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Research article

Corrector results for a class of elliptic problems with nonlinear Robin conditions and L^1 data

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Abstract: In this paper, we consider a class of elliptic problems in a periodically perforated domain with L^1 data and nonlinear Robin conditions on the boundary of the holes. Using the framework of renormalized solutions, which is well adapted to this situation, we show a convergence result for the truncated energy in the quasilinear case. When the operator is linear, we also prove a corrector result. Since we cannot expect to have solutions belonging to H^1, the main difficulty is to express the corrector result through the truncations of the solutions, together with the fact that the definition of a renormalized solution contains test functions which are nonlinear functions of the solution itself.

Keywords: homogenization; periodic unfolding; nonlinear Robin condition; correctors; renormalized solutions; integrable data

1. Introduction

The aim of this work is to prove convergence of energies and corrector results for an elliptic problem with nonlinear Robin conditions and L^1 data in a periodically perforated domain. This study completes the homogenization results given by the authors in [11], where the convergence of the solutions to a limit problem explicitly described, including its unfolded version, is proved.

More precisely, we consider the following elliptic problem in a periodically perforated domain Ω_ε^*:

$$\begin{cases} -\operatorname{div}\left(A(\tfrac{x}{\varepsilon}, u^\varepsilon)\nabla u^\varepsilon\right) = f & \text{in} \quad \Omega_\varepsilon^*, \\ u^\varepsilon = 0 & \text{on} \quad \Gamma_0^\varepsilon, \\ A(\tfrac{x}{\varepsilon}, u^\varepsilon)\nabla u^\varepsilon \cdot n + \varepsilon^\gamma \tau(\tfrac{x}{\varepsilon}) h(u^\varepsilon) = \varepsilon g(\tfrac{x}{\varepsilon}) & \text{on} \quad \Gamma_1^\varepsilon, \end{cases} \tag{1.1}$$

where $f \in L^1(\Omega)$, $g \in L^1(\partial T)$, $A(\cdot, t)$ is a coercive Y-periodic matrix which is bounded where t is bounded, h is a monotone continuous function verifying a sign condition and τ is a positive function in $L^\infty(\partial T)$.

Here, as in [11] (see also [4] and the references therein), the perforated domain is obtained by removing from a fixed domain Ω a set of ε-periodic holes of size ε. Its boundary consists of two parts on which we prescribe two different boundary conditions. Roughly speaking, on the boundary of the holes which are completely contained in Ω, we prescribe the nonlinear Robin condition, and on the remaining part of the boundary is a homogeneous Dirichlet condition. We refer the reader to Section 2 for a rigorous definition of the domain.

Heterogeneous media are widely studied since they have many interesting applications in sciences, in industry and, more recently, even in biology and environmental sciences. Let us recall that the mathematical homogenization theory (see, for instance, [3, 8]) allows to describe the microscopical behavior of a problem with periodic oscillations in the coefficients and/or in the domain. Theory provides a limit homogenized problem described through a problem posed in the periodicity reference cell. It represents a good approximation of the initial problem, and it is easier to compute since it does not present oscillations anymore.

It is well known already, in the classical case where $f \in L^2$, that the gradient of the solution u^ε converges weakly (never strongly) in L^2 to that of the solution u^0 of the homogenized problem. This is the reason why one looks for corrector results improving the weak convergence. To do that, one replaces ∇u^0 by $C^\varepsilon \nabla u^0$, where C^ε is the corrector matrix field, described via the cell problem. Hence, one proves that $\nabla u^\varepsilon - C^\varepsilon \nabla u^0$ strongly converges to zero in L^1. As a first step of the proof, one has to prove the convergence of the energy of the problem to the one of the homogenized problem.

In this paper, we prove similar results in the more delicate case where f is only in L^1. For physical motivations and references of related works, we refer the reader to [11].

As usual, the presence of the L^1 data requires a specific framework: we use here that of renormalized solutions (see [9] and the references therein and Definition 2.1 below). Since, in this case, the solution does not belong to H^1, the notion of a renormalized solution consists first of imposing the regularity of the truncations of the solution, and second, of making use of test functions of the type $S(u)\varphi$ where $S \in C^1(\mathbb{R})$ has a compact support and $\varphi \in L^\infty \cap H^1$. The test functions depend on the solution and vanish for large values of the solution. To counterbalance the lack of information where $|u|$ is large, a decay of the truncated energy is imposed.

The existence and the uniqueness of a renormalized solution of this problem have been proved in [12]. Successively, in [11], the authors, using the periodic unfolding method introduced in [5] (see for a complete presentation the book [7]), studied the homogenization of problem (1.1), proving that the renormalized solution converges to the renormalized solution of a homogenized problem posed in the whole domain.

We prove first the convergence of the energies for problem (1.1) (see Theorem 3.1 and Theorem 3.3), and then a corrector result (Theorem 4.1) for the corresponding linear equation, where the matrix field does not depend on the solution that is $A(y,t) = A(y)$ (see Remark 4.2). As far as we know, the results presented here are new, even in the case of a fixed domain (where there are no holes, so that $\Omega^*_\varepsilon = \Omega$). With respect to the classical situation with L^2 data, since the solutions are not in H^1, we cannot expect to have for the renormalized solutions a convergence result for $\nabla u^\varepsilon - C^\varepsilon \nabla u^0$, and we can only describe the convergences in terms of the truncated solutions and of the truncated limit function (at a fixed level). Our corrector result states the following convergence:

$$\lim_{\varepsilon\to 0}\left\| \nabla T_k(u^\varepsilon) - C^\varepsilon \nabla T_k(u^0) \right\|_{L^1(\Omega^*_\varepsilon)} = 0, \tag{1.2}$$

where u^0 is the solution of the homogenized problem, T_k is the truncation at level k and C^ε is the corrector matrix of the classical linear case in perforated domains (see [10]). This is not surprising, since, in the homogenization results proved in [11] for the case $f \in L^1$, all of the convergences concern the truncations $T_k(u^\varepsilon)$. In fact, the use of the truncation is standard in the literature of renormalized solutions.

The proofs are quite technical, since one cannot merely replace the solutions by their truncations and follow the usual arguments because the definition of a renormalized solution (see (2.21) in Definition 2.1) contains test functions which are nonlinear functions of the solution itself. This is the main difficulty all along the proofs. In addition, since the truncation function is not differentiable, we need to approach it by using suitable and more regular functions.

In Section 2, we introduce the problem and we recall some results on the periodic unfolding method, as well as the homogenization results from [11]. In Section 3, we prove the convergence of both (unfolded and not) types of truncated energy to those of the homogenized problem. Section 4 and 5 are devoted to the statement of the corrector result, and to the related proofs.

2. Position of the problem and preliminaries

In this paper, we study some corrector results for an elliptic problem with nonlinear Robin conditions and L^1 data, in a periodically perforated domain Ω^*_ε.

In Subsection 2.1, we define the perforated domain and set the problem, together with its variational formulation. In Subsection 2.2, we recall the definition of the periodic unfolding operator and the homogenization results obtained in [11].

2.1. Position of the problem

Let us introduce the geometrical framework used in [11] (see also [4]). In what follows, Ω is a connected open bounded subset of $\mathbb{R}^N$ ($N \geq 2$) with a Lipschitz-continuous boundary and $\mathbf{b} = (b_1, \ldots, b_N)$ as a given basis of $\mathbb{R}^N$.

We define the reference periodicity cell Y by

$$Y = \Big\{ \ell \in \mathbb{R}^N : \ell = \sum_{i=1}^N l_i b_i \,,\ (l_1, \ldots, l_N) \in (0,1)^N \Big\},$$

and denote by $\{\varepsilon\}_{\varepsilon>0}$ a positive sequence converging to zero. We set

$$\mathbf{G} = \Big\{ \xi \in \mathbb{R}^N : \xi = \sum_{i=1}^N k_i b_i \,,\ (k_1, \ldots, k_N) \in \mathbb{Z}^N \Big\}.$$

As is usual in the periodic unfolding method, (see for instance [6], and the exhaustive book [7]), we construct the interior of the largest union of cells $\varepsilon(\xi + \overline{Y})$ contained in Ω, as well as its complement, that is,

$$\widehat{\Omega}_\varepsilon = \text{interior}\left\{\bigcup_{\xi\in\Xi_\varepsilon} \varepsilon(\xi+\overline{Y}) \subset \Omega\right\},\ \Lambda_\varepsilon = \Omega\backslash\widehat{\Omega}_\varepsilon,\ \text{where}\ \ \boldsymbol{\Xi}_\varepsilon = \{\xi\in\mathbf{G} : \varepsilon(\xi+Y)\subset\Omega\}. \tag{2.1}$$

We now denote by T the reference hole, which is a compact subset of Y, and by $Y^* = Y\backslash T$ the perforated reference cell. We suppose that the boundary ∂T is Lipschitz-continuous with a finite number of connected components.

Then, the holes and the perforated domain Ω^*_ε (see Figure 1) are defined by

$$T_\varepsilon = \bigcup_{\xi\in\mathbf{G}} \varepsilon(\xi+T), \qquad \Omega^*_\varepsilon = \Omega\backslash T_\varepsilon, \tag{2.2}$$

respectively, while the perforated sets corresponding to (2.1) are

$$\widehat{\Omega}^*_\varepsilon = \widehat{\Omega}_\varepsilon\backslash T_\varepsilon \quad \text{and} \quad \Lambda^*_\varepsilon = \Omega^*_\varepsilon\backslash\,\widehat{\Omega}^*_\varepsilon. \tag{2.3}$$

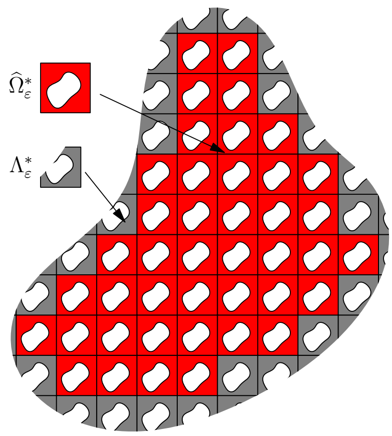

Figure 1. The reference cell Y and the perforated domain Ω^*_ε.

Finally, we decompose the boundary of the perforated domain Ω^*_ε as

$$\partial\Omega^*_\varepsilon = \Gamma^\varepsilon_0 \cup \Gamma^\varepsilon_1, \quad \text{where} \quad \Gamma^\varepsilon_1 = \partial\widehat{\Omega}^*_\varepsilon \cap \partial T_\varepsilon \ \ \text{and} \ \ \Gamma^\varepsilon_0 = \partial\Omega^*_\varepsilon\backslash\,\Gamma^\varepsilon_1. \tag{2.4}$$

In the sequel, we denote by

- $\widetilde{v}$, the extension by zero outside B of a function v defined on any set B,
- $\theta = \dfrac{|Y^*|}{|Y|}$, the proportion of the material,
- χ_A, the characteristic function of a measurable set A,
- $\mathcal{M}_{\partial T}(v) = \dfrac{1}{|\partial T|}\displaystyle\int_{\partial T} v(y)d\sigma_y$, the mean value over ∂T of a function $v\in L^1(\partial T)$.

