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Research article

BV regularity of the adapted entropy solutions for conservation laws with infinitely many spatial discontinuities

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Abstract: In this article, we focus on the BV regularity of the adapted entropy solutions of the conservation laws whose flux function contains infinitely many discontinuities with possible accumulation points. It is well known that due to discontinuities of the flux function in the space variable, the total variation of the solution can blow up to infinity in finite time. We establish the existence of total variation bounds for certain classes of fluxes and the initial data. Furthermore, we construct two counterexamples, which exhibit BV blow-up of the entropy solution. These counterexamples not only demonstrate that these assumptions are essential, but also show that the BV-regularity result of [S. S. Ghoshal, J. Differential Equations, 258 (3), 980–1014, 2015] does not hold true when the spatial discontinuities of the flux are infinite.

Keywords: conservation laws with discontinuous flux; adapted entropy solution; BV-regularity; BV blow-up

1. Introduction

In this article, we consider the following initial value problem for scalar conservation laws in one space dimension:

$$u_t + A(x,u)_x = 0 \qquad \text{for } (x,t) \in \mathbb{R}\times(0,\infty), \tag{1.1}$$

$$u(x,0) = u_0(x) \qquad \text{for } x \in \mathbb{R}. \tag{1.2}$$

It is assumed that the flux function $A(x,\cdot)$ is C^2 and uniformly convex, i.e., there exists a positive constant C such that $A_{uu}(x,\cdot) \geq C > 0$ for all $x \in \mathbb{R}$. It is to be noted that as a function of the space variable, the function $A(x,u)$ can have infinitely many spatial discontinuities with possible accumulation points.

Conservation laws with discontinuous flux [2–5, 7, 9, 11, 12, 14, 15, 23–25, 29, 30] has been a subject of immense interest to the mathematical community, as it is known to model various real-world problems. Examples include, the hydrodynamic limit of interacting particle systems with discontinuous speed parameter [13], sedimentation [14], petroleum industry and polymer flooding [29], two-phase flow in heterogeneous porous medium [6], clarifier thickener units used in wastewater treatment plants and mineral processing [10], and traffic flow on highways with changing surface conditions [15], etc. One of the classical studies of this IVP with the flux having spatial heterogeneities come from the seminal paper of Kruzkov [26], which studies its well-posedness for sufficiently smooth flux A. However, in practice, the flux function A appearing in the IVP modelling real-life applications, may not be smooth and may have spatial discontinuities. Since the flux contains spatial discontinuities, the Kruzkov entropy condition does not make sense and hence a different notion for uniqueness, incorporating the non-smooth structure of the flux is required. Owing to this, the notion of an adapted entropy solution is used to single out a unique solution for IVPs with non-smooth flux. The notion of adapted entropy condition was introduced by Audusse and Perthame in [8] for unimodal and monotone fluxes and later it was generalized in [27] to a more general class of fluxes of the type $A(x,u) = g(\beta(x,u))$ with g, a continuous function and β, Caratheodory function (i.e. β measurable in the x variable and continuous in the u variable). The existence of the entropy solutions was established via the convergence of numerical approximations in, [28] using front tracking approximations for monotone convex fluxes, and in [31] using finite volume approximations for possibly non–convex monotone fluxes. Additionally, the convergence of finite volume approximations for unimodal fluxes was studied in [18] using a singular mapping technique. In addition, Ghoshal et al. [20, 21] extended the results for a Locally Lipschitz g and BV spatial β, using a $\beta-$ TVD (Total Variation Diminishing) property. Further, Ghoshal et al. [22] also extended the results using the front tracking approximations for fluxes with flat regions. In this article, we describe the existence and non-existence of total variation bounds on the solution for uniformly convex flux $A(x,\cdot)$.

Total variation plays a pivotal role in the study of hyperbolic conservation laws. In the case of homogeneous flux i.e., $A(x,u) = f(u)$, using the Lax–Oleinik formula, it can be shown that if the flux function is C^2 and uniformly convex, i.e., $f'' \geq C > 0$, then the entropy solution $u(t,\cdot) \in \mathrm{BV}_{loc}(\mathbb{R})$ for any initial data $u_0 \in L^\infty(\mathbb{R})$. In other words, a C^2 uniformly convex flux induces a BV regularizing effect on the entropy solution. For a locally Lipschitz flux (homogeneous), this result is not true in general. However, front tracking and finite volume approximations indicate that, $t \mapsto \mathrm{TV}(u(t,\cdot))$ is non-increasing. Thus, the total variation of the solution at any time $t > 0$ cannot be greater than the total variation of the initial data. On the other hand, for a two flux problem with $A(x,u) = H(-x)g(u) + H(x)f(u)$, where f and g are C^2 and convex, the authors in [1] construct counterexamples to show that the total variation of the entropy solution (see [1, Def. 2.4] with $A = \theta_g$ and $B = \theta_f$) blows up in finite time, even for BV initial data.

Furthermore, results produced by the finite volume method [9] suggest that the total variation of the entropy solution away from the interface $x = 0$ remains bounded, i.e., $u(t,\cdot) \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R} \setminus \{0\})$. Also, the BV smoothing effect away from the interface for two flux problems with uniformly convex flux is discussed in [17]. Surprisingly, if $f(\theta_f) = g(\theta_g)$, the existence of BV bounds was proved in [16] using an explicit formula for the two flux problem obtained in [5]. Furthermore, the regularity of the solutions in BV^s space for conservation laws with discontinuous flux were recently investigated in [19].

These results can be naturally extended for the case of finitely many discontinuities and hence we have the existence of uniform BV bounds on the solution.

In the present article, we will show that results on BV bounds for the case when F is unimodal and has infinitely many spatial discontinuities are completely different from the case of finitely many spatial discontinuities. At this point, we would like to mention that the case where $A(x,\cdot)$ does not contain critical points is relatively simpler and BV bounds can be easily obtained using finite volume [31] and front tracking [28] approximations provided $u_0 \in \mathrm{BV}(\mathbb{R})$.

Hence, the question of whether a BV bound on the solution for fluxes having critical points in the u-variable and infinitely many spatial discontinuities is interesting and nontrivial. In this article, we provide a sufficient condition on the initial data to obtain a BV bound for the entropy solution for uniformly convex fluxes. In the later part of this article, we construct two counterexamples to demonstrate that the conditions are essential. Note that the flux $A(x,\cdot)$ considered in the current article has the same minimum for each x and is uniformly convex. By providing two examples in Sections 3.2 and 3.3, we show that the BV-regularity results of [16] can not hold when the flux F has infinitely many discontinuities. The example in Section 3.3 further shows that even if we assume uniform convexity of the flux, the TV (Total Variation) of the entropy solution can blow up. The construction of the example in Section 3.3 is novel and it contains new ideas which differ from previous constructions [1, 16].

