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Research article

Stability results of a swelling porous-elastic system with two nonlinear variable exponent damping

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Abstract: In this paper, a swelling soil system with two nonlinear dampings of variable exponent-type is considered. The stability analysis of this system is investigated and it is proved that the system is stable under a natural condition on the parameters of the system and the variable exponents. It is noticed that one variable damping is enough to achieve polynomial and exponential decay and the decay is not necessarily improved if the system has two variable dampings.

Keywords: swelling soil; Enemy method; asymptotic behavior; variable exponents

1. Introduction

Our aim for this work was to investigate the stability analysis for a swelling soil through the application of theory of the porous media. Precisely, we consider the following nonlinear swelling soil system:

$$\begin{cases} \rho_z z_{tt} - a_1 z_{xx} - a_2 u_{xx} + \gamma |z_t|^{p(\cdot)-2} z_t = 0, & \text{in } (0,1)\times(0,\infty), \\ \rho_u u_{tt} - a_3 u_{xx} - a_2 z_{xx} + \beta |u_t|^{q(\cdot)-2} u_t = 0, & \text{in } (0,1)\times(0,\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad z(x,0) = z_0(x), z_t(x,0) = z_1(x) & x \in [0,1], \\ z(0,t) = z(1,t) = u(0,t) = u(1,t) = 0 & t \geq 0\,, \end{cases} \tag{1.1}$$

where $\gamma, \beta \geq 0$ and the components z and u indicate the displacements of the fluid and the elastic solid material, respectively. The densities of each component are represented by the positive constant coefficients ρ_u and ρ_z. The coefficients $a_2 \neq 0$, $a_1 > 0$, and $a_3 > 0$ are positive constants that meet some particular requirements. The variables $p(\cdot)$ and $q(\cdot)$ are exponent functions that satisfy additional requirements that will be stated later.

This problem was proposed for the first time by Iecsan [1] and simplified by Quintanilla [2], as follows:

$$\begin{cases} \rho_z z_{tt} = P_{1x} - G_1 + F_1 \\ \rho_u u_{tt} = P_{2x} + G_2 + F_2, \end{cases} \tag{1.2}$$

where the functions (P_1, G_1, F_1), in that order, stand for the partial tension, internal body forces, and external forces, respectively, that are operating on the displacement. For (P_2, G_2, F_2), but in the case of acting on the elastic solid, the definition is analogous. The constitutive equations for partial tensions are also provided by

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \underbrace{\begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix}}_{A} \begin{bmatrix} z_x \\ u_x \end{bmatrix}, \tag{1.3}$$

where the matrix A has the positive definite property in the sense of $a_1 a_3 \geq a_2^2$. For more information about swelling soils, we refer the reader to [3–7]. Regarding the stability, Quintanilla [2] established an exponential decay for the system (1.2) where

$$G_1 = G_2 = \xi(z_t - u_t), \quad F_1 = a_3 z_{xxt}, \quad F_2 = 0,$$

and $\xi > 0$ is the gain feedback. By using the spectral approach, Wang and Guo [8] obtained the exponential stability result for the system (1.2) with

$$G_1 = G_2 = 0, \quad F_1 = -\rho_z \gamma(x) z_t, \quad F_2 = 0,$$

where $\gamma(x)$ is an internal viscous damping function with a positive mean. After that, Ramos et al. [9] proved that the system (1.2) with

$$G_1 = G_2 = F_1 = 0, F_2 = -\gamma(t)\, g(u_t)$$

is exponentially stable provided that the wave speeds of the system are equal. Regarding viscoelastic swelling systems, Al-Mahdi and Al-Gharabli [10] and Apalara [11] obtained general decay results for Systems (1.2) with

$$G_1 = G_2 = F_1 = 0, F_2 = -\int_0^t g(t-s)\, u_{xx}(x,s)\, ds$$

for different classes of the relaxation function g. Similarly, Youkana et al. [12] considered the system (1.2) with

$$G_1 = G_2 = F_2 = 0,\, F_1 = -\int_0^t g(t-s)\, z_{xx}(x,s)\, ds$$

and they came up with a general decay result without imposing the system's wave speed. Apalara et al. [13] established a general decay result for the system (1.2) with

$$G_1 = \xi_1 z_t(x,t) + \xi_2 z_t(x,t-\tau),\, G_2 = 0,\, F_1 = -\int_0^t g(t-s)\, z_{xx}(x,s)\, ds,\, F_2 = 0,$$

without imposing the system's wave speed. The reader is referred to related research for other outcomes in porous elasticity systems, thermo-porous-elastic systems, Timoshenko systems, and other systems [2, 8, 14–27].

Equations with varying exponents of nonlinearity have drawn increasing amounts of attention in recent years. The applications to the mathematical modeling of non-Newtonian fluids are what have sparked such strong interest. These fluids include electro rheological fluid, which can undergo significant changes in response to an external electromagnetic field. A number of factors, including density, temperature, saturation, electric field, and others, affect the variable exponent of nonlinearity. We cite [28, 29] for further details on the electro-rheological fluids mathematical model. We briefly mention a few of the many references [30–36] that discuss the existence, blow-up, and stability of viscoelastic systems with variable exponents. Regarding swelling systems with variable exponents, Al-Mahdi et al. [37] proved that the system (1.1) (with $\beta = 0$) is exponentially and polynomially stable based on the range of the variable exponents. In the present work, we study the interaction between the two nonlinear dampings of variable exponent type in the system (1.1). We prove that one damping is enough to have exponential stability and two dampings do not improve the decay rates. In addition to the stability analysis, we present some numerical examples to illustrate the stability theory.

2. Preliminary and assumptions

In this section, we take into account the following **hypotheses**:

- (**A1**): $p, q : [0,1] \to [1,\infty)$ is a continuous function such that

$$p_1 := \text{essinf}_{x\in[0,1]} p(x), \quad p_2 := \text{esssup}_{x\in[0,1]} p(x).$$

where $1 < p_1 \le p(x) \le p_2 < \infty$.

$$q_1 := \text{essinf}_{x\in[0,1]} q(x), \quad q_2 := \text{esssup}_{x\in[0,1]} q(x).$$

where $1 < q_1 \le q(x) \le q_2 < \infty$. Additionally, by satisfying the log-Hölder continuity condition that is, for any λ with $0 < \lambda < 1$, there exists a constant $\delta > 0$ such that,

$$|f(x) - f(y)| \le -\frac{\delta}{\log|x-y|}, \text{ for all } x, y \in \Omega, \text{ with } |x-y| < \lambda. \tag{2.1}$$

- (**A2**): The coefficients denoted by $a_i,\ i = 1, ..., 3$ satisfy that $a_1 a_3 - a_2^2 > 0$.

Throughout the paper, $\Omega = [0,1]$ and $\overline{c}$ is a positive constant that depends on the coefficients of the system (1.1).

Lemma 2.1. *The energy of the problem* (1.1) *is defined by*

$$E(t) \;=\; \frac{1}{2}\int_0^1 \left[\rho_z z_t^2 + \rho_u u_t^2 + a_3 u_x^2 + a_1 z_x^2 + 2a_2 z_x u_x\right] dx, \tag{2.2}$$

and it satisfies the following

$$E'(t) = -\gamma \int_0^1 |z_t|^{p(\cdot)} dx - \beta \int_0^1 |u_t|^{q(\cdot)} dx \le 0. \tag{2.3}$$

Proof. The proof of Eq (2.3) is straightforward by multiplying (1.1) by z_t and u_t respectively, integrating over the interval $(0,1)$, using integration by parts, and performing some modifications. □

3. The main results

In this section, we state our decay results in the following theorems:

Theorem 3.1. *Assume that* (***A1–A2***) *hold and* $1 < p_1, q_1 < 2$*. Then, the energy functional* (2.2) *satisfies the for positive constants denoted by* $C_i,\ i = 1,2,3,$ *and for any* $t > 0$

$$\begin{cases} E(t) < \dfrac{C_1}{(t+1)^{\left(\frac{q_1-1}{2-q_1}\right)}}, & \text{if } \gamma = 0 \text{ and } \beta \neq 0; \\[2ex] E(t) < \dfrac{C_2}{(t+1)^{\left(\frac{p_1-1}{2-p_1}\right)}}, & \text{if } \gamma \neq 0 \text{ and } \beta = 0; \\[2ex] E(t) < \dfrac{C_3}{(t+1)^{\left(\frac{\overline{p}_1-1}{2-\overline{p}_1}\right)}}, & \text{if } \gamma \neq 0 \text{ and} \beta \neq 0. \end{cases} \tag{3.1}$$

where $\overline{p}_1 = \min\{p_1, q_1\}$.

Theorem 3.2. *Assume that* (***A1–A2***) *hold and* $p_1, q_1 \ge 2$*. Then, the energy functional* (2.2) *satisfies the for some positive constants* $\lambda_i, \sigma_i, \mu_i > 0,\ i = 1,2,3$ *and for any* $t \ge 0$

$$\begin{cases} E(t) < \mu_1 e^{-\lambda_1 t}, & \text{if } \gamma = 0,\ \beta \neq 0 \text{ and } q_2 = 2; \\[1ex] E(t) < \mu_2 e^{-\lambda_2 t}, & \text{if } \gamma \neq 0,\ \beta = 0 \text{ and } p_2 = 2; \\[1ex] E(t) < \mu_3 e^{-\lambda_3 t}, & \text{if } \gamma \neq 0,\ \beta \neq 0 \text{ and } p_2 = q_2 = 2, \end{cases} \tag{3.2}$$

and

$$\begin{cases} E(t) < \dfrac{\sigma_1}{(t+1)^{\left(\frac{q_2-2}{2}\right)}}, & \text{if } \gamma = 0,\ \beta \neq 0 \text{ and } q_2 > 2; \\[2ex] E(t) < \dfrac{\sigma_2}{(t+1)^{\left(\frac{p_2-2}{2}\right)}}, & \text{if } \gamma \neq 0,\ \beta = 0 \text{ and } p_2 > 2; \\[2ex] E(t) < \dfrac{\sigma_3}{(t+1)^{\left(\frac{\overline{p}_2-2}{2}\right)}}, & \text{if } \gamma \neq 0,\ \beta \neq 0 \text{ and } p_2, q_2 > 2. \end{cases} \tag{3.3}$$

where $\overline{p}_2 = \min\{p_2, q_2\}$.