Let us recall that, as $\varepsilon\to 0$,

$$\chi_{\Omega^*_\varepsilon} \rightharpoonup \theta \quad \text{weakly} \star \ \text{in}\ L^\infty(\Omega). \tag{2.5}$$

We are concerned with the following problem:

$$\begin{cases} -\operatorname{div}(A^\varepsilon(x,u^\varepsilon)\nabla u^\varepsilon) = f & \text{in} \quad \Omega^*_\varepsilon, \\ u^\varepsilon = 0 & \text{on} \quad \Gamma^\varepsilon_0, \\ A^\varepsilon(x,u^\varepsilon)\nabla u^\varepsilon \cdot n + \varepsilon^\gamma \tau^\varepsilon(x) h(u^\varepsilon) = g^\varepsilon & \text{on} \quad \Gamma^\varepsilon_1, \end{cases} \tag{2.6}$$

where $\gamma \geq 1$ and n is the unit exterior normal to Ω^*_ε.

We suppose that the following assumptions hold true:

- The functions f, g, h and τ are such that
  1. $f \in L^1(\Omega)$. (2.7)
  2. $h : \mathbb{R} \longrightarrow \mathbb{R}$ is an increasing continuous function, with $h(0) = 0$. (2.8)
  3. τ is a positive Y-periodic function in $L^\infty(\partial T)$ with (2.9)

  $$\tau^\varepsilon(x) = \tau\left(\frac{x}{\varepsilon}\right).$$

  4. Either (2.10)

  (i) $g^\varepsilon(x) = \varepsilon g\left(\frac{x}{\varepsilon}\right)$, with $g \in L^1(\partial T)$ Y-periodic with $\mathcal{M}_{\partial T}(g) \neq 0$

  or

  (ii) $g^\varepsilon \equiv 0$.

- Let $A : (y,t) \in Y \times \mathbb{R} \longmapsto A(y,t) \in \mathbb{R}^{N^2}$ be a real matrix field such that the matrix field $A(\cdot,t) = \{a_{ij}(\cdot,t)\}_{i,j=1\ldots N}$ is Y-periodic for every t.
  We suppose that A is a Carathéodory function, i.e., for almost every $y \in Y$, the map $t \mapsto A(y,t)$ is continuous, and for every $t \in \mathbb{R}$, the map $y \mapsto A(y,t)$ is measurable.
  For some constant $\alpha > 0$, we suppose further that the matrix A satisfies the following:
  1. $A(y,t)\xi\,\xi \geq \alpha|\xi|^2$, for a.e. $y \in Y$, $\forall\, t \in \mathbb{R}$, $\forall\, \xi \in \mathbb{R}^N$, (2.11)
  2. $\forall\, k > 0$, $A(y,t) \in L^\infty(Y \times (-k,k))^{N\times N}$, (2.12)
  3. The matrix field $A(y,t)$ is locally Lipschitz-continuous with respect to the second variable, that is, for every $r > 0$, there exists a positive constant M_r such that

  $$|A(y,s) - A(y,t)| < M_r|s-t|, \quad \forall s,t \in [-r,r], \quad \forall y \in Y, \tag{2.13}$$

  and we set

  $$A^\varepsilon(x,t) = A\left(\frac{x}{\varepsilon}, t\right) \quad \text{for every } (x,t) \in \Omega \times \mathbb{R}. \tag{2.14}$$

In order to define a renormalized solution of problem (2.6), let us introduce the space

$$V^\varepsilon = \{\, v \in H^1(\Omega^*_\varepsilon) : \ v = 0 \text{ on } \Gamma^\varepsilon_0 \,\}, \tag{2.15}$$

equipped with the norm

$$\|\, v \,\|_{V^\varepsilon} = \|\, \nabla v \,\|_{L^2(\Omega^*_\varepsilon)} \quad \text{for all } v \in V^\varepsilon. \tag{2.16}$$

Observe that (2.16) defines a norm since a Poincaré inequality holds in V^ε, namely,

$$\|\, u \,\|_{L^2(\Omega^*_\varepsilon)} \leq C \,\|\, \nabla u \,\|_{L^2(\Omega^*_\varepsilon)} \quad \forall u \in V^\varepsilon, \tag{2.17}$$

where the constant C is independent of ε. Also, the Sobolev continuous and compact embedding theorems on V^ε hold with constants independent of ε.

We recall now the definition of the truncation, which plays a crucial role in our work. For any $k > 0$, the truncation function $T_k : \mathbb{R} \to \mathbb{R}$ at height $\pm k$ is given by

$$T_k(t) = \min(k, \max(t, -k)) \tag{2.18}$$

for all $t \in \mathbb{R}$ (see Figure 2).

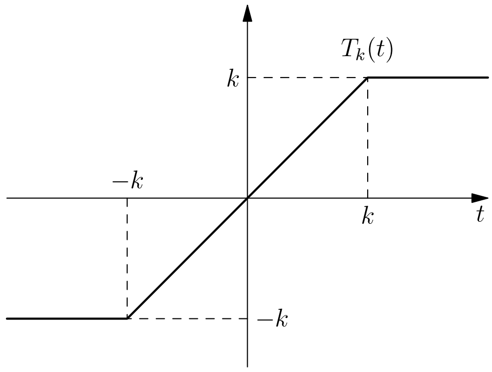

Figure 2. The function T_k.

Let us now present the definition of a renormalized solution to our problem, introduced in [12].

Definition 2.1. *We say that u^ε is a renormalized solution of* (2.6) *if*

$$T_k(u^\varepsilon) \in V^\varepsilon \qquad \textit{for any } k > 0, \tag{2.19}$$

$$\lim_{n\to+\infty} \frac{1}{n} \int_{\{x\in\Omega_\varepsilon^* : |u^\varepsilon|<n\}} A^\varepsilon(x, u^\varepsilon)\nabla u^\varepsilon \nabla u^\varepsilon \, dx = 0, \tag{2.20}$$

and for any $\psi \in C^1(\mathbb{R})$ (or equivalently for any $\psi \in W^{1,\infty}(\mathbb{R})$) with compact support, u^ε satisfies

$$\int_{\Omega_\varepsilon^*} \psi(u^\varepsilon)A^\varepsilon(x, u^\varepsilon)\nabla u^\varepsilon \nabla v \, dx + \int_{\Omega_\varepsilon^*} \psi'(u^\varepsilon)A^\varepsilon(x, u^\varepsilon)\nabla u^\varepsilon \nabla u^\varepsilon v \, dx + \int_{\Gamma_1^\varepsilon} \varepsilon^\gamma \tau^\varepsilon(x)\psi(u^\varepsilon)h(u^\varepsilon)v \, d\sigma_x = \int_{\Omega_\varepsilon^*} f\psi(u^\varepsilon)v \, dx + \int_{\Gamma_1^\varepsilon} g^\varepsilon \psi(u^\varepsilon)v \, d\sigma_x \tag{2.21}$$

for all $v \in V^\varepsilon \cap L^\infty(\Omega_\varepsilon^*)$.

Remark 2.2.

1. *Proposition 2.3 in [12] (see also [2]) guarantees that the gradient and the trace along the boundaries of any function verifying* (2.19) *and* (2.20) *are well defined almost everywhere in Ω_ε^* and Γ_1^ε, respectively. This shows that every term in* (2.21) *is well defined.*
2. *Observe that, for every $k > 0$, we have*

$$\nabla v \, \nabla T_k(v) = \nabla T_k(v) \, \nabla T_k(v) \tag{2.22}$$

for any function v such that $T_k(v) \in V^\varepsilon$ for all $k > 0$.

3. *It has been proved in [12] that, under assumptions* (2.7)–(2.12)*, there exists a renormalized solution to* (2.6) *in the sense of Definition 2.1. Moreover, assumption* (2.13) *provides the uniqueness of a solution.*

2.2. Review of homogenization results

In this subsection, we recall the homogenization results proved in [11] by using the periodic unfolding method, and we state them in the particular case where assumption (2.13) holds. This condition is needed in the following sections, since it provides the uniqueness of the solution to the problem we consider here.

Let us start by recalling the definitions of the unfolding operator and the boundary unfolding operator. For a detailed and extensive presentation of the method, see [6, 4, 7]. For the properties used in this paper, we refer the reader to [11, Section 3].

For a.e. $z \in \mathbb{R}^N$, we denote by $[z]_Y = \sum_{i=1}^N l_i b_i$, $l_i \in \mathbb{Z}$ for $i = 1, \dots, n$, the unique integer combination such that $z - [z]_Y \in Y$ and set $\{z\}_Y = z - [z]_Y \in Y$.

Thus, for a positive ε, we can write

$$x = \varepsilon\left(\left\{\frac{x}{\varepsilon}\right\}_Y + \left[\frac{x}{\varepsilon}\right]_Y\right) \quad \text{for a.e. } x \in \mathbb{R}^N.$$

Definition 2.3. *Suppose φ is a Lebesgue-measurable function. The unfolding operator $\mathcal{T}_\varepsilon^*$ is defined as*

$$\mathcal{T}_\varepsilon^*(\varphi)(x,y) = \begin{cases} \varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) & \text{for a.e. } (x,y) \in \widehat{\Omega}_\varepsilon \times Y^*, \\ 0 & \text{for a.e. } (x,y) \in \Lambda_\varepsilon \times Y^*. \end{cases} \tag{2.23}$$

Definition 2.4. *Suppose that φ is a Lebesgue-measurable function on $\partial\widehat{\Omega}_\varepsilon^* \cap \partial T_\varepsilon$. The boundary unfolding operator $\mathcal{T}_\varepsilon^b$ is defined as*

$$\mathcal{T}_\varepsilon^b(\varphi)(x,y) = \begin{cases} \varphi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) & \text{for a.e. } (x,y) \in \widehat{\Omega}_\varepsilon \cap \partial T, \\ 0 & \text{for a.e. } (x,y) \in \Lambda_\varepsilon \times \partial T. \end{cases} \tag{2.24}$$

Remark 2.5. *For a given a continuous function $r(x)$, with $r(0) = 0$, one has*

$$\mathcal{T}_\varepsilon^*(r(u^\varepsilon)) = r\left(\mathcal{T}_\varepsilon^*(u^\varepsilon)\right) \tag{2.25}$$

in $\Omega \times Y^*$.

Nevertheless, for any Lebesgue measurable function φ, we can write

$$\mathcal{T}_\varepsilon^*(r(u^\varepsilon))\mathcal{T}_\varepsilon^*(\varphi) = r\left(\mathcal{T}_\varepsilon^*(u^\varepsilon)\right)\mathcal{T}_\varepsilon^*(\varphi) \tag{2.26}$$

even if $r(0) \neq 0$. This is due to the fact that, if $(x,y) \in \Lambda_\varepsilon \times Y^*$, equality is still obtained since $\mathcal{T}_\varepsilon^*(\varphi)(x,y) = 0$. Further, this implies that (2.26) *holds for all $(x,y) \in \Omega \times Y^*$.*

Similar properties hold for $\mathcal{T}_\varepsilon^b$.

Let us state now the homogenization results proved in [11].

