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Research article

On Kuramoto-Sakaguchi-type Fokker-Planck equation with delay

Hirotada Honda*

Faculty of Information and Networking for Innovation and Design, Toyo university, Akabanedai 1-7-11, Kita-Ku, Tokyo, Japan

* **Correspondence:** honda.hirotada@iniad.org; Tel: +81-3-5924-2658; Fax: +81-3-5924-2600.

Abstract: Recently, the Kuramoto model with transmission delay has been attracting increasing attention, accompanied by the increase in its practical applications. In this paper, we studied the Kuramoto-Sakaguchi-type Fokker-Planck equation of the above model proposed by Lee et al., in 2009. We proved the global-in-time solvability of the equation under some conditions on the initial data and distribution of delay.

Keywords: Kuramoto–Sakaguchi equation, synchronization, dynamic system, Fokker-Planck equation, Sobolevel–Slobodetskii space

1. Introduction

The study of synchronization phenomena has a rather long history. One of the most sophisticated formulations is the Kuramoto model, which describes the temporal behavior of oscillator phases. A range of models related to the Kuramoto model, which offers vast applications, have been considered and discussed to date.

An important factor in practical applications is the consideration of transmission delay, as pointed out in past arguments (see, for instance, [23]). For example, when discussing the neural networks of the human brain, the effect of the transmission delay of the synaptic propagation cannot be disregarded.

A general method among the models that takes into account the effect of transmission delay is that proposed by Lee et al. [17] because it approaches delay as a random variable that is subjected to a certain probability density. The discussion in this work [17] begins with a system of ordinary differential equations with a delay imposed on each pair of oscillators.

$$\frac{\mathrm{d}}{\mathrm{d}t}\theta_i(t) = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin[\theta_j(t-\tau_{ij}) - \theta_i(t)] \quad (i = 1, 2, \dots, N), \tag{1.1}$$

where $\theta_i(t)$ is the phase of oscillator i, ω_i is the natural frequency of i, K is the coupling strength between oscillators, and N is the total number of oscillators. τ_{ij}, which denotes the transmission delay on the link between oscillators i and j, satisfies $\tau_{ii} = 0$ ($i = 1, 2, \dots, N$).

As in the original Kuramoto model, Lee et al. [17] also introduced a measure of phase synchronization known as the *order parameter*:

$$N^{-1} \sum_{j=1}^{N} \sin[\theta_j(t - \tau_{ij}) - \theta_i(t)] = \mathrm{Im}[r_i \mathrm{e}^{-\mathrm{i}\theta_i(t)}], \tag{1.2}$$

$$r_i(t) = N^{-1} \sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}\theta_j(t-\tau_{ij})}. \tag{1.3}$$

Hereafter, we use the notation for imaginary units $\mathrm{i} = \sqrt{-1}$. Using Eqs (1.2) and (1.3), we can rewrite Eq (1.1) as a simpler system of ordinary differential equations, then by adding further white noise, we consider

$$\frac{\mathrm{d}}{\mathrm{d}t}\theta_i(t) = \omega_i + \frac{K}{2\mathrm{i}}\Big(r_i(t)\mathrm{e}^{-\mathrm{i}\theta_i(t)} - \bar{r}_i(t)\mathrm{e}^{\mathrm{i}\theta_i(t)}\Big) + \xi_i(t), \tag{1.4}$$

where we use the fact that $\mathrm{Im}[z] = \frac{z-\bar{z}}{2\mathrm{i}}$ for $z \in \mathbb{C}$. Here, $\bar{z}$ denotes the complex conjugate of z. We also used the notation $\{\xi_i(t)\}_{i=1}^N$, which are the independent Wiener processes satisfying $\langle \xi_i(t) \rangle = 0$ and $\langle \xi_j(t)\xi_k(\tau) \rangle = 2\varepsilon\delta_{jk}\delta(t - \tau)$, in which $\delta(\cdot)$ denotes the Dirac measure. Then, as a continuum limit of the infinite oscillator population $N \to +\infty$, the Fokker-Planck equation of (1.4) as a limit of $\varepsilon \to 0$ [6], which describes the temporal behavior of the probability density of the oscillators, can be obtained:

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \theta}\left\{\Big[\omega + \frac{k}{2\mathrm{i}}(\mathrm{e}^{-\mathrm{i}\theta} r - \mathrm{e}^{\mathrm{i}\theta}\bar{r})\Big] f\right\} = 0, \tag{1.5}$$

where

$$r(t) = \int_0^\infty \xi(t - \tau)h(\tau)\, \mathrm{d}\tau, \quad \xi(t) = \int_{-\infty}^{\infty}\int_0^{2\pi} f(\theta, t; \omega)\mathrm{e}^{\mathrm{i}\theta}\, \mathrm{d}\theta \mathrm{d}\omega. \tag{1.6}$$

Here, the unknown function $f = f(\theta, t; \omega)$ is the distribution function of the single oscillator and $h(\tau)$ is that of the delay.

In this study, we rigorously examine the well-posedness of models (1.5) and (1.6) under suitable initial and boundary conditions, and discuss the following:

(i) We propose a Fokker-Planck equation [19] corresponding to the model developed by Lee et al. with additional noise.

(ii) We demonstrate the existence and uniqueness of a global-in-time solution in a suitable function space for the aforementioned model.

Some difficulties are encountered owing to the delay τ_{ij} in Eq (1.2) and its density $h(\cdot)$ in Eq (1.6). We demonstrate that, when using a method developed in the study of fluid mechanics, we can obtain an *a-priori* estimate of the solution and its global existence. The novel contributions of this paper are summarized as follows.

(i) We consider the uniqueness and existence of a global-in-time solution to the parabolic equation that corresponds to the formulation of Lee et al. [17]. To the best of our knowledge, this is the first study that rigorously discusses their formulation.

(ii) Unlike other models, our formulation includes the random delay, which causes some difficulties. We overcome this issue with the aid of the approach used in the mathematical analyses of fluid mechanics.

The remainder of this paper is organized as follows. In the following section, we discuss the central objective of our analysis. The problem formulation is outlined in Section 2. Section 3 provides an overview of existing mathematical arguments. Section 4 introduces the notation used throughout this study. The main results of this study are presented in Section 5, followed by proofs in Sections 6 and 7. In Section 8, we discuss other properties of the solution. Section 9 concludes the paper.

2. Formulation

In this section, we discuss the problems identified in this study. For simplicity, hereafter, we use notations $\Omega = (0, 2\pi)$, $\Omega_T = \Omega\times(0,T)$ with an arbitrary $T \in (0, +\infty]$, and $\mathbb{R}_+ = (0, +\infty)$. By substituting Eqs (1.5) and (1.6) into Eq (1.4) and imposing the periodic boundary conditions and initial condition, we obtain

$$\begin{cases} \dfrac{\partial f}{\partial t} + \omega\dfrac{\partial f}{\partial \theta} + K\dfrac{\partial}{\partial \theta}(F[f,f]) = 0 \quad \text{in } \Omega_\infty \times \mathbb{R}, \\ f^{(j,0)}(0,t;\omega) = f^{(j,0)}(2\pi,t;\omega)\ (j=0,1) \quad \text{on } \mathbb{R}_+ \times \mathbb{R}, \\ f(\theta,t;\omega) = f_0(\theta;\omega) \quad \text{on } \Omega \times (-\infty, 0] \times \mathbb{R}. \end{cases} \tag{2.1}$$

where

$$F[f_1, f_2] \equiv f_1(\theta,t;\omega)\int_0^\infty h(\tau)\mathrm{d}\tau \int_{\mathbb{R}} \mathrm{d}\omega' \int f_2(\theta', t-\tau;\omega')\sin(\theta'-\theta)\,\mathrm{d}\theta'.$$

Note that in Eq (2.1), the initial condition is provided on the interval $(-\infty, 0]$ with respect to t owing to the presence of a delay. In addition, in Eq (2.1), the initial condition is replaced by that over the half-infinite interval $(-\infty, 0]$ owing to the existence of a delay. Instead of Eq (2.1), we first consider the parabolic regularization:

$$\begin{cases} \dfrac{\partial f}{\partial t} + \omega\dfrac{\partial f}{\partial \theta} - \varepsilon\dfrac{\partial^2 f}{\partial \theta^2} + K\dfrac{\partial}{\partial \theta}(F[f,f]) = 0 \quad \text{in } \Omega_\infty \times \mathbb{R}, \\ f^{(j,0)}(0,t;\omega) = f^{(j,0)}(2\pi,t;\omega)\ (j=0,1) \quad \text{on } \mathbb{R}_+ \times \mathbb{R}, \\ f(\theta,0;\omega) = f_0(\theta;\omega) \quad \text{on } \Omega \times (-\infty, 0] \times \mathbb{R}. \end{cases} \tag{2.2}$$

Note that $\bar{f} \equiv g(\omega)/2\pi$ is a trivial stationary solution to Eq (2.2), where $g(\omega)$ is the probability density of the natural frequency ω. Moreover, based on appropriate assumptions for f_0 and h, the following properties of f are derived (see, for instance, Lemma 2.1 in [11], Lemmas 1.1 and 1.2 in [24], and Lemma 8.1 of this paper):

$$f(\theta,t;\omega) \geq 0.$$

However, in the case of the typical Kuramoto–Sakaguchi equation [10], the unknown phase probability density satisfies $\int_0^{2\pi} f(\theta, t; \omega) = 1$, whereas for Eq (2.2), f satisfies $\int_0^{2\pi} f(\theta, t; \omega) d\theta = const$, and $\int_{\mathbb{R}} \mathrm{d}\omega \int_0^{2\pi} f(\theta, t; \omega)\, \mathrm{d}\theta = 1$. Therefore, we find a suitable decay rate must be imposed with respect to ω on the initial data f_0, which is conserved at all times.

Similar to case of the original Kuramoto–Sakaguchi equation [9], we introduce the transformation of coordinates as

$$\theta = \theta' + \omega t, \quad t = t',$$

and

$$f(\theta, t; \omega) = f(\theta' + \omega t, t'; \omega) = \hat{f}(\theta', t'; \omega),$$

then, $\hat{f}$ satisfies

$$\begin{cases} \dfrac{\partial \hat{f}}{\partial t} - \varepsilon \dfrac{\partial^2 \hat{f}}{\partial \theta^2} + K \dfrac{\partial}{\partial \theta}\Big(F[\hat{f}, \hat{f}]\Big) = 0, \quad \text{in } \Omega_\infty \times \mathbb{R}, \\ \hat{f}^{(j,0)}(0, t; \omega) = \hat{f}^{(j,0)}(2\pi, t; \omega) \ (j = 0, 1) \quad \text{on } \mathbb{R}_+ \times \mathbb{R}, \\ \hat{f}(\theta, 0; \omega) = f_0(\theta; \omega) \quad \text{on } \Omega \times (-\infty, 0] \times \mathbb{R}. \end{cases} \tag{2.3}$$

Next, we introduce a *weight* function with $\delta > 1/2$:

$$w_{(\delta)}(\omega) = \begin{cases} 1 + |\omega|^{1/2+\delta} \quad (|\omega| > 1), \\ 2 \quad (|\omega| \leq 1). \end{cases}$$

After multiplying $w_{(\delta)}(\omega)$ and Eq (2.3), we introduce $\tilde{f}(\theta, t; \omega) \equiv w_{(\delta)}(\omega) \hat{f}(\theta, t; \omega)$. This satisfies

$$\begin{cases} \dfrac{\partial \tilde{f}}{\partial t} - \varepsilon \dfrac{\partial^2 \tilde{f}}{\partial \theta^2} + K \dfrac{\partial}{\partial \theta}\Big(F\Big[\tilde{f}, \dfrac{\tilde{f}}{w_{(\delta)}}\Big]\Big) = 0 \quad \text{in } \Omega_\infty \times \mathbb{R}, \\ \tilde{f}^{(j,0)}(0, t; \omega) = \tilde{f}^{(j,0)}(2\pi, t; \omega) \ (j = 0, 1) \quad \text{on } \mathbb{R}_+ \times \mathbb{R}, \\ \tilde{f}(\theta, 0; \omega) = \tilde{f}_0(\theta; \omega) \equiv w_{(\delta)}(\omega) f_0(\theta; \omega) \quad \text{on } \Omega \times (-\infty, 0] \times \mathbb{R}. \end{cases} \tag{2.4}$$

In this study, we discuss the solvability of Eq (2.4).