Theorem 3.3. *Assume that (**A1–A2**) hold and* $p_1 \geq 2$ *and* $1 < q_1 < 2$. *Then, the energy functional* (2.2) *satisfies the for some positive constants denoted by* $\vartheta_i > 0,\ i = 1, ..., 6,$ *and for any* $t \geq 0$

$$\begin{cases} E(t) < \frac{\vartheta_1}{(t+1)^{\left(\frac{q_1-1}{2-q_1}\right)}} & \text{if } \gamma = 0,\ \beta \neq 0 \text{ and } q_2 \geq q_1; \\ E(t) < \vartheta_2 e^{-\vartheta_3 t}, & \text{if } \gamma \neq 0,\ \beta = 0 \text{ and } p_2 = 2; \\ E(t) < \frac{\vartheta_4}{(t+1)^{\left(\frac{p_2-2}{2}\right)}}, & \text{if } \gamma \neq 0,\ \beta = 0 \text{ and } p_2 > 2. \\ E(t) < \frac{\vartheta_5}{(t+1)^{\left(\frac{q_1-1}{2-q_1}\right)}} & \text{if } \gamma \neq 0,\ \beta \neq 0,\ p_2 = 2 \text{ and } q_2 \geq q_1; \\ E(t) < \frac{\vartheta_6}{(t+1)^{\left(\frac{p_2-2}{2}\right)}}, & \text{if } \gamma \neq 0,\ \beta \neq 0,\ p_2 > 2 \text{ and } q_2 \geq q_1. \end{cases} \tag{3.4}$$

4. Technical lemmas

In this section, we establish several lemmas needed for the proofs of our main results.

Lemma 4.1. *Assume that (**A1–A2**) hold. The functional*

$$\chi_1(t) = \rho_z \int_0^1 z z_t\, dx - \frac{a_2}{a_3}\rho_u \int_0^1 u_t z\, dx \tag{4.1}$$

satisfies the for $p_1, q_1 \geq 2$ *and any* $\varepsilon_1 > 0$

$$\begin{aligned} \chi_1'(t) \quad &\leq \quad -\frac{\alpha_0}{2a_3}\int_0^1 z_x^2 dx + \frac{\overline{c}}{\varepsilon_1}\int_0^1 z_t^2 dx + \varepsilon_1 \int_0^1 u_t^2 dx + c\gamma^2 \int_0^1 |z_t|^{p(x)} dx \\ &+ \quad c\beta^2 \int_0^1 |u_t|^{q(x)} dx, \end{aligned} \tag{4.2}$$

and for $1 < p_1, q_1 < 2$, *the functional satisfies*

$$\begin{aligned} \chi_1'(t) \quad &\leq \quad -\frac{\alpha_0}{2a_3}\int_0^1 z_x^2 dx + \frac{\overline{c}}{\varepsilon_1}\int_0^1 z_t^2 dx + \varepsilon_1 \int_0^1 u_t^2 dx + c\gamma^2 \int_0^1 |z_t|^{p(x)} dx \\ &+ \quad c\beta^2 \int_0^1 |u_t|^{q(x)} dx + c\gamma^2 \left(\int_0^1 |z_t|^{p(x)} dx\right)^{p_1-1} + c\beta^2 \left(\int_0^1 |u_t|^{q(x)} dx\right)^{q_1-1}, \end{aligned} \tag{4.3}$$

where $\alpha_0 = a_1 a_3 - a_2^2 > 0$ *and* $\overline{c} > 0$ *depends on* $a_1, a_2, a_3, \rho_u, \rho_z$.

Proof. By considering Eq (1.1) and integrating by parts, we obtain

$$\chi_1'(t) \quad = \quad \rho_z \int_0^1 z_t^2 dx - \left[a_1 - \frac{a_2^2}{a_3}\right]\int_0^1 z_x^2 dx - \frac{a_2}{a_3}\rho_u \int_0^1 u_t z_t dx$$

$$+ \quad a_2 \int_0^1 z_x u_x dx - a_2 \int_0^1 z_x u_x dx - \gamma \int_0^1 |z_t|^{p(\cdot)-2} z_t z dx$$
$$- \quad \frac{a_2}{a_3}\beta \int_0^1 |u_t|^{q(\cdot)-2} u_t z dx. \tag{4.4}$$

Using Young's inequality, we get the for any $\varepsilon_1 > 0$

$$-\frac{a_2}{a_3}\rho_u \int_0^1 u_t z_t \, dx \le \varepsilon_1 \int_0^1 u_t^2 dx + \frac{a_2^2}{4\varepsilon_1 a_3^2}\rho_u^2 \int_0^1 z_t^2 dx. \tag{4.5}$$

Applying Young's inequality with $\zeta(x) = \frac{p(x)}{p(x)-1}$ and $\zeta^*(x) = p(x)$ helps to estimate the last two terms in (4.4) as follows: For a.e $x \in \Omega$ and any $\delta_1 > 0$, we have

$$|z_t|^{p(x)-2} z_t z \le \delta_1 |z|^{p(x)} + c_{\delta_1}(x)|z_t|^{p(x)},$$

where

$$c_{\delta_1}(x) = \delta_1^{1-p(x)}(p(x))^{-p(x)}(p(x)-1)^{p(x)-1}.$$

Hence,

$$-\int_\Omega z|z_t|^{p(x)-2} z_t dx \le \delta_1 \int_\Omega |z|^{p(x)} dx + \int_\Omega c_{\delta_1}(x)|z_t|^{p(x)} dx. \tag{4.6}$$

Next, using Eqs (2.2) and (2.3), Poincaré's inequality and the embedding property, we get

$$\begin{aligned}
\int_\Omega |z|^{p(x)} dx &= \int_{\Omega_+} |z|^{p(x)} dx + \int_{\Omega_-} |z|^{p(x)} dx \\
&\le \int_{\Omega_+} |z|^{p_2} dx + \int_{\Omega_-} |z|^{p_1} dx \\
&\le \int_\Omega |z|^{p_2} dx + \int_\Omega |z|^{p_1} dx \\
&\le c_e^{p_1}\|z_x\|_2^{p_1} + c_e^{p_2}\|z_x\|_2^{p_2} \\
&\le \left(c_e^{p_1}\|z_x\|_2^{p_1-2} + c_e^{p_2}\|z_x\|_2^{p_2-2}\right)\|z_x\|_2^2 \\
&\le \left(c_e^{p_1}\left(\frac{2}{a_1}E(0)\right)^{p_1-2} + c_e^{p_2}\left(\frac{2}{a_1}E(0)\right)^{p_2-2}\right)\|z_x\|_2^2 \\
&\le c_1\|z_x\|_2^2,
\end{aligned} \tag{4.7}$$

where c_e is the embedding constant,

$$\Omega_+ = \{x \in \Omega : |z(x,t)| \ge 1\}, \quad \Omega_- = \{x \in \Omega : |z(x,t)| < 1\}$$

and

$$c_1 = \left(c_e^{p_1}\left(\frac{2}{a_1}E(0)\right)^{p_1-2} + c_e^{p_2}\left(\frac{2}{a_1}E(0)\right)^{p_2-2}\right). \tag{4.8}$$

Then, Eqs (4.6) and (4.7) yield

$$-\gamma \int_\Omega z|z_t|^{p(x)-2} z_t dx \le \delta_1 c_1 \|z_x\|_2^2 + \gamma^2 \int_\Omega c_{\delta_1}(x)|z_t|^{p(x)} dx. \tag{4.9}$$

Similarly, we can have

$$-\frac{a_2}{a_3}\beta\int_\Omega z|u_t|^{q(x)-2}u_t dx \le \delta_1 c_1\|z_x\|_2^2 + \frac{a_2^2\beta^2}{a_3^2}\int_\Omega c_{\delta_1}(x)|u_t|^{q(x)}dx. \tag{4.10}$$

By combining all estimates (4.4)–(4.10), and selecting $\delta_1 = \frac{\alpha_0}{4a_3c_1}$, it follows that $c_\delta(x)$ remains bounded; then, estimate (4.2) is established.

To prove Eq (4.3), we re-estimate the last two terms in Eq (4.4) as follows:

First, we set

$$\Omega_1 = \{x \in \Omega : p(x) < 2\} \text{ and } \Omega_2 = \{x \in \Omega : p(x) \ge 2\}.$$

Then, we have

$$-\int_\Omega z|z_t|^{p(x)-2}z_t dx = -\int_{\Omega_1} z|z_t|^{p(x)-2}z_t dx - \int_{\Omega_2} z|z_t|^{p(x)-2}z_t dx. \tag{4.11}$$

We notice that on Ω_1, we have

$$2p(x) - 2 < p(x), \text{ and } 2p(x) - 2 \ge 2p_1 - 2. \tag{4.12}$$

Therefore, by using Young's and Poincaré's inequalities, then (4.12) leads to

$$\begin{aligned}
-\int_{\Omega_1} z|z_t|^{p(x)-2}z_t dx &\le \eta\int_{\Omega_1}|z|^2dx + \frac{1}{4\eta}\int_{\Omega_1}|z_t|^{2p(x)-2}dx \\
&\le \eta\|z_x\|_2^2 + c_\eta\Big[\int_{\Omega_1^+}|z_t|^{2p(x)-2}dx + \int_{\Omega_1^-}|z_t|^{2p(x)-2}dx\Big] \\
&\le \eta\|z_x\|_2^2 + c_\eta\Big[\int_{\Omega_1^+}|z_t|^{p(x)}dx + \int_{\Omega_1^-}|z_t|^{2p_1-2}dx\Big] \\
&\le \eta\|z_x\|_2^2 + c_\eta\Big[\int_{\Omega}|z_t|^{p(x)}dx + \Big(\int_{\Omega_1^-}|z_t|^{2}dx\Big)^{p_1-1}\Big] \\
&\le \eta\|z_x\|_2^2 + c_\eta\Big[\int_{\Omega}|z_t|^{p(x)}dx + \Big(\int_{\Omega_1^-}|z_t|^{p(x)}dx\Big)^{p_1-1}\Big] \\
&\le \eta\|z_x\|_2^2 + c_\eta\Big[\int_{\Omega}|z_t|^{p(x)}dx + \Big(\int_{\Omega}|z_t|^{p(x)}dx\Big)^{p_1-1}\Big],
\end{aligned} \tag{4.13}$$

where

$$\Omega_1^+ = \{x \in \Omega_1 : |z_t(x,t)| \ge 1\} \text{ and } \Omega_1^- = \{x \in \Omega_1 : |z_t(x,t)| < 1\}. \tag{4.14}$$