2. Preliminaries

Throughout this article, we assume that the flux function $A(x,u)$ satisfies the following assumptions.

A-1 The function $x \mapsto A(x,u)$ is measurable, $u \mapsto A(x,u)$ is uniformly convex and there exists $u_M \in \mathrm{BV}(\mathbb{R})$ such that the following holds:

$$\min_{u\in\mathbb{R}} A(x,u) = A(x,u_M(x)) = 0.$$

A-2 There exists $(h_1,h_2) \in C^0(\mathbb{R})^2$ such that for all $x \in \mathbb{R}$, we have $h_1(u) \le A(x,u) \le h_2(u)$. In addition, h_i for $i = 1,2$ are strictly decreasing and then increasing functions with $|h_1(\pm\infty)| = +\infty$.

A-3 There exists a continuous function $\eta : \mathbb{R} \to \mathbb{R}$ and a BV function $a : \mathbb{R} \to \mathbb{R}$ such that

$$|A(x,u) - A(y,u)| \le \eta(u)|a(x) - a(y)| \quad \forall x,y,u \in \mathbb{R}.$$

Remark 2.1. A typical example of a flux function satisfying the above hypothesis (see [8]) is given by

$$A(x,u) = (u - u_M(x))^2\,.$$

2.1. Adapted entropy condition and uniqueness

In view of the above hypothesis, we define the following:

Definition 2.1. *A function $k : \mathbb{R} \to \mathbb{R}$ is said to be a stationary state, if $u(x,t) = k(x)$ is the weak solution to the IVP* (1.1) *and* (1.2)*, with $u_0(x) = k(x)$. For $\alpha > 0$, we work with two types of stationary* *states namely,* $k^\pm_\alpha : \mathbb{R} \to \mathbb{R}$ *such that* $A(x, k^\pm_\alpha(x)) = \alpha$ *with* $k^+_\alpha(x) \in (u_M(x), \infty)$ *and* $k^-_\alpha(x) \in (-\infty, u_M(x))$. *On the other hand, for* $\alpha = 0$, *we have* $k^+_0 = k^-_0 = u_M$. *We define* $\mathscr{S}_\alpha$ *to be the set of all stationary states corresponding to height* $\alpha \geq 0$.

Definition 2.2 (Adapted Entropy Condition, [8])**.** *A function* $u \in L^\infty(\mathbb{R} \times [0, T]) \cap C([0, T], L^1_{loc}(\mathbb{R}))$ *is an adapted entropy solution of the Cauchy problem if it satisfies the following inequality in the sense of distribution:*

$$\partial_t |u(x, t) - k^\pm_\alpha(x)| + \partial_x \left[\operatorname{sgn}(u - k^\pm_\alpha(x))(A(x, u) - \alpha)\right] \leq 0, \tag{2.1}$$

for $\alpha \geq 0$. *Or equivalently, for all* $0 \leq \phi \in C^\infty_c(\mathbb{R} \times [0, T))$

$$\int\limits_{\mathbb{R}\times[0,T]} |u(x, t) - k_\alpha(x)|\phi_t(x, t) + \operatorname{sgn}(u(x, t) - k_\alpha(x))(A(x, u(x, t)) - \alpha)\phi_x(x, t)\, dxdt$$
$$+ \int\limits_{\mathbb{R}} |u_0(x) - k_\alpha(x)|\phi(0, x)\, dx \geq 0.$$

Theorem 2.1. *[Uniqueness [8, 27]] Let* u, v *be entropy solutions to the IVP* (1.1) *and* (1.2) *with initial data* $u_0, v_0 \in L^\infty(\mathbb{R})$. *Assume that the flux A(x,u) satisfies the hypotheses* (**A-1**) *and* (**A-2**). *Then, there exists* $M > 0$ *such that the following holds,*

$$\int\limits_a^b |u(x, t) - v(x, t)| dx \leq \int\limits_{a-Mt}^{b+Mt} |u_0(x) - v_0(x)| dx, \tag{2.2}$$

for $-\infty \leq a < b \leq \infty$ *and* $t \in [0, T]$.

2.2. Front tracking algorithm and existence

For our BV regularity result established in Theorem 3.1, we employ approximate solutions constructed using a wave front tracking algorithm. The front tracking algorithm can be summarized as follows:

Step-I Approximate the initial data u_0 by a piecewise constant function u^δ_0 with finitely many discontinuities such that $u^\delta_0 \to u_0$ in $L^1_{loc}(\mathbb{R})$ as $\delta \to 0$ and

$$\operatorname{TV}(u^\delta_0) \leq C,$$

for some $C > 0$.

Step-II Approximate $A(x, u)$ by $A^\delta(x, u)$ such that $x \mapsto A^\delta(x, \cdot)$ is piecewise constant with finitely many discontinuities and for fixed x, the function $u \to A^\delta(x, u)$ is piecewise linear convex such that $A^\delta(x, u) \to A(x, u)$ for a.e. $x \in \mathbb{R}$. In addition, the stationary states $k^{\delta,\pm}_\alpha$ corresponding to the flux A^δ should satisfy $k^{\delta,\pm}_\alpha(x) \to k^\pm_\alpha(x)$ for a.e. $x \in \mathbb{R}$.

Step-III Solve the following initial value problem,

$$u_t + \left(A^\delta(x, u)\right)_x = 0 \qquad \text{for } (x, t) \in \mathbb{R} \times (0, \infty), \tag{2.3}$$

$$u(x,0) = u_0^\delta(x) \quad \text{for } x \in \mathbb{R}, \tag{2.4}$$

and denote the solution by u^δ. More specifically, at each discontinuity point of u_0^δ, and at each discontinuity point of $x \mapsto A^\delta(u,x)$, use the Riemann solver to obtain a solution $u^\delta(\cdot,\tau)$ until the time τ at which the a discontinuity (front) of u^δ interacts with other discontinuities of u^δ or with the spatial discontinuities of the flux A^δ. Solve the initial value problem (IVP) given by Eq (2.3), at this interaction time $t = \tau$, using $u^\delta(\cdot,\tau)$ as the initial data. Repeat this procedure until the next interaction, and continue in this manner.