Theorem 2.6 ([11]). *Let u^ε be the renormalized solution of* (2.6) *under assumptions* (2.7)–(2.14)*, with* $\gamma \geq 1$. *Set* $J(\gamma)$

$$J(\gamma) = \begin{cases} |\partial T| & \text{if } \gamma = 1\,, \\ 0 & \text{if } \gamma > 1\,. \end{cases} \tag{2.27}$$

Then, as ε tends to zero, there exists $u^0 : \Omega \to \mathbb{R}$, measurable and finite almost everywhere, and for every $k \in \mathbb{N}$, $\widehat{u_k} \in L^2(\Omega, H^1_{per}(Y^*))$ with $\mathcal{M}_{Y^*}(\widehat{u_k}) = 0$ satisfying

$$\begin{cases} (i). \quad \mathcal{T}^*_\varepsilon(u^\varepsilon) \longrightarrow u^0 \quad \text{a.e. in } \Omega \times Y^*, \\ (ii). \quad \mathcal{T}^b_\varepsilon(u^\varepsilon) \longrightarrow u^0 \quad \text{a.e. in } \Omega \times \partial T, \end{cases} \tag{2.28}$$

and

$$\begin{cases} (i). \quad \mathcal{T}^*_\varepsilon(T_k(u^\varepsilon)) \to T_k(u^0) & \text{strongly in } L^2(\Omega, H^1(Y^*)), \\ (ii). \quad \mathcal{T}^*_\varepsilon(\nabla T_k(u^\varepsilon)) \rightharpoonup \nabla T_k(u^0) + \nabla_y \widehat{u_k} & \text{weakly in } L^2(\Omega \times Y^*), \\ (iii). \quad T_k(\widetilde{u^\varepsilon}) = \widetilde{T_k(u^\varepsilon)} \rightharpoonup \theta T_k(u^0) & \text{weakly in } L^2(\Omega), \\ (iv). \quad \left\| T_k(u^\varepsilon) - T_k(u^0) \right\|_{L^2(\Omega^*_\varepsilon)} \to 0. \end{cases} \tag{2.29}$$

Further, there exists a unique measurable function $\widehat{u} : \Omega \times Y^* \to \mathbb{R}$ such that, for every function $R \in W^{1,\infty}(\mathbb{R})$ with compact support such that $\operatorname{supp} R \subset [-n, n]$ *for some $n \in \mathbb{N}$, we have*

$$R(u^0)\, \widehat{u_k} = R(u^0)\, \widehat{u} \quad \text{a.e. in } \Omega \times Y^*, \quad \forall k \geq n. \tag{2.30}$$

Moreover, if S, S_1 are functions in $C^1(\mathbb{R})$ with compact supports, then the pair $(u^0, \widehat{u})$ is the unique solution of the limit problem

$$\begin{cases} \displaystyle\int_{\Omega\times Y^*} A(y, u^0)\left(\nabla u^0 + \nabla_y \widehat{u}\right)\left(\nabla \left(S(u^0)\eta_0\right) + S_1(u^0)\nabla_y \Psi(x,y) \right) dx\, dy \\ \qquad + J(\gamma)\mathcal{M}_{\partial T}(\tau) \displaystyle\int_\Omega h(u^0) S(u^0)\eta_0\, dx \\ \qquad = |Y^*| \displaystyle\int_\Omega f S(u^0)\eta_0\, dx + |\partial T| \mathcal{M}_{\partial T}(g) \int_\Omega S(u^0)\eta_0\, dx, \\ \text{for every } \eta_0 \in H^1_0(\Omega) \cap L^\infty(\Omega) \text{ and for every } \Psi \in L^2(\Omega, H^1_{per}(Y^*)). \end{cases} \tag{2.31}$$

We also have the convergence

$$\lim_{k\to\infty} \frac{1}{k} \int_{\Omega\times Y^*} A(y, u^0)\left(\nabla T_k(u^0) + \nabla_y \widehat{u_k}\right)\left(\nabla T_k(u^0) + \nabla_y \widehat{u_k}\right) dx\, dy = 0. \tag{2.32}$$

As a consequence, we prove the following result, used in the sequel:

Corollary 2.7. *Under the assumptions of Theorem 2.6, for any bounded continuous function $H : \mathbb{R} \to \mathbb{R}$ such that $H(0) = 0$, we have*

$$\|H(u_\varepsilon) - H(u_0)\|_{L^2(\Omega^*_\varepsilon)} \to 0. \tag{2.33}$$

Consequently, using (2.5),

$$\widetilde{H(u_\varepsilon)} \rightharpoonup \theta H(u_0) \quad \text{in } L^\infty(\Omega)\text{-weak*}. \tag{2.34}$$

Proof. Using the properties of H, from Remark 2.5, convergence (2.28)(i) and the dominated convergence Lebesgue theorem, we get

$$\mathcal{T}_\varepsilon^*(H(u^\varepsilon)) = H(\mathcal{T}_\varepsilon^*(u^\varepsilon)) \longrightarrow wH(u^0) \quad \text{strongly in } L^2(\Omega \times Y^*).$$

Applying Corollary 1.19 of [4], from the boundedness of H we derive convergence (2.33). Convergence (2.34) is then straightforward. □

We also recall the next theorem, which identifies $\widehat{u}$ in terms of the limit function u^0.

Theorem 2.8 ([11]). *Under the same assumptions and notations of Theorem 2.6, the function $\widehat{u}$ can be expressed as*

$$\widehat{u}(y, x) = -\sum_{j=1}^{N} \widehat{\chi_{e_j}}(y, u^0(x)) \frac{\partial u^0}{\partial x_j}(x), \tag{2.35}$$

where $(e_j)_{j=1}^N$ is the canonical basis of $\mathbb{R}^N$ and $\widehat{\chi_{e_j}}(\cdot, t)$ is the solution of

$$\begin{cases} -\operatorname{div}\left(A(\cdot, t)\nabla_y \widehat{\chi_\lambda}(\cdot, t)\right) = -\operatorname{div}(A(\cdot, t)\lambda) & \text{in } Y^*, \\ A(\cdot, t)\left(\lambda - \nabla_y \widehat{\chi_\lambda}(\cdot, t)\right) \cdot n = 0 & \text{on } \partial T, \\ \widehat{\chi_\lambda}(\cdot, t) \qquad Y\text{-periodic}, \\ \mathcal{M}_{Y^*}\left(\widehat{\chi_\lambda}(\cdot, t)\right) = 0 \end{cases} \tag{2.36}$$

for every $t \in \mathbb{R}$ and $\lambda \in \mathbb{R}^N$.

The next result shows that u^0 is a renormalized solution to a homogenized elliptic problem corresponding to the homogenized matrix A^0.

Theorem 2.9 ([11]). *Let u^0 be the function given in Theorem 2.6. Then, u^0 is the renormalized solution of the problem*

$$\begin{cases} -\operatorname{div}\left(A^0(u^0)\nabla u^0\right) + \dfrac{J(\gamma)}{|Y|}\mathcal{M}_{\partial T}(\tau)h(u^0) = \theta f + \dfrac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g) & \text{in} \quad \Omega, \\ u^0 = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.37}$$

that is, u^0 satisfies

$$T_k(u^0) \in H_0^1(\Omega) \qquad \text{for any } k > 0, \tag{2.38}$$

$$\lim_{n\to+\infty} \frac{1}{n} \int_{\{x\in\Omega \,:\, |u^0|<n\}} A^0(u^0)\nabla u^0 \nabla u^0 \, dx = 0, \tag{2.39}$$

and for every $S \in C^1(\mathbb{R})$ with compact support, u^0 satisfies

$$\int_\Omega S(u^0)A^0(u^0)\nabla u^0 \nabla \eta_0 \, dx + \int_\Omega S_0'(u^0)A^0(u^0)\nabla u^0 \nabla u^0 \eta_0 \, dx + \frac{J(\gamma)}{|Y|}\mathcal{M}_{\partial T}(\tau) \int_\Omega S(u^0)h(u^0)\eta_0 \, dx$$

$$= \theta \int_{\Omega} f S(u^0)\eta_0 \, dx + \frac{|\partial T|}{|Y|} \mathcal{M}_{\partial T}(g) \int_{\Omega} S(u^0)\eta_0 \, dx \quad (2.40)$$

for all $\eta_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

The homogenized matrix $A^0(t)$ *is defined, for every fixed* $t \in \mathbb{R}$, *as*

$$A^0(t)\lambda = \frac{1}{|Y|} \int_{Y^*} A(y,t)\nabla_y \widehat{w_\lambda}(y,t)\, dy \qquad \forall \lambda \in \mathbb{R}^N, \quad (2.41)$$

in which

$$\widehat{w_\lambda}(y,t) = \lambda y - \widehat{\chi_\lambda}(y,t), \quad (2.42)$$

and where the function $\widehat{\chi_\lambda}(\cdot, t)$ *is the solution of the problem* (2.36).

Consequently, in view of Theorem 2.8,

$$A^0(u^0)\nabla u^0 = \frac{1}{|Y|} \int_{Y^*} A(y,u^0)\left(\nabla u^0 + \nabla_y \widehat{u}\right) dy, \quad \textit{a.e. in } \Omega. \quad (2.43)$$

The result below, proved in [11] (Proposition 6.1), plays an important role in the proof of the corrector results to our problem.

Proposition 2.10. *Under the assumptions of Theorem 2.6, for every* $k \in \mathbb{N}$ *and* $\varepsilon > 0$, *we have*

$$\lim_{k\to+\infty} \limsup_{\varepsilon\to 0} \frac{1}{k} \int_{\{|u^\varepsilon|<k\}} A^\varepsilon(x,u^\varepsilon)\nabla u^\varepsilon \nabla u^\varepsilon \, dx = 0. \quad (2.44)$$

3. Convergence of the energies

The first result of this section states the convergence of the truncated energies associated with our problem. This convergence is important in itself, and it is essential in the proof of our corrector results.

To this aim, for $n \in \mathbb{N}$, we define the function ψ_n (see Figure 3) by

$$\psi_n(x) = \begin{cases} \dfrac{x}{n} + 2 & , \quad -2n \le x \le -n \\ 1 & , \quad -n \le x \le n \\ -\dfrac{x}{n} + 2 & , \quad n \le x \le 2n \\ 0 & , \quad |x| \ge 2n, \end{cases} \quad (3.1)$$

which is Lipschitz-continuous and has a compact support given by supp $\psi_n = [-2n, 2n]$.

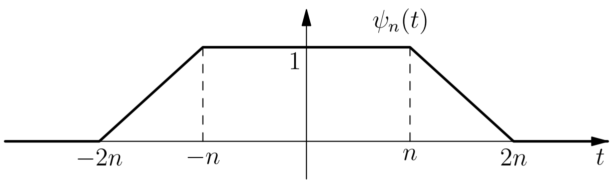

Figure 3. The function ψ_n.