Next, we have the following for the case $p(x) \ge 2$

$$-\int_{\Omega_2} z|z_t|^{p(x)}z_t dx \le \eta\|z_x\|_2^2 + \int_\Omega c_\eta(x)|z_t|^{p(x)}dx. \tag{4.15}$$

Therefore, we conclude that

$$-\gamma\int_\Omega z|z_t|^{p(x)}z_t dx \le 2\eta\|z_x\|_2^2 + \gamma^2 c_\eta\Big[\int_\Omega |z_t|^{p(x)}dx + \Big(\int_\Omega |z_t|^{p(x)}dx\Big)^{p_1-1}\Big]. \tag{4.16}$$

Similarly, we can get

$$-\frac{a_2\beta}{a_3}\int_\Omega z|u_t|^{q(x)-2}u_t dx \le 2\eta\|z_x\|_2^2 + \beta^2 c_\eta\Big[\int_\Omega |u_t|^{q(x)}dx + \Big(\int_\Omega |u_t|^{q(x)}dx\Big)^{q_1-1}\Big]. \tag{4.17}$$

Selecting $\eta = \frac{\alpha}{8a_3}$, that $c_\delta(x)$ remains bounded; and, then, combining Eqs (4.11)–(4.17), estimate (4.3) is established. □

Lemma 4.2. *Assume that (**A1–A2**) hold. The functional*

$$\chi_2(t) = -\rho_z \int_0^1 z_t z dx \tag{4.18}$$

satisfies the for $p_1 \ge 2$ and any $\varepsilon_2, \delta_2 > 0$:

$$\begin{aligned} \chi_2'(t) \;&\le\; -\rho_z \int_0^1 z_t^2 dx + \left[a_1 + \frac{a_2^2}{4\varepsilon_2} + c_1\delta_2\right]\int_0^1 z_x^2 dx + \varepsilon_2 \int_0^1 u_x^2 dx \\ &+\; \gamma^2 \int_\Omega c_{\delta_2}(x)|z_t|^{p(x)}dx, \end{aligned} \tag{4.19}$$

and for $1 < p_1 < 2$, the functional satisfies

$$\begin{aligned} \chi_2'(t) \;&\le\; -\rho_z \int_0^1 z_t^2 dx + \left[a_1 + \frac{a_2^2}{4\varepsilon_2} + c_1\delta_2\right]\int_0^1 z_x^2 dx + \varepsilon_2 \int_0^1 u_x^2 dx \\ &+\; \gamma^2 \int_\Omega c_{\delta_2}(x)|z_t|^{p(x)}dx + c\gamma^2\left(\int_0^1 |z_t|^{p(x)}dx\right)^{p_1-1}, \end{aligned} \tag{4.20}$$

where c_1 is defined in Eq (4.8).

Proof. Direct computations using Eq (1.1) give

$$\chi_2'(t) \;=\; -\rho_z \int_0^1 z_t^2 dx + a_1 \int_0^1 z_x^2 dx + a_2 \int_0^1 z_x u_x dx - \gamma \int_\Omega z|z_t|^{p(x)-2}z_t dx \tag{4.21}$$

Hence, Young's inequality and the same estimates for the last term in Eq (4.21) yield Eqs (4.19) and (4.20). □

Lemma 4.3. *Assume that (**A1–A2**) hold. The functional*

$$\chi_3(t) = a_2\rho_z\rho_u \int_0^1 u z_t\, dx - a_2\rho_u\rho_z \int_0^1 z u_t\, dx \tag{4.22}$$

satisfies the for $p_1, q_1 \ge 2$ and any $\eta_1 > 0$:

$$\chi_3'(t) \;\le\; -\frac{a_2^2\rho_u}{4}\int_0^1 u_x^2 dx + c\gamma^2 \int_0^1 |z_t|^{p(\cdot)}dx + \overline{c}\int_0^1 z_x^2 dx + c\beta^2 \int_0^1 |u_t|^{q(\cdot)}dx, \tag{4.23}$$

and for $1 < p_1, q_1 < 2$*, the functional satisfies*

$$\begin{aligned}\chi_3'(t) \quad &\leq \quad -\frac{a_2^2\rho_u}{4}\int_0^1 u_x^2 dx + c\gamma^2\int_0^1 |z_t|^{p(\cdot)}dx + \overline{c}\int_0^1 z_x^2 dx + c\beta^2\int_0^1 |u_t|^{q(\cdot)}dx \\ &+ \quad c\beta^2\left(\int_0^1 |u_t|^{q(x)}dx\right)^{q_1-1} + c\gamma^2\left(\int_0^1 |z_t|^{p(x)}dx\right)^{p_1-1}, \end{aligned} \tag{4.24}$$

where $\overline{c} > 0$ *depends on* $a_1, a_2, a_3, \rho_u, \rho_z$.

Proof. By exploiting (1.1), we have

$$\begin{aligned}\chi_3'(t) \quad &= \quad a_2\rho_z\rho_u\int_0^1 u_t z_t\, dx - a_2\rho_u\rho_z\int_0^1 u_t z_t\, dx - a_2 a_1\rho_u\int_0^1 u_x z_x\, dx - a_2^2\rho_u\int_0^1 u_x^2\, dx \\ &+ \quad a_2 a_3\rho_z\int_0^1 z_x u_x dx + a_2^2\rho_z\int_0^1 z_x^2 dx - a_2\gamma\rho_u\int_0^1 |z_t|^{p(x)-2}z_t u dx \\ &+ \quad a_2\beta\rho_z\int_0^1 |u_t|^{q(x)-2}u_t z dx. \end{aligned} \tag{4.25}$$

Using Young's inequality, we get

$$a_2(a_3\rho_z - a_1\rho_u)\int_0^1 u_x z_x\, dx \leq \frac{a_2^2\rho_u}{2}\int_0^1 u_x^2\, dx + \overline{c}\int_0^1 z_x^2 dx, \tag{4.26}$$

where $\overline{c} > 0$ depends on $a_1, a_2, a_3, \rho_u, \rho_z$. To estimate the last two terms in (4.25), we apply Young's inequality with $\zeta(x) = \frac{p(x)}{p(x)-1}$ and $\zeta^*(x) = p(x)$. So, for a.e $x \in \Omega$ and any $\delta_3 > 0$, we have

$$|z_t|^{p(x)-2}z_t u \leq \delta_3|u|^{p(x)} + c_{\delta_3}(x)|z_t|^{p(x)},$$

where

$$c_{\delta_3}(x) = \delta_3^{1-p(x)}(p(x))^{-p(x)}(p(x)-1)^{p(x)-1}.$$

Hence,

$$\rho_u a_2\int_\Omega u|z_t|^{p(x)-2}z_t dx \leq \delta_3\int_\Omega |u|^{p(x)}dx + a_2^2\rho_u^2\int_\Omega c_{\delta_3}(x)|z_t|^{p(x)}dx. \tag{4.27}$$

Using Eqs (2.2) and (2.3), Poincaré's inequality and the embedding property, we find that

$$\begin{aligned}\int_\Omega |u|^{p(x)}dx &= \int_{\Omega_+}|u|^{p(x)}dx + \int_{\Omega_-}|u|^{p(x)}dx \\ &\leq \int_{\Omega_+}|u|^{p_2}dx + \int_{\Omega_-}|u|^{p_1}dx \\ &\leq \int_\Omega |u|^{p_2}dx + \int_\Omega |u|^{p_1}dx \\ &\leq c_e^{p_1}\|u_x\|_2^{p_1} + c_e^{p_2}\|u_x\|_2^{p_2} \\ &\leq \left(c_e^{p_1}\|u_x\|_2^{p_1-2} + c_e^{p_2}\|u_x\|_2^{p_2-2}\right)\|u_x\|_2^2 \\ &\leq \left(c_e^{p_1}\left(\frac{2}{a_3}E(0)\right)^{p_1-2} + c_e^{p_2}\left(\frac{2}{a_3}E(0)\right)^{p_2-2}\right)\|u_x\|_2^2 \\ &\leq c_3\|u_x\|_2^2, \end{aligned} \tag{4.28}$$

where c_e is the embedding constant,

$$\Omega_+ = \{x \in \Omega : |u(x,t)| \geq 1\}, \;\; \Omega_- = \{x \in \Omega : |u(x,t)| < 1\}$$

and

$$c_3 = \left(c_e^{p_1}\left(\frac{2}{a_3}E(0)\right)^{p_1-2} + c_e^{p_2}\left(\frac{2}{a_3}E(0)\right)^{p_2-2} \right). \tag{4.29}$$

Then, Eqs (4.27) and (4.28) yield

$$a_2\rho_u\gamma \int_\Omega u|z_t|^{p(x)-2}z_t dx \leq \delta_3 c_3 \|u_x\|_2^2 + \gamma^2 \int_\Omega c_{\delta_3}(x)|z_t|^{p(x)}dx. \tag{4.30}$$

In the same way, we get

$$a_2\rho_z\beta \int_\Omega z|u_t|^{q(x)-2}u_t dx \leq \omega_3 c_1 \|z_x\|_2^2 + \beta^2 \int_\Omega c_{\omega_3}(x)|u_t|^{q(x)}dx, \tag{4.31}$$

where c_1, c_3 have been defined in Eqs (4.8) and (4.29).

Combining all of the above estimates and selecting $\delta_3 = \frac{a_2^2\rho_u}{4c_3}$ and $\omega_3 = \frac{1}{c_1}$ we arrive at Eq (4.23).