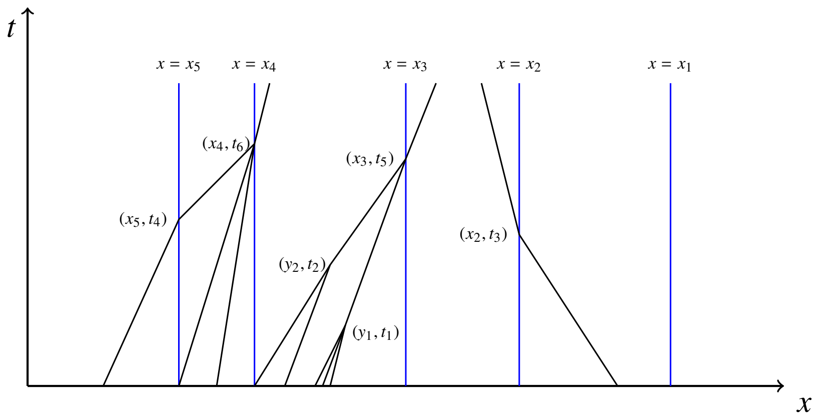

Figure 1. This illustrates possible wave interactions for a front tracking approximation with piecewise constant flux with spatial discontinuities at $x = x_i$, $i = 1, \cdots, 5$. At the points (y_1, t_1) and (y_2, t_2) two or more fronts interact away from the interface and only one front emanates after interaction. Similarly two or more fronts at the discontinuities x_1, x_5, x_3 and x_4 at the time t_3, t_4, t_5 and t_6 respectively and one front emanates.

Theorem 2.2. *Suppose* $u_0 \in BV(\mathbb{R}) \cap L^1(\mathbb{R})$ *and* $A(\cdot,\cdot)$ *satisfies the assumptions* (**A-1**)–(**A-3**). *Then, there exists a unique adapted entropy solution to Eq* (1.1) *corresponding to the initial data* u_0.

Proof. The proof can be done as in [22]. Since $A(x,\cdot)$ is uniformly convex, the proof is even simpler. Due to the strict convexity assumption, the number of fronts does not increase at any interactions (interactions in the interior as well as on the interface). This implies that the front tracking approximations can be continued for all time $t > 0$. On the other hand, the singular map Ψ is invertible and the TVD property of $\Psi(\cdot, u(t,\cdot))$ gives the desired compactness. Thus, repeating the arguments in [22], the entire sequence of front tracking approximations converge to the unique entropy solution. □

3. Study of BV regularity

This section is devoted to the study of BV regularity of adapted entropy solution. In the next subsection, we introduce a condition on initial data to get a TV bound on entropy solution when the fluxes are uniformly convex. Later, we give two counter-examples to show that neither the additional condition on initial data ($\Lambda(u_0) < \infty$) nor the uniform convexity assumption on flux can be relaxed.

3.1. Regularity in BV space

To obtain a BV regularity of entropy solutions, we need an assumption on initial data which is stronger than BV. This is introduced below:
Let $b \in L^\infty(\mathbb{R})$ and $-\infty < x_1 < \cdots < x_{n+1} < \infty$. We define a sequence $\{\Delta w_j\}_{1\le j\le n}$ as follows,

$$\Delta w_j := \begin{cases} \left|b(x_j) - b(x_{j+1})\right| & \text{if } A(x_j,\cdot) = A(x_{j+1},\cdot), \\ \left|b(x_j) - u_M(x_j)\right| + \left|b(x_{j+1}) - u_M(x_{j+1})\right| & \text{if } A(x_j,\cdot) \neq A(x_{j+1},\cdot), \end{cases} \tag{3.1}$$

for $1 \le j \le n$. We define $\Lambda(b)$ as follows,

$$\Lambda(b) := \sup\left\{\sum_{j=1}^{n} \Delta w_j;\ -\infty < x_1 < \cdots < x_{n+1} < \infty \text{ and } \Delta w_j \text{ is defined as in (3.1)}\right\} \tag{3.2}$$

and the set $X := \{b \in L^\infty(\mathbb{R}); \Lambda(b) < \infty\}$. Since $u_M \in \mathrm{BV}(\mathbb{R})$, we observe that $X \subset \mathrm{BV}(\mathbb{R})$. If $x \mapsto A(x,\cdot)$ is piecewise constant then Eq (3.2) can be simplified in the following way: let $-\infty = z_0 < z_1 < \cdots < z_n < z_{n+1} = \infty$ be discontinuity points of $x \mapsto A(x,u)$. We can write

$$\Lambda(b) = \sum_{k=0}^{n} TV_{(z_k,z_{k+1})}(b) + \sum_{k=1}^{n} \left[|b(z_k-) - u_M(z_k-)| + |b(z_k+) - u_M(z_k+)|\right].$$

Remark 3.1. Suppose $u_M \in \mathrm{BV}(\mathbb{R})$, then $\Lambda(b) < \infty$ implies $\mathrm{TV}(b) < \infty$ but the converse is not true. Furthermore, if A is homogeneous then $\Lambda(b) = \mathrm{TV}(b)$.

Theorem 3.1. *Let $A(\cdot,\cdot)$ satisfy* (**A-1**)–(**A-3**). *Additionally, we assume that $A(\cdot,x) \in C^2(\mathbb{R})$ and $\partial_{uu}A(\cdot,x) \ge C_1$ for all $x \in \mathbb{R}$ where $C_1 > 0$ does not depend on x. Let u be an adapted entropy solution to Eqs* (1.1) *and* (1.2) *with initial data u_0 such that $\Lambda(u_0) < \infty$ where Λ is defined as in Eq* (3.2). *Then we have*

$$TV(u(\cdot,t)) \le TV(u_M) + C_3\Lambda(u_0) \text{ for } t > 0,$$

where $C_2 = \sup\{\|A(\cdot,x)\|_{C^2[-M,M]},\ x \in \mathbb{R}\}$ and $C_3 := \left(\frac{C_2}{C_1}\right)^{\frac{3}{2}}$.

Before the proof of Theorem 3.1, here we prove a property of uniform convex flux in the following lemma.

Lemma 3.1. *For a closed interval $I \subset \mathbb{R}$, let $f, g \in C^2(I)$ such that $f'', g'' \in [C_1, C_2]$ for $C_2 > C_1 > 0$. Additionally, assume that $f(\theta_f) = g(\theta_g) = 0$. Then we have*

$$\left|g_+^{-1}(a) - g_+^{-1}(b)\right| \le C_3 \left|f_+^{-1}(a) - f_+^{-1}(b)\right| \text{ for } a, b \in I \text{ where } C_3 = \left(\frac{C_2}{C_1}\right)^{\frac{3}{2}}.$$

Proof. Suppose $f(u_1) = g(u_2)$ with $u_1 > \theta_f, u_2 > \theta_g$. Now, we observe that

$$f(u_1) = f(\theta_f) + f'(\theta_f)(u_1 - \theta_f) + \frac{f''(u_*)}{2}\left|u_1 - \theta_f\right|^2 \ge \frac{C_1}{2}\left|u_1 - \theta_f\right|^2,$$
$$g(u_2) = g(\theta_g) + g'(\theta_g)(u_2 - \theta_g) + \frac{g''(u^*)}{2}\left|u_2 - \theta_g\right|^2 \le \frac{C_2}{2}\left|u_2 - \theta_g\right|^2.$$