3. Related works

Mathematical arguments concerning the solvability of the Kuramoto–Sakaguchi equation, which corresponds to the original Kuramoto model, were first presented by Lavrentiev et al. [15, 16]. In their earlier work [15], they constructed a classical global-in-time solution wherein the support of $g(\omega)$ is compact.

They later removed this restriction [16] by applying *a priori* estimates derived from the energy method. They also studied the regularity of an unknown function with respect to ω.

In terms of stability, pioneering work was conducted by Strogatz and Mirollo [21] who focused on the linear stability of the trivial stationary solution $\bar{\varrho} = 1/2\pi$. By investigating the spectrum of the linearized operator, they verified the existence of a critical coupling strength, over which the coherent state became stable.

Remarkably, Ha and Xiao [7] discussed the nonlinear stability of $\bar{\varrho}$ and the convergence of the solution as D tends to zero. Their estimate was obtained in the space L_∞ with respect to θ. They also verified the instability of $\bar{\varrho}$ when the support from $g(\omega)$ is sufficiently narrow [8].

In the case of vanishing diffusion (2.1), Chiba [4] argued for the nonlinear stability of a trivial stationary solution under the assumption of unbounded support for $g(\omega)$.

Several contributions relating to the Kuramoto model with delay were made prior to the work by Lee et al. [17]. To the best of our knowledge, one of the earliest works in this direction was presented by Niebur et al. [18]. They investigated the behavior of a system with two oscillators and found that the collective frequency decreases with the increase in the delay.

Subsequently, Kim et al. [12] discussed a Kuramoto model with a constant delay and pinning forces. Based on thorough numerical computations, they clarified the existence of a multi-stable state that depends on the delay and the coefficient of the pinning forces (in fact, the coexistence of coherent and incoherent states had also been reported in oscillator networks under nonlocal coupling [13]). In addition, they verified that the system may exhibit a hysteresis of the collective frequency when the coefficient of the pinning forces increases.

Their motivation was the fact that the behavior of a neural network in the human brain is also significantly affected by the presence of delay [3]. Thereafter, Yeung and Strogatz [23] formulated a Kuramoto model with a constant delay τ, and analyzed the stability of coherent and incoherent states. They demonstrated that the incoherence becomes stable within certain regions of K and τ. They also showed that coherence and incoherence coexist in certain K–τ regions.

This work was continued by Choi et al. [5], who derived the Fokker–Planck equation and its solution that corresponds to the coherent state. For each phase ϕ_i of each oscillator, they began with the transform $\psi_i = \phi_i - \Omega' t$ with a constant Ω'. Subsequently, they derived the approximate solution using the asymptotic expansion of Ω under the assumption $\Omega \ll K\triangle \ll 1$, where $\triangle$ is the magnitude of the order parameter. The restriction in their model was the convexity on the distribution $g(\omega)$ of natural frequencies at $\omega = 0$; that is, $g''(0) < 0$.

As in the original Kuramoto model, the shape of the natural frequency density drastically affects the results, as reported by Strogatz and Mirillo [21]. Moreover, motivated by the study of the human brain, several recent works have investigated the pattern of the behavior of the oscillators under the presence of heterogeneous transmission delay and network structure [2, 24].

However, none of these studies considered models with diffusion. In addition, to the best of our knowledge, no mathematical arguments have been made regarding the solvability of the Kuramoto model with delays. In this regard, the novelty of this paper is characterized as follows. First, we consider the Kuramoto–Sakaguchi type Fokker-Planck equation with transmission delay. Second, by using the approaches developed in mathematical fluid analysis, we establish the global-in-time solvability of the equation. It is noteworthy that under the presence of a transmission delay, the standard method used in our previous works based on the energy method cannot be applied in the present case.

4. Notations

The functional spaces used throughout this study are described in this subsection. Let $T > 0$ and $\mathcal{G}$ be an open set in $\mathbb{R}$. Hereafter, $L_2(\mathcal{G})$ represents a set of square-integrable functions defined on $\mathcal{G}$, equipped with the norm

$$\|u\| \equiv \int_{\mathcal{G}} |u(x)|^2 \, \mathrm{d}x.$$

The inner product is defined as follows:

$$(u_1, u_2) \equiv \int_{\mathcal{G}} u_1(x)\overline{u_2(x)} \, \mathrm{d}x,$$

where $\bar{z}$ denotes the complex conjugate of $z \in \mathbf{C}$. We denote the L_2-norm of the function $f(\theta, t; \omega)$ with respect to θ as $\|f\|_{L_2(\Omega)}$. We use $C(\mathcal{G})$ and $C^k(\mathcal{G})$ $(k \in \mathbb{N} \bigcup \{+\infty\})$ to denote the spaces of real continuous and k-times continuously differentiable functions on Ω, respectively. The notation $C_0^\infty(\mathcal{G})$ refers to the set of $C^\infty(\mathcal{G})$ functions with compact support in $\mathcal{G}$.

For a Banach space E with norm $\|\cdot\|_E$, we denote the space of E-valued measurable functions $u(t)$ in the interval (a, b) as $L_p(a, b; E)$, the norm of which is defined by

$$\|u\|_{L_p(a,b;E)} \equiv \begin{cases} \left(\int_a^b \|u(t)\|_E^p \, \mathrm{d}t\right)^{1/p} & (1 \le p < \infty), \\ \operatorname*{ess\,sup}_{a \le t \le b} \|u(t)\|_E & p = \infty. \end{cases}$$

Similarly, we denote the space of continuous functions as $C(a, b; E)$ (and $C^k(a, b; E)$), (resp. k continuously differentiable functions) from (a, b) to E. $W_2^l(\mathcal{G})$ denotes a space of functions $u(\theta)$, $\theta \in \mathcal{G}$ equipped with the norm $\|u\|^2_{W_2^l(\mathcal{G})} = \sum_{|\alpha|<l} \left\|\frac{\partial^\alpha u}{\partial \theta^\alpha}\right\|^2_{L_2(\mathcal{G})} + \|u\|^2_{\dot{W}_2^l(\mathcal{G})}$, where

$$\begin{cases} \|u\|^2_{\dot{W}_2^l(\mathcal{G})} = \sum_{|\alpha|=l} \left\|\frac{\partial^\alpha u}{\partial \theta^\alpha}\right\|^2_{L_2(\mathcal{G})} = \sum_{|\alpha|=l} \int_{\mathcal{G}} \left|\frac{\partial^\alpha u}{\partial \theta^\alpha}\right|^2 \mathrm{d}\theta \quad \text{if } l \text{ is an integer}, \\ \|u\|^2_{\dot{W}_2^l(\mathcal{G})} = \sum_{|\alpha|=[l]} \int_{\mathcal{G}} \int_{\mathcal{G}} \frac{\left|\frac{\partial^\alpha u(x)}{\partial \theta^\alpha} - \frac{\partial^\alpha u(y)}{\partial \theta^\alpha}\right|^2}{|x-y|^{n+2\{l\}}} \, \mathrm{d}x\mathrm{d}y \\ \qquad \text{if } l \text{ is a non-integer}, \ l = [l] + \{l\}, \ 0 < \{l\} < 1. \end{cases}$$

Subsequently, we introduce anisotropic Sobolev–Slobodetskiĭ spaces [22] $W_2^{l,\frac{l}{2}}(\mathcal{G}_{T_1}) \equiv W_2^{l,0}(\mathcal{G}_{T_1}) \cap W_2^{0,\frac{l}{2}}(\mathcal{G}_{T_1})$ $(\mathcal{G}_{T_1} \equiv \mathcal{G} \times (0, T_1))$, the norms of which are defined as

$$\|u\|^2_{W_2^{l,\frac{l}{2}}(\mathcal{G}_{T_1})} = \int_0^{T_1} \|u(\cdot, t)\|^2_{W_2^l(\mathcal{G})} \mathrm{d}t + \int_{\mathcal{G}} \|u(x, \cdot)\|^2_{W_2^{\frac{l}{2}}(0,T_1)} \mathrm{d}x \equiv \|u\|^2_{W_2^{l,0}(\mathcal{G}_{T_1})} + \|u\|^2_{W_2^{0,\frac{l}{2}}(\mathcal{G}_{T_1})}.$$

The set of functions with vanishing initial data $\overset{\circ}{W}{}_2^{r,\frac{r}{2}}(\mathcal{G}_T)$ is defined as [14]:

$$\overset{\circ}{W}{}_2^{r,\frac{r}{2}}(\mathcal{G}_T) = \left\{ f \in W_2^{r,\frac{r}{2}}(\mathcal{G}_T) \middle| \frac{\partial^k f}{\partial t^k}\bigg|_{t=0} = 0 \left(k = 0, 1, 2, \dots, \left[\frac{r}{2}\right]\right) \right\}.$$

The norms of these spaces are denoted by $\|\cdot\|_{\overset{\circ}{W}{}_2^{r,\frac{r}{2}}(\mathcal{G}_T)}$, etc. We define the function spaces for $m > 0$ as

$$\mathcal{V}_T^m \equiv \left\{ u(\theta, t;\omega) \in W_2^{m,m/2}(\Omega_T) \middle| \left\| u \right\|_{(m,\frac{m}{2});T} \equiv \sup_{\omega\in\mathbb{R}} \|u(\omega)\|^2_{W_2^{m,\frac{m}{2}}(\Omega_T)} < +\infty \right\},$$

$$\overset{\circ}{\mathcal{V}}{}_T^m \equiv \left\{ u(\theta, t;\omega) \in \overset{\circ}{W}{}_2^{m,m/2}(\Omega_T) \middle| \left\| u \right\|_{(m,\frac{m}{2});T} < +\infty \right\},$$

$$\mathcal{V}^m \equiv \left\{ u(\theta;\omega) \in W_2^m(\Omega) \middle| \left\| u \right\|_{(m)} \equiv \sup_{\omega\in\mathbb{R}} \|u(\omega)\|^2_{W_2^m(\Omega)} < +\infty \right\}.$$

For brevity, we use the notation $\left\| u \right\|^2 = \sup_{\omega\in\mathbb{R}} \left\| u(\omega) \right\|^2_{L_2(\Omega)}$, $\left\| u \right\|_{(0,0);T} = \left\| u \right\|_T$ with an arbitrary $T \in (0, +\infty]$. Finally, we introduce the following notations:

$$L_1^{(1)}(\Omega) \equiv \left\{ u(\cdot;\omega) \in L_1(\Omega) \middle| u \geq 0, \int_\Omega u(\theta;\omega)\, \mathrm{d}\theta = 1 \ \forall\omega \in \mathbb{R} \right\},$$

$$L_1^{(1)}(\mathbb{R}) \equiv \left\{ u(\omega) \in L_1(\mathbb{R}) \int_\mathbb{R} u(\omega)\, \mathrm{d}\omega = 1 \right\},$$

$$L_1^{(1)}(T) \equiv \left\{ u(\cdot, t;\omega) \in L_1(\Omega) \middle| u \geq 0, \int_0^{2\pi} u(\theta, t;\omega)\, \mathrm{d}\theta \leq c_{41}/(1 + |\omega|^{2+\delta})\ t \in (0, T),\ \forall\omega \in \mathbb{R} \right\},$$

where $T > 0$ denotes an arbitrary number.

