^\infty)^\alpha} \|V(t_1)\|^{2-\alpha} \\
&< X^\infty.
\end{aligned} \tag{3.9}$$

Accordingly, we can find

$$\|X(\bar{S}-)\| < X^\infty,$$

and it gives a contradiction to (3.8). Hence, we get $\bar{S} > t_3$; moreover, for $t \geq t_3$, we derive

$$\|V(t)\| \leq \left(\|V(t_1)\|^{1-\alpha} - \frac{\kappa_1 \phi(\sqrt{2}X^\infty)^\alpha (1-\alpha)}{(2N)^{\frac{1-\alpha}{2}} (T^\infty)^\alpha} \cdot (t - t_1) \right)^{\frac{1}{1-\alpha}} = 0.$$

Therefore, $\bar{S} = \infty$ because the following contradiction can be observed in the same way as (3.9):

$$\sup_{t \geq t_1} \|X(t)\| \leq \|X(t_1)\| + \int_{t_1}^{\infty} \|V(s)\| ds = \|X(t_1)\| + \int_{t_1}^{t_3} \|V(s)\| ds < X^\infty.$$

Finally, we conclude the following desired finite-in-time flocking: there exists a finite time $t_3 \in \mathbb{R}_+$ satisfying for $i \in [N]$ and $t \geq t_3$,

$$\sup_{t \in \mathbb{R}_+} \|X(t)\| < \max \left(\sup_{t \in [0, t_1)} \|X(t)\|, X^\infty \right) < \infty, \qquad v_i(t) = 0, \qquad T_i(t) = T^\infty.$$

□

4. Numerical simulations

In this section, we describe several numerical examples on $\mathbb{R}^3 \times \mathbb{R}^3 \times (\mathbb{R}_+ - \{0\})$ using the first-order forward Euler scheme, concerning the finite-time flocking of (1.3), and then compare them with the analytical results studied in Section 3. For all simulations, we set

$$N = 20, \quad \Delta t = h = 10^{-2}, \quad t = nh \in [0, 1000], \quad n \in \{0, 1, 2, \ldots, 100000\}.$$

Next, we select the following communication weights and system parameters:

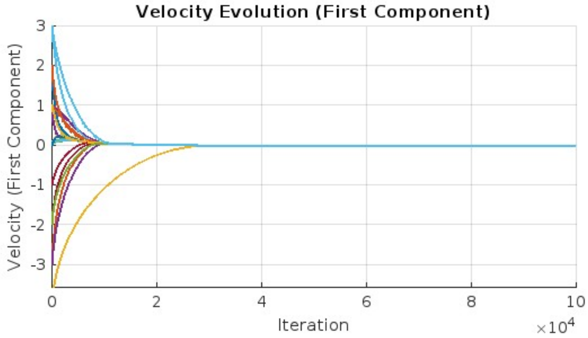
$$\phi(s) = \frac{1}{1+s}, \ s \in \mathbb{R}_+, \qquad \zeta = 1 + \frac{1}{1+s}, \ s \in \mathbb{R}_+, \qquad \kappa_1 = \kappa_2 = 1, \qquad \alpha = \beta = \frac{1}{2},$$

and initial data satisfying

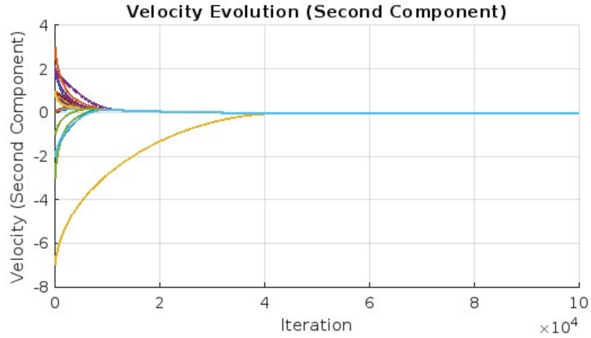
$$\begin{aligned} \{x_i^0\}_{i=1}^{20} = \Big\{ &(1,-1,-2), (-1,-1,2), (2,1,-1), (-2,1,1), (1,-2,1), (-1,2,-1), (2,1,-3), \\ &(-2,-3,3), (1,-1,2), (-1,3,-2), (3,2,-5), (-3,-3,5), (2,-2,1), (-2,3,-4), \\ &(3,1,-1), (-3,-1,4), (1,3,-3), (-1,-3,4), (4,1,-4), (-4,-1,3) \Big\}, \\ \{v_i^0\}_{i=1}^{20} = \Big\{ &(2,0,-2), (-3,0,1), (0,1,-1), (1,2,4), (0,-3,-2), (3,1,-1), (-2,1,2), (1,2,0), \\ &(2,3,5), (-4,-7,-6), (1,0,0), (0,1,0), (0,0,1), (-1,1,0), (0,-2,-1), (1,0,-3), \\ &(1,1,1), (-3,2,-1), (-2,-1,3), (3,-2,0) \Big\}, \\ \{T_i^0\}_{i=1}^{20} = \{ &2, 7, 19, 20, 3, 8, 9, 10, 4, 13, 1, 14, 5, 16, 17, 18, 6, 11, 12, 15 \}. \end{aligned}$$