Subsequently, we have

$$\frac{C_1}{2}\left|u_1-\theta_f\right|^2 \le \frac{C_2}{2}\left|u_2-\theta_g\right|^2.$$

Therefore, we have

$$(u_1-\theta_f) \le \left(\frac{C_2}{C_1}\right)^{\frac{1}{2}}(u_2-\theta_g).$$

Notice that

$$\begin{aligned}
f'(f_+^{-1}(p)) = f'(f_+^{-1}(p)) - f'(\theta_f) &= (f_+^{-1}(p)-\theta_f)\int_0^1 f''(\lambda f_+^{-1}(p)+(1-\lambda)\theta_f)\,d\lambda \\
&\le \left(\frac{C_2}{C_1}\right)^{\frac{3}{2}}(g_+^{-1}(p)-\theta_g)\int_0^1 g''(\lambda g_+^{-1}(p)+(1-\lambda)\theta_g)\,d\lambda \\
&= \left(\frac{C_2}{C_1}\right)^{\frac{3}{2}} g'(g_+^{-1}(p)).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
g_+^{-1}(a)-g_+^{-1}(b) &= (a-b)\int_0^1 \frac{1}{g'(g_+^{-1}(\lambda a+(1-\lambda)b))}\,d\lambda \\
&\le \left(\frac{C_2}{C_1}\right)^{\frac{3}{2}}(a-b)\int_0^1 \frac{1}{f'(f_+^{-1}(\lambda a+(1-\lambda)b))}\,d\lambda \\
&= \left(\frac{C_2}{C_1}\right)^{\frac{3}{2}}(f_+^{-1}(a)-f_+^{-1}(b)).
\end{aligned}$$

□

Proof of Theorem 3.1. We prove the estimate for the approximate solution u^δ, and then we show that the estimates are uniform over δ. Let u^δ be the approximate solution to Eq (1.1) obtained via wave front tracking. Note that there is no affine part on $A(\cdot,x)$ for all $x\in\mathbb{R}$ due to $\partial_{uu}A(x,u)\ge C_1$. Due to uniform convexity no new state arises away from the interface for time $t>0$. Since there is no linear part in the flux, no rarefaction wave occurs at the interface for $t>0$. Therefore, for a time $t_0>0$ and $x\in\mathbb{R}$, there exists $y(x)\in\mathbb{R}$ such that $A^\delta(u^\delta(x,t_0),x) = A^\delta(u_0^\delta(y(x)),y(x))$. Suppose $-\infty = x_0 < x_1 < \cdots < x_m < x_{m+1} = \infty$ are discontinuity points of $x\mapsto A(\cdot,x)$. Let $z_1,z_2\in(x_i,x_{i+1})$ for some $0\le i\le m$. Now, there are two cases

1. $y(z_1), y(z_2)\in(x_j,x_{j+1})$ for some $0\le j\le m$. Then by Lemma 3.1 we have
$$\left|u^\delta(z_1,t_0)-u^\delta(z_2,t_0)\right| \le C_3\left|u_0^\delta(y(z_1))-u_0^\delta(y(z_2))\right|. \tag{3.3}$$

2. $y(z_1)\in(x_j,x_{j+1}), y(z_2)\in(x_k,x_{k+1})$ for some $0\le j\ne k\le m$. Then by Lemma 3.1 and the triangle inequality, we have
$$\left|u^\delta(z_1,t_0)-u^\delta(z_2,t_0)\right| \le \left|u^\delta(z_1,t_0)-u_M^\delta(z_1)\right| + \left|u^\delta(z_2,t_0)-u_M^\delta(z_2)\right| + \left|u_M^\delta(z_1)-u_M^\delta(z_2)\right|$$

$$\leq C_3\left[\left|u_0^\delta(y(z_1))-u_M^\delta(y(z_1))\right|+\left|u_0^\delta(y(z_2))-u_M^\delta(y(z_2))\right|\right] + \left|u_M^\delta(z_1)-u_M^\delta(z_2)\right|. \tag{3.4}$$

Suppose $z\in(x_i,x_{i+1})$ for $0\leq i\leq m-1$, then by Lemma 3.1, we have

$$\left|u^\delta(z,t_0)-u_M^\delta(z)\right|\leq C_3\left|u_0^\delta(y(z))-u_M^\delta(y(z))\right|. \tag{3.5}$$

Combining Eqs (3.3), (3.4) and (3.5) with definition Eq (3.2) of Λ we conclude Theorem 3.1. □

3.2. Counterexample-I

In this section, we wish to construct an example of an adapted entropy solution such that $u(\cdot,1)\notin \mathrm{BV}_{loc}(\mathbb{R})$ for a flux satisfying (**A-1**)–(**A-3**) and an initial datum $u_0\in\mathrm{BV}(\mathbb{R})$ with $\Lambda(u_0)<\infty$. We also assume that the flux possesses C^2 regularity in the state variable, that is, for a.e. $x\in\mathbb{R}$, $u\mapsto A(x,u)$ is a C^2-function (but not necessarily uniformly convex). Note that this result is special for fluxes having infinitely many spatial discontinuities since for a flux A with finitely many discontinuities and satisfying (**A-1**)–(**A-3**), we have $TV(u(\cdot,t))<\infty$ for BV initial data (see [16]).

Proposition 3.1. *There exists a flux A satisfying* (**A-1**)–(**A-3**) *and an initial datum* u_0 *such that* $\Lambda(u_0)<\infty$ *such that the corresponding entropy solution to* (1.1) *has TV blow up at some finite time* $T_0>0$.

Construction: Let us consider two sequences $\{y_n\}_{n\geq0}$ and $\{z_n\}_{n\geq1}$ such that they satisfy

$$y_{n-1}<z_n<y_n \text{ for } n\geq1 \text{ and } \lim_{n\to\infty}y_n=\lim_{n\to\infty}z_n=z^* \text{ for some } z^*\in\mathbb{R}. \tag{3.6}$$

Now we consider the following flux,

$$A(x,u):=\sum_{n=1}^{\infty}f_n(u)\chi_{(y_{n-1},z_n]}(x)+\sum_{n=1}^{\infty}g_n(u)\chi_{(z_n,y_n]}(x)+f_1\chi_{(-\infty,y_0]}(x)+u^2\chi_{(z^*,\infty]}(x) \tag{3.7}$$

where f_n,g_n are defined as follows

$$f_n(u)=\frac{2}{n^{3/2}}u^2+u^4 \text{ and } g_n(u)=u^4.$$

Note that $u\mapsto A(x,u)$ is a C^2-function for a.e. $x\in\mathbb{R}$. Hence (**A-1**) is satisfied. Observe that $A(x,u)$ satisfies (**A-2**), with $h_1(u)=u^4,h_2(u)=2u^4$ and $u_M\equiv0$. Note that (**A-3**) is verified for $A(x,u)$ with the following choice of η,a

$$\eta(u):=2u^2 \text{ and } a(x):=\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}\chi_{(y_{n-1},z_n]}(x)+1\chi_{(-\infty,y_0]}(x).$$

Loosely speaking, $\left|(g_n)_+^{-1}(f_n(u))\right|\geq u^{\frac12}n^{-\frac38}$ gives a possible blow up in the solution.