Hereafter, c with suffixes represents constants in the approximation of certain quantities. We also denote $c(t)$ using suffixes depending on t. Furthermore, we use the notation $u^{(j,k)} \equiv \left(\frac{\partial}{\partial\theta}\right)^j \left(\frac{\partial}{\partial t}\right)^k u$ $(j, k = 0, 1, 2, \ldots)$ for the function $u = u(\theta, t)$. We also use the notation

$$F^{(1)}[f_1, f_2] \equiv f_1(\theta, t;\omega) \int_0^\infty h(\tau)\mathrm{d}\tau \int_\mathbb{R} \mathrm{d}\omega' \int f_2(\theta', t-\tau;\omega') \cos(\theta' - \theta)\, \mathrm{d}\theta'.$$

5. Main results

The following subsections present our main results. For the proof, we discuss the estimate of f in Eq (2.4) for the local-in-time (Theorem 5.1) and global-in-time (Theorem 5.2) solvabilities. We estimate $\tilde{f} = f - \frac{g(\omega)}{2\pi}$. First, we state the existence and uniqueness of the local-in-time solution to problem (2.4).

Theorem 5.1. *Let us assume that* $l \in (1/2, 1)$, $T > 0$, *and the following hold true:*

(i) *The natural frequency* ω *follows a distribution with probability density function* $g(\omega)$, *which satisfies* $g \in L_1^{(1)}(\mathbb{R}) \bigcap L_\infty(\mathbb{R})$.

(ii) $f_0 \in \mathcal{V}^{2+l} \bigcap L_1^{(1)}(\Omega)$, *and* $\int_0^{2\pi} f_0(\theta;\omega)\, \mathrm{d}\theta \leq c_{51}/(1 + |\omega|^{2+\delta})$ *with a certain constant* $c_{51} > 0$ *and* $\delta \in (0, 1)$.

(iii) $h \in L_2(\mathbb{R}_+) \bigcap L_\infty(\mathbb{R}_+)$.

Then, there exists a certain $T_* \in (0, T]$ *and solution* $\tilde{f}(\theta, t;\omega)$ *to* Eq (2.4) *(and consequently,* Eq (2.3)*) on* $(0, T_*)$ *such that*

$$\tilde{f} \in \mathcal{V}^{3+l}(T_*).$$

Based on Theorem 5.1, we obtain the global-in-time solvability of Eq (2.4).

Theorem 5.2. *In addition to the assumptions in* Theorem 5.1*, we assume that*

(i) $\left\| \tilde{f}_0 - \bar{f} \right\|_{2+l} \leq \delta_1$ *with sufficiently small* $\delta_1 > 0$*;*

(ii) $h(0) = h'(0) = 0$*, where* $h'(t)$ *is the first derivative of* h*;*

Then, there exists a solution f *to* Eq (2.4) *(and consequently,* Eq (2.3)*) that satisfies* $\mathrm{e}^{-kt} f(\theta, t; \omega) \in \mathcal{V}_\infty^{3+l}$ *for a certain* $k > 0$*.*

6. Local-in-time solvability (proof of Theorem 5.1)

In this section, we examine the solvability of Equation (2.4) (and consequently, (2.3)). We first discuss the local-in-time solvability of Eq (2.2); that is, the proof of Theorem 5.1: We construct the successive approximation of Eq (2.4), $\{\tilde{f}_{(m)}\}_{m=0}^\infty$, which is defined for $m = 0$ by $\tilde{f}_{(0)} = \tilde{f}_0$ and $m \geq 1$ using

$$\begin{cases} \dfrac{\partial \tilde{f}_{(m)}}{\partial t} - \varepsilon \dfrac{\partial^2 \tilde{f}_{(m)}}{\partial \theta^2} = -K \dfrac{\partial}{\partial \theta}\left(F\left[\tilde{f}_{(m-1)}, \dfrac{\tilde{f}_{(m-1)}}{w_{(\delta)}} \right] \right) & \text{in } \Omega_\infty \times \mathbb{R}, \\ \tilde{f}_{(m)}^{(j,0)}(0, t; \omega) = \tilde{f}_{(m)}^{(j,0)}(2\pi, t; \omega) \ (j = 0, 1) & \text{on } \mathbb{R}_+ \times \mathbb{R}, \\ \tilde{f}_{(m)}(\theta, 0; \omega) = \tilde{f}_0(\theta; \omega) & \text{on } \Omega \times (-\infty, 0] \times \mathbb{R}. \end{cases} \tag{6.1}$$

For $T > 0$, we introduce the following notation for $m \geq 1$:

$$E_{(m)}(T; \omega) \equiv \left\| \tilde{f}_{(m)}(\cdot; \omega) \right\|_{W_2^{3+l, \frac{3+l}{2}}(\Omega_T)}, \quad K_{(m)}(T) \equiv \left\| \tilde{f}_{(m)} \right\|^2_{\left(3+l, \frac{3+l}{2}\right); T}.$$

We also define

$$E_{(0)}(\omega) \equiv \left\| \tilde{f}_0(\cdot; \omega) \right\|_{W_2^{2+l}(\Omega)}, \quad K_{(0)} \equiv \sup_{\omega \in \mathbf{R}} \left\| \tilde{f}_0(\cdot; \omega) \right\|^2_{W_2^{2+l}(\Omega)} = \left\| \tilde{f}_0 \right\|^2_{(2+l)}.$$

Owing to the elementary results for the initial and boundary value problems of the heat equation [14], we observe

$$E_{(m)}(T; \omega) \leq c_{61} \left\{ \left\| \frac{\partial}{\partial \theta} \left(F\left[\tilde{f}_{(m-1)}, \frac{\tilde{f}_{(m-1)}}{w_{(\delta)}} \right] \right) \right\|_{W_2^{1+l, \frac{1+l}{2}}(\Omega_T)} \right\} + E_0(\omega) \quad (\forall \omega \in \mathbb{R}). \tag{6.2}$$

At this point, we state the following:

Lemma 6.1. *Let* $T > 0$ *and assume that* $l \in (1/2, 1)$ *and* $\alpha \in (0, 1)$*. then for* $u \in W_2^{2+l, \frac{2+l}{2}}(\Omega_T)$*, we obtain the following estimates.*

$$\begin{aligned} &\left\| \int_0^\infty h(\tau) \mathrm{d}\tau \int_{\mathbb{R}} \frac{\mathrm{d}\omega}{w_{(\delta)}(\omega)} \int_\Omega u(\theta', t - \tau; \omega) \sin(\theta' - \theta) \mathrm{d}\theta' \right\|_{W_2^{2+l, \frac{2+l}{2}}(\Omega_T)} \\ &\qquad \leq c_{62} \phi(T) \|h\|_{W_2^{1+\alpha}(0,T)} \left(\left\| u \right\|_T + \left\| u^{(0,1)} \right\|_T \right), \end{aligned} \tag{6.3}$$

where $\phi(T)$ *is a polynomial of its argument with a degree not exceeding* $(3 - l)/2$*, and* $c_{62} > 0$ *is a constant that is independent of* T*.*

Proof. For simplicity, we introduce the following notation:

$$S(\theta,t)\equiv\int_0^\infty h(\tau)\mathrm{d}\tau\int_{\mathbb{R}}\frac{\mathrm{d}\omega'}{w_{(\delta)}(\omega')}\int_\Omega u(\phi,t-\tau;\omega')\sin(\phi-\theta)\mathrm{d}\phi.$$

We first estimate $\left\|S(\cdot,t)\right\|^2_{L_2(\Omega)}$ for each t. By an elementary variable change, it can be observed that

$$\begin{aligned}
&\left|\int_0^\infty h(\tau)\mathrm{d}\tau\int_{\mathbb{R}}\frac{\mathrm{d}\omega'}{w_{(\delta)}(\omega')}\int_\Omega u(\phi,t-\tau;\omega')\sin(\phi-\theta)\mathrm{d}\phi\right|^2\\
&\qquad=\left|\int_0^t h(t-t')\mathrm{d}\tau\int_{\mathbb{R}}\frac{\mathrm{d}\omega'}{w_{(\delta)}(\omega')}\int_\Omega u(\phi,t';\omega')\sin(\phi-\theta)\mathrm{d}\phi\right|^2\\
&\qquad\le\left(\int_0^t|h(t-t')|^2\,\mathrm{d}t'\right)\left\{\int_0^t\left|\int_{\mathbb{R}}\frac{\mathrm{d}\omega'}{w_{(\delta)}(\omega')}\int_\Omega u(\phi,t';\omega')\sin(\phi-\theta)\mathrm{d}\phi\right|^2\mathrm{d}t'\right\}\\
&\qquad\equiv J_0(t).
\end{aligned}\tag{6.4}$$

We apply the Cauchy–Schwartz inequality results in

$$\left|\int_{\mathbb{R}}\frac{\mathrm{d}\omega'}{w_{(\delta)}(\omega')}\int_\Omega u(\phi,t';\omega')\sin(\phi-\theta)\mathrm{d}\phi\right|^2\le c_{63}\left(\int_{\mathbb{R}}\frac{\mathrm{d}\omega'}{w_{(\delta)}(\omega')}\right)\left\|u(t')\right\|^2_{(0)}\le c_{64}\left\|u(t')\right\|^2_{(0)}.\tag{6.5}$$

Thus, Eqs (6.4) and (6.5) yield

$$J_0(t)\le c_{65}\|h\|^2_{L_2(0,T)}\left\|u\right\|^2_T\quad\forall t\in(0,T].$$

Next, we estimate $\|S^{(2,0)}\|_{W_2^{l,0}(\Omega_T)}$. It is worth noting that $S^{(2,0)}=-S$ holds. Thus, we obtain

$$\begin{aligned}
&S^{(2,0)}(\theta_1,t)-S^{(2,0)}(\theta_2,t)\\
&\qquad=-\int_0^t h(t-t')\mathrm{d}t'\int_{\mathbb{R}}\frac{\mathrm{d}\omega'}{w_{(\delta)}(\omega')}\int_\Omega u(\phi,t';\omega')\Big\{\sin(\phi-\theta_1)-\sin(\phi-\theta_2)\Big\}\mathrm{d}\phi.
\end{aligned}\tag{6.6}$$