To prove Eq (4.24), we re-estimate the last two terms in Eq (4.25) as in the above calculations to obtain

$$-a_2\rho_u\gamma \int_\Omega u|z_t|^{p(x)}z_t dx \leq 2\eta c_3\|u_x\|_2^2 + \gamma^2 \int_\Omega c_\eta(x)|z_t|^{p(x)}dx + \gamma^2 c_\eta \left(\int_0^1 |z_t|^{p(x)}dx\right)^{p_1-1}, \tag{4.32}$$

and

$$a_2\rho_z\beta \int_\Omega z|u_t|^{q(x)}u_t dx \leq 2\lambda c_1\|z_x\|_2^2 + \beta^2 \int_\Omega c_\lambda(x)|u_t|^{q(x)}dx + \beta^2 c_\lambda \left(\int_0^1 |u_t|^{q(x)}dx\right)^{q_1-1}, \tag{4.33}$$

Then, by selecting $\eta = \frac{a_2^2\rho_u}{8c_3}$ and $\lambda = \frac{1}{2c_1}$, estimate (4.24) is established. □

Lemma 4.4. *Assume that (**A1–A2**) hold. The functional*

$$\chi_4(t) = -\rho_u\varepsilon \int_0^1 u_t u \, dx \tag{4.34}$$

satisfies the for some $\varepsilon > 0$, *and* $q_1 \geq 2$

$$\chi_4'(t) \leq -\varepsilon\rho_u \int_0^1 u_t^2 \, dx + \frac{3\varepsilon a_3}{2}\int_0^1 u_x^2 dx + \frac{\varepsilon a_2^2}{a_3}\int_0^1 z_x^2 dx + c\beta^2 \int_0^1 |u_t|^{q(x)}dx, \tag{4.35}$$

and for $1 < q_1 < 2$,

$$\begin{aligned} \chi_4'(t) \;\; &\leq \;\; -\varepsilon\rho_u \int_0^1 u_t^2 \, dx + \frac{3\varepsilon a_3}{2}\int_0^1 u_x^2 dx + \frac{\varepsilon a_2^2}{a_3}\int_0^1 z_x^2 dx + c\beta^2 \int_0^1 |u_t|^{q(x)}dx \\ &+ \;\; c\beta^2 \left(\int_0^1 |u_t|^{q(x)}dx\right)^{q_1-1} dx. \end{aligned} \tag{4.36}$$

Proof. Direct computations using Eq (1.1) yield

$$\chi_4'(t) = -\rho_u\varepsilon\int_0^1 u_t^2\,dx + \varepsilon a_3\int_0^1 u_x^2\,dx + \varepsilon a_2\int_0^1 u_x z_x dx - \varepsilon\beta\int_0^1 u|u_t|^{q(x)}u_t dx. \tag{4.37}$$

Estimates (4.35) and (4.36) can be established in a similar manner as for the above estimations. □

Lemma 4.5. *Assume that (**A1–A2**) hold. If $p_1, q_1 \geq 2$, then*

$$\begin{aligned}&\int_0^1 z_t^2 dx \leq -E'(t), \quad \text{if } p_2 = 2,\\ &\int_0^1 u_t^2 dx \leq -E'(t), \quad \text{if } q_2 = 2,\end{aligned} \tag{4.38}$$

and

$$\begin{aligned}&\int_0^1 z_t^2 dx \leq -E'(t) + c\left(-E'(t)\right)^{\frac{2}{p_2}}, \quad \text{if } p_2 > 2\\ &\int_0^1 u_t^2 dx \leq -E'(t) + c\left(-E'(t)\right)^{\frac{2}{q_2}}, \quad \text{if } q_2 > 2.\end{aligned} \tag{4.39}$$

Proof. By recalling Eq (2.3), it is easy to establish Eq (4.38). To prove the first estimate in Eq (4.39), we set the following partitions

$$\Omega_1 = \{x \in \Omega : |z_t| \geq 1\} \quad \text{and} \quad \Omega_2 = \{x \in \Omega : |z_t| < 1\}. \tag{4.40}$$

Using the Hölder and Young inequalities and Eq (2.2), we obtain the following for Ω_1,

$$\int_{\Omega_1} z_t^2 dx \leq \int_\Omega |z_t|^{p(x)}\,dx = -E'(t), \tag{4.41}$$

and for Ω_2, we get

$$\begin{aligned}\int_{\Omega_2} z_t^2 \mathrm{d}x &\leq c\left(\int_{\Omega_2} |z_t|^{p_2}\,\mathrm{d}x\right)^{\frac{2}{p_2}}\\ &\leq c\left(\int_{\Omega_2} |z_t|^{p(x)}\,\mathrm{d}x\right)^{\frac{2}{p_2}} \leq c\left(\int_{\Omega} |z_t|^{p(x)}\,\mathrm{d}x\right)^{\frac{2}{p_2}} = c\left(-E'(t)\right)^{\frac{2}{p_2}}.\end{aligned} \tag{4.42}$$

Combining Eqs (4.41) and (4.42), the first estimate in Eq (4.39) can be established ; also, repeat the same steps to establish the second estimate in Eq (4.39). □

5. Proofs of the main results

In this section, we prove our decay results in Theorems 3.1, 3.2 and 3.3.

5.1. Proof of Theorem 3.1

Proof. To prove Theorem 3.1, let

$$\mathcal{L}(t) = \mu E(t) + \mu_1\chi_1(t) + \mu_2\chi_2(t) + \mu_3\chi_3(t) + \mu_4\chi_4(t) \tag{5.1}$$

where $\mu, \mu_1, \mu_2, \mu_3, \mu_4$ are positive constants to be properly chosen. By taking the derivative of the functional $\mathcal{L}$ and using all of the above estimates (4.2)–(4.35), we obtain

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\left(\mu_3\frac{\alpha}{2} - \mu_2\varepsilon_2 - \frac{3\varepsilon a_3\mu_4}{2}\right)\int_0^1 u_x^2 dx \\
& - \left(\mu_1\frac{\alpha}{2a_3} - \mu_2\frac{\overline{c}}{\varepsilon_2} - c_1\delta_2\mu_2 - \overline{c}\mu_3 - \frac{\varepsilon a_2^2\mu_4}{a_3}\right)\int_0^1 z_x^2\, dx \\
& - \left(\rho_z\mu_2 - \mu_1\frac{\overline{c}}{\varepsilon_1}\right)\int_0^1 z_t^2 dx - (\varepsilon\mu_4\rho_u - \varepsilon_1\mu_1)\int_0^1 u_t^2 dx \\
& - \left[\gamma\mu - c\gamma^2\mu_1 - c\gamma^2\mu_2 - c\gamma^2\mu_3\right]\int_0^1 |z_t|^{p(\cdot)}dx + c\gamma^2\left(\int_0^1 |z_t|^{p(x)}dx\right)^{p_1-1} \\
& - \left[\beta\mu - c\beta^2\mu_1 - c\beta^2\mu_3 - c\beta^2\mu_4\right]\int_0^1 |u_t|^{q(\cdot)}dx + c\beta^2\left(\int_0^1 |u_t|^{q(x)}dx\right)^{q_1-1}
\end{aligned}$$

Choosing $\varepsilon_i = \mu_i,\ i = 1, 2$, and $\delta_2 = \frac{1}{\mu_2}$, the above estimate becomes

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\left(\mu_3\frac{\alpha}{2} - \mu_2^2 - \frac{3\varepsilon a_3\mu_4}{2}\right)\int_0^1 u_x^2 dx \\
& -\left(\mu_1\frac{\alpha}{2a_3} - \overline{c} - c_1 - \overline{c}\mu_3 - \frac{\varepsilon a_2^2\mu_4}{a_3}\right)\int_0^1 z_x^2 dx \\
& - (\rho_z\mu_2 - \overline{c})\int_0^1 z_t^2 dx - \left(\varepsilon\mu_4\rho_u - \mu_1^2\right)\int_0^1 u_t^2 dx \\
& -\left[\gamma\mu - c\gamma^2\mu_1 - c\gamma^2\mu_2 - c\gamma^2\mu_3\right]\int_0^1 |z_t|^{p(\cdot)}dx + c\gamma^2\left(\int_0^1 |z_t|^{p(x)}dx\right)^{p_1-1} \\
& -\left[\beta\mu - c\beta^2\mu_1 - c\beta^2\mu_2 - c\beta^2\mu_4\right]\int_0^1 |u_t|^{q(\cdot)}dx + c\beta^2\left(\int_0^1 |u_t|^{q(x)}dx\right)^{q_1-1}.
\end{aligned}$$

First, we select μ_2 such that

$$\rho_z\mu_2 - \overline{c} > 1.$$

Then, we choose μ_3 large enough such that

$$\Lambda_1 := \mu_3\frac{\alpha}{2} - \mu_2^2 > 0.$$

Next, we choose μ_1 large enough such that

$$\Lambda_2 := \mu_1\frac{\alpha}{2a_3} - \overline{c} - c_1 - \overline{c}\mu_3 > 0.$$

Now, we choose μ_4 such that

$$\varepsilon\mu_4\rho_u - \mu_1^2 > 1.$$

Select ε such that

$$\varepsilon = \min\left[\frac{2\Lambda_1}{3a_3\mu_4}, \frac{a_3\Lambda_2}{3a_2^2\mu_4}\right]$$

After fixing μ_i, where $i = 1, 2, 3, 4$, we select μ large enough such that

$$\gamma\mu - c\gamma^2\mu_1 - c\gamma^2\mu_2 - c\gamma^2\mu_3 > 1,$$

$$\beta\mu - c\beta^2\mu_1 - c\beta^2\mu_2 - c\beta^2\mu_4 > 1,$$

and $\mathcal{L} \sim E$. That is, we can find two positive constants α_1 and α_2 such that

$$\alpha_1 E(t) \leq \mathcal{L}(t) \leq \alpha_2 E(t), \tag{5.2}$$

On the other hand, Young's inequality and (2.2) allow us to obtain

$$E(t) \leq \bar{c} \int_0^1 \left(u_t^2 + u_x^2 + z_t^2 + z_x^2\right) dx. \tag{5.3}$$

Hence, estimate (5.2) becomes as follows for any $t \geq 0$ and some positive constant α_3,

$$\begin{aligned}\mathcal{L}'(t) \leq {}& -\alpha_3 \int_0^1 \left(u_t^2 + u_x^2 + z_t^2 + z_x^2\right) dx + c\gamma^2 \left(\int_0^1 |z_t|^{p(x)} dx\right)^{p_1-1} \\ &+ c\beta^2 \left(\int_0^1 |u_t|^{q(x)} dx\right)^{q_1-1}.\end{aligned} \tag{5.4}$$

Then, from Eqs (5.3) and (5.4), we get the following for some positive constant α_4,

$$\mathcal{L}'(t) \leq -\alpha_4 E(t) + c\gamma^2 \left(\int_0^1 |z_t|^{p(x)} dx\right)^{p_1-1} + c\beta^2 \left(\int_0^1 |u_t|^{q(x)} dx\right)^{q_1-1}, \ t \geq 0. \tag{5.5}$$

Thanks to Eq (5.2), we get the following for any $t \geq 0$ and some positive constant α_5,

$$\mathcal{L}'(t) \leq -\alpha_5 \mathcal{L}(t) + c\gamma^2 \left(\int_0^1 |z_t|^{p(x)} dx\right)^{p_1-1} + c\beta^2 \left(\int_0^1 |u_t|^{q(x)} dx\right)^{q_1-1}.$$