We consider an initial datum defined as follows,

$$u_0^n(x):=\begin{cases}0 & \text{if } y_{n-1}<x<x_n^1,\\ a_n & \text{if } x_n^1<x<x_n^2,\\ 0 & \text{if } x_n^2<x<z_n,\\ 0 & \text{if } z_n<x<y_n.\end{cases}$$

We observe that rarefaction wave arises at x_n^1 and shock wave is generated at x_n^2 with shock speed λ_n where λ_n is calculated as follows

$$\lambda_n = \frac{f_n(a_n) - f_n(0)}{a_n - 0} = \frac{2a_n}{n^{\frac{3}{2}}} + a_n^3. \tag{3.8}$$

Assume that at time $t = t_n$ the shock curve hits the interface; therefore, we have $x_n^2 + t_n\lambda_n = z_n$. Since the speed of the extreme right characteristic of the rarefaction wave $f_n'(a_n)$ is bigger than a_n, we need to give sufficient gap between x_n^1 and x_n^2 such that before time $t = 1$ it does not hit the shock wave and the interface at z_n. Hence, the following inequality must hold:

$$z_n - x_n^1 \geq f_n'(a_n). \tag{3.9}$$

Now after the shock wave from x_n^2 hits the interface at z_n it generates another shock wave from $(z_n, t_n) \in \mathbb{R} \times \mathbb{R}_+$. Note that this new shock wave has states $b_n, 0$ on its left and right respectively where $b_n = (g_n)_+^{-1}(f_n(a_n))$. We want this shock wave not to meet the interface at y_n before time $t = 1$. Hence we must have

$$\xi_n(1 - t_n) + z_n < y_n, \tag{3.10}$$

where ξ_n is the shock speed between $b_n, 0$, that is,

$$\xi_n = \frac{g_n(b_n) - g_n(0)}{b_n - 0} = b_n^3. \tag{3.11}$$

If we are able to choose such x_n^1, x_n^2, z_n, y_n then the solution has the following structure up to time $t = 1$, (see Figure 2)

1. For $t \in [0, t_n]$ we have

$$u^n(x,t) := \begin{cases} 0 & \text{if } x < x_n^1, \\ (f_n')^{-1}(x - x_n^1/t) & \text{if } x_n^1 < x < x_n^1 + f_n'(a_n)t, \\ a_n & \text{if } x_n^1 + f_n'(a_n)t < x < x_n^2 + \lambda_n t, \\ 0 & \text{if } x_n^2 + \lambda_n t < x < z_n, \\ 0 & \text{if } z_n < x < y_n, \end{cases}$$

where λ_n is defined as in Eq (3.8).

2. For $t \in [t_n, 1]$ we have

$$u^n(x,t) := \begin{cases} 0 & \text{if } x < x_n^1, \\ (f_n')^{-1}(x - x_n^1/t) & \text{if } x_n^1 < x < x_n^2 + f_n'(a_n)t, \\ a_n & \text{if } x_n^2 + f_n'(a_n)t < x < z_n, \\ b_n & \text{if } z_n < x < z_n + \xi_n(t - t_n), \\ 0 & \text{if } z_n + \xi_n(t - t_n) < x < y_n \end{cases}$$

where ξ_n is defined as in Eq (3.11) and b_n is defined as follows,

$$b_n = (g_n)_+^{-1}(f_n(a_n)).$$

Finally, we choose

$$a_n = \frac{1}{n^{1+\delta}},\ x_n^2 - x_n^1 = \frac{8}{n^{\delta+\frac{5}{2}}} + \frac{8}{n^{3+3\delta}} \text{ and } z_n - x_n^2 = \left(\frac{4}{n^{\delta+\frac{5}{2}}} + \frac{2}{n^{3+3\delta}}\right)\left(1 - \frac{1}{n^2}\right),$$

where $\delta > 0$. Next we choose

$$y_n - z_n = 2\left(\frac{2}{n^{2\delta+\frac{7}{2}}} + \frac{1}{n^{4+4\delta}}\right)^{\frac{3}{4}} \leq 6\left(\frac{1}{n^{2\delta+\frac{7}{2}}}\right)^{\frac{3}{4}} \text{ and } x_n^1 = y_{n-1} + \frac{1}{n^2}.$$

Note that

$$y_{n-1} < x_n^1 < x_n^2 < z_n < y_n.$$

Note that $x_n^2 - x_n^1 = f_n'(a_n)$. Hence, we have $z_n - x_n^1 = z_n - x_n^2 + x_n^2 - x_n^1 > f'(a_n)$ since $z_n - x_n^2 > 0$. Therefore, Eq (3.9) is satisfied. Observe that $y_n - z_n = 2((g_n)_+^{-1}(f_n(a_n)))^3 = 2\xi_n > \xi_n(1-t_n)$ since $\xi_n > 0$ and $0 < 1 - t_n < 1$. Hence, the condition (3.10) is satisfied. Now we note that

$$\begin{aligned} y_n - y_{n-1} &= y_n - z_n + z_n - x_n^2 + x_n^2 - x_n^1 + x_n^1 - y_{n-1} \\ &\leq 6\left(\frac{1}{n^{2\delta+\frac{7}{2}}}\right)^{\frac{3}{4}} + \left(\frac{4}{n^{\delta+\frac{5}{2}}} + \frac{2}{n^{3+3\delta}}\right)\left(1 - \frac{1}{n^2}\right) + \frac{8}{n^{\delta+\frac{5}{2}}} + \frac{8}{n^{3+3\delta}} + \frac{1}{n^2} \\ &\leq \frac{1}{n^2} + \frac{22}{n^{\delta+\frac{5}{2}}} + \frac{8}{n^{\frac{21}{8}}}. \end{aligned}$$

Thus, there exists an y_∞ such that $y_n \to y_\infty$. Set $z^* = y_\infty$ and $y_0 = 0$ in Eq (3.7).