By virtue of the mean value theorem, we obtain

$$\begin{aligned}
\sin(\theta'-\theta_1)-\sin(\theta'-\theta_2)&=\int_0^1\frac{\mathrm{d}}{\mathrm{d}\sigma}\sin(\sigma(\phi-\theta_1)+(1-\sigma)(\phi-\theta_2))\,\mathrm{d}\sigma\\
&=(\theta_2-\theta_1)\int_0^1\cos(\sigma(\phi-\theta_1)+(1-\sigma)(\phi-\theta_2))\,\mathrm{d}\sigma.
\end{aligned}$$

From these, we obtain

$$\left|S^{(2,0)}(\theta_1,t)-S^{(2,0)}(\theta_2,t)\right|^2\le c_{66}|\theta_1-\theta_2|^2\|h\|^2_{L_2(0,T)}\left\|u\right\|^2_T.\tag{6.7}$$

The approximation (6.7) and some elementary calculations yield

$$\|S^{(2,0)}\|^2_{W_2^{l,0}(\Omega_T)}\le c_{67}T\|h\|^2_{L_2(0,T)}\left\|u\right\|^2_T.\tag{6.8}$$

Next, we estimate $\|S^{(0,1)}\|_{W_2^{0,\frac{l}{2}}(\Omega_T)}$. By noting that

$$S^{(0,1)} = \int_0^\infty h(\tau)\mathrm{d}\tau \int_{\mathbb{R}} \mathrm{d}\omega' \int_\Omega f^{(0,1)}(\phi, t'; \omega') \sin(\phi - \theta)\, \mathrm{d}\phi,$$

we have

$$\begin{aligned} &S^{(0,1)}(\theta, t_1; \omega) - S^{(0,1)}(\theta, t_2; \omega) \\ &\quad = \int_0^{t_1} \{h(t_1 - t') - h(t_2 - t')\}\mathrm{d}t' \int_{\mathbb{R}} \frac{\mathrm{d}\omega'}{w_{(\delta)}(\omega')} \int_\Omega u^{(0,1)}(\phi, t'; \omega') \sin(\phi - \theta)\, \mathrm{d}\phi \\ &\qquad\quad + \int_{t_2}^{t_1} h(t_2 - t') \int_{\mathbb{R}} \mathrm{d}\omega' \int_\Omega u^{(0,1)}(\phi, t'; \omega') \sin(\phi - \theta)\, \mathrm{d}\phi \\ &\quad \equiv J_1 + J_2. \end{aligned}$$

Therefore, for each $\theta \in \Omega$, we obtain

$$\begin{aligned} &\int_0^T \mathrm{d}t_1 \int_0^T \frac{\left|S^{(0,1)}(\theta, t_1) - S^{(0,1)}(\theta, t_2)\right|^2}{|t_1 - t_2|^{1+l}}\, \mathrm{d}t_2 \\ &\qquad \le c_{68}\left(\int_0^T \mathrm{d}t_1 \int_0^T \frac{\left|J_1\right|^2}{|t_1 - t_2|^{1+l}}\, \mathrm{d}t_2 + \int_0^T \mathrm{d}t_1 \int_0^T \frac{\left|J_2\right|^2}{|t_1 - t_2|^{1+l}}\, \mathrm{d}t_2\right). \end{aligned} \tag{6.9}$$

To estimate the first term, we again apply the mean value theorem to obtain

$$\begin{aligned} &\int_0^{t_1} \{h(t_1 - t') - h(t_2 - t')\}\mathrm{d}t' \int_{\mathbb{R}} \frac{\mathrm{d}\omega'}{w_{(\delta)}(\omega')} \int_\Omega u^{(0,1)}(\phi, t'; \omega') \sin(\phi - \theta)\, \mathrm{d}\phi \\ &\quad = (t_2 - t_1) \int_0^{t_1} \Big(\int_0^1 h'((t_1 - t') + \sigma(t_2 - t'))\mathrm{d}\sigma\Big)\mathrm{d}t' \\ &\qquad\quad \times \int_{\mathbb{R}} \frac{\mathrm{d}\omega'}{w_{(\delta)}(\omega')} \int_\Omega u^{(0,1)}(\phi, t'; \omega') \sin(\phi - \theta)\, \mathrm{d}\phi \end{aligned}$$

Thus, we have

$$\begin{aligned} &\int_0^T \mathrm{d}t_1 \int_0^T \frac{\left|J_1\right|^2}{|t_1 - t_2|^{1+l}}\, \mathrm{d}t_2 \\ &\quad \le c_{69} \int_0^T \mathrm{d}t_1 \int_0^T \frac{|t_1 - t_2|^2}{|t_1 - t_2|^{1+l}} \left(\int_0^{t_1} \Big|\int_0^1 h'((t_1 - t') + \sigma(t_2 - t'))\mathrm{d}\sigma\Big|^2 \mathrm{d}t'\right) \\ &\qquad\qquad \times \left(\int_0^{t_1} \Big|\int_{\mathbb{R}} \frac{\mathrm{d}\omega'}{w_{(\delta)}(\omega')} \int_\Omega u^{(0,1)}(\phi, t'; \omega') \sin(\phi - \theta)\, \mathrm{d}\phi\Big|^2 \mathrm{d}t'\right) \mathrm{d}t_2 \\ &\quad \le c_{610} \|h'\|^2_{L_2(0,T)} \int_0^T \mathrm{d}t_1 \int_0^T |t_1 - t_2|^{1-l} \\ &\qquad\qquad \times \left(\int_0^{t_1} \Big|\int_{\mathbb{R}} \frac{\mathrm{d}\omega'}{w_{(\delta)}(\omega')} \int_\Omega u^{(0,1)}(\phi, t'; \omega') \sin(\phi - \theta)\, \mathrm{d}\phi\Big|^2 \mathrm{d}t'\right) \mathrm{d}t_2. \end{aligned} \tag{6.10}$$

To estimate the rightmost side of Eq (6.10) by virtue of the Cauchy–Schwartz inequality, we obtain

$$\begin{aligned}\Big|\int_{\mathbb{R}} \frac{\mathrm{d}\omega'}{w_{(\delta)}(\omega')} \int_{\Omega} u^{(0,1)}(\phi, t'; \omega') \sin(\phi - \theta)\, \mathrm{d}\phi\Big|^2 \\ &\le c_{611}\left(\int_{\mathbb{R}} \frac{\mathrm{d}\omega'}{w_{(\delta)}(\omega')}\right)\left(\sup_{\omega' \in \mathbf{R}}\left\|u^{(0,1)}(\cdot, t'; \omega')\right\|^2_{L_2(\Omega)}\right) \\ &\le c_{612}\left\| u^{(0,1)}(t')\right\|^2.\end{aligned}$$

Together with Eq (6.10), this yields

$$\begin{aligned}\int_0^T \mathrm{d}t_1 \int_0^T \frac{|J_1|^2}{|t_1 - t_2|^{1+l}}\, \mathrm{d}t_2 &\le c_{610}c_{612}\|h'\|^2_{L_2(0,T)}\left\| u^{(0,1)}\right\|^2_T \int_0^T \mathrm{d}t_1 \int_0^T |t_1 - t_2|^{1-l}\, \mathrm{d}t_2 \\ &= \frac{2c_{610}c_{612}T^{3-l}}{(2-l)(3-l)}\|h'\|^2_{L_2(0,T)}\left\| u^{(0,1)}\right\|^2_T. \end{aligned} \tag{6.11}$$

J_2 is approximated in a simpler manner using the fact that

$$\begin{aligned}\int_{t_1}^{t_2} |h(t_2 - t')|^2\, \mathrm{d}t' &\le |t_1 - t_2|^2\|h\|^2_{L_\infty(\mathbb{R}_+)} \\ &\le c_{613}T^2\|h\|^2_{W_2^{1+\alpha}(\mathbb{R}_+)}.\end{aligned}$$

Here, the Sobolev embedding theorem is applied. From these considerations, we obtain:

$$\left\|S^{(0,1)}\right\|^2_{W_2^{0,\frac{l}{2}}(\Omega_T)} \le c_{614}(T^{3-l}\|h'\|^2_{L_2(\mathbb{R})} + T^2\|h\|^2_{W_2^{1+\alpha}(\mathbb{R})})\left\| u^{(0,1)}\right\|^2_T. \tag{6.12}$$

This completes the proof of Eq (6.3). □

We also prepare the following lemmas (see, for instance, [20]).

Lemma 6.2. *Letting $l \in (1/2, 1)$, the following holds:*

(i) *For $g_1, g_2 \in W_2^{1+l}(\Omega)$ in general,*

$$\|g_1g_2\|_{W_2^{1+l}(\Omega)} \le c_{615}\|g_1\|_{W_2^{1+l}(\Omega)}\|g_2\|_{W_2^{1+l}(\Omega)}$$

holds with some constant $c_{615} > 0$.

(ii) *For $g_1 \in W_2^{1+l}(\Omega)$, $g_2 \in W_2^{l}(\Omega)$ in general,*

$$\|g_1g_2\|_{W_2^{l}(\Omega)} \le c_{616}\|g_1\|_{W_2^{1+l}(\Omega)}\|g_2\|_{W_2^{l}(\Omega)}$$

holds with some constant $c_{616} > 0$.

Lemma 6.3. *Let $l \in (1/2, 1)$, $m \ge 2$ and $m \ge k$, then for $f \in W_2^{m+l,\frac{m+l}{2}}(\Omega_\infty)$ and $g \in W_2^{k+l,\frac{k+l}{2}}(\Omega_\infty)$ in general, $fg \in W_2^{k+l,\frac{k+l}{2}}(\Omega_\infty)$ and*

$$\|fg\|_{W_2^{k+l,\frac{k+l}{2}}(\Omega_\infty)} \le c_{617}\|f\|_{W_2^{m+l,\frac{m+l}{2}}(\Omega_\infty)}\|g\|_{W_2^{k+l,\frac{k+l}{2}}(\Omega_\infty)},$$

with some constant $c_{617} > 0$.