Further, ψ_n satisfies

$$0 \le \psi_n \le 1, \qquad |\psi_n'(s)| \le \frac{1}{n} \ \text{ for } |s| \le 2n, \ \text{a.e. in } \mathbb{R}. \quad (3.2)$$

Theorem 3.1. *Under assumptions* (2.7)–(2.14)*, let* u^ε *be the renormalized solution to* (2.6)*. Let also* $G \in W^{1,\infty}(\mathbb{R})$ *be a nondecreasing function such that* G' *has a compact support and* $G(0) = 0$*.*

Then,

$$\lim_{\varepsilon\to 0}\int_{\Omega^*_\varepsilon} A^\varepsilon(x,u^\varepsilon)\nabla u^\varepsilon\,\nabla G(u^\varepsilon)\,dx \longrightarrow \int_\Omega A^0(u^0)\nabla u^0\,\nabla G(u^0)\,dx \tag{3.3}$$

as ε *tends to zero, where* u^0 *and* A^0 *are given by Theorem 2.6.*

In particular, for every fixed $k \in \mathbb{N}$*,*

$$\begin{aligned}\int_{\Omega^*_\varepsilon} A^\varepsilon(x,u^\varepsilon)\nabla T_k(u^\varepsilon)\nabla T_k(u^\varepsilon)\,dx &= \int_{\Omega^*_\varepsilon} A^\varepsilon(x,u^\varepsilon)\nabla u^\varepsilon\nabla T_k(u^\varepsilon)\,dx\\ &\longrightarrow \int_\Omega A^0(u^0)\nabla T_k(u^0)\nabla u^0\,dx = \int_\Omega A^0(u_0)\nabla T_k(u^0)\nabla T_k(u^0)\end{aligned} \tag{3.4}$$

as ε *tends to zero.*

Proof. Let $G \in W^{1,\infty}(\mathbb{R})$ be a nondecreasing function such that, for some $k \in \mathbb{N}$, $\operatorname{supp} G' \subset [-k, k]$. Since, for $n \geq k$,

$$\int_{\Omega^*_\varepsilon} A^\varepsilon(x,u^\varepsilon)\nabla u^\varepsilon\,\nabla G(u^\varepsilon)\,dx = \int_{\Omega^*_\varepsilon} \psi_n(u^\varepsilon)A^\varepsilon(x,u^\varepsilon)\nabla u^\varepsilon\,\nabla G(u^\varepsilon)\,dx; \tag{3.5}$$

it suffices to prove that

$$\lim_{n\to+\infty}\lim_{\varepsilon\to 0}\int_{\Omega^*_\varepsilon} \psi_n(u^\varepsilon)A^\varepsilon(x,u^\varepsilon)\nabla u^\varepsilon\nabla G(u^\varepsilon)\,dx = \int_\Omega A^0(u^0)\nabla u^0\,\nabla G(u^0)\,dx, \tag{3.6}$$

where ψ_n is defined by (3.1). Using $\psi = \psi_n$ and $v = G(u^\varepsilon)$ in (2.21), we have

$$\begin{aligned}\int_{\Omega^*_\varepsilon} \psi_n(u^\varepsilon)A^\varepsilon(x,u^\varepsilon)\nabla u^\varepsilon\nabla G(u^\varepsilon)\,dx &= \int_{\Omega^*_\varepsilon} f\psi_n(u^\varepsilon)G(u^\varepsilon)\,dx + \int_{\Gamma^\varepsilon_1} g^\varepsilon\psi_n(u^\varepsilon)G(u^\varepsilon)\,d\sigma_x\\ &- \int_{\Gamma^\varepsilon_1} \varepsilon^\gamma\tau^\varepsilon(x)\psi_n(u^\varepsilon)h(u^\varepsilon)G(u^\varepsilon)\,d\sigma_x - \int_{\Omega^*_\varepsilon} \psi_n'(u^\varepsilon)A^\varepsilon(x,u^\varepsilon)\nabla u^\varepsilon\nabla u^\varepsilon\,G(u^\varepsilon)\,dx.\end{aligned} \tag{3.7}$$

Let us first prove that, for any $n \in \mathbb{N}$,

$$\begin{aligned}\lim_{\varepsilon\to 0}\Bigg(\int_{\Omega^*_\varepsilon} f\psi_n(u^\varepsilon)G(u^\varepsilon)\,dx &+ \int_{\Gamma^\varepsilon_1} g^\varepsilon\psi_n(u^\varepsilon)G(u^\varepsilon)\,d\sigma_x - \int_{\Gamma^\varepsilon_1} \varepsilon^\gamma\tau^\varepsilon(x)\psi_n(u^\varepsilon)h(u^\varepsilon)G(u^\varepsilon)\,d\sigma_x\Bigg)\\ &= \theta\int_\Omega f\psi_n(u^0)G(u_0)\,dx + \frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)\int_\Omega \psi_n(u^0)G(u_0)\,dx\\ &\quad - \frac{J(\gamma)}{|Y|}\mathcal{M}_{\partial T}(\tau)\int_\Omega \psi_n(u^0)h(u^0)G(u_0)\,dx,\end{aligned} \tag{3.8}$$

where $J(\gamma)$ is given by (2.27).

Corollary 2.7 applied to $\psi_n G$, and the first convergence in (2.28), give

$$\lim_{\varepsilon\to 0}\int_{\Omega^*_\varepsilon} f\psi_n(u^\varepsilon)G(u^\varepsilon)\,dx = \theta\int_\Omega f\psi_n(u^0)G(u^0)\,dx. \tag{3.9}$$

From the properties of the boundary unfolding operator (see [4]) and Remark 2.5, we have

$$\begin{aligned}\varepsilon \int_{\Gamma_1^\varepsilon} \tau^\varepsilon(x)\psi_n(u^\varepsilon)h(u^\varepsilon)G(u_\varepsilon)\,d\sigma_x &= \frac{1}{|Y|}\int_{\Omega\times\partial T} \tau(y)\mathcal{T}_\varepsilon^b\left(\psi_n(u^\varepsilon)\right)\mathcal{T}_\varepsilon^b\left(h(u^\varepsilon)\right)\mathcal{T}_\varepsilon^b\left(G(u_\varepsilon)\right)\,dx\,d\sigma_y \\ &= \frac{1}{|Y|}\int_{\Omega\times\partial T} \tau(y)\psi_n(\mathcal{T}_\varepsilon^b(u^\varepsilon))h(\mathcal{T}_\varepsilon^b(u^\varepsilon))G(\mathcal{T}_\varepsilon^b(u_\varepsilon))\,dx\,d\sigma_y.\end{aligned} \tag{3.10}$$

Using convergence (ii) of (2.28), we obtain

$$\psi_n(\mathcal{T}_\varepsilon^b(u^\varepsilon)) \longrightarrow \psi_n(u^0) \quad \text{a.e. in } \Omega\times\partial T, \tag{3.11}$$

$$G(\mathcal{T}_\varepsilon^b(u^\varepsilon)) \longrightarrow G(u^0) \quad \text{a.e. in } \Omega\times\partial T. \tag{3.12}$$

Also, from the assumptions on h and Remark 2.5, we have

$$\mathcal{T}_\varepsilon^b(h(u^\varepsilon)) = h(\mathcal{T}_\varepsilon^b(u^\varepsilon)) \longrightarrow h(u^0) \quad \text{a.e. in } \Omega\times\partial T. \tag{3.13}$$

Thus, combining the convergences above, equality (3.10) gives

$$\begin{aligned}\lim_{\varepsilon\to 0} \varepsilon \int_{\Gamma_1^\varepsilon} \tau^\varepsilon(x)\psi_n(u^\varepsilon)h(u^\varepsilon)G(u^\varepsilon)\,d\sigma_x &= \frac{1}{|Y|}\int_{\Omega\times\partial T} \tau(y)\psi_n(u^0)h(u^0)G(u^0)\,dx\,d\sigma_y \\ &= \frac{1}{|Y|}\left(\int_{\partial T}\tau(y)\,d\sigma_y\right)\left(\int_\Omega \psi_n(u^0)h(u^0)G(u^0)\,dx\right) \\ &= \frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(\tau)\int_\Omega \psi_n(u^0)h(u^0)\varphi\,dx,\end{aligned} \tag{3.14}$$

since u^0 is independent of y. When $\gamma > 1$, we deduce from (3.14) that

$$\lim_{\varepsilon\to 0} \varepsilon^\gamma \int_{\Gamma_1^\varepsilon} \tau^\varepsilon(x)\psi_n(u^\varepsilon)h(u^\varepsilon)G(u^\varepsilon)\,d\sigma_x = 0. \tag{3.15}$$

Concerning the second integral on the left-hand side of (3.8), for the case $\mathcal{M}_{\partial T}(g) \neq 0$, we again use (2.10), Remark 2.5 and properties of the boundary unfolding operator to write

$$\begin{aligned}\varepsilon \int_{\Gamma_1^\varepsilon} g\left(\frac{x}{\varepsilon}\right)\psi_n(u^\varepsilon)G(u^\varepsilon)\,d\sigma_x &= \frac{1}{|Y|}\int_{\Omega\times\partial T} g(y)\mathcal{T}_\varepsilon^b\left(\psi_n(u^\varepsilon)\right)\mathcal{T}_\varepsilon^b\left(G(u^\varepsilon)\right)\,dx\,d\sigma_y \\ &= \frac{1}{|Y|}\int_{\Omega\times\partial T} g(y)\psi_n(\mathcal{T}_\varepsilon^b(u^\varepsilon))G(u^\varepsilon)\left(\mathcal{T}_\varepsilon^b\right)\,dx\,d\sigma_y.\end{aligned}$$

Arguing as above, we obtain

$$\lim_{\varepsilon\to 0}\varepsilon \int_{\Gamma_1^\varepsilon} g\left(\frac{x}{\varepsilon}\right)\psi_n(u^\varepsilon)G(u^\varepsilon)\,d\sigma_x = \frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)\int_\Omega \psi_n(u^0)\varphi(x)\,dx, \tag{3.16}$$

which completes the proof of (3.8).