Note that $TV(u_0^n) \leq n^{-1-\delta}$ and $TV(u^n(\cdot, 1)) \geq |b_n|$. Observe that

$$|b_n| = \left[\frac{2}{n^{2\delta+\frac{7}{2}}} + \frac{1}{n^{4+4\delta}}\right]^{\frac{1}{4}} \geq \frac{1}{n^{\frac{7}{8}+\frac{\delta}{2}}}.$$

Note if we take an initial datum $u_0(x) = \sum_{k=1}^{\infty} u_0^k(x)\mathbb{1}_{(y_{k-1},y_k)}(x)$, then $u(x) = \sum_{k=1}^{\infty} u_k(x)\mathbb{1}_{(y_{k-1},y_k)}(x)$ will be the adapted entropy solution to Eq (1.1). Since $TV(u_0^k) \leq k^{-1-\delta}$ and $\text{supp}(u_0^k), k \geq 1$ are mutually disjoint, we have $TV(u_0) < \infty$. We observe that $\text{supp}(u^k(\cdot, 1)), k \geq 1$ are mutually disjoint. Therefore, we have,

$$TV(u(\cdot, 1)) \geq \sum_{k\geq 1} \frac{1}{k^{\frac{7}{8}+\frac{\delta}{2}}}. \tag{3.12}$$

Now we choose, $\delta \in (0, 4^{-1})$. Then, we have $\frac{7}{8} + \frac{\delta}{2} < 1$. Hence, $TV(u(\cdot, 1)) = \infty$.

3.3. Counterexample-II

In Section 3.2, we presented an example exhibiting a BV blow-up of entropy solutions corresponding to an initial datum u_0 for which the quantity $\Lambda(u_0)$ is finite but the flux as a function of the state variable is not necessarily uniformly convex. In contrast to the previous section, we now construct an example starting from BV initial data and a uniformly convex flux such that the entropy solution has a TV blow-up at some finite time T. More precisely, we construct a flux and a BV initial data for which the quantity $\Lambda(u_0)$ is not finite.

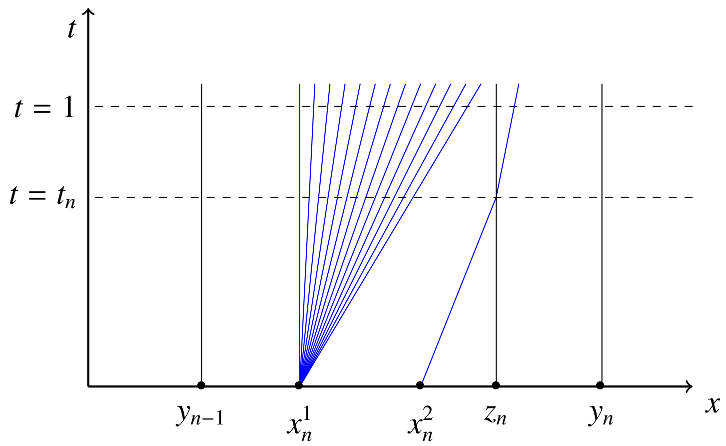

Figure 2. This illustrates the structure of the solution in $[y_{n-1}, y_n]\times[0,1]$. This is the building block for the solution giving TV blow up at time $t = 1$ for strictly convex flux. Here $x = y_{n-1}$, $x = z_n$, $x = y_n$ represent interfaces. Note that rarefaction and shock are generated at $(x_n^1, 0)$ and $(x_n^2, 0)$ respectively. The shock line hits the interface $x = z_n$ at time $t = t_n < 1$.

Proposition 3.2. *There exists a flux A satisfying* (**A-1**)–(**A-3**) *and an initial datum u_0 such that $\partial_{uu}A(x,u) \in [C_1, C_2]$ for some $C_2 > C_1 > 0$ and the corresponding entropy solution to Eq* (1.1) *has TV blow up at some finite time $T_0 > 0$.*

Proof. We start by constructing a flux A satisfying conditions (**A-1**)–(**A-3**) and the initial datum, for which the above proposition holds true: Let $g(u) = u^2$ and for any $n \in \mathbb{N}$, let

$$f_n(u) := \begin{cases} u^2 & \text{if } u \le -\dfrac{1}{n^{2/3}}, \\ u^2 + \dfrac{u^2}{n^{1/4}} - 3n^{13/12}u^4 + 3n^{29/12}u^6 - n^{15/4}u^8 & \text{if } -\dfrac{1}{n^{2/3}} \le u \le \dfrac{1}{n^{2/3}}, \\ u^2 & \text{if } u \ge \dfrac{1}{n^{2/3}}, \end{cases} ,$$

which implies that

$$f_n'(u) = \begin{cases} 2u & \text{if } u \le -\dfrac{1}{n^{2/3}}, \\ 2u + \dfrac{2u}{n^{1/4}} - 12n^{13/12}u^3 + 18n^{29/12}u^5 - 8n^{15/4}u^7 & \text{if } -\dfrac{1}{n^{2/3}} \le u \le \dfrac{1}{n^{2/3}}, \\ 2u & \text{if } u \ge \dfrac{1}{n^{2/3}}. \end{cases}$$

and

$$f_n''(u) = \begin{cases} 2 & \text{if } u \le -\dfrac{1}{n^{2/3}}, \\ 2 + \dfrac{2}{n^{1/4}} - 36n^{13/12}u^2 + 90n^{29/12}u^4 - 56n^{15/4}u^6 & \text{if } -\dfrac{1}{n^{2/3}} \le u \le \dfrac{1}{n^{2/3}}, \\ 2 & \text{if } u \ge \dfrac{1}{n^{2/3}}. \end{cases} .$$

Hence $f_n \in C^2(\mathbb{R})$ for $n \ge 1$. Note that there exists $n_0 \in \mathbb{N}$ such that $f_n''(u) \ge 1$ for $n \ge n_0$. Suppose

$\{a_n\}, \{b_n\}$ are two sequences such that $0 < a_{n+1} < b_{n+1} < a_n$. We consider a flux A as follows

$$A(x,u) := \begin{cases} f_n(u) & \text{for } a_n \le x \le b_n \text{ for } n \ge n_0, \\ g(u) & \text{otherwise.} \end{cases}$$

We choose $a_n = \dfrac{1}{n^{2/3}}$ and $b_n = a_n + \dfrac{1}{n^{2n}}$. We approximate A by A^N as follows

$$A^N(x,u) := \begin{cases} f_n(u) & \text{for } a_n \le x \le b_n \text{ with } n_0 \le n \le N, \\ g(u) & \text{otherwise.} \end{cases}$$

By the choice of $f_n(u) = g(u)$ for $|u| \ge n^{-2/3}$. Consider the following data

$$u_0(x) := \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } x > 0. \end{cases} \tag{3.13}$$

Now, we analyze the characteristics. We track the paths traveled by the characteristics emanating from $x = 0, t = 0$ to get structural information about adapted entropy solution to Eq (1.1) corresponding to flux A^N and data u_0 as in Eq (3.13).