We estimate the first term on the right side of inequality (6.2). Considering that

$$\left\|\frac{\partial}{\partial\theta}\Big(F[\tilde{f}_{(m-1)},\frac{\tilde{f}_{(m-1)}}{w_{(\delta)}}]\Big)\right\|_{W_2^{1+l,\frac{1+l}{2}}(\Omega_T)} \leq \left\|F\Big[\tilde{f}^{(1,0)}_{(m-1)},\frac{\tilde{f}_{(m-1)}}{w_{(\delta)}}\Big]\right\|_{W_2^{1+l,\frac{1+l}{2}}(\Omega_T)} + \left\|F^{(1)}\Big[\tilde{f}_{(m-1)},\frac{\tilde{f}_{(m-1)}}{w_{(\delta)}}\Big]\right\|_{W_2^{1+l,\frac{1+l}{2}}(\Omega_T)},$$

we only estimate the first term. Using Lemma 6.3, we estimate

$$\begin{aligned}&\left\|F\Big[\tilde{f}^{(1,0)}_{(m-1)},\frac{\tilde{f}_{(m-1)}}{w_{(\delta)}}\Big]\right\|_{W_2^{1+l,\frac{1+l}{2}}(\Omega_T)}\\ &\quad\leq \left\|\tilde{f}^{(1,0)}_{(m-1)}(\omega)\right\|_{W_2^{1+l,\frac{1+l}{2}}(\Omega_T)}\left\|\int h(\tau)\mathrm{d}\tau\int_{\mathbb{R}}\frac{\mathrm{d}\omega'}{w_{(\delta)}(\omega')}\int_{\Omega}\tilde{f}_{(m-1)}(\theta',t-\tau;\omega')\sin(\theta'-\theta)\mathrm{d}\theta'\right\|_{W_2^{2+l,\frac{2+l}{2}}(\Omega_T)},\end{aligned} \tag{6.13}$$

then, by applying Lemma 6.1, we obtain

$$\left\|F\Big[\tilde{f}^{(1,0)}_{(m-1)},\frac{\tilde{f}_{(m-1)}}{w_{(\delta)}}\Big](\omega)\right\|_{W_2^{1+l,\frac{1+l}{2}}(\Omega_T)} \leq c_{618}\phi(T)\|\tilde{f}_{(m-1)}(\omega)\|_{W_2^{1+l,\frac{1+l}{2}}(\Omega_T)}\|\tilde{f}_{(m-1)}(\omega)\|_{(2+l,\frac{2+l}{2});T}. \tag{6.14}$$

We can then estimate the supremum of the function with respect to ω, as follows:

$$\sup_{\omega\in\mathbf{R}}\left\|F\Big[\tilde{f}^{(1,0)}_{(m-1)},\frac{\tilde{f}_{(m-1)}}{w_{(\delta)}}\Big](\omega)\right\|^2_{W_2^{1+l,\frac{1+l}{2}}(\Omega_T)} \leq c_{618}\phi(T)\big\|\tilde{f}_{(m-1)}\big\|^2_{(1+l,\frac{1+l}{2});T}\big\|\tilde{f}_{(m-1)}\big\|^2_{(2+l,\frac{2+l}{2});T}.$$

Thus, we integrate the square of Eq (6.2) with respect to t and take the supremum with respect to ω to obtain the following:

$$\begin{aligned}K_{(m)}(T) &\leq c_{619}\Big\{\phi(T)\big\|\tilde{f}_{(m-1)}\big\|^2_{(2+l,\frac{2+l}{2});T}\big\|f_{(m-1)}\big\|^2_{(2+l,\frac{2+l}{2});T}+K_0\Big\}\\ &\leq c_{619}(\eta + C_\eta T)\Big\{K_{(m-1)}(T) + (K_{(m-1)}(T))^2 + K_0\Big\},\end{aligned} \tag{6.15}$$

where $\eta > 0$ is a constant that is determined later and C_η is a constant that is decreasingly dependent on η.

We now inductively demonstrate that $\{K_m(T)\}_{m\geq 0}$ is bounded in $\mathcal{V}^{3+l}$. Consider a constant M_{69} that satisfies

$$K_0 < M_{620}.$$

We also assume that, for a certain $m-1$,

$$K_{(m-1)}(T) < M_{620}$$

holds. By virtue of Eq (6.15), we initially consider η so that

$$c_{613}\eta(2M_{620} + M_{620}^2) < M_{620}.$$

Next, let T_{621} be sufficiently small so that

$$c_{619}C_\eta T_{621}(2M_{620}+M_{620}^2) < M_{620} - c_{619}\eta(2M_{620}+M_{620}^2).$$

Subsequently, based on Eq (6.15), we obtain $K_{(m)}(T_{621}) < M_{619}$. By induction, $\{K_{(m)}(T_{621})\}_m$ is a bounded sequence in $\mathcal{V}^{3+l}_{T_{621}}$.

Next, we demonstrate that $\tilde{f}_{(m)}$ converges to some element in $\mathcal{V}^{3+l}_{T_{622}}$ with a certain $T_{622} \in (0, T_{621}]$. To demonstrate this, we define

$$\tilde{\tilde{f}}_{(m)} = \tilde{f}_{(m)} - \tilde{f}_{(m-1)}$$

for $m \geq 1$ and subtract Eq (6.1) with m replaced by $m+1$ from itself. This satisfies

$$\begin{cases} \dfrac{\partial \tilde{\tilde{f}}_{(m)}}{\partial t} - \varepsilon \dfrac{\partial^2 \tilde{\tilde{f}}_{(m)}}{\partial \theta^2} = -K\left[\dfrac{\partial}{\partial\theta}\Big(F\Big[\tilde{f}_{(m)}, \dfrac{\tilde{f}_{(m)}}{w_{(\delta)}}\Big]\Big) - \dfrac{\partial}{\partial\theta}\Big(F\Big[\tilde{f}_{(m-1)}, \dfrac{\tilde{f}_{(m-1)}}{w_{(\delta)}}\Big]\Big)\right], \\ \tilde{\tilde{f}}^{(j,0)}_{(m)}(0,t;\omega) = \tilde{\tilde{f}}^{(j,0)}_{(m)}(2\pi,t;\omega) \ (j=0,1), \\ \tilde{\tilde{f}}_{(m)}(\theta,0;\omega) = 0. \end{cases} \tag{6.16}$$

Note that

$$\begin{aligned} \frac{\partial}{\partial\theta}\left(F\Big[\tilde{f}_{(m)}, \frac{\tilde{f}_{(m)}}{w_{(\delta)}}\Big]\right) - \frac{\partial}{\partial\theta}\left(F\Big[\tilde{f}_{(m-1)}, \frac{\tilde{f}_{(m-1)}}{w_{(\delta)}}\Big]\right) &= F\Big[\tilde{\tilde{f}}^{(1,0)}_{(m)}, \frac{\tilde{f}_{(m)}}{w_{(\delta)}}\Big] + F\Big[\tilde{f}^{(1,0)}_{(m-1)}, \frac{\tilde{\tilde{f}}_{(m)}}{w_{(\delta)}}\Big] \\ &\quad + F^{(1)}\Big[\tilde{\tilde{f}}_{(m)}, \frac{\tilde{f}_{(m)}}{w_{(\delta)}}\Big] + F^{(1)}\Big[\tilde{f}_{(m-1)}, \frac{\tilde{\tilde{f}}_{(m)}}{w_{(\delta)}}\Big] \\ &\equiv \sum_{j=1}^{4} L'_j(\theta,t;\omega). \end{aligned}$$

Using multiplicative and interpolation inequalities, we estimate the first term on the righthand side. By introducing

$$S_{(m)}(\theta,t) \equiv \int_0^t h(\tau)\mathrm{d}\tau \int_{\mathbb{R}} \mathrm{d}\omega' \int_\Omega \tilde{f}_{(m)}(\theta', t-\tau;\omega')\sin(\theta'-\theta)\,\mathrm{d}\theta',$$

we can write

$$F\left[\tilde{\tilde{f}}^{(1,0)}_{(m)}, \frac{\tilde{f}_{(m)}}{w_{(\delta)}}\right] = \tilde{\tilde{f}}^{(1,0)}_{(m)}(\theta,t;\omega)S_{(m)}(\theta,t).$$

Using Lemma 6.3 again, we can observe that

$$\begin{aligned} \left\|L'_1(\cdot;\omega)\right\|^2_{W_2^{1+l,\frac{1+l}{2}}(\Omega_T)} &\leq \left\|\tilde{\tilde{f}}_{(m)}(\cdot;\omega)\right\|^2_{W_2^{1+l,\frac{1+l}{2}}(\Omega_T)} \left\|S_{(m)}\right\|^2_{W_2^{2+l,\frac{2+l}{2}}(\Omega_T)} \\ &\leq c_{623}\phi(T)\left\|\tilde{\tilde{f}}_{(m)}(\cdot;\omega)\right\|^2_{W_2^{1+l,\frac{1+l}{2}}(\Omega_T)} \left[\!\left[\tilde{f}_{(m)}\right]\!\right]^2_{(2+l,\frac{2+l}{2});T} \\ &\leq c_{623}\phi(T)(\eta+C_\eta T)^2\left\|\tilde{\tilde{f}}_{(m)}(\cdot;\omega)\right\|^2_{W_2^{2+l,\frac{2+l}{2}}(\Omega_T)} \left[\!\left[\tilde{f}_{(m)}\right]\!\right]^2_{(2+l,\frac{2+l}{2});T}, \end{aligned}$$

where $\phi(\cdot)$ is a polynomial of its argument. Thus, we obtain

$$\sup_{\omega\in\mathbf{R}}\left\|L_1'(\cdot;\omega)\right\|^2_{W_2^{1+l,\frac{1+l}{2}}(\Omega_T)} \le c_{623}\phi(T)(\eta + C_\eta T)^2 \left\|\tilde{\tilde{f}}_{(m)}\right\|^2_{(2+l,\frac{2+l}{2});T}\left\|\tilde{f}_{(m)}\right\|^2_{(2+l,\frac{2+l}{2});T}.$$

The other terms $L_j'(\cdot;\omega)$ $(j = 2, 3, 4)$ are estimated similarly, and we obtain

$$\sup_{\omega\in\mathbb{R}}\left|L_j'(\omega)\right|^2 \le c_{623}(\eta + C_\eta T)^2\phi(T)\Big(\left\|\tilde{f}_{(m-1)}\right\|^2_{(2+l,\frac{2+l}{2});T} + \left\|\tilde{f}_{(m)}\right\|^2_{(2+l,\frac{2+l}{2});T}\Big)\left\|\tilde{\tilde{f}}_{(m)}\right\|^2_{(3+l,\frac{3+l}{2});T}. \tag{6.17}$$

We introduce the following notation:

$$\widetilde{K}_{(m)}(T) \equiv \sup_{\omega\in\mathbb{R}}\left\|\tilde{f}_{(m)}(\omega)\right\|_{W_2^{3+l,\frac{3+l}{2}}(\Omega_T)} = \left\|\tilde{f}_{(m)}\right\|_{(3+l,\frac{3+l}{2});T}.$$

Then, we obtain

$$\widetilde{K}_{(m)}(t) \le c_{623}\Big(\eta' + C_{\eta'}t\Big)\phi(t)\left\{1 + \sum_{j=m-1}^{m} K_{(j)}(t)\right\}\widetilde{K}_{(m)}(t).$$

Let η' be sufficiently small so that the following holds:

$$c_{617}\eta'\phi(T_{622})(1 + 2M_{620}) < 1,$$

then, for this η', we take a sufficiently small $t = T_{624} \in (0, T_{622}]$ so that

$$c_{623}C_{\eta'}\phi(T_{622})\{1 + 2M_{619}\} < 1 - c_{623}\eta'\phi(T_{621})$$

holds. We can now observe that

$$\widetilde{K}_{(m+1)}(t) < r\widetilde{K}_{(m)}(t),$$

with $r = c_{623}\Big(\eta' + C_{\eta'}T_{624}\Big)\phi(T_{624})\Big(1 + 2M_{620}\Big) \in (0, 1)$. We can conclude that $\{\tilde{f}_{(m)}\}_{m=1}^{\infty}$ forms a Cauchy sequence in $\mathcal{V}^{3+l}_{T_{624}}$, and the limitations

$$f = \lim_{m\to+\infty}\tilde{f}_{(m)}$$

exist in the same function space. This is the desired solution. As the uniqueness of this solution can be verified in a similar manner, we omit the detailed discussion. This completes the proof of Theorem 5.1.