Now, using the properties of ψ_n and setting $m_G = \max_{\mathbb{R}}|G|$, we obtain

$$\begin{aligned}\left|\int_{\Omega_\varepsilon^*}\psi_n'(u^\varepsilon)A^\varepsilon(x,u^\varepsilon)\nabla u^\varepsilon\nabla u^\varepsilon G(u^\varepsilon)\,dx\right| &= \left|\int_{\{|u^\varepsilon|<2n\}}\psi_n'(u^\varepsilon)A^\varepsilon(x,u^\varepsilon)\nabla u^\varepsilon\nabla u^\varepsilon G(u^\varepsilon)\,dx\right| \\ &\le \frac{m_G}{n}\int_{\{|u^\varepsilon|<2n\}} A^\varepsilon(x,u^\varepsilon)\nabla u^\varepsilon\nabla u^\varepsilon\,dx.\end{aligned}$$

Then, from Proposition 2.10, we deduce that

$$\limsup_{\varepsilon\to 0}\left|\int_{\Omega_\varepsilon^*}\psi_n'(u^\varepsilon)A^\varepsilon(x,u^\varepsilon)\nabla(u^\varepsilon)\nabla(u^\varepsilon)G(u^\varepsilon)\,dx\right| = \omega_1(n), \tag{3.17}$$

where $\omega_1(n)$ goes to zero as $n\to\infty$. On the other hand, taking $\eta_0 = G(u^0)$ and $S = \psi_n$ as test functions in (2.40) gives

$$\begin{aligned}\int_\Omega \psi_n(u^0)A^0(u^0)\nabla u^0\nabla G(u_0)\,dx &= \theta\int_\Omega f\psi_n(u^0)G(u_0)\,dx\\ &+\frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)\int_\Omega\psi_n(u^0)G(u_0)\,dx - \frac{J(\gamma)}{|Y|}\mathcal{M}_{\partial T}(\tau)\int_\Omega\psi_n(u^0)h(u^0)G(u_0)\,dx\\ &-\int_\Omega\psi_n'(u^0)A^0(u^0)\nabla u^0\nabla u^0 G(u_0)\,dx. \end{aligned}\tag{3.18}$$

This, combined with (3.7), and using (3.8) and (3.17), yields

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^*}\psi_n(u^\varepsilon)A^\varepsilon(x,u^\varepsilon)\nabla u^\varepsilon\nabla G(u^\varepsilon)\,dx = \int_\Omega\psi_n(u^0)A^0(u^0)\nabla u^0\nabla G(u_0)\,dx + \omega_1(n).$$

Now, since $\psi_n\to 1$ as $n\to\infty$, by the Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty}\int_\Omega\psi_n(u^0)A^0(u^0)\nabla u^0\nabla G(u^0)\,dx = \int_\Omega A^0(u^0)\nabla u^0\nabla G(u^0)\,dx. \tag{3.19}$$

Therefore, passing to the limit as $n\to+\infty$ in (3.19), we get (3.6), which, in view of (3.5), proves (3.3). □

Proposition 3.2. *Under the assumptions of Theorem 3.1, for every $k\in\mathbb{N}$, we have*

$$\int_\Omega A^0(u^0)\nabla T_k(u^0)\nabla T_k(u^0)\,dx = \frac{1}{|Y|}\int_{\Omega\times Y^*}A(y,u^0)\left(\nabla T_k(u^0)+\nabla_y\widehat{u_k}\right)\left(\nabla T_k(u^0)+\nabla_y\widehat{u_k}\right)dx\,dy \tag{3.20}$$

and

$$\begin{aligned}&\int_{\Omega\times Y^*}A(y,u^\varepsilon)\mathcal{T}_\varepsilon^*(\nabla T_k(u^\varepsilon))\,\mathcal{T}_\varepsilon^*(\nabla T_k(u^\varepsilon))\,dx\,dy\\ &\qquad\longrightarrow\frac{1}{|Y|}\int_{\Omega\times Y^*}A(y,u^0)\left(\nabla T_k(u^0)+\nabla_y\widehat{u_k}\right)\left(\nabla T_k(u^0)+\nabla_y\widehat{u_k}\right)dx\,dy\end{aligned}\tag{3.21}$$

as ε tends to zero.

Proof. We prove first the following inequality for the two energies:

$$\begin{aligned}&\int_\Omega A^0(u^0)\nabla T_k(u^0)\nabla T_k(u^0)\,dx\\ &=\frac{1}{|Y|}\int_{\Omega\times Y^*}\chi_{\{|u_0|<k\}}A(y,u^0)\left(\nabla T_k(u^0)+\nabla_y\widehat{u_k}\right)\left(\nabla T_k(u^0)+\nabla_y\widehat{u_k}\right)dx\,dy \qquad (3.22)\\ &\le\frac{1}{|Y|}\int_{\Omega\times Y^*}A(y,u^0)\left(\nabla T_k(u^0)+\nabla_y\widehat{u_k}\right)\left(\nabla T_k(u^0)+\nabla_y\widehat{u_k}\right)dx\,dy.\end{aligned}$$

Observe that

$$\nabla T_k(u_0) = \nabla u_0 \chi_{\{|u_0|<k\}} \quad \text{a.e in } \Omega. \tag{3.23}$$

Then, using (2.30) and equality (2.43) from Theorem 2.8, we can write

$$\begin{aligned}\frac{1}{|Y|}\int_{\Omega\times Y^*} \chi_{\{|u_0|<k\}} A(y,u^0)\left(\nabla T_k(u^0)+\nabla_y\widehat{u_k}\right)\nabla T_k(u^0)\,dx\,dy \\ = \frac{1}{|Y|}\int_{\Omega\times Y^*} A(y,u^0)\left(\nabla u^0+\nabla_y\widehat{u}\right)\nabla T_k(u^0)\,dx\,dy \\ = \int_\Omega A^0(u^0)\nabla u^0\nabla T_k(u^0)\,dx = \int_\Omega A^0(u^0)\nabla T_k(u^0)\nabla T_k(u^0)\,dx.\end{aligned}$$

Hence, to prove the equality in (3.22), it suffices to show that

$$\begin{aligned}\frac{1}{|Y|}\int_{\Omega\times Y^*} \chi_{\{|u_0|<k\}} A(y,u^0)\left(\nabla T_k(u^0)+\nabla_y\widehat{u_k}\right)\nabla_y\widehat{u_k}\,dx\,dy \\ = \frac{1}{|Y|}\int_{\Omega\times Y^*} \chi_{\{|u_0|<k\}} A(y,u^0)\left(\nabla T_k(u^0)+\nabla_y\widehat{u}\right)\nabla_y\widehat{u_k}\,dx\,dy = 0,\end{aligned} \tag{3.24}$$

where, again, we used (2.30) in the first equality.

To do that, for any $\delta > 0$, let $S^1_\delta \in C^1(\mathbb{R})$ be a bounded sequence function with compact support contained in $[-k,k]$, and such that

$$0 \le S^1_\delta(r) \le 1, \quad \text{and} \quad \lim_{\delta\to 0} S^1_\delta(r) \to \chi_{\{|r|<k\}} \quad \text{for every } r\in\mathbb{R}. \tag{3.25}$$

Then, choosing in (2.31) $\eta_0 = 0$, $\Psi = \widehat{u_k}$ and $S_1 = S^1_\delta$, we obtain

$$\int_{\Omega\times Y^*} A(y,u^0)\left(\nabla T_k(u^0)+\nabla_y\widehat{u}\right)S^1_\delta(u^0)\nabla_y\widehat{u_k}\,dx\,dy = \int_{\Omega\times Y^*} A(y,u^0)\left(\nabla u^0+\nabla_y\widehat{u}\right)S^1_\delta(u^0)\nabla_y\widehat{u_k}\,dx\,dy = 0.$$

Passing to the limit as $\delta\to 0$, from (3.25), we deduce (3.24), which concludes the proof of (3.22).

Let us prove now convergence (3.21). From (2.11)–(2.12), by the lower semi-continuity of the limit and using convergence (2.29)(ii), the properties of the unfolding operator and convergence (3.4) from Theorem 3.1, we have

$$\begin{aligned}&\int_{\Omega\times Y^*} A(y,u^0)\left(\nabla T_k(u^0)+\nabla_y\widehat{u_k}\right)\left(\nabla T_k(u^0)+\nabla_y\widehat{u_k}\right)\,dx\,dy \\ &\le \liminf_{\varepsilon\to 0}\int_{\Omega\times Y^*} A(y,u^\varepsilon)\mathcal{T}^*_\varepsilon\left(\nabla T_k(u^\varepsilon)\right)\mathcal{T}^*_\varepsilon\left(\nabla T_k(u^\varepsilon)\right)\,dx\,dy \\ &\le \limsup_{\varepsilon\to 0}\int_{\Omega\times Y^*} A(y,u^0)\mathcal{T}^*_\varepsilon\left(\nabla T_k(u^\varepsilon)\right)\mathcal{T}^*_\varepsilon\left(\nabla T_k(u^\varepsilon)\right)\,dx\,dy \\ &= \limsup_{\varepsilon\to 0}|Y|\int_{\widehat{\Omega}_\varepsilon} A^\varepsilon(x,u^\varepsilon)\nabla T_k(u^\varepsilon)\nabla T_k(u^\varepsilon)\,dx \\ &= |Y|\int_\Omega A^0(u^0)\nabla T_k(u^0)\nabla T_k(u^0)\,dx \\ &\le \int_{\Omega\times Y^*} A(y,u^0)\left(\nabla T_k(u^0)+\nabla_y\widehat{u_k}\right)\left(\nabla T_k(u^0)+\nabla_y\widehat{u_k}\right)\,dx\,dy,\end{aligned}$$

where we also used (3.22). This implies the equality of all terms above and proves both equality (3.20) and convergence (3.21). □

The following result shows that convergence (ii) of (2.29) is actually strong.

Theorem 3.3. *Under the assumptions of Theorem 3.1, for all $k \in \mathbb{N}$,*

$$\mathcal{T}^*_\varepsilon(\nabla T_k(u^\varepsilon)) \longrightarrow \nabla T_k(u^0) + \nabla_y \widehat{u_k} \quad \text{strongly in } L^2(\Omega \times Y^*). \tag{3.26}$$

Proof. By the ellipticity of A, we have

$$\alpha \left\| \mathcal{T}^*_\varepsilon(\nabla T_k(u^\varepsilon)) - (\nabla T_k(u^0) + \nabla_y \widehat{u_k}) \right\|^2_{L^2(\Omega\times Y^*)} \leq J_\varepsilon, \tag{3.27}$$

where

$$J_\varepsilon = \int_{\Omega\times Y^*} A(y,u^\varepsilon)\left(\mathcal{T}^*_\varepsilon(\nabla T_k(u^\varepsilon)) - (\nabla T_k(u^0) + \nabla_y \widehat{u_k})\right) \cdot \left(\mathcal{T}^*_\varepsilon(\nabla T_k(u^\varepsilon)) - (\nabla T_k(u^0) + \nabla_y \widehat{u_k})\right) dx\,dy. \tag{3.28}$$

It suffices to show that

$$J_\varepsilon \to 0 \text{ as } \varepsilon \to 0. \tag{3.29}$$

Now, from (3.28), we can write J_ε as

$$J_\varepsilon = J_{\varepsilon,1} - J_{\varepsilon,2} - J_{\varepsilon,3} + J_{\varepsilon,4}, \tag{3.30}$$

where

$$\begin{aligned}
J_{\varepsilon,1} &= \int_{\Omega\times Y^*} A(y,u^\varepsilon)\mathcal{T}^*_\varepsilon(\nabla T_k(u^\varepsilon))\,\mathcal{T}^*_\varepsilon(\nabla T_k(u^\varepsilon))\,dx\,dy,\\
J_{\varepsilon,2} &= \int_{\Omega\times Y^*} A(y,u^\varepsilon)\mathcal{T}^*_\varepsilon(\nabla T_k(u^\varepsilon))\left(\nabla T_k(u^0) + \nabla_y \widehat{u_k}\right) dx\,dy,\\
J_{\varepsilon,3} &= \int_{\Omega\times Y^*} A(y,u^\varepsilon)\left(\nabla T_k(u^0) + \nabla_y \widehat{u_k}\right)\mathcal{T}^*_\varepsilon(\nabla T_k(u^\varepsilon))\,dx\,dy,\\
J_{\varepsilon,4} &= \int_{\Omega\times Y^*} A(y,u^\varepsilon)\left(\nabla T_k(u^0) + \nabla_y \widehat{u_k}\right)\left(\nabla T_k(u^0) + \nabla_y \widehat{u_k}\right) dx\,dy.
\end{aligned}$$