Recalling Eq (2.3) and multiplying the above equation by $E^\alpha(t)$, where $\alpha > 0$, we obtain

$$E^\alpha(t)\mathcal{L}'(t) \leq -\alpha_5 E^{\alpha+1}(t) + c\gamma^2 E^\alpha(t)(-E'(t))^{p_1-1} + c\beta^2 E^\alpha(t)(-E'(t))^{q_1-1}. \tag{5.6}$$

- If $\gamma = 0$ and $\beta \neq 0$, then we have

$$E^{\alpha}(t)\mathcal{L}'(t) \leq -\alpha_5 E^{\alpha+1}(t) + c\beta^2 E^{\alpha}(t)(-E'(t))^{q_1-1}. \tag{5.7}$$

Using Young's inequality with $\zeta = \frac{1}{q_1-1}$ and $\zeta^* = \frac{1}{2-q_1}$, for any $\varepsilon > 0$, we have

$$E^{\alpha}(t)\mathcal{L}'(t) \leq -\alpha_5 E^{\alpha+1}(t) + \alpha_5 \varepsilon E^{\frac{\alpha}{2-q_1}}(t) + c_\varepsilon(-E'(t)). \tag{5.8}$$

Taking $\alpha = \frac{2-q_1}{q_1-1} > 0$, we have

$$E^{\alpha}(t)\mathcal{L}'(t) \leq -\alpha_5(1-\varepsilon)E^{\alpha+1}(t) + c_\varepsilon(-E'(t)). \tag{5.9}$$

By taking ε small enough Eq (5.9) becomes:

$$\mathcal{L}_1'(t) \leq -\alpha_6 E^{\alpha+1}(t), \qquad \forall t \geq 0, \tag{5.10}$$

where $\mathcal{L}_1 = E^{\alpha}\mathcal{L} + cE \sim E$. Integrating (5.10) over $(0, t)$, we obtain

$$E(t) < \frac{c_{q_1}}{(t+1)^{\frac{1}{\alpha}}}, \ \forall \ t > 0, \tag{5.11}$$

where $\alpha = \frac{2-q_1}{q_1-1}$. Then the first estimate in Eq (3.1) is proved.

- If $\gamma \neq 0$ and $\beta = 0$, then we have

$$E^{\alpha}(t)\mathcal{L}'(t) \leq -\alpha_5 E^{\alpha+1}(t) + c\gamma^2 E^{\alpha}(t)(-E'(t))^{p_1-1}. \tag{5.12}$$

The proof of the second estimate in (3.1) is straightforward. obtained in a similar manner as for the above one.

- If $\gamma \neq 0$ and $\beta \neq 0$, then we have

$$E^{\alpha}(t)\mathcal{L}'(t) \leq -\alpha_5 E^{\alpha+1}(t) + \alpha_5 \varepsilon E^{\frac{\alpha}{2-p_1}} + \alpha_5 \varepsilon E^{\frac{\alpha}{2-q_1}} + c_\varepsilon(-E'(t)). \tag{5.13}$$

Now, we discuss two cases:
Case 1: If $p_1 > q_1$, then

$$E^{\alpha}(t)\mathcal{L}'(t) \leq -\alpha_5 E^{\alpha+1}(t) + \alpha_5 \varepsilon E^{\frac{\alpha}{2-p_1}} + \alpha_5 \varepsilon E^{\frac{\alpha}{2-p_1}} E^{\frac{\alpha(q_1-p_1)}{(2-p_1)(2-q_1)}} + c_\varepsilon(-E'(t)). \tag{5.14}$$

Since E is non-increasing, then we get

$$E^{\alpha}(t)\mathcal{L}'(t) \leq -\alpha_5 E^{\alpha+1}(t) + \alpha_5 \varepsilon E^{\frac{\alpha}{2-p_1}} + \alpha_5 \varepsilon E^{\frac{\alpha}{2-p_1}} E(0)^{\frac{\alpha(q_1-p_1)}{(2-p_1)(2-q_1)}} + c_\varepsilon(-E'(t)). \tag{5.15}$$

Then, Eq (5.15) becomes

$$E^{\alpha}(t)\mathcal{L}'(t) \leq -\alpha_5 E^{\alpha+1}(t) + \alpha_5 \varepsilon E^{\frac{\alpha}{2-p_1}} + c\alpha_5 \varepsilon E^{\frac{\alpha}{2-p_1}} + 2c_\varepsilon(-E'(t)). \tag{5.16}$$

From Eq (5.16), we have

$$E^{\alpha}(t)\mathcal{L}'(t) \leq -\alpha_5(1-\varepsilon-c\varepsilon)E^{\alpha+1}(t) + c_\varepsilon(-E'(t)). \tag{5.17}$$

By taking ε small enough Eq (5.17) becomes:

$$\mathcal{L}_1'(t) \leq -\alpha_6 E^{\alpha+1}(t), \qquad \forall t \geq 0, \tag{5.18}$$

where $\mathcal{L}_1 = E^{\alpha}\mathcal{L} + cE \sim E$. Integrating (5.25) over $(0, t)$, we get

$$E(t) < \frac{c_{p_1}}{(t+1)^{\frac{1}{\alpha}}}, \ \forall\ t > 0, \tag{5.19}$$

where $\alpha = \frac{2-p_1}{p_1-1}$.

Case 2: If $q_1 < p_1$, we will get

$$E(t) < \frac{c_{q_1}}{(t+1)^{\frac{1}{\alpha}}}, \ \forall\ t > 0, \tag{5.20}$$

where $\alpha = \frac{2-q_1}{q_1-1} > 0$. So, by taking $\overline{p}_1 = \min\{p_1, q_1\}$, the proof of the last estimate in Eq (3.1) is completed.

□

5.2. *Proof of Theorem 3.2*

Proof. To prove Theorem 3.2, we reformulate the integrals $\int_0^1 z_t^2 dx$ and $\int_0^1 u_t^2 dx$ in Eq (5.2) and recall that the integrals $\left(\int_0^1 |z_t|^{p(x)} dx\right)^{p_1-1}$ *and* $\left(\int_0^1 |u_t|^{q(x)} dx\right)^{q_1-1}$ are not relevant in this situation; thus, we have

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\left(\mu_3 \frac{\alpha}{2} - \mu_2^2 - \frac{3\varepsilon a_3 \mu_4}{2}\right) \int_0^1 u_x^2 dx \\
& - \left(\mu_1 \frac{\alpha}{2a_3} - \overline{c} - c_1 - \overline{c}\mu_3 - \frac{\varepsilon a_2^2 \mu_4}{a_3}\right) \int_0^1 z_x^2 dx \\
& - \rho_z \mu_2 \int_0^1 z_t^2 dx - \varepsilon \mu_4 \rho_u \int_0^1 u_t^2 dx + \overline{c} \int_0^1 z_t^2 dx + \mu_1^2 \int_0^1 u_t^2 dx \\
& - \left[\gamma\mu - c\gamma^2\mu_1 - c\gamma^2\mu_2 - c\gamma^2\mu_3\right] \int_0^1 |z_t|^{p(\cdot)} dx \\
& - \left[\beta\mu - c\beta^2\mu_1 - c\beta^2\mu_3 - c\beta^2\mu_4\right] \int_0^1 |u_t|^{q(\cdot)} dx.
\end{aligned} \tag{5.21}$$

We shall prove the case that $\gamma, \beta \neq 0$ and the other cases will be straightforward by letting either $\gamma = 0$ or $\beta = 0$. Let us select $\mu_2 = 1$ and $\varepsilon\mu_4 = 1$. Then it is easy to select μ_3 and then μ_1; finally, we can select μ large enough such that estimate (5.21) becomes

$$\mathcal{L}'(t) \leq -\beta_1 \int_0^1 \left(u_t^2 + u_x^2 + z_t^2 + z_x^2\right) dx + \overline{c} \int_0^1 z_t^2 dx + \overline{c} \int_0^1 u_t^2 dx, \quad \forall t \geq 0 \tag{5.22}$$

and for two positive constants β_2 and β_3,

$$\beta_2 E(t) \leq \mathcal{L}(t) \leq \beta_3 E(t), \tag{5.23}$$

By recalling Poincaré's inequality and the energy functional defined in Eq (2.2), estimate Eq (5.22) becomes, for a positive constant β_4,

$$\mathcal{L}'(t) \leq -\beta_4 E(t) + \overline{c}\int_0^1 z_t^2 dx + \overline{c}\int_0^1 u_t^2 dx, \quad \forall t \geq 0, \tag{5.24}$$

and thanks to Eq (5.23), we get the following for any $t \geq 0$

$$\mathcal{L}'(t) \leq -\beta_5 \mathcal{L}(t) + \overline{c}\int_0^1 z_t^2 dx + \overline{c}\int_0^1 u_t^2 dx.$$

Here, we will discuss two cases:

Case I: If $p_2 = q_2 = 2$, then by using Lemma 4.5, we have

$$\mathcal{L}'(t) \leq -\beta_5 \mathcal{L}(t) + c(-E'(t)).$$

This gives

$$\mathcal{L}_1'(t) \leq -\beta_5 \mathcal{L}(t).$$

where $\mathcal{L}_1 = (\mathcal{L} + cE) \sim E$. Integrating the last estimate over the interval $(0, t)$ and using the equivalence properties $\mathcal{L}_1, \mathcal{L} \sim E$, the proof of the last estimate in (3.4) is completed.