7. Global-in-time solvability

We now prove Theorem 5.2. We examine the temporal behavior of the solution around a trivial stationary solution $\bar{f} = g(\omega)/2\pi$. For this purpose, we subtract $\bar{f}$ from Eq (2.4) to obtain the problem for $\check{f} \equiv \tilde{f} - \bar{f}$, which satisfies

$$\begin{cases} \dfrac{\partial\check{f}}{\partial t} - \varepsilon\dfrac{\partial^2\check{f}}{\partial\theta^2} + K\dfrac{\partial}{\partial\theta}\left(F\Big[\check{f} + \bar{f}, \dfrac{\check{f}}{w_{(\delta)}}\Big]\right) = 0 \quad \text{in } \Omega_\infty\times\mathbb{R}, \\ \check{f}^{(j,0)}(0, t;\omega) = \check{f}^{(j,0)}(2\pi, t;\omega)\ (j = 0, 1) \quad \text{on } \mathbb{R}_+\times\mathbb{R}, \\ \check{f}(\theta, t;\omega) = \check{f}_0(\theta;\omega) \equiv \tilde{f}_0 - \bar{f} \quad \text{on } \Omega\times(-\infty, 0]\times\mathbb{R}. \end{cases} \tag{7.1}$$

We begin with the linear problem that corresponds to Eq (7.1):

$$\begin{cases} \dfrac{\partial \breve{\breve{f}}}{\partial t} - \varepsilon \dfrac{\partial^2 \breve{\breve{f}}}{\partial \theta^2} + K \dfrac{\partial}{\partial \theta}\Big(F\Big[\bar{f}, \dfrac{\breve{\breve{f}}}{w_{(\delta)}}\Big]\Big) = G \quad \text{in } \Omega_\infty \times \mathbb{R}, \\ \breve{\breve{f}}^{(j,0)}(0,t;\omega) = \breve{\breve{f}}^{(j,0)}(2\pi,t;\omega)\ (j=0,1) \quad \text{on } \mathbb{R}_+ \times \mathbb{R}, \\ \breve{\breve{f}}(\theta,t;\omega) = \breve{\breve{f}}_0(\theta,t;\omega) \quad \text{on } \Omega \times (-\infty,0] \times \mathbb{R}. \end{cases} \tag{7.2}$$

We introduce the new variable $v(\theta,t;\omega) \equiv \mathrm{e}^{-kt}\breve{\breve{f}}(\theta,t;\omega)$, where $\breve{\breve{f}}$ is the solution of Eq (7.2), which solves

$$\begin{cases} \dfrac{\partial v}{\partial t} + kv - \varepsilon \dfrac{\partial^2 v}{\partial \theta^2} \\ \qquad + K\bar{f}\dfrac{\partial}{\partial \theta}\Big(\displaystyle\int_0^\infty h(\tau)\mathrm{e}^{-k\tau}\mathrm{d}\tau \int_{\mathbb{R}} \mathrm{d}\omega' \int_\Omega \dfrac{v(\theta', t-\tau;\omega')}{w_{(\delta)}(\omega')} \sin(\theta'-\theta)\,\mathrm{d}\theta' \Big) \\ \qquad = e^{-kt}G \\ \qquad\qquad \text{in } \Omega_\infty \times \mathbb{R}, \\ v^{(j,0)}(0,t;\omega) = v^{(j,0)}(2\pi,t;\omega)\ (j=0,1) \quad \text{on } \mathbb{R}_+ \times \mathbb{R}, \\ v(\theta,t;\omega) = \breve{\breve{f}}_0 \quad \text{on } \Omega \times (-\infty,0] \times \mathbb{R}. \end{cases} \tag{7.3}$$

7.1. A-priori estimate

We now obtain an *a-priori* estimate of the solution under a small initial data size. We first approximate the linear problem and then discuss the nonlinear model.

7.2. Linear problem

First, we consider the linearized problem of Eq (7.3) with zero initial data.

$$\begin{cases} \dfrac{\partial v}{\partial t} + kv - \varepsilon \dfrac{\partial^2 v}{\partial \theta^2} \\ \qquad + K\bar{f}\dfrac{\partial}{\partial \theta}\Big(\displaystyle\int_0^\infty h(\tau)\mathrm{e}^{-k\tau}\mathrm{d}\tau \int_{\mathbb{R}} \mathrm{d}\omega' \int_\Omega \dfrac{v(\theta', t-\tau;\omega')}{w_{(\delta)}(\omega')} \sin(\theta'-\theta)\,\mathrm{d}\theta' \Big) \\ \qquad = e^{-kt}G \\ \qquad\qquad \text{in } \Omega_\infty \times \mathbb{R}, \\ v^{(j,0)}(0,t;\omega) = v^{(j,0)}(2\pi,t;\omega)\ (j=0,1) \quad \text{on } \mathbb{R}_+ \times \mathbb{R}, \\ v(\theta,t;\omega) = 0 \quad \text{on } \Omega \times (-\infty,0] \times \mathbb{R}. \end{cases} \tag{7.4}$$

Lemma 7.1. *Let $T > 0$ be an arbitrary number, and $G(\theta,t;\omega) \in \mathcal{V}_\infty^{(1+l)}$ be periodic with respect to $\theta \in \Omega$. In addition, suppose that the assumptions on $h(t)$ in* Theorem 5.1 *and* Theorem 5.2 *holds, then there exists a solution to* Eq (7.4) *for* $(0,+\infty)$ *that satisfies the estimate in the form of*

$$\big\|v\big\|_{(3+l,\frac{3+l}{2});T} \le c_{71}\|e^{-kt}G\|_{(1+l,\frac{1+l}{2});T}. \tag{7.5}$$

Proof. First, we consider the problem with vanishing initial data.

$$\begin{cases} \dfrac{\partial v}{\partial t} + kv - \varepsilon \dfrac{\partial^2 v}{\partial \theta^2} \\ \qquad + K\bar{f}\dfrac{\partial}{\partial \theta}\Big(\displaystyle\int_0^\infty h(\tau)\mathrm{e}^{-k\tau}\mathrm{d}\tau \int_{\mathbb{R}} \mathrm{d}\omega' \int_{\Omega} \frac{v(\theta', t-\tau; \omega')}{w_{(\delta)}(\omega')} \sin(\theta' - \theta)\, \mathrm{d}\theta' \Big) \\ \qquad = e^{-kt} G \quad \text{in } \Omega_\infty \times \mathbb{R}, \\ v^{(j,0)}(0, t; \omega) = v^{(j,0)}(2\pi, t; \omega)\ (j = 0, 1) \quad \text{on } \mathbb{R}_+ \times \mathbb{R}, \\ v(\theta, t; \omega) = 0 \quad \text{on } \Omega \times (-\infty, 0] \times \mathbb{R}. \end{cases} \tag{7.6}$$

We now expand v and G using the Fourier series with respect to θ as follows:

$$v(\theta, t; \omega) = \sum_{n=-\infty}^{+\infty} \tilde{a}_n(t; \omega)\mathrm{e}^{\mathrm{i}n\theta}, \quad G(\theta, t; \omega) = \sum_{n=-\infty}^{+\infty} b_n(t; \omega)\mathrm{e}^{\mathrm{i}n\theta},$$

then, Eq (7.6) becomes

$$\sum_n \frac{\partial \tilde{a}_n}{\partial t}\mathrm{e}^{\mathrm{i}n\theta} + \sum_n (k + \varepsilon n^2)\tilde{a}_n \mathrm{e}^{\mathrm{i}n\theta}$$
$$- \frac{Kg(\omega)}{2}\Big[\int_0^{+\infty} h(\tau)e^{-k\tau}\mathrm{d}\tau \int_{\mathbb{R}} \frac{1}{w_{(\delta)}(\omega')} \Big\{ \tilde{a}_1(t-\tau; \omega')\mathrm{e}^{\mathrm{i}\theta} + \tilde{a}_{-1}(t-\tau; \omega')\mathrm{e}^{-\mathrm{i}\theta} \Big\} \mathrm{d}\omega' \Big] = \sum_n b_n \mathrm{e}^{-kt + \mathrm{i}\theta}.$$

A comparison of the coefficients of $\mathrm{e}^{\mathrm{i}\theta}$ yields:

$$\frac{\partial \tilde{a}_1}{\partial t} + (k + \varepsilon)\tilde{a}_1 - \frac{Kg(\omega)}{2} \int_0^{+\infty} h(\tau)e^{-k\tau}\mathrm{d}\tau \int_{\mathbb{R}} \frac{\tilde{a}_1(t-\tau; \omega')}{w_{(\delta)}(\omega')}\, \mathrm{d}\omega' = b_1 \mathrm{e}^{-kt}, \tag{7.7}$$

$$\frac{\partial \tilde{a}_n}{\partial t} + (k + \varepsilon n^2)\tilde{a}_n = b_n \mathrm{e}^{-kt} \quad (n \pm 2, \pm 3, \ldots). \tag{7.8}$$

Next, we apply a transform similar to that of Beale [1] to Eqs (7.7) and (7.8).

$$\mathcal{F}[f] \equiv \hat{f}(\xi) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\xi t} f(t)\, \mathrm{d}t.$$

Note that the following equality holds:

$$\mathcal{F}\Big[\int_{\mathbb{R}} \tilde{a}_1(t-s)g(s)ds \Big] = \int_{\mathbb{R}} e^{-\mathrm{i}\xi t}\Big\{ \int_{\mathbb{R}} \tilde{a}_1(t-s)g(s)ds \Big\}\mathrm{d}t = \int_{\mathbb{R}}\Big(\int_{\mathbb{R}} e^{-\mathrm{i}\xi(t-s)}\tilde{a}_1(t-s)dt \Big) e^{-\mathrm{i}\xi s} g(s)ds,$$

then from the assumption that $\tilde{a}_n\big|_{t=0} = 0$, we have

$$(\mathrm{i}\xi + k + \varepsilon)\hat{a}_{\pm 1}(\xi; \omega) - \frac{Kg(\omega)}{2} \int_{\mathbb{R}} \frac{\hat{a}_{\pm 1}(\xi; \omega')\hat{h}(\xi - \mathrm{i}k)}{w_{(\delta)}(\omega')}\, \mathrm{d}\omega' = \hat{b}_{\pm 1}(\xi - \mathrm{i}k; \omega), \tag{7.9}$$

$$(\mathrm{i}\xi + k + \varepsilon n^2)\hat{a}_n(\tau; \omega) = \hat{b}_n(\xi - \mathrm{i}k; \omega) \quad (n \neq \pm 1). \tag{7.10}$$

Now, we introduce a region $D_{(+)} \equiv \{z \in \mathbb{C} \big| \mathrm{Re}(z) > 0\}$ and $\lambda \in D_{(+)}$. For Eq (7.9), by letting $\xi = -\mathrm{i}\lambda$, we obtain

$$(\lambda + k + \varepsilon)\hat{a}_1(-\mathrm{i}\lambda; \omega) - \frac{Kg(\omega)}{2}\hat{h}(-\mathrm{i}(\lambda + k)) \int \frac{\hat{a}_1(-\mathrm{i}\lambda; \omega')}{w_{(\delta)}(\omega')} \mathrm{d}\omega' = \hat{b}_1(-\mathrm{i}(\lambda + k); \omega).$$

Multiplying this by $\overline{(\hat{a}_1(-\mathrm{i}\lambda; \omega))}$ results in

$$(\lambda + k + \varepsilon)\left|\hat{a}_1(-\mathrm{i}\lambda; \omega)\right|^2 - \frac{Kg(\omega)}{2}\hat{h}(-\mathrm{i}(\lambda + k))\overline{\hat{a}_1(-\mathrm{i}\lambda; \omega)} \int \frac{\hat{a}_1(-\mathrm{i}\lambda; \omega')}{w_{(\delta)}(\omega')} \mathrm{d}\omega'$$
$$= \overline{\hat{a}_1(-\mathrm{i}\lambda; \omega)}\hat{b}_1(-\mathrm{i}(\lambda + k); \omega).$$