By Proposition 3.2, we get

$$J_{\varepsilon,1} \longrightarrow J_{\varepsilon,4}. \tag{3.31}$$

From the convergences of (2.29) in Theorem 2.6, we also have

$$J_{\varepsilon,2} \longrightarrow J_{\varepsilon,4}, \tag{3.32}$$

$$J_{\varepsilon,3} \longrightarrow J_{\varepsilon,4}. \tag{3.33}$$

Hence, using (3.31)–(3.33), from (3.30), we get (3.29). □

4. The corrector result

In this section, we suppose that the equation is linear, that is, $A(y,t) = A(y)$. Then, (2.12) reads as $A \in (L^\infty(Y))^{N\times N}$, and we set

$$\beta = \|A\|_{L^\infty(Y)}. \tag{4.1}$$

Moreover, the homogenized matrix A^0 is constant and is given by

$$A^0\lambda = \frac{1}{|Y|}\int_{Y^*} A(y)\nabla_y \widehat{w_\lambda}(y)\,dy \qquad \forall \lambda \in \mathbb{R}^N. \tag{4.2}$$

We recall the well-known inequality (see, for instance, [8, Prop. 8.3])

$$\|A^0\|_{L^\infty(Y)} \leq \frac{\beta^2}{\alpha}. \tag{4.3}$$

In this case, the functions $\widehat{\chi_\lambda}$ and $\widehat{w_\lambda}$ defined in (2.36) and (2.42), respectively, are the classical functions used in the linear homogenization. That is, for any $\lambda \in \mathbb{R}^N$, the function $\widehat{\chi_\lambda}$ is the unique solution of

$$\begin{cases} -\operatorname{div}(A\nabla\widehat{\chi_\lambda}) = -\operatorname{div}(A\lambda) & \text{in } Y^*, \\ A\,(\lambda - \nabla\widehat{\chi_\lambda})\cdot n = 0 & \text{on } \partial T, \\ \widehat{\chi_\lambda} \quad Y\text{-periodic}, \\ \mathcal{M}_{Y^*}(\widehat{\chi_\lambda}) = 0, \end{cases} \tag{4.4}$$

and $\widehat{w_\lambda}$ is defined in Y^* by

$$\widehat{w_\lambda}(y) = \lambda y - \widehat{\chi_\lambda}(y). \tag{4.5}$$

Moreover, setting

$$\widehat{w^\varepsilon_\lambda}(x) = \varepsilon\widehat{w}_\lambda(\frac{x}{\varepsilon}) \tag{4.6}$$

in Ω^*_ε, one has

$$A^\varepsilon\nabla\widetilde{w^\varepsilon_\lambda} \rightharpoonup A^0\lambda \quad \text{weakly in } (L^2(\Omega))^n, \tag{4.7}$$

and

$$\int_{\Omega^*_\varepsilon} A^\varepsilon\nabla w^\varepsilon_\lambda \nabla v\,dx = 0, \quad \forall v \in V^\varepsilon. \tag{4.8}$$

For any ε, the corrector matrix for perforated domain $C^\varepsilon = (C^\varepsilon_{ij})_{1\leq i,j\leq N}$, introduced in [10], is defined by

$$\begin{cases} C^\varepsilon(x) = C\left(\dfrac{x}{\varepsilon}\right) & \text{a.e. in } \Omega^*_\varepsilon, \\ C_{ij}(y) = \dfrac{\partial\widehat{w_j}}{\partial y_i}(y) \quad i,j = 1,\dots,N & \text{a.e. on } Y, \end{cases} \tag{4.9}$$

where $\widehat{w_j} = \widehat{w_{e_j}}$ and $\{e_j\}_{j=1}^N$ is the canonical basis of $\mathbb{R}^{\mathbb{N}}$.

We are now ready to present our main corrector result.

Theorem 4.1. *Let u^ε be the renormalized solution to* (2.6) *under the assumptions of Theorem 2.6. Then, for any fixed $k \in \mathbb{N}$, we have*

$$\lim_{\varepsilon\to 0} \left\| \nabla T_k(u^\varepsilon) - C^\varepsilon \nabla T_k(u^0) \right\|_{L^1(\Omega_\varepsilon^*)} = 0. \tag{4.10}$$

The proof of Theorem 4.1 is given at the end of this section and makes use of the following results whose proof is given in next section. It makes use of a somehow "regularized truncation" function H_k^δ (see (4.12)–(4.13)).

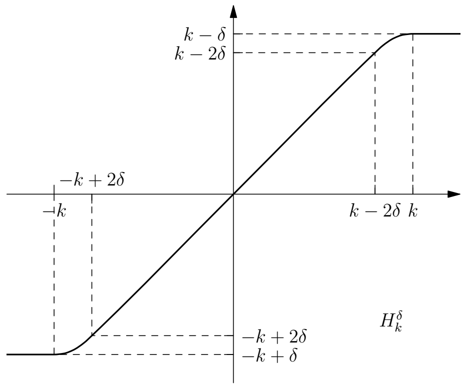

Figure 4. The function H_k^δ.

Remark 4.2. *Let us observe that, in the quasilinear case, the corrector would have the form $C^\varepsilon(x,t) = C\left(\frac{x}{\varepsilon}, t\right)$, and then, in the corrector result, one should replace t by a function of x. To do that, it is necessary to have at least the measurability of C with respect to t, and the proof requires that C (hence, each derivative of w_j) is Lipschitz-continuous in t. This is not the case under our assumptions on A, since this regularity is essentially true only under very strong and global regularity assumptions on A, as proved in [1]. This is not adapted to homogenization, and this is why, in this section, we suppose that A is independent of t.*

Theorem 4.3. *Let u^ε be the renormalized solution to* (2.6) *under the assumptions of Theorem 2.6. Let $k \in \mathbb{N}$ be fixed, and let H be a nondecreasing function in $C^2(\mathbb{R})$ such that $H(0) = 0$, and such that H' has a compact support included in $[-k, k]$. Then, for any $\Phi = (\Phi_1, \Phi_2, \ldots, \Phi_N) \in (\mathcal{D}(\Omega))^N$,*

$$\limsup_{\varepsilon\to 0} \| \nabla H(u^\varepsilon) - C^\varepsilon \Phi \|_{L^2(\Omega_\varepsilon^*)} \leq \frac{\beta}{\alpha} \left\| \nabla H(u^0) - \Phi \right\|_{L^2(\Omega)},$$

where α and β are given by (2.11) *and* (4.1)*, respectively.*

Proof of Theorem 4.1. For a fixed k, for all $\delta > 0$, there exists $\Phi_\delta \in (\mathcal{D}(\Omega))^N$ such that

$$\left\| \Phi_\delta - \nabla T_k(u^0) \right\|_{L^2(\Omega)} \leq \delta. \tag{4.11}$$

Further, for any $\delta > 0$, let $H_k^\delta \in C^2(\mathbb{R})$ be a smooth approximation of T_k verifying (see Figure 4)

$$H_k^\delta(r) = \begin{cases} r, & \text{if } |r| \leq k - 2\delta, \\ k - \delta, & \text{if } r \geq k, \\ -k + \delta, & \text{if } r \leq -k \end{cases} \tag{4.12}$$

with

$$0 \le (H_k^\delta)' \le 2 \text{ in } \mathbb{R}. \tag{4.13}$$

Observe that, by construction, H_k^δ satisfies the assumptions of Theorem 4.3; in particular, its support is contained in $[-k, k]$.

In view of the regularity of $T_k(u^0)$, and by construction of the function H_k^δ, we have

$$\left\| \nabla H_k^\delta(u^0) - \nabla T_k(u^0) \right\|_{L^2(\Omega)} = \omega_2(\delta), \tag{4.14}$$

with $\lim_{\delta\to 0} \omega_2(\delta) = 0$. Let us prove that

$$\limsup_{\varepsilon\to 0} \left\| \nabla H_k^\delta(u^\varepsilon) - \nabla T_k(u^\varepsilon) \right\|_{L^2(\Omega_\varepsilon^*)} \le \frac{\beta}{\alpha}\omega_2(\delta). \tag{4.15}$$

By the definitions of H_k^δ and T_k, and using the ellipticity condition (2.11), we have

$$\begin{aligned} \left\| \nabla H_k^\delta(u^\varepsilon) - \nabla T_k(u^\varepsilon) \right\|^2_{L^2(\Omega_\varepsilon^*)} &= \int_{\Omega_\varepsilon^*} \nabla u^\varepsilon \nabla G_k^\delta(u^\varepsilon)\, dx = \int_{\Omega_\varepsilon^*} (G_k^\delta)'(u_\varepsilon) |\nabla u^\varepsilon|^2\, dx \\ &\le \frac{1}{\alpha} \int_{\Omega_\varepsilon^*} (G_k^\delta)' A^\varepsilon \nabla u^\varepsilon \nabla u^\varepsilon\, dx = \frac{1}{\alpha} \int_{\Omega_\varepsilon^*} A^\varepsilon \nabla u^\varepsilon \nabla G_k^\delta(u^\varepsilon)\, dx, \end{aligned} \tag{4.16}$$

where

$$G_k^\delta = \int_0^r \left((H_k^\delta)' - \chi_{\{|s|\le k\}} \right)^2 ds.$$

Since the function $G = G_k^\delta$ satisfies the assumptions of Theorem 3.1, from (2.12) and (4.16), and by using (4.3), we obtain

$$\begin{aligned} \limsup_{\varepsilon\to 0} \left\| \nabla H_k^\delta(u^\varepsilon) - \nabla T_k(u^\varepsilon) \right\|^2_{L^2(\Omega_\varepsilon^*)} &\le \frac{1}{\alpha} \int_\Omega A^0 \nabla u^0\, \nabla G(u^0)\, dx \\ &\le \frac{\beta^2}{\alpha^2} \left\| \nabla H_k^\delta(u^0) - \nabla T_k(u^0) \right\|^2_{L^2(\Omega)}, \end{aligned}$$

which proves (4.15).