Case II: If $p_2, q_2 > 2$, then by using Lemma 4.5, we have

$$\mathcal{L}'(t) \leq -\beta_5 \mathcal{L}(t) + (-E'(t))^{\frac{2}{p_2}} + (-E'(t))^{\frac{2}{q_2}}.$$

Multiplying the last equation by E^α where $\alpha = \frac{p_2-2}{2} > 0$, we obtain

$$E^q \mathcal{L}'(t) \leq -\beta_5 E^\alpha \mathcal{L}(t) + E^\alpha (-E'(t))^{\frac{2}{p_2}} + E^\alpha (-E'(t))^{\frac{2}{q_2}}.$$

Applying Young's inequality twice, we obtain the following for $\varepsilon > 0$

$$E^\alpha \mathcal{L}'(t) \leq -\alpha_5 E^{\alpha+1} \mathcal{L}(t) + \varepsilon E^{\frac{\alpha p_2}{p_2-2}} + \varepsilon E^{\frac{\alpha q_2}{q_2-2}} + C_\varepsilon (-E'(t)).$$

We will discuss two cases:

Case A: If $p_2 < q_2$, we will have

$$E^\alpha \mathcal{L}'(t) \leq -\alpha_5 E^{\alpha+1} \mathcal{L}(t) + \varepsilon E^{\frac{\alpha p_2}{p_2-2}} + \varepsilon E^{\frac{\alpha p_2}{p_2-2}} E^{\frac{2\alpha(p_2-q_2)}{(p_2-2)(q_2-2)}} + C_\varepsilon (-E'(t)).$$

Using the non-increasing property of E, we get

$$E^\alpha \mathcal{L}'(t) \leq -(\alpha_5 - \varepsilon - c\varepsilon) E^{\alpha+1} \mathcal{L}(t) + C_\varepsilon (-E'(t)).$$

Taking ε small enough, the above estimate becomes:

$$\mathcal{L}_2(t) \leq -\beta_6 E^{\alpha+1}(t), \qquad \forall t \geq 0, \tag{5.25}$$

where $\mathcal{L}_2 = E^\alpha \mathcal{L} + cE \sim E$.
Integration over $(0, t)$, using $E \sim \mathcal{L}_2$ gives

$$E(t) < \frac{c_{p_2}}{(t+1)^{1/\alpha}}, \; \forall\, t > 0, \tag{5.26}$$

where $\alpha = \frac{p_2-2}{2}$.

Case B: If $q_2 < p_2$, we will get

$$E(t) < \frac{c_{q_2}}{(t+1)^{1/\alpha}}, \ \forall\, t > 0, \tag{5.27}$$

where $\alpha = \frac{q_2-2}{2}$. So, by taking $\overline{p}_2 = \min\{p_2, q_2\}$, the proof of the last estimate in Eq (3.3) is completed. □

5.3. *Proof of Theorem 3.3*

The proof of this theorem can be obtained by repeating proofs similar to those in Theorem 3.1 and Theorem 3.2.

6. Numerical Tests

In the numerical part of this paper, we computationally justify our theoretical results form Theorems 3.1, 3.2 and 3.3. We examine the suggested fourteen cases according to our theorems. For the spatial and temporal discretization of the system (1.6) , we use a second-order finite difference method in time and space for the space-time domain $[0, L] \times [0, T_e] = [0, 1] \times [0, 10]$. Thereafter, we implement the conservative Lax-Wendroff scheme. Finally, we discuss the computational confirmation of our theoretical results. Moreover, we compare these fourteen tests accordingly. We would also like to mention that, for references for similar construction, we invite the readers to see [36–39]. According to the assumptions and conditions of our theorems, we chose to simulate the temporal evolution of the waves for the following tests:

From Theorem 3.1, we examine the following cases

- **TEST 1**:
  (i) $\rho_u = 1; \rho_z = 1; \gamma = 0; \beta = 1; a_1 = 2; a_2 = 0.5$; and $a_3 = 2$.
  (ii) $p(x) = q(x) = 2 - \dfrac{1}{1+x}$.
- **TEST 2**:
  (i) $\rho_u = 1; \rho_z = 1; \gamma = 1; \beta = 0; a_1 = 2; a_2 = 0.5$; and $a_3 = 2$.
  (ii) $p(x) = q(x) = 2 - \dfrac{1}{1+x}$.
- **TEST 3**:
  (i) $\rho_u = 1; \rho_z = 1; \gamma = 1; \beta = 1; a_1 = 2; a_2 = 0.5$; and $a_3 = 2$.
  (ii) $p(x) = q(x) = 2 - \dfrac{1}{1+x}$.

From Theorem 3.2, we examine the following cases

- **TEST 4**:
  (i) $\rho_u = 1; \rho_z = 1; \gamma = 0; \beta = 1; a_1 = 2; a_2 = 0.5$; and $a_3 = 2$.
  (ii) $p(x) = 2 + \dfrac{1}{1+x}$ and $q(x) = 2$.
- **TEST 5**:

(i) $\rho_u = 1; \rho_z = 1; \gamma = 1; \beta = 0; a_1 = 2; a_2 = 0.5$; and $a_3 = 2$.
(ii) $q(x) = 2 + \dfrac{1}{1+x}$ and $p(x) = 2$.

- **TEST 6**:
  (i) $\rho_u = 1; \rho_z = 1; \gamma = 1; \beta = 1; a_1 = 2; a_2 = 0.5$; and $a_3 = 2$.
  (ii) $q(x) = p(x) = 2$.

- **TEST 7**:
  (i) $\rho_u = 1; \rho_z = 1; \gamma = 0; \beta = 1; a_1 = 2; a_2 = 0.5$; and $a_3 = 2$.
  (ii) $p(x) = 2$ and $q(x) = 2 + \dfrac{1}{1+x}$.
- **TEST 8**:
  (i) $\rho_u = 1; \rho_z = 1; \gamma = 1; \beta = 0; a_1 = 2; a_2 = 0.5$; and $a_3 = 2$.
  (ii) $q(x) = 2$ and $p(x) = 2 + \dfrac{1}{1+x}$.
- **TEST 9**:
  (i) $\rho_u = 1; \rho_z = 1; \gamma = 1; \beta = 1; a_1 = 2; a_2 = 0.5$; and $a_3 = 2$.
  (ii) $q(x) = p(x) = 2 + \dfrac{1}{1+x}$.

From Theorem 3.3, we examine the following cases

- **TEST 10**:
  (i) $\rho_u = 1; \rho_z = 1; \gamma = 0; \beta = 1; a_1 = 2; a_2 = 0.5$; and $a_3 = 2$.
  (ii) $p(x) = 2 + \dfrac{1}{1+x}$ and $q(x) = 2 - \dfrac{1}{1+x}$.
- **TEST 11**:
  (i) $\rho_u = 1; \rho_z = 1; \gamma = 1; \beta = 0; a_1 = 2; a_2 = 0.5$; and $a_3 = 2$.
  (ii) $q(x) = 2 - \dfrac{1}{1+x}$ and $p(x) = 2$.
- **TEST 12**:
  (i) $\rho_u = 1; \rho_z = 1; \gamma = 1; \beta = 0; a_1 = 2; a_2 = 0.5$; and $a_3 = 2$.
  (ii) $q(x) = 2 - \dfrac{1}{1+x}$ and $p(x) = 2 + \dfrac{1}{1+x}$.
- **TEST 13**:
  (i) $\rho_u = 1; \rho_z = 1; \gamma = 1; \beta = 1; a_1 = 2; a_2 = 0.5$; and $a_3 = 2$.
  (ii) $q(x) = 2 + \dfrac{1}{1+x}$ and $p(x) = 2$.
- **TEST 14**:
  (i) $\rho_u = 1; \rho_z = 1; \gamma = 1; \beta = 1; a_1 = 2; a_2 = 0.5$; and $a_3 = 2$.
  (ii) $q(x) = 2 + \dfrac{1.1}{1+x}$ and $p(x) = 2 + \dfrac{1}{1+x}$.

To ensure the numerical stability of the implemented numerical scheme, we chose to design our code to satisfy the spatiotemporal Courant-Friedrichs-Lewy (CFL) condition, given as $\Delta t < 0.5\Delta x$, where Δt represents the time step and Δx is the spatial step. The spatial interval $[0, 1]$ has been subdivided into 500 subintervals, whereas the temporal interval $[0, T_e] = [0, 10]$ was deduced from the stability condition above. We ran our code for 20000 time steps by using the following initial conditions:

$$u(x, 0) = (1 - x)x \text{ and } u_t(x, 0) = 0 \quad \text{in } [0, 1]. \tag{6.1}$$

$$z(x, 0) = \sin(\pi x) \text{ and } z_t(x, 0) = 0 \quad \text{in } [0, 1].$$

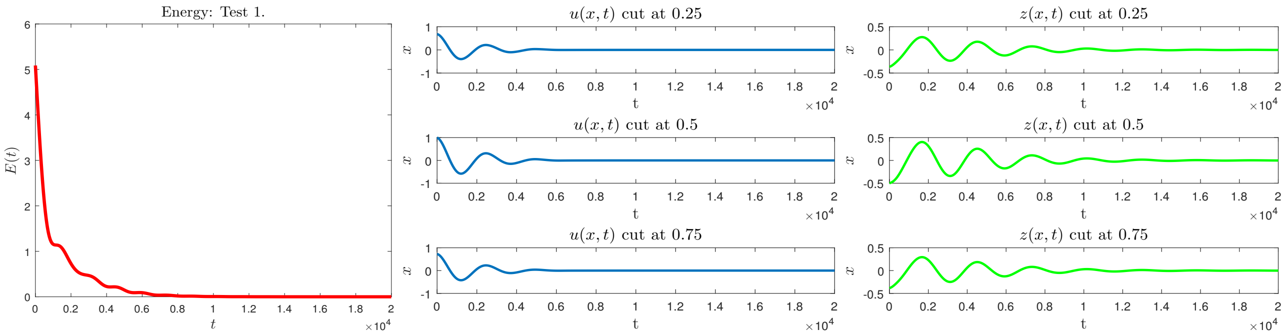

Figure 1. TEST 1, The cross sections for temporal behavior of the solutions u, z and the corresponding polynomial decay of the energy function.

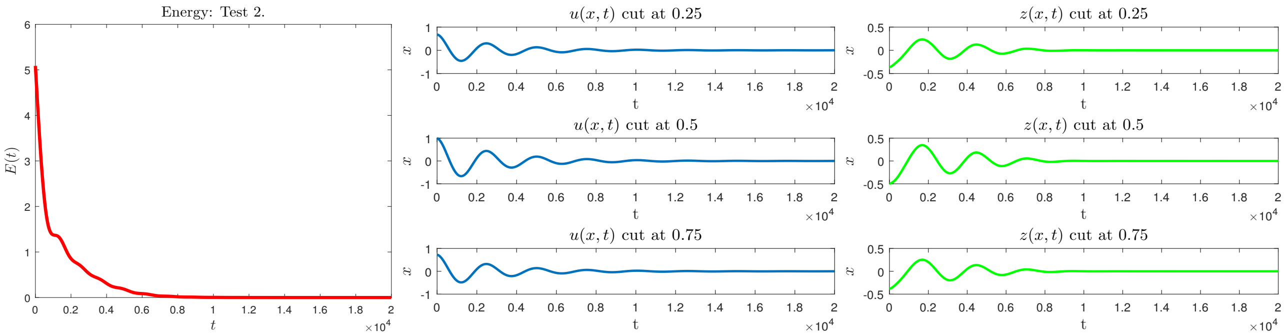

Figure 2. TEST 2, The cross sections of the behavior for waves u, z and the corresponding polynomial decay of the energy function.