Let u^N be the adapted entropy solution to Eq (1.1) with flux A^N and data u_0. Note that initially a rarefaction arises at $x = 0$ and then it travels through the interfaces, and there is no interaction between waves. We observe that since $n_0 \ge 4$, $f_n = g = u^2$ for $u = 1$, no wave arises from interfaces at $t = 0+$ for $x > 0$. Let $L_N : [0, 2] \to \mathbb{R}$ be the path traveled by the characteristic arising from $x = 0$ with speed $2w_N$ with $w_N := (N + 1)^{-2/3}$.

Note that L_N is a piecewise linear curve originating from $x = 0$ (see Figure 3). We get $L_N(0) = 0$ and wish to estimate $L_N(1)$. Let $v_{N,k} \in [0, \infty)$ be defined such that $g(w_N) = f_m(v_{N,k})$. Suppose $L_N(t) \in$

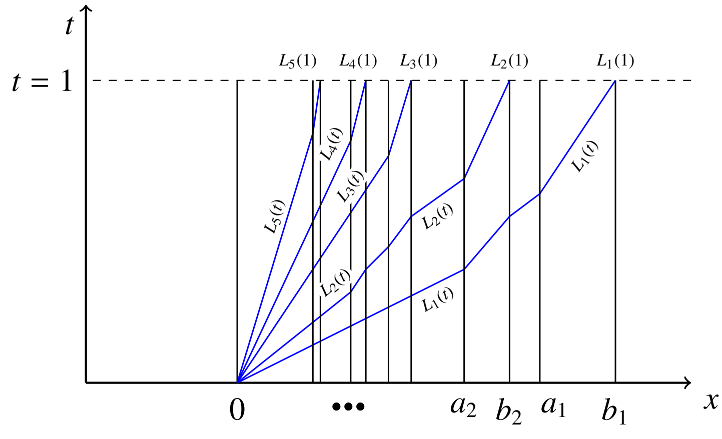

Figure 3. This figure shows the structure of the entropy solution for approximated flux A^{N+5}. Here the curves $L_k(t)$ represent the path traveled by characteristic arising from origin with speed $2(N + k)^{-2/3}$ for $k = 1, 2, 3, 4, 5$.

$[a_k, b_k]$ for $t \in [t_k, t_k']$. Therefore, we have $f_k'(v_{N,k})(t_k' - t_k) = b_k - a_k$. Suppose $t_k'' > t_k$ is such that $b_k - a_k = (t_k'' - t_k)g'(w_N)$. We obtain

$$\left|t_k'' - t_k'\right| = (b_k - a_k)\frac{\left|f_k'(v_{N,k}) - g'(w_N)\right|}{f_k'(v_{N,k})g'(w_N)}.$$

Subsequently, we have

$$g'(w_N)\left|t_k''-t_k'\right| = (b_k-a_k)\frac{\left|f_k'(v_{N,k})-g'(w_N)\right|}{f_k'(v_{N,k})}.$$

Let $L(\tilde{t}_N)=a_{N/2}$; then we have

$$\left|1-\tilde{t}_N\right| \le \sum_{k=\left[\frac{N}{2}\right]+1}^{N}(b_k-a_k)\frac{\left|f_k'(v_{N,k})-g'(w_N)\right|}{f_k'(v_{N,k})g'(w_N)}.$$

Note that $\dfrac{1}{(N+1)^{2/3}} \ge \dfrac{1}{2(N/2)^{2/3}}$ for $N \ge n_1$ for some $n_1>1$. Subsequently, we have $\dfrac{k^{2/3}}{(N+1)^{2/3}} \ge \dfrac{1}{2}$. Now we observe that

$$\begin{aligned} f_k\left(\frac{1}{2(N+1)^{2/3}}\right) &= \frac{1}{2(N+1)^{4/3}}\left(1+\frac{1}{k^{1/4}}\left(1-\frac{k^{4/3}}{4(N+1)^{4/3}}\right)^3\right)\\ &\le g(w_N)\frac{1}{2^{4/3}}(1+\frac{1}{16k^{1/4}})\\ &\le g(w_N)=f_k(v_{N,k}). \end{aligned}$$

Hence, we get $\dfrac{w_N}{2} \le v_{N,k}$ for $N/2+1 \le k \le N$. Since $(1-k^{4/3}u^2)^3 \ge 0$ for $|u| \le k^{-2/3}$ we have $f_k(w_N) \ge f_k(v_{N,k})$, therefore $w_N \ge v_{N,k}$. This shows that

$$\left|f'(v_{N,k})-g'(w_N)\right| \le \left|w_N-v_{N,k}\right| + \frac{\bar{C}}{k^{1/4}}v_{N,k} \le \tilde{C}w_N.$$

Hence,

$$\begin{aligned} \left|1-\tilde{t}_N\right| \le \sum_{k=\frac{N}{2}+1}^{N}\frac{\tilde{C}}{k^{2k}w_N} = \tilde{C}(N+1)^{2/3}\sum_{k=\frac{N}{2}+1}^{N}\frac{1}{k^{2k}} &\le \tilde{C}_1(N+1)^{2/3}\frac{2^N}{N^{2N+1}}\\ &\le \tilde{C}_2\frac{2^N}{N^{N+\frac{1}{3}}}. \end{aligned} \tag{3.14}$$

Therefore,

$$\left|L(1)-\frac{2}{(N+1)^{2/3}}\right| \le \tilde{C}\left|1-\tilde{t}_N\right|w_N \le \tilde{C}_3\frac{2^N}{N^{N+1}}. \tag{3.15}$$

Let u^m be the entropy solution to Eq (1.1) with data u_0 as in Eq (3.13) and flux A^m. It will be now shown that the adapted entropy solution u^m agrees with u^N in some subset of $\mathbb{R}\times[0,1]$ for sufficiently large m. Due to the fact that $f_m=g$ for $u \ge (N+1)^{-2/3}$, then characteristic with speed $2(N+1)^{-2/3}$ travels through $L_N(1)$ for $m \ge N+1$ and we observe the following:

$$u^m(x,t)=u^N(x,t) \qquad \text{for } (x,t)\in D_N \tag{3.16}$$

where D_N is defined as (see Figure 3)

$$D_N := (-\infty,0]\times[0,1]\cup\{(x,t),x\ge L_N(t),t\in[0,1]\}.$$

We now calculate the variation $\left|u^N(a_m+,1) - u^N(a_m-,1)\right|$, where u^N is the adapted entropy solution to Eq (1.1) corresponding to flux A^N and data u_0.