Hence, from Theorem 4.3, (4.11) and (4.15), we have

$$\begin{aligned} 0 &\le \liminf_{\varepsilon\to 0} \left\| \nabla T_k(u^\varepsilon) - C^\varepsilon \nabla T_k(u^0) \right\|_{L^1(\Omega_\varepsilon^*)} \\ &\le \limsup_{\varepsilon\to 0} \left\| \nabla T_k(u^\varepsilon) - C^\varepsilon \nabla T_k(u^0) \right\|_{L^1(\Omega_\varepsilon^*)} \\ &\le \limsup_{\varepsilon\to 0} \left\| \nabla T_k(u^\varepsilon) - \nabla H_k^\delta(u^\varepsilon) \right\|_{L^1(\Omega_\varepsilon^*)} + \limsup_{\varepsilon\to 0} \left\| \nabla H_k^\delta(u^\varepsilon) - C^\varepsilon \Phi_\delta \right\|_{L^1(\Omega_\varepsilon^*)} \\ &\qquad + \limsup_{\varepsilon\to 0} \left\| C^\varepsilon \Phi_\delta - C^\varepsilon \nabla T_k(u^0) \right\|_{L^1(\Omega_\varepsilon^*)} \\ &\le c_1 \limsup_{\varepsilon\to 0} \left\| \nabla T_k(u^\varepsilon) - \nabla H_k^\delta(u^\varepsilon) \right\|_{L^2(\Omega_\varepsilon^*)} + c_1 \limsup_{\varepsilon\to 0} \left\| \nabla H_k^\delta(u^\varepsilon) - C^\varepsilon \Phi_\delta \right\|_{L^2(\Omega_\varepsilon^*)} \\ &\qquad + \| C^\varepsilon \|_{L^2(\Omega)} \left\| \nabla T_k(u^0) - \Phi_\delta \right\|_{L^2(\Omega)} \\ &\le c_1 \frac{\beta}{\alpha} \omega_2(\delta) + c_1 \frac{\beta}{\alpha} \left\| \nabla H_k^\delta(u^0) - \Phi_\delta \right\|_{L^2(\Omega)} + c_2 \left\| \nabla T_k(u^0) - \Phi_\delta \right\|_{L^2(\Omega)} \\ &\qquad \le c\,\omega_3(\delta), \end{aligned} \tag{4.17}$$

where, for a fixed k, $\lim_{\delta\to 0} \omega_3(\delta) = 0$. This proves (4.10). □

5. Proof of Theorem 4.3

For any $\varepsilon > 0$, let u^ε be a renormalized solution of (2.6). Let $k \in \mathbb{N}$ be fixed, and let H be a nondecreasing function in $C^2(\mathbb{R})$ such that $H(0) = 0$ and H' has a compact support included in $[-k, k]$; it follows that $H(u^\varepsilon) = H(T_k(u^\varepsilon))$ belongs to $L^\infty(\Omega_\varepsilon^\star) \cap H_0^1(\Omega_\varepsilon^\star)$.

In the whole proof of Theorem 4.3, to shorten the notations, we set

$$u_k^\varepsilon = T_k(u^\varepsilon) \text{ and } u_k^0 = T_k(u^0).$$

We consider the quantity $\alpha \parallel \nabla H(u^\varepsilon) - C^\varepsilon \Phi \parallel^2_{L^2(\Omega^*_\varepsilon)}$, where $\Phi = (\Phi_1, \Phi_2, \dots, \Phi_N) \in (\mathcal{D}(\Omega))^N$.

Since A^ε is uniformly coercive,

$$\begin{aligned} \alpha \parallel \nabla H(u^\varepsilon) - C^\varepsilon \Phi \parallel^2_{L^2(\Omega^*_\varepsilon)} &\leq \int_{\Omega^*_\varepsilon} A^\varepsilon (\nabla H(u^\varepsilon) - C^\varepsilon \Phi)(\nabla H(u^\varepsilon) - C^\varepsilon \Phi)\, dx \\ &= \int_{\Omega^*_\varepsilon} A^\varepsilon \nabla H(u^\varepsilon) \nabla H(u^\varepsilon)\, dx - \int_{\Omega^*_\varepsilon} A^\varepsilon \nabla H(u^\varepsilon)(C^\varepsilon \Phi)\, dx \\ &\quad - \int_{\Omega^*_\varepsilon} A^\varepsilon (C^\varepsilon \Phi) \nabla H(u^\varepsilon)\, dx + \int_{\Omega^*_\varepsilon} A^\varepsilon (C^\varepsilon \Phi)(C^\varepsilon \Phi)\, dx \\ &\doteq I_\varepsilon^1 - I_\varepsilon^2 - I_\varepsilon^3 + I_\varepsilon^4. \end{aligned} \tag{5.1}$$

We will pass to the limit in (5.1) in each term as $\varepsilon \to 0$.

Let us point out that the difficulties in our situation concern the first three terms studied below in Step 1, Step 2 and Step 3, respectively. In particular, Step 2 requires the most delicate arguments due to the fact that we are dealing with renormalized solutions. Passing to the limit in the last term is standard.

Step 1. Limit of I_ε^1

By defining $G(r) = \int_0^r H'(s)^2 ds$ and recalling that the support of H' is included in $[-k, k]$, we can write

$$\begin{aligned} I_\varepsilon^1 &= \int_{\Omega^*_\varepsilon} A^\varepsilon \nabla H(u^\varepsilon) \nabla H(u^\varepsilon)\, dx = \int_{\Omega^*_\varepsilon} A^\varepsilon \nabla u^\varepsilon \nabla u^\varepsilon (H'(u^\varepsilon))^2\, dx \\ &= \int_{\Omega^*_\varepsilon} A^\varepsilon \nabla u^\varepsilon \nabla G(u^\varepsilon)\, dx = \int_{\Omega^*_\varepsilon} A^\varepsilon \nabla u^\varepsilon \nabla G(u^\varepsilon)\, dx. \end{aligned}$$

Since G is a nondecreasing element of $W^{1,\infty}(\mathbb{R})$ such that $G(0) = 0$, Theorem 3.1 leads to

$$\lim_{\varepsilon \to 0} I_\varepsilon^1 = \int_\Omega A^0 \nabla u^0 \nabla G(u^0)\, dx = \int_\Omega A^0 \nabla H(u^0) \nabla H(u^0)\, dx. \tag{5.2}$$

Step 2. Limit of I_ε^2

For the second integral I_ε^2 on the right-hand side of (5.1), by the definition (4.9) of C^ε, we have

$$\begin{aligned} I_\varepsilon^2 &= \int_{\Omega^\star_\varepsilon} A^\varepsilon \nabla H(u^\varepsilon)(C^\varepsilon \Phi)\, dx = \sum_{i=1}^N \int_{\Omega^\star_\varepsilon} A^\varepsilon \nabla H(u^\varepsilon)(\Phi_i \nabla \widehat{w_i^\varepsilon})\, dx \\ &= \sum_{i=1}^N \left(\int_{\Omega^\star_\varepsilon} A^\varepsilon \nabla H(u^\varepsilon) \nabla (\Phi_i \widehat{w_i^\varepsilon})\, dx - \int_{\Omega^\star_\varepsilon} A^\varepsilon \nabla H(u^\varepsilon) \nabla \Phi_i \widehat{w_i^\varepsilon}\, dx \right). \end{aligned} \tag{5.3}$$

Since H' belongs to $C^1(\mathbb{R})$ and has a compact support, we have $\nabla H(u^\varepsilon) = H'(u^\varepsilon)\nabla u^\varepsilon$ almost everywhere in $\Omega_\varepsilon^\star$. On the other hand, the function $\widehat{w_i^\varepsilon}$ given by (4.6) belongs to $L^\infty(\Omega_\varepsilon^\star) \cap H^1(\Omega_\varepsilon^\star)$, so that $\Phi_i\widehat{w_i^\varepsilon} \in L^\infty(\Omega_\varepsilon^\star) \cap H_0^1(\Omega_\varepsilon^\star)$. Then, choosing $\Phi_i\widehat{w_i^\varepsilon}$ as a test function and $\psi = H'$ in (2.21), we get, for $1 \le i \le N$,

$$\int_{\Omega_\varepsilon^\star} H'(u^\varepsilon)A^\varepsilon(x)\nabla u^\varepsilon\nabla(\Phi_i\widehat{w_i^\varepsilon})\,dx = \int_{\Omega_\varepsilon^\star} fH'(u^\varepsilon)\Phi_i\widehat{w_i^\varepsilon}\,dx + \int_{\Gamma_1^\varepsilon} g^\varepsilon H'(u^\varepsilon)\Phi_i\widehat{w_i^\varepsilon}\,d\sigma_x$$
$$- \int_{\Gamma_1^\varepsilon} \varepsilon^\gamma\tau^\varepsilon(x)H'(u^\varepsilon)h(u^\varepsilon)\Phi_i\widehat{w_i^\varepsilon}\,d\sigma_x - \int_{\Omega_\varepsilon^\star} H''(u^\varepsilon)A^\varepsilon(x)\nabla u^\varepsilon\nabla u^\varepsilon\Phi_i\widehat{w_i^\varepsilon}\,dx. \quad (5.4)$$

Thus, to study the behavior of I_ε^2, it remains to determine the limit of $\int_{\Omega_\varepsilon^\star} A^\varepsilon\nabla H(u^\varepsilon)\nabla\Phi_i\widehat{w_i^\varepsilon}dx$ and the limit of the right hand-side of (5.4) as ε goes to zero.

By the properties of the unfolding operator, Remark 2.5, the convergences in (2.29) and definition (4.6), we compute

$$\begin{aligned}
&\lim_{\varepsilon\to0}\int_{\Omega_\varepsilon^\star} A^\varepsilon\nabla H(u^\varepsilon)\nabla\Phi_i\widehat{w_i^\varepsilon}\,dx\\
=\ &\lim_{\varepsilon\to0}\int_{\Omega_\varepsilon^\star} H'(u^\varepsilon)A^\varepsilon\nabla T_k(u^\varepsilon)\nabla\Phi_i\widehat{w_i^\varepsilon}\,dx\\
=\ &\lim_{\varepsilon\to0}\frac{1}{|Y|}\int_{\Omega\times Y^*} H'(\mathcal{T}_\varepsilon^*(u^\varepsilon))\mathcal{T}_\varepsilon^*(A^\varepsilon)\,\mathcal{T}_\varepsilon^*(\nabla T_k(u^\varepsilon))\,\mathcal{T}_\varepsilon^*(\nabla\Phi_i)\,\mathcal{T}_\varepsilon^*\left(\widehat{w_i^\varepsilon}\right)dx\,dy\\
=\ &\lim_{\varepsilon\to0}\frac{1}{|Y|}\int_{\Omega\times Y^*} H'(\mathcal{T}_\varepsilon^*(u^\varepsilon))A(y)\mathcal{T}_\varepsilon^*(\nabla T_k(u^\varepsilon))\,\mathcal{T}_\varepsilon^*(\nabla\Phi_i)\ \varepsilon\widehat{w}_i(y)\,dx\,dy\\
=\ &\lim_{\varepsilon\to0}\frac{1}{|Y|}\int_{\Omega\times Y^*} H'(\mathcal{T}_\varepsilon^*(u^\varepsilon))A(y)\mathcal{T}_\varepsilon^*(\nabla T_k(u^\varepsilon))\,\mathcal{T}_\varepsilon^*(\nabla\Phi_i)\ (x_i-\varepsilon\widehat{\chi}_i(y))\,dx\,dy\\
=\ &\frac{1}{|Y|}\int_{\Omega\times Y^*} H'(u^0)A(y)\left(\nabla T_k(u^0)+\nabla_y\widehat{u_k}\right)\nabla\Phi_i x_i\,dx\,dy.
\end{aligned} \quad (5.5)$$

We now study the behavior of the terms on the right-hand side of (5.4) when ε goes to zero. In view of (4.6) and the properties of H', and applying Corollary 2.7, we have

$$\lim_{\varepsilon\to0}\int_{\Omega_\varepsilon^\star} fH'(u^\varepsilon)\Phi_i\widehat{w_i^\varepsilon}\,dx = \frac{|Y^*|}{|Y|}\int_\Omega fH'(u^0)\Phi_i x_i\,dx. \quad (5.6)$$