Then, it is easy to check that the above setting satisfies the assumptions of Theorem 3.2.

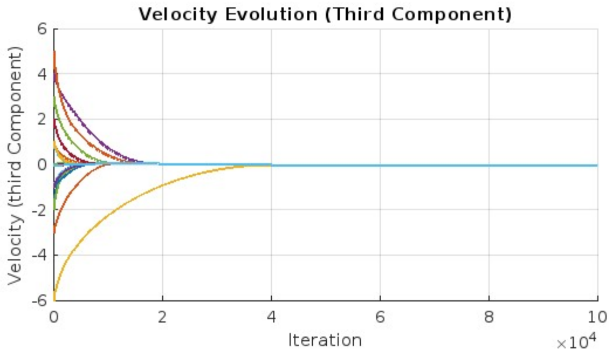
Now, we present several numerical results depicting the finite-time flocking of the system (1.3).



(a) $\{v_i^1\}_{i=1}^{20}$ vs n



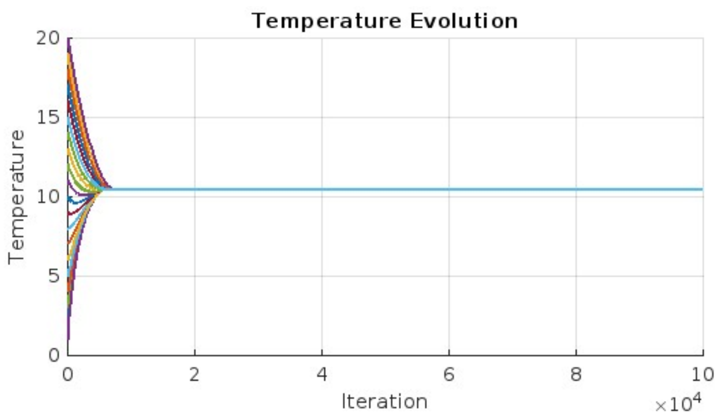
(b) $\{v_i^2\}_{i=1}^{20}$ vs n



(c) $\{v_i^3\}_{i=1}^{20}$ vs n

Figure 1. Velocity evolution.

First, we observe that Figure 1a, Figure 1b, and Figure 1c illustrate the emergence of the finite-time velocity alignment, supporting the validity of the analytical result demonstrated in the second assertion of Theorem 3.2.



(a) $\{T_i\}_{i=1}^{20}$ vs n

Figure 2. Temperature evolution.

In the following, it becomes apparent that Figure 2 illustrates the emergence of the finite-time temperature equilibrium, and this helps to confirm the validity of the analytical result described in the third assertion of Theorem 3.2.

5. Conclusion

In this paper, we have demonstrated several sufficient frameworks for the finite-in-time flocking of the generalized TCS model with a vector field that is not locally Lipschitz continuous. To achieve this, using basic estimates for temperatures and the propagation of conserved quantities, we first derived suitable dissipative inequalities regarding the *position–velocity–temperature* and divided it into two situations when the communication weight ϕ either had a positive lower bound or when it satisfied the nonnegativity and monotonicity. For each case, we proved the desired finite-in-time flocking under suitable assumptions. Therefore, considering that the mathematical community has primarily focused on Cucker–Smale-type models, which do not account for thermodynamics, to represent finite-in-time flocking, this paper is meaningful in addressing the TCS-type model that incorporates thermodynamics into the study of finite-in-time flocking. However, the current work only studied the finite-in-time convergence analysis of multi-agent systems with a finite size. Thus, it would be interesting to extend the current work to a finite-in-time consensus of appropriate kinetic systems with an infinite size thorough the mean-field limit (i.e., $N \to \infty$). This will be left for future work.

Author contributions

Hyunjin Ahn: Methodology, formal analysis, and writing original draft; Se Eun Noh: Review, editing, supervision, and validation.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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