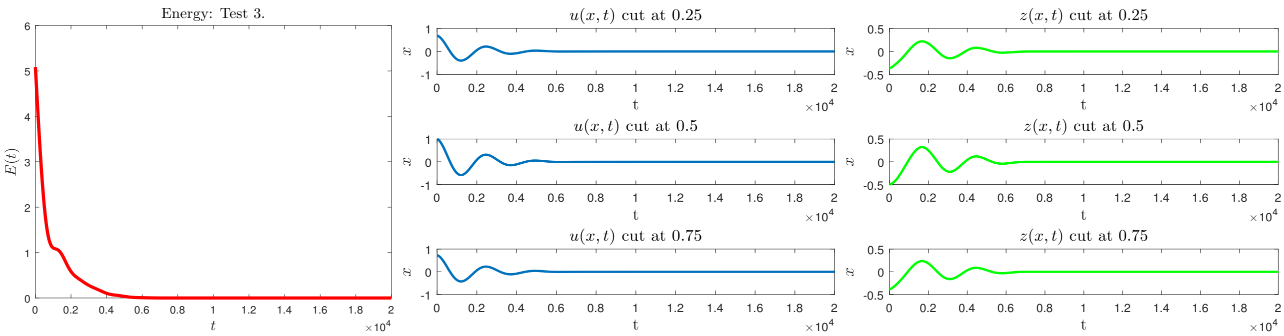

Figure 3. TEST 3, The cross sections of the wave behavior for u, z and the corresponding polynomial decay of the energy function.

In the first block of the numerical Tests 1–3, we examined the polynomial decay of the energy derived from u and z. These results were proved in Theorem 3.1. Given the initial and boundary conditions 6.1 and the parameters mentioned above (see **TEST 1–3**, (i) and (ii)), in Figures 1–3,we have plotted the energy function and the three cross sections at $x = 0.25, 0.5$ and at 0.75, where the polynomial decay of the energy is clearly assured.

In the second block of the numerical Tests 4–6, we evaluated the polynomial decay of the energy. These results were proved in Theorem 3.2. Given the same initial and boundary conditions in 6.1 for **TEST 4–6**, (i) and (ii), in Figures 4–6, we have plotted the energy function and the three cross sections at $x = 0.25, 0.5$ and 0.75, where the exponential decay of the energy has been numerically proved.

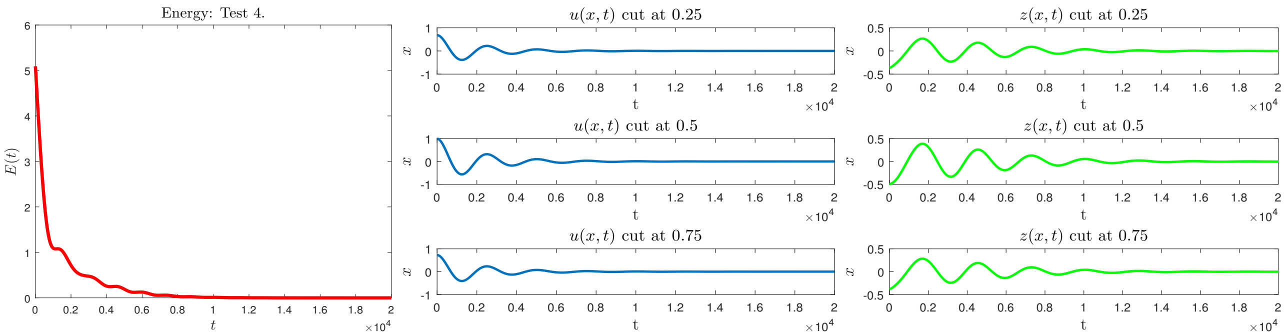

Figure 4. TEST 4, The cross sections of the wave behavior for u, z and the corresponding exponential decay of the energy function.

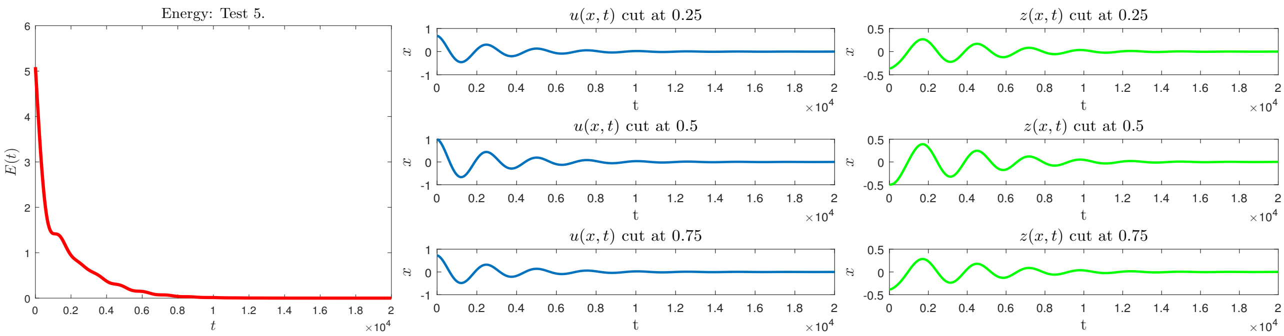

Figure 5. TEST 5, The cross sections of the wave behavior for u, z and the corresponding exponential decay of the energy function.

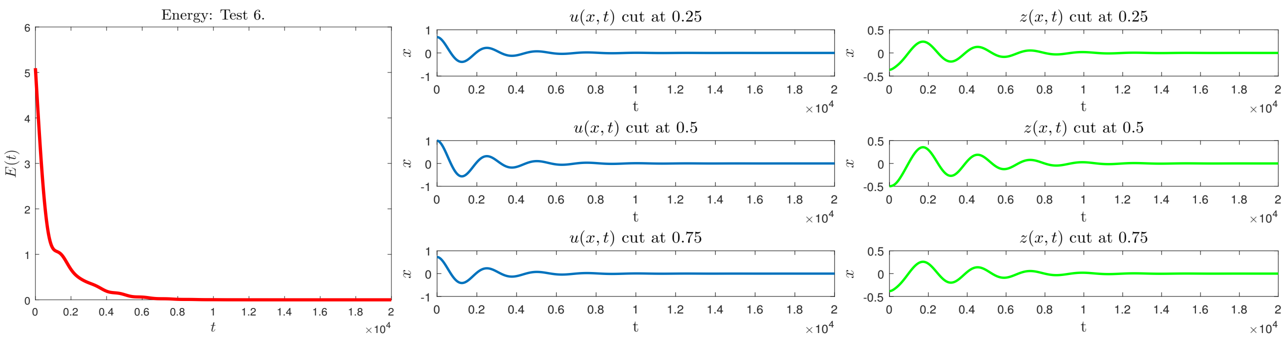

Figure 6. TEST 6, The cross sections of the wave behavior for u, z and the corresponding exponential decay of the energy function.

In the third block of the numerical Tests 7–9, we examine again the polynomial decay of the energy.

These results have been proven in the last three cases of Theorem 3.2. Given the same initial and boundary conditions in 6.1 and the parameters mentioned above (see **TEST 7–9**, (i) and (ii)), in Figures 7–9, we have plotted the energy function and the three cross sections $x = 0.25, 0.5$, and 0.75, where the polynomial decay of the energy has been numerically proved.

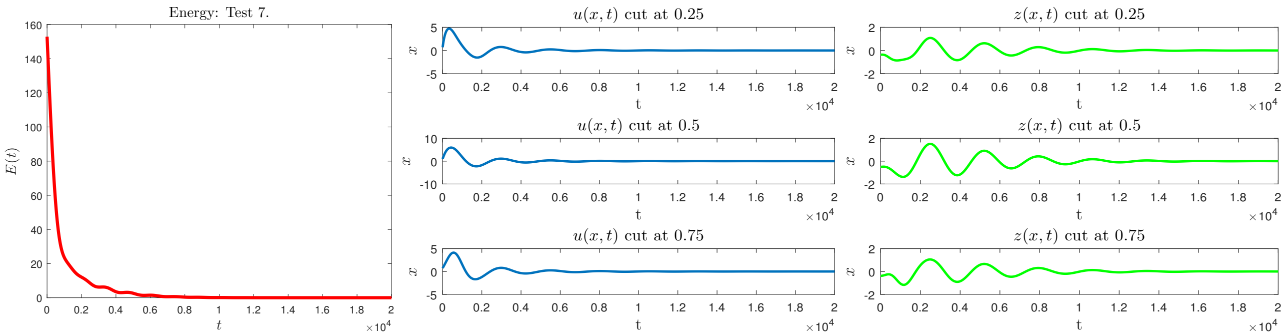

Figure 7. TEST 7, The cross sections of the wave behavior for u, z and the corresponding polynomial decay of the energy function.

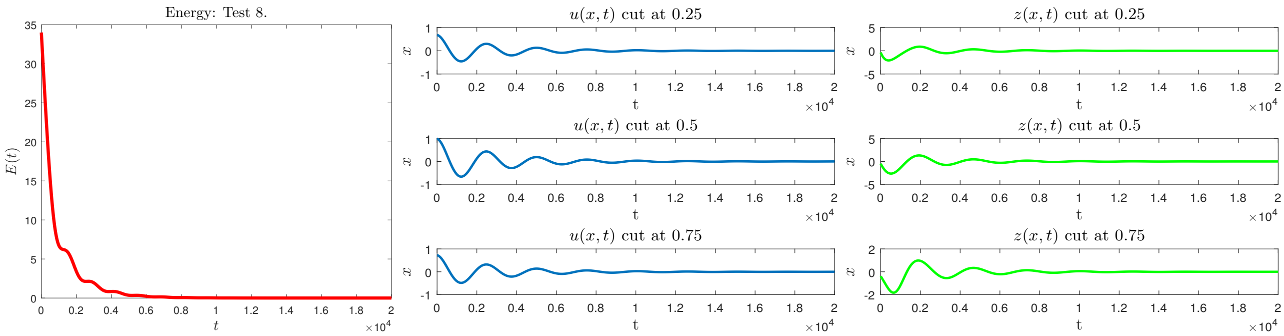

Figure 8. TEST 8, The cross sections of the wave behavior for u, z and the corresponding polynomial decay of the energy function.