The boundary unfolding operator properties and Remark 2.5 give

$$\begin{aligned}
\int_{\Gamma_1^\varepsilon} g^\varepsilon H'(u^\varepsilon)\Phi_i\widehat{w_i^\varepsilon}\,d\sigma_x &= \varepsilon\int_{\Gamma_1^\varepsilon} g\left(\frac{x}{\varepsilon}\right)H'(u^\varepsilon)\Phi_i\widehat{w_i^\varepsilon}\,d\sigma_x\\
&= \frac{1}{|Y|}\int_{\Omega\times\partial T} g(y)\mathcal{T}_\varepsilon^b(H'(u^\varepsilon))\,\mathcal{T}_\varepsilon^b(\Phi_i)\,\mathcal{T}_\varepsilon^b(\widehat{w_i^\varepsilon})\,dx\,d\sigma_y\\
&= \frac{1}{|Y|}\int_{\Omega\times\partial T} g(y)H'(\mathcal{T}_\varepsilon^b(u^\varepsilon))\mathcal{T}_\varepsilon^b(\Phi_i)\,(x_i-\varepsilon\widehat{\xi}(y))\,dx\,d\sigma_y.
\end{aligned}$$

Next, due to (3.11) and the properties of the boundary unfolding operator, we obtain

$$\lim_{\varepsilon\to0}\int_{\Gamma_1^\varepsilon} g^\varepsilon H'(u^\varepsilon)\Phi_i(x)\widehat{w_i^\varepsilon}\,d\sigma_x = \frac{1}{|Y|}\int_{\Omega\times\partial T} g(y)H'(u^0)\Phi_i x_i\,dx\,d\sigma_y.$$

Since $H'(u^0)$, Φ_i and $x \mapsto x_i$ are independent of y, we get, at last,

$$\lim_{\varepsilon\to 0}\int_{\Gamma_1^\varepsilon} g^\varepsilon H'(u^\varepsilon)\Phi_i(x)\widehat{w_i^\varepsilon}\,d\sigma_x = \frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)\int_\Omega H'(u^0)\Phi_i(x)x_i\,dx. \tag{5.7}$$

Using similar arguments, we get

$$\lim_{\varepsilon\to 0}\int_{\Gamma_1^\varepsilon} \varepsilon^\gamma \tau^\varepsilon(x)H'(u^\varepsilon)h(u^\varepsilon)\Phi_i\widehat{w_i^\varepsilon}\,d\sigma_x = \frac{J(\gamma)}{|Y|}\mathcal{M}_{\partial T}(\tau)\int_\Omega H'(u^0)h(u^0)\Phi_i x_i\,dx, \tag{5.8}$$

where $J(\gamma)$ is given by (2.27).

We now turn to the last term of the right-hand side of (5.4). Again, the unfolding operator and Remark 2.5 allow us to write

$$\begin{aligned}
&\int_{\Omega_\varepsilon^\star} H''(u^\varepsilon)A^\varepsilon(x)\nabla u^\varepsilon\nabla u^\varepsilon\Phi_i\widehat{w_i^\varepsilon}\,dx\\
&\quad= \frac{1}{|Y|}\int_{\Omega\times Y^*} H''(\mathcal{T}_\varepsilon^*(u^\varepsilon))\mathcal{T}_\varepsilon^*(A^\varepsilon)\,\mathcal{T}_\varepsilon^*(\nabla T_k(u^\varepsilon))\,\mathcal{T}_\varepsilon^*(\nabla T_k(u^\varepsilon))\,\mathcal{T}_\varepsilon^*(\Phi_i)\,\mathcal{T}_\varepsilon^*\left(\widehat{w_i^\varepsilon}\right)dx\,dy\\
&\quad= \frac{1}{|Y|}\int_{\Omega\times Y^*} H''(\mathcal{T}_\varepsilon^*(u^\varepsilon))A(y)\mathcal{T}_\varepsilon^*(\nabla T_k(u^\varepsilon))\,\mathcal{T}_\varepsilon^*(\nabla T_k(u^\varepsilon))\,\mathcal{T}_\varepsilon^*(\Phi_i)\,(x_i-\varepsilon\widehat{\chi_i}(y))\,dx\,dy.
\end{aligned} \tag{5.9}$$

From Theorem 3.3, we have

$$\mathcal{T}_\varepsilon^*(\nabla T_k(u^\varepsilon)) \longrightarrow \nabla T_k(u^0)+\nabla_y\widehat{u_k} \quad \text{strongly in } L^2(\Omega\times Y^*) \text{ as } \varepsilon\to 0,$$

and since H'' is a continuous and bounded function, convergence (2.28) implies that

$$H''(\mathcal{T}_\varepsilon^*(u^\varepsilon)) \longrightarrow H''(u^0) \quad \text{in } L^\infty(\Omega\times Y^*) \text{ weak star as } \varepsilon\to 0.$$

By the properties of the unfolding operator, the function $\mathcal{T}_\varepsilon^*(\Phi_i)$ goes to Φ in $L^\infty(\Omega\times Y^*)$ weak star as $\varepsilon\to 0$. It follows that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon^\star} H''(u^\varepsilon)A^\varepsilon(x)\nabla u^\varepsilon\nabla u^\varepsilon\Phi_i\widehat{w_i^\varepsilon}\,dx\\
&\qquad= \frac{1}{|Y|}\int_{\Omega\times Y^*} H''(u^0)A(y)(\nabla T_k(u^0)+\nabla_y\widehat{u_k})(\nabla T_k(u^0)+\nabla_y\widehat{u_k})\Phi(x)x_i\,dx\,dy.
\end{aligned} \tag{5.10}$$

Gathering (5.3), (5.4), (5.5), (5.6), (5.7), (5.8) and (5.10), we obtain

$$\begin{aligned}
\lim_{\varepsilon\to 0} I_\varepsilon^2 = \sum_{i=1}^N\Bigg(&\frac{|Y^*|}{|Y|}\int_\Omega fH'(u^0)\Phi_i x_i\,dx\\
&+\frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)\int_\Omega H'(u^0)\Phi_i(x)x_i\,dx - \frac{1}{|Y|}J(\gamma)\mathcal{M}_{\partial T}(\tau)\int_\Omega H'(u^0)h(u^0)\Phi_i x_i\,dx\\
&-\frac{1}{|Y|}\int_{\Omega\times Y^*} H''(u^0)A(y)(\nabla T_k(u^0)+\nabla_y\widehat{u_k})(\nabla T_k(u^0)+\nabla_y\widehat{u_k})\Phi(x)x_i\,dx\,dy\\
&-\frac{1}{|Y|}\int_{\Omega\times Y^*} H'(u^0)A(y)\left(\nabla T_k(u^0)+\nabla_y\widehat{u_k}\right)\nabla\Phi_i x_i\,dx\,dy\Bigg).
\end{aligned} \tag{5.11}$$

For $1 \leq i \leq N$, using $\eta_0 = \Phi_i x_i$ and $\Psi = \widehat{u_k}\Phi_i x_i$ as test functions in (2.31) with $S = H'$ and $S_1 = H''$, and recalling that $\mathrm{supp}(H') \subset [-k, k]$, we get

$$\int_{\Omega\times Y^*} H''(u^0)A(y)(\nabla T_k(u^0) + \nabla_y\widehat{u_k})(\nabla T_k(u^0) + \nabla_y\widehat{u_k})\Phi(x)x_i\,dx\,dy$$
$$+ \int_{\Omega\times Y^*} H'(u^0)A(y)\left(\nabla T_k(u^0) + \nabla_y\widehat{u_k}\right)\nabla(\Phi_i x_i)\,dx\,dy + J(\gamma)\mathcal{M}_{\partial T}(\tau)\int_\Omega H'(u^0)h(u^0)\Phi_i x_i\,dx$$
$$= |Y^*|\int_\Omega f H'(u^0)\Phi_i x_i\,dx + |\partial T|\mathcal{M}_{\partial T}(g)\int_\Omega H'(u^0)\Phi_i(x)x_i\,dx.$$

Because $\nabla(\Phi_i x_i) = \nabla\Phi_i x_i + \Phi_i e_i$, using (2.43) written for $T_k(u^0)$, we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0} I^2_\varepsilon &= \sum_{i=1}^N\left(\frac{1}{|Y|}\int_{\Omega\times Y^*} H'(u^0)A(y)\left(\nabla T_k(u^0) + \nabla_y\widehat{u_k}\right)e_i\Phi_i\,dx\,dy\right)\\ &= \sum_{i=1}^N\left(\int_\Omega H'(u^0)A^0\nabla T_k(u^0)e_i\Phi_i\,dx\right)\\ &= \int_\Omega H'(u^0)A^0\nabla T_k(u^0)\Phi\,dx = \int_\Omega A^0\nabla H(u^0)\Phi\,dx.\end{aligned} \tag{5.12}$$

Step 3. Limit of I^3_ε and I^4_ε

From (4.8), we have

$$I^3_\varepsilon = \sum_{i=1}^n\int_{\Omega^*_\varepsilon} A^\varepsilon\nabla w^\varepsilon_\lambda\Phi_i\nabla H(u^\varepsilon)\,dx = -\sum_{i=1}^n\int_{\Omega^*_\varepsilon} A^\varepsilon\nabla w^\varepsilon_\lambda H(u^\varepsilon)\nabla\Phi_i\,dx. \tag{5.13}$$

Using convergence (2.34) given in Corollary 2.7, and (4.7), we can pass to the limit in (5.13). Since, here, A^0 is constant, we obtain

$$\lim_{\varepsilon\to 0} I^3_\varepsilon = -\int_\Omega A^0\,H(u^0)\nabla\Phi\,dx = \int_\Omega A^0\,\Phi\,\nabla H(u^0)\,dx. \tag{5.14}$$

On the other hand, it has been proved in [10] that

$$\lim_{\varepsilon\to 0} I^4_\varepsilon = \int_\Omega A^0\Phi\Phi\,dx, \tag{5.15}$$

which ends this step.

Step 4. Conclusion

Collecting (5.2), (5.12), (5.14) and (5.15), we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\left(I^1_\varepsilon - I^2_\varepsilon - I^3_\varepsilon + I^4_\varepsilon\right) &= \int_\Omega A^0\nabla H(u^0)\nabla H(u^0)\,dx - \int_\Omega A^0\nabla H(u^0)\Phi\,dx\\ &\quad - \int_\Omega A^0\,\Phi\,\nabla H(u^0)\,dx + \int_\Omega A^0\Phi\Phi\,dx\\ &= \int_\Omega A^0\left(\nabla H(u^0) - \Phi\right)\left(\nabla H(u^0) - \Phi\right)dx\\ &= \int_\Omega A^0\left(\nabla H(u^0) - \Phi\right)\left(\nabla H(u^0) - \Phi\right)dx.\end{aligned} \tag{5.16}$$

From (5.1), the ellipticity of the matrix A (see (2.11)) and (4.3), we obtain

$$\begin{aligned}\alpha \limsup_{\varepsilon\to 0} \| \nabla H(u^\varepsilon) - C^\varepsilon \Phi \|^2_{L^2(\Omega)} &\leq \limsup_{\varepsilon\to 0} \left(I^1_\varepsilon - I^2_\varepsilon - I^3_\varepsilon + I^4_\varepsilon \right) \\ &\leq \frac{\beta^2}{\alpha} \int_\Omega \left| \nabla H(u^0) - \Phi \right|^2 \, dx,\end{aligned} \tag{5.17}$$

which concludes the proof. □

Conflict of interest

The authors declare that there is no conflict of interest.

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