Let the characteristic corresponding to $u = \frac{1}{2m^{2/3}}$, originating from $x = 0$ move through path $Q(t)$. Note that $Q(t)$ is analogous to $L(t)$. By a similar argument as in Eq (3.15), we obtain

$$|Q(1) - a_m| \leq \frac{\tilde{C}_2}{m^{2m+1}}.$$

Suppose $Q(t)$ hits $x = b_{m-1}$ and $x = a_m$ at time $t = \tilde{t}_1$ and $t = \tilde{t}_2$ respectively. Suppose $u^N(a_m-) = \tilde{w}_N$, then we have,

$$a_m - b_{m-1} = 2w_N(\tilde{t}_2 - \tilde{t}_1) \text{ and } a_m - b_{m-1} = 2\tilde{w}_N(1 - \tilde{t}_1).$$

Since $\tilde{t}_2 \geq 1$ we get $w_N \leq \tilde{w}_N$. Then we have

$$\left|\frac{1}{w_N} - \frac{1}{\tilde{w}_N}\right| \leq \frac{\tilde{t}_2 - 1}{a_m - b_{m-1}}. \tag{3.17}$$

Note that

$$a_m - b_{m-1} = \frac{1}{2n^{2/3}} - \frac{1}{2(n+1)^{2/3}} - \frac{1}{(n+1)^{2n+2}} \geq \frac{1}{3(n+1)^{5/3}} - \frac{1}{(n+1)^{2n+2}} \geq \frac{1}{6(n+1)^{5/3}}.$$

By a similar argument as in Eq (3.14) we have

$$\left|\tilde{t}_2 - 1\right| \leq \frac{B}{m^{2m+1}}.$$

Hence, from Eq (3.17), we obtain

$$1 - \frac{w_N}{\tilde{w}_N} \leq \frac{24Bm^{5/3}}{m^{2m+1}} \frac{1}{2m^{2/3}} \leq \frac{12B}{m^{2m}}.$$

Then, we get

$$1 - \frac{12B}{m^{2m}} \leq \frac{w_m}{\tilde{w}_m}.$$

For sufficiently large m, that is there exists an $m_0 > 1$ such that $1 - \frac{12B}{m^{2m}} \geq \frac{1}{2}$ for $m \geq m_0$. Therefore, $\tilde{w}_m \leq 2w_m$. Hence, we get

$$|w_m - \tilde{w}_m| \leq 2w_m^2 \frac{\tilde{t}_2 - 1}{a_m - b_{m-1}} \leq \frac{24B}{m^{2m+\frac{2}{3}}}.$$

Suppose $f_m(\tilde{v}_m) = g(\tilde{w}_m)$. Note that there exists $m_1 > 1$ such that $|\tilde{w}_m - w_m| \leq \frac{1}{6m^{2/3}}$. Then we note that $\frac{1}{3m^{2/3}} \leq \tilde{w}_m \leq \frac{1}{2m^{2/3}}$. Subsequently, we get

$$f_m(\tilde{w}_m) = f_m\left(\frac{1}{2m^{2/3}}\right) = \frac{1}{4m^{4/3}}\left(1 + \frac{1}{8m^{1/4}}\right) \geq \frac{1}{4m^{4/3}} = g(\tilde{w}_m) = f_m(\tilde{v}_m).$$

Thus, we get $\tilde{w}_m \geq \tilde{v}_m$. Note that

$$f_m\left(\frac{1}{3n^{2/3}}\right) = \frac{1}{9m^{4/3}}\left(1 + \frac{4}{9m^{1/4}}\right) \leq \frac{2}{9m^{4/3}} \leq g(\tilde{w}_m) = f_m(\tilde{v}_m).$$

Hence, $1/3 \leq \tilde{v}_m m^{2/3} \leq 1/2$. Therefore,

$$\tilde{w}_m^2 = \frac{1}{4m^{4/3}} = f_m(\tilde{v}_m) = v_m^2\left(1 + \frac{(1 - m^{2/3}v_m)^3}{m^{1/4}}\right) \leq v_m^2\left(1 + \frac{8}{27m^{1/4}}\right).$$

Hence,

$$|\tilde{w}_m - \tilde{v}_m| = \tilde{w}_m - \tilde{v}_m \geq \tilde{w}_m\left(\frac{\sqrt{1 + \frac{8}{27m^{1/4}}} - 1}{\sqrt{1 + \frac{8}{27m^{1/4}}}}\right) \geq \frac{1}{27m^{1/4}}\left(\frac{1}{2m^{2/3}} - \frac{B}{m^{2m+\frac{2}{3}}}\right).$$

Therefore, we obtain for $N \geq 2m + 1$,

$$\left|u^N(a_m+, 1) - u^N(a_m-, 1)\right| \geq \frac{1}{27m^{1/4}}\left(\frac{1}{2m^{2/3}} - \frac{B}{m^{2m+\frac{2}{3}}}\right). \tag{3.18}$$

Finally, we show that u^m converges to an adapted entropy solution to Eq (1.1) for initial data u_0 defined as in Eq (3.13) as $m \to \infty$. Note that $\left\|u^N - u^m\right\|_{L^1(\mathbb{R}\times[0,1])} \leq \tilde{B}N^{-2/3}$ for all $m \geq N$. Therefore, $\{u^m\}_{m\geq 1}$ is a Cauchy sequence in $L^1(\mathbb{R} \times [0, 1])$. Hence, there exists a $u \in L^1(\mathbb{R} \times [0, 1])$ such that $u^m \to u$ in $L^1(\mathbb{R} \times [0.1])$. We also note that $A^N(x, \cdot) = A(x, \cdot)$ for $x \in \mathbb{R} \setminus [0, b_N]$. This implies $k^{\pm}_{\alpha,N}(x) = k^{\pm}_{\alpha}(x)$ for $x \in \mathbb{R} \setminus [0, b_N]$ and we get $k^{\pm}_{\alpha,N} \to 0$ as $N \to \infty$. Since u^N is an adapted entropy solution, $u^N \to u$ in $L^1(\mathbb{R}\times[0, T])$, $k^{\pm}_{\alpha,N} \to k^{\pm}_{\alpha}, A^N(x, u) \to A(x, u)$ for a.e. $x \in \mathbb{R}$, we get u as the adapted entropy solution to Eq (1.1) for data u_0 as in Eq (3.13) with flux $A(x, u)$. By using Eq (3.16), we note that $u(x, t) = u^N(x, t)$ for $(x, t) \in D^N$. From Eq (3.18) we get

$$|u(a_m+, 1) - u(a_m-, 1)| \geq \frac{1}{27m^{1/4}}\left(\frac{1}{2m^{2/3}} - \frac{B}{m^{2m+\frac{2}{3}}}\right).$$

Hence we have $TV(u(\cdot, 1)) = \infty$.

□

Remark 3.2. *Let u_0 be defined as in Eq* (3.13). *Due to the finite speed of propagation we can consider a data $\tilde{u}_0 = u_0\chi_{[-M,M]}$ for some large $M > 0$ and the solution gives a TV blow up at time $t = 1$.*

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that there are no conflicts of interest.

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