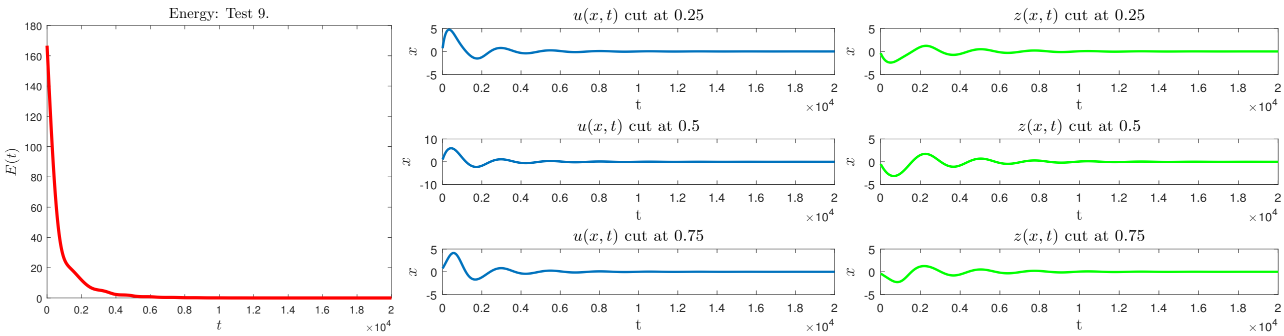

Figure 9. TEST 9, The cross sections of the wave behavior for u, z and the corresponding polynomial decay of the energy function.

In the fourth block of the numerical Tests 10–14, we examined other cases leading to the polynomial decay of the energy. These results have been proven in the last three cases of Theorem 3.3. Under the same initial and boundary conditions in 6.1 and the parameters mentioned above (see **TEST 10–14**,

(i) and (ii)), in Figures 10–14, we have plotted the energy function and the three cross sections at $x = 0.25, 0.5$ and 0.75, where the polynomial decay of the energy has been numerically proved.

Finally, it should be stressed that our numerical simulations show the energy decay that was proved in Theorems 3.1, 3.2 and 3.3. Obviously, in some cases the polynomial decay could be easily deduced from the exponential decay behavior of the energy. This result can be accepted, since the required and expected result is the polynomial one. We are pretty sure that for other choices of the initial solutions and a rigorous choice of the functional parameters, we could get a clear discrepancy between the energy functions reflecting the polynomial and exponential decays.

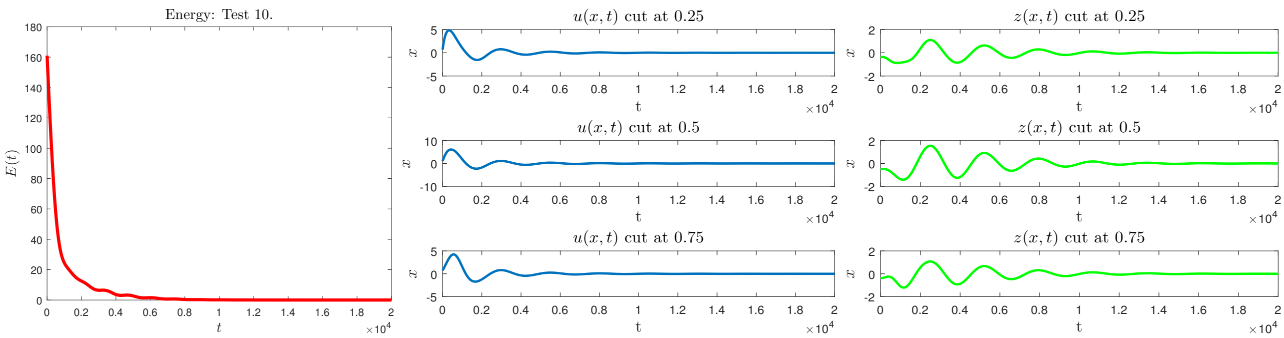

Figure 10. TEST 10, The cross sections of the wave behavior for u, z and the corresponding polynomial decay of the energy function.

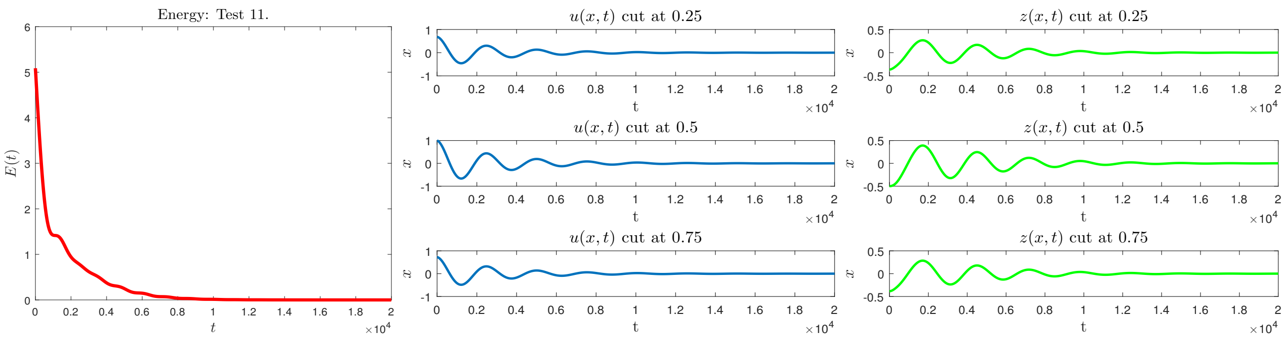

Figure 11. TEST 11, The cross sections of the wave behavior for u, z and the corresponding polynomial decay of the energy function.

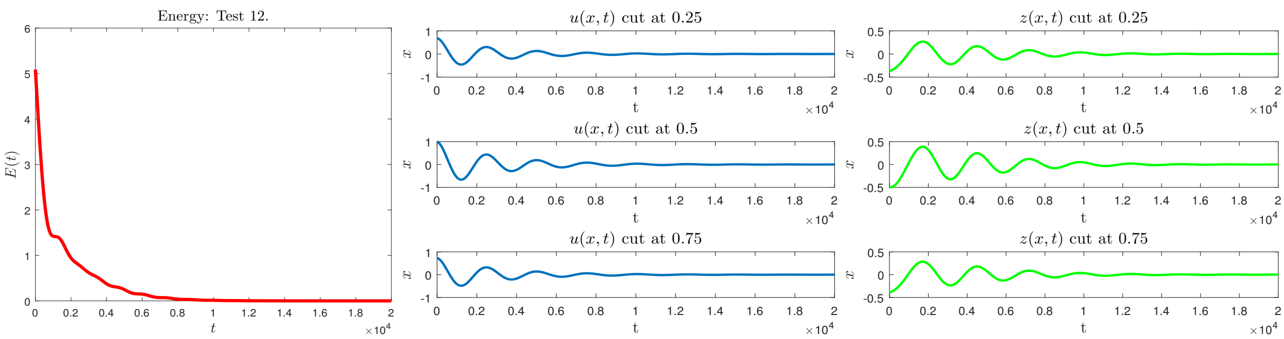

Figure 12. TEST 12, The cross sections of the wave behavior for u, z and the corresponding polynomial decay of the energy function.

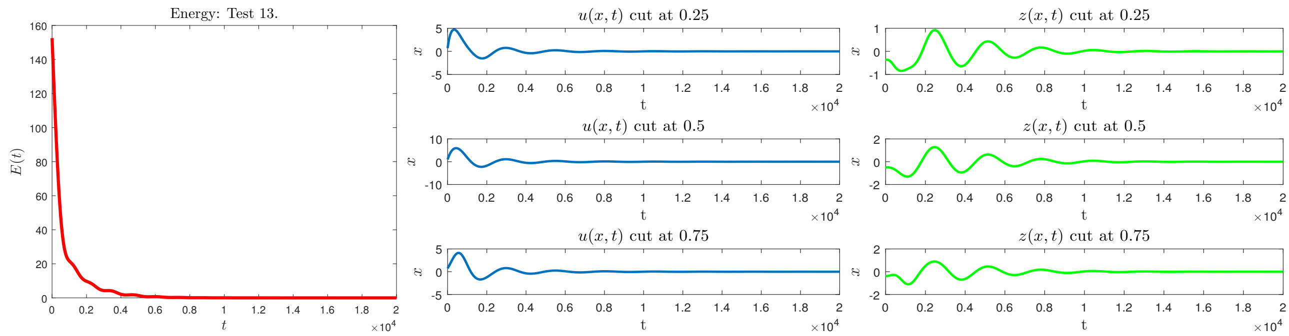

Figure 13. TEST 13, The cross sections of the wave behavior for u, z and the corresponding polynomial decay of the energy function.

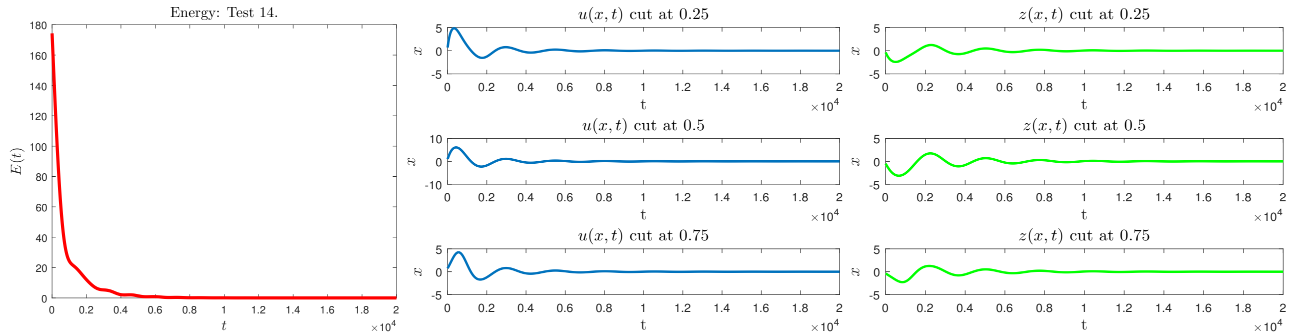

Figure 14. TEST 14, The cross sections of the wave behavior for u, z and the corresponding polynomial decay of the energy function.

7. Conclusion

In this study, we considered a swelling elastic system with two nonzero dampings of the variable exponent type. We discussed different cases and proved that the system is exponentially and polynomially stable, and that the stability results depend on the values of p_1, p_2, q_1, q_2. In addition, we conclude that the decay estimate is not necessarily improved if the system has two dampings.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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