From this, we obtain

$$(\lambda + k + \varepsilon)\left|\hat{a}_1(-\mathrm{i}\lambda; \omega)\right|^2 \le \frac{1}{2}\left[\left|\hat{a}_1(-\mathrm{i}\lambda; \omega)\right|^2 + \left|\frac{Kg(\omega)}{2}\hat{h}(-\mathrm{i}(\lambda + k))\left(\int_{\mathbb{R}} \frac{\hat{a}_1(-\mathrm{i}\lambda; \omega')}{w_{(\delta)}(\omega')} \mathrm{d}\omega'\right)\right|^2\right],$$

then by taking the supremum with respect to ω, we obtain

$$(\lambda + k + \varepsilon) \sup_{\omega}\left|\hat{a}_1(-\mathrm{i}\lambda; \omega)\right|^2 \le \frac{1}{2} \sup_{\omega}\left|\hat{a}_1(-\mathrm{i}\lambda; \omega)\right|^2 + \frac{K^2}{4} \sup_{\omega} |g(\omega)||\hat{h}(-\mathrm{i}(\lambda + k))| \sup_{\omega}\left|\hat{a}_1(-\mathrm{i}\lambda; \omega)\right|^2$$
$$+ \sup_{\omega}\left|\hat{a}_1(-\mathrm{i}\lambda; \omega)\right| \sup_{\omega}\left|\hat{b}_1(-\mathrm{i}(\lambda + k); \omega)\right|.$$

By estimating $\frac{K^2}{4} \sup_{\omega} |g(\omega)||\hat{h}(-\mathrm{i}(\lambda + k))| \le \tilde{c}_0$ and letting $k > 0$ be sufficiently large, we have

$$\left(\lambda + k + \varepsilon - \tilde{c}_0 - \frac{1}{2}\right) \sup_{\omega}\left|\hat{a}_1(-\mathrm{i}\lambda; \omega)\right|^2 \le \sup_{\omega}\left|\hat{a}_1(-\mathrm{i}\lambda; \omega)\right| \sup_{\omega}\left|\hat{b}_1(-\mathrm{i}(\lambda + k); \omega)\right|,$$

which yields

$$\sup_{\omega}\left|\hat{a}_1(-\mathrm{i}\lambda; \omega)\right| \le \frac{\sup_{\omega}\left|\hat{b}_1(-\mathrm{i}(\lambda + k); \omega)\right|}{(\lambda + k + \varepsilon - \tilde{c}_0)}. \tag{7.11}$$

This implies that

$$(1 + |\lambda|^{\frac{3+l}{2}}) \sup_{\omega}\left|\hat{a}_1(-\mathrm{i}\lambda; \omega)\right| \le (1 + |\lambda|^{\frac{1+l}{2}}) \sup_{\omega}\left|\hat{b}_1(-\mathrm{i}(\lambda + k); \omega)\right|. \tag{7.12}$$

For $n \neq \pm 1$, we have

$$\hat{a}_n(\xi; \omega) = \frac{\hat{b}_n(\xi; \omega)}{\mathrm{i}\xi + k + \varepsilon n^2}.$$

By setting $\xi = -\mathrm{i}\lambda$ to $\lambda = \sigma_0 + \mathrm{i}\sigma_1$ and integrating with respect to σ_1, we obtain:

$$\int_{\mathbb{R}}\left|\hat{a}_n(-\mathrm{i}\lambda; \omega)\right|^2 \mathrm{d}\sigma_1 = \int_{\mathbb{R}} \frac{|\hat{b}_n(-\mathrm{i}\lambda; \omega)|^2}{|\lambda + k + \varepsilon n^2|^2} \mathrm{d}\sigma_1.$$

As $|\lambda + k + \varepsilon n^2| \geq k + \varepsilon n^2$,

$$\frac{|\lambda + k + \varepsilon n^2|}{1 + n^2} \geq c_{72}$$

holds with c_{72} independent of n. Therefore, for an arbitrary $\alpha \in \mathbb{R}$,

$$\begin{aligned}\int_{\mathbb{R}} (1+n^2)^\alpha \left|\hat{a}_n(-\mathrm{i}\lambda;\omega)\right|^2 \mathrm{d}\sigma_1 &= \int_{\mathbb{R}} \frac{(1+n^2)^\alpha |\hat{b}_n(-\mathrm{i}\lambda;\omega)|^2}{|\lambda + k + \varepsilon n^2|^2} \mathrm{d}\sigma_1 \\ &= \int_{\mathbb{R}} \frac{(1+n^2)^\alpha |\hat{b}_n(-\mathrm{i}(\eta - k);\omega)|^2}{|\eta + \varepsilon n^2|^2} \mathrm{d}\eta_1,\end{aligned}$$

where $\eta = \lambda + k$. From these, we eventually obtain

$$|\hat{a}_n(-\mathrm{i}\lambda;\omega)| \leq \frac{|\hat{b}_n(-\mathrm{i}\lambda;\omega)|}{\lambda + k + \varepsilon n^2} \quad (n \neq \pm 1).$$

Next, by multiplying (7.11) by $|\lambda|^{3+l}$, we obtain

$$\begin{aligned}|\lambda|^{3+l} \sup_{\omega\in\mathbb{R}} |\hat{a}_1(-\mathrm{i}\lambda;\omega)|^2 &\leq \frac{|\lambda|^{3+l} \sup_\omega \hat{b}_1(-\mathrm{i}(\lambda+k);\omega)}{|\lambda + k + \varepsilon - \tilde{c}_0|^2} \\ &\leq |\lambda|^{1+l} \sup_{\omega\in\mathbb{R}} |\hat{b}_1(-\mathrm{i}(\lambda+k);\omega)|^2.\end{aligned}$$

Thus, we have

$$\begin{aligned}&(1 + n^{3+l} + |\lambda|^{\frac{3+l}{2}}) \sup_\omega \left|\hat{a}_n(-\mathrm{i}\lambda;\omega)\right| \\ &\quad \leq (1 + n^{1+l} + |\lambda|^{\frac{1+l}{2}}) \sup_\omega \left|\hat{b}_n(-\mathrm{i}(\lambda+k);\omega)\right| \quad \forall \lambda \in D_{(+)}\ (n = \pm 1, \pm 2, \ldots).\end{aligned} \tag{7.13}$$

Let us denote $\lambda = \sigma_0 + \mathrm{i}\sigma_1$ with $\sigma_j \in \mathbb{R}$ $(j = 0, 1)$, then $\hat{b}_1(-\mathrm{i}(\lambda+k);\omega) = \int_{\mathbb{R}} e^{-\lambda t} b_1(t;\omega)\,\mathrm{d}t$ is the Fourier transform of $e^{-\sigma_0 t} b_1(t;\omega)$ with respect to σ_1. We obtain

$$\begin{aligned}\hat{b}_1(-\mathrm{i}(\lambda+k);\omega) &= \int_{\mathbb{R}} \mathrm{e}^{-(\lambda+k)t} b_1(t;\omega)\,\mathrm{d}t \\ &= \int_{\mathbb{R}} \mathrm{e}^{-kt} \mathrm{e}^{-(\sigma_0 + \mathrm{i}\sigma_1)t} b_1(t;\omega)\,\mathrm{d}t \\ &= \mathcal{F}[\mathrm{e}^{-(\sigma_0+k)t} b_1(t;\omega)].\end{aligned}$$

Thus, by virtue of the Plancherel theorem, we obtain

$$\int_{\mathbb{R}} \left|\hat{b}_1(-\mathrm{i}(\lambda+k);\omega)\right|^2 \mathrm{d}\sigma_1 = \int_{\mathbb{R}} \left|\mathrm{e}^{-(\sigma_0+k)t} b_1(t;\omega)\right|^2 \mathrm{d}t. \tag{7.14}$$

Note that the righthand side of Eq (7.12) is finite for all $\sigma_0 \in \mathbb{R}$. By letting $\sigma_0 \to 0$ there, the right side of Eq (7.12) tends to

$$\int_{\mathbb{R}} \left|\mathrm{e}^{-kt} b_1(t;\omega)\right|^2 \mathrm{d}t = \left\|\mathrm{e}^{-kt} b_1\right\|^2_{L_2(0,+\infty)}.$$

Likewise, we obtain

$$\int_{\mathbb{R}} \left|\hat{a}_1(-\mathrm{i}\lambda;\omega)\right|^2 \,\mathrm{d}\sigma_1 = \int_{\mathbb{R}} \left|\mathrm{e}^{-\sigma_0 t} a_1(t;\omega)\right|^2 \,\mathrm{d}t \to \int_{\mathbb{R}} \left|a_1(t;\omega)\right|^2 \,\mathrm{d}t \quad (\sigma_0 \to 0).$$

Thus, if we take a sufficiently large k, we have

$$\left\| v \right\|_{(m,\frac{m}{2});\infty} \leq \left\| \mathrm{e}^{-kt} G \right\|_{(m,\frac{m}{2});\infty}.$$

The uniqueness of the solution is guaranteed by construction; from Eqs (7.9) and (7.10), it is obvious that the solution $\{a_n\}$ is uniquely determined. Thus, the proof is complete. □

7.3. Nonlinear problem

Now, we consider problem (7.1). We use the following notations:

$$L\breve{f} \equiv -k\breve{f} + \varepsilon \frac{\partial^2 \breve{f}}{\partial \theta^2} + K\bar{f}\frac{\partial}{\partial \theta}\Big(\int_0^\infty h(s)\mathrm{e}^{-ks}\mathrm{d}s \int_{\mathbb{R}} \mathrm{d}\omega' \int_{\Omega} \frac{\breve{f}(\theta', t-s;\omega')}{w_{(\delta)}(\omega')} \sin(\theta'-\theta)\,\mathrm{d}\theta'\Big),$$

$$\mathcal{F}[\breve{f}] \equiv -K\mathrm{e}^{kt}\frac{\partial}{\partial \theta}\Big[\breve{f}(\theta,t;\omega)\int_0^\infty h(\tau)\mathrm{e}^{-ks}\mathrm{d}s \int_{\mathbb{R}} \frac{\mathrm{d}\omega'}{w_{(\delta)}(\omega')} \int_{\Omega} \breve{f}(\theta', t-s;\omega')\sin(\theta'-\theta)\mathrm{d}\theta'\Big],$$

then, the problem is expressed as

$$\mathcal{A}\breve{f} = \mathcal{F}[\breve{f}], \tag{7.15}$$

where $\mathcal{A}$ is a linear operator defined on $\mathcal{V}_\infty^{3+l}$, which associates $\breve{f}$ with $\frac{\partial \breve{f}}{\partial t} - L\breve{f}$ under the periodic boundary conditions. From Lemma 6.3, we obtain

$$\left\| \mathcal{F}[\breve{f}] \right\|_{(1+l,\frac{1+l}{2});\infty} \leq c_{73} \left\| \breve{f} \right\|_{(3+l,\frac{3+l}{2});\infty} \tag{7.16}$$

for v in a bounded set of $\mathcal{V}_\infty^{3+l}$. Similarly, it is apparent that for $\breve{f}_j \in \overset{\circ}{\mathcal{V}}{}_\infty^{3+l}$ $(j = 1,2)$,

$$\left\| \mathcal{F}[\breve{f}_1] - \mathcal{F}[\breve{f}_2] \right\|_{(1+l,\frac{1+l}{2});\infty} \leq c_{74} \left\| \breve{f}_1 - \breve{f}_2 \right\|_{(3+l,\frac{3+l}{2});\infty} \sum_{j=1}^{2} \left\| \breve{f}_j \right\|_{(3+l,\frac{3+l}{2});\infty}. \tag{7.17}$$

To solve Eq (7.15) iteratively, we first determine $\breve{f}^{(0)} \in \mathcal{V}_\infty^{3+l}$, which satisfies Eq (7.3) at $t = 0$ and

$$\left\| \breve{f}^{(0)} \right\|_{(3+l,\frac{3+l}{2});\infty} \leq c_{75} \|\breve{f}_0\|_{W_2^{2+l}(\Omega)}.$$

This is obtained by tracing the classical argument ([14], Theorem IV. 4.3). Next, we rewrite the problem for the new variable $\breve{f}^{(1)} \equiv \breve{f} - \breve{f}^{(0)}$:

$$\mathcal{A}\breve{f}^{(1)} = \mathcal{F}[\breve{f}^{(0)} + \breve{f}^{(1)}] - \mathcal{A}\breve{f}^{(0)}. \tag{7.18}$$

If $\breve{f}^{(1)} \in \overset{\circ}{\mathcal{V}}{}_\infty^{3+l}$, using our method of constructing $\breve{f}^{(0)}$, $\mathcal{A}\breve{f}^{(0)} = \mathcal{F}[\breve{f}^{(0)}] = \mathcal{F}[\breve{f}^{(0)} + \breve{f}^{(1)}]$ at $t = 0$. Thus, the right-hand side of Equation (7.18) belongs to $\overset{\circ}{\mathcal{V}}{}_\infty^{1+l}$. Let $\mathcal{A}_0$ be a solution operator of Lemma 7.1 for a linear problem with zero initial data. Then, by virtue of Lemma 7.1, if

$$\breve{f}^{(1)} = \mathcal{A}_0^{-1}\Big[\mathcal{F}[\breve{f}^{(0)} + \breve{f}^{(1)}] - \mathcal{A}\breve{f}^{(0)}\Big], \tag{7.19}$$

the above assumption is satisfied because Lemma 7.1 indicates that $\mathcal{A}_0^{-1}$ is a bounded operator from $\overset{\circ}{\mathcal{V}}{}^{1+l}_\infty$ to $\overset{\circ}{\mathcal{V}}{}^{3+l}_\infty$. To demonstrate the solvability of Eq (7.19), we define a map

$$\mathcal{M}[\breve{f}^{(1)}] \equiv \mathcal{A}_0^{-1}\Big[\mathcal{F}[\breve{f}^{(0)} + \breve{f}^{(1)}] - \mathcal{A}\breve{f}^{(0)}\Big],$$

and show that it has a fixed point assuming that

$$\big\|\breve{f}_0\big\|_{(1+l)} \leq \delta_0$$

with sufficiently small $\delta_0 > 0$. We obtain

$$\big\|\breve{f}^{(0)}\big\|_{(3+l,\frac{3+l}{2});\infty} \leq c_{75}\delta_0$$

and $\big\|\mathcal{A}\breve{f}^{(0)}\big\|_{(1+l,\frac{1+l}{2});\infty} \leq c_{76}\delta_0$. Then, using Eq (7.16), we obtain

$$\big\|\mathcal{F}[\breve{f}^{(0)} + \breve{f}^{(1)}]\big\|_{(1+l,\frac{1+l}{2});\infty} \leq c_{77}\Big(\delta_0^2 + \big\|\breve{f}^{(1)}\big\|^2_{(3+l,\frac{3+l}{2});\infty}\Big). \tag{7.20}$$

Combining this result with the boundedness of $\mathcal{A}_0^{-1}$ yields

$$\big\|\mathcal{M}[\breve{f}^{(1)}]\big\|_{(1+l,\frac{1+l}{2});\infty} \leq c_{78}\Big(\big\|\breve{f}^{(1)}\big\|^2_{(3+l,\frac{3+l}{2});\infty} + \delta_0^2 + \delta_0\Big). \tag{7.21}$$

Thus, if we consider $\overline{B} \equiv \left\{\breve{f}^{(1)} \in \overset{\circ}{\mathcal{V}}{}^{3+l}_\infty \,\middle|\, \|\breve{f}^{(1)}\|_{\overset{\circ}{\mathcal{V}}{}^{3+l}_\infty} \leq 2c_{78}\delta_0\right\}$, $\mathcal{M}$ maps $\overline{B}$ to itself provided that δ_0 is sufficiently small, satisfying $(4c_{78}^2 + 1)\delta_0 \leq 1$. Similarly, we obtain the following from Eq (7.17):

$$\big\|\mathcal{M}[v^{(1)}] - \mathcal{M}[v^{(2)}]\big\|_{(3+l,\frac{3+l}{2});\infty} \leq c_{79}\delta_0\big\|v^{(1)} - v^{(2)}\big\|_{(3+l,\frac{3+l}{2});\infty}. \tag{7.22}$$

Thus, if we consider $\delta_0 < 1/c_{79}$, $\mathcal{M}$ is a contraction map of $\overline{B}$ and has a unique fixed point. This establishes the existence of a solution $v \in \overset{\circ}{\mathcal{V}}{}^{3+l}_\infty$.

Because our method is based on the assumptions of Theorems 5.1 and 5.2, this approach will be available to use as long as those assumptions are satisfied.

8. Other properties of the solution

In this section, we prove other properties of the solution obtained thus far. We first prove that the solution satisfies the requirements of a probability density.

Lemma 8.1. *The solution $f(\theta, t; \omega)$ of* Eq (2.2) *satisfies the following:*

(i) $\int_{\mathbb{R}} \mathrm{d}\omega \int_\Omega f(\theta, t; \omega)\mathrm{d}\theta = 1 \quad \forall t \in \mathbb{R}_+$.

(ii) $f(\theta, t; \omega) \geq 0 \ \forall(\theta, t, \omega) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}$.

Proof. Note that it is sufficient to consider Eq (2.3). The first statement can be easily proven by considering

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \mathrm{d}\omega \int_\Omega \hat{f}(\theta, t; \omega)\, \mathrm{d}\theta = \varepsilon \int_{\mathbb{R}} \mathrm{d}\omega \int_\Omega \frac{\partial^2 \hat{f}}{\partial\theta^2}\, \mathrm{d}\theta - K \int_{\mathbb{R}} \mathrm{d}\omega \int_\Omega \frac{\partial}{\partial\theta}\Big(F[\hat{f}, \hat{f}]\Big)\, \mathrm{d}\theta. \tag{8.1}$$

Clearly, the righthand side of Eq (8.1) vanishes owing to the periodicity of $\hat{f}$ with respect to θ. □

Next, we prove the following lemma.

Lemma 8.2. *Suppose that* $f_0(\theta;\omega) \geq 0 \forall(\theta,\omega) \in \Omega\times\mathbb{R}$. *Then, the solution* $f(\theta,t;\omega)$ *to* Eq (2.2) *satisfies* $f(\theta,t;\omega) \geq 0 \quad \forall(\theta,t,\omega) \in \Omega\times\mathbb{R}_+\times\mathbb{R}$.

Proof. Again, if is sufficient to consider Eq (2.3). This time, we employ Stampaccia's truncation method. By introducing

$$f_+ \equiv \frac{(|f|+f)}{2} \geq 0,\ f_- \equiv \frac{(|f|-f)}{2} \geq 0,$$

which are the positive and negative parts of $\hat{f}$ at each point, we can decompose $\hat{f}$ into $\hat{f} = \hat{f}_+ - \hat{f}_-$. Then, if we multiply (2.3) by $\hat{f}_-$, and integrate with respect to θ over Ω, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{f}_-(t;\omega)\|^2_{L_2(\Omega)} + \varepsilon\left\|\frac{\partial\hat{f}_-}{\partial\theta}(t;\omega)\right\|^2_{L_2(\Omega)} = K\left|\int_\Omega f_-\frac{\partial}{\partial\theta}\Big(F(\hat{f},\hat{f})\Big)\,\mathrm{d}\theta\right|.$$

By virtue of Lemmas 8.1 and 8.2, we obtain

$$\begin{aligned}\int_\Omega f_-\frac{\partial}{\partial\theta}\Big(F(\hat{f},\hat{f})\Big)\,\mathrm{d}\theta &= \int_\Omega f_-\frac{\partial}{\partial\theta}\Big[(f_+-f_-)\Big\{\int_0^\infty h(\tau)\mathrm{d}\tau\int_\mathbb{R}\mathrm{d}\omega'\int_\Omega f(\theta',t-\tau;\omega')\sin(\theta'-\theta)\mathrm{d}\theta'\Big\}\Big]\mathrm{d}\theta\\ &= -\int_\Omega f_-\frac{\partial f_-}{\partial\theta}\Big\{\int_0^\infty h(\tau)\mathrm{d}\tau\int_\mathbb{R}\mathrm{d}\omega'\int_\Omega f(\theta',t-\tau;\omega')\sin(\theta'-\theta)\mathrm{d}\theta'\Big\}\mathrm{d}\theta.\end{aligned}$$

Indeed, if we introduce notations $\Omega_-(t,\omega) \equiv \{\theta \in \Omega\big|f(\theta,t;\omega) \leq 0\}$, and $g(\theta,t) \equiv \int_0^\infty h(\tau)\mathrm{d}\tau\int\mathrm{d}\omega'\int_\Omega f(\theta',t-\tau;\omega')\sin(\theta'-\theta)\mathrm{d}\theta'$, we have

$$\int_\Omega f_-(\theta,t;\omega)\frac{\partial}{\partial\theta}\Big(f_+(\theta,t;\omega)g(\theta,t)\Big)\,\mathrm{d}\theta = \int_{\Omega_-(t,\omega)} f_-(\theta,t;\omega)\frac{\partial}{\partial\theta}\Big(f_+(\theta,t;\omega)g(\theta,t)\Big)\,\mathrm{d}\theta$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{f}_-(t;\omega)\|^2_{L_2(\Omega)} + \varepsilon\left\|\frac{\partial\hat{f}_-}{\partial\theta}(t;\omega)\right\|^2_{L_2(\Omega)} \leq K\|\hat{f}_-(t;\omega)\|^2_{L_2(\Omega)}.$$

As $\hat{f}_-(\theta,0) = 0\ \forall\theta\in\Omega$, we obtain $\hat{f}\equiv 0$ by virtue of the Gronwall's inequality. □

9. Conclusion

We have proven the existence and uniqueness of a global-in-time solution to a parabolic-regularized Fokker–Planck equation that corresponds to the Kuramoto model with delay proposed by Lee et al. [17].

This argument theoretically demonstrates the validity of the model. However, our approach has some limitations. First, we proved the existence of a global-in-time solution under small initial data. We will relax this restriction in our future work.

Second, we will discuss the existence and structure of the invariant set or inertial manifold of the proposed model. This will be interesting because past arguments have implied the existence of multiple stable states under the presence of delay. Finally, we have considered only the parabolic-regularized problem. In our future work, we will consider the vanishing diffusion limit of the diffusion coefficient, which corresponds to the original model proposed by Lee et al. [17].

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict

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