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Research article

Isoperimetric planar clusters with infinitely many regions

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Abstract: In this paper we study infinite isoperimetric clusters. An infinite cluster $\mathbf{E}$ in $\mathbb{R}^d$ is a sequence of disjoint measurable sets $E_k \subset \mathbb{R}^d$, called regions of the cluster, $k = 1, 2, 3, \dots$ A natural question is the existence of a cluster $\mathbf{E}$ with given volumes $a_k \geq 0$ of the regions E_k, having finite perimeter $P(\mathbf{E})$, which is minimal among all the clusters with regions having the same volumes. We prove that such a cluster exists in the planar case $d = 2$, for any choice of the areas a_k with $\sum \sqrt{a_k} < \infty$. We also show the existence of a bounded minimizer with the property $P(\mathbf{E}) = \mathcal{H}^1(\tilde{\partial}\mathbf{E})$, where $\tilde{\partial}\mathbf{E}$ denotes the measure theoretic boundary of the cluster. Finally, we provide several examples of infinite isoperimetric clusters for anisotropic and fractional perimeters.

Keywords: isoperimetric clusters; isoperimetric sets; regularity

1. Introduction

A finite cluster $\mathbf{E}$ is a sequence $\mathbf{E} = (E_1, \dots E_k, \dots, E_N)$ of Lebesgue measurable sets in $\mathbb{R}^d$, such that $|E_k \cap E_j| = 0$ for $k \neq j$, where $|\cdot|$ denotes the Lebesgue measure (usually called *volume*). The sets E_j are called *regions* of the cluster $\mathbf{E}$, and $E_0 := \mathbb{R}^d \setminus \bigcup_{k=1}^N E_k$ is called the *external region*. We denote the sequence of volumes of the regions of the cluster $\mathbf{E}$ as

$$\mathbf{m}(\mathbf{E}) := (|E_1|, |E_2|, \dots, |E_N|) \tag{1.1}$$

and call the perimeter of the cluster the quantity

$$P(\mathbf{E}) := \frac{1}{2}\left[P(E_0) + \sum_{k=1}^{N} P(E_k)\right], \tag{1.2}$$

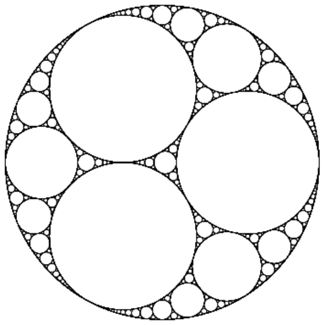 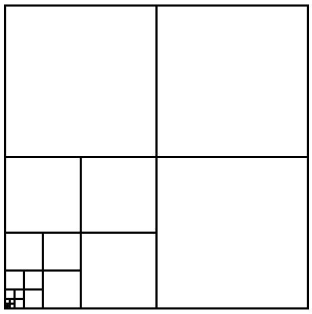

Figure 1. The Apollonian gasket; on the left, is a cluster with a minimal fractional perimeter. On the right is a similar construction with squares: this is a minimal cluster with respect to the perimeter induced by the Manhattan distance.

where $P(A)$ stands for the Caccioppoli perimeter of the set A. A cluster $\mathbf{E}$ is called *minimal*, or *isoperimetric*, if

$$P(\mathbf{E}) = \min\{P(\mathbf{F})\colon \mathbf{m}(\mathbf{F}) = \mathbf{m}(\mathbf{E})\}.$$

In this paper we consider infinite clusters, i.e., infinite sequences $\mathbf{E} = (E_k)_{k\geq 1}$ of essentially pairwise disjoint regions: $\left|E_j \cap E_i\right| = 0$ for $i \neq j$ (this can be interpreted as a model for a soap foam). We define $E_0 := \mathbb{R}^d \setminus \bigcup_{k=1}^{\infty} E_k$, i.e., the external region of the cluster $\mathbf{E}$. The perimeter of an infinite cluster $\mathbf{E}$ is defined by Eq (1.2) with $N := +\infty$. Note that a finite cluster with N regions can also be considered a particular case of an infinite cluster, for example by posing $E_k := \emptyset$ for $k > N$. Clusters with infinitely many regions of equal area were considered in [13], where it has been shown that the honeycomb cluster is the unique minimizer with respect to compact perturbations. Infinite clusters have been considered also in [4, 14, 16], dealing with Apollonian packing, in [18] where variational curvatures are prescribed rather than volumes, and in [24], where the existence of generalized minimizers for both finite and infinite isoperimetric clusters has been proven in the general setting of homogeneous metric measure spaces.

Note that very few explicit examples of minimal clusters are known [11,15,20,26,27], and only with a finite (and quite small) number of regions. An example of an infinite minimal cluster, described in details in Example 4.1, is the Apollonian packing of a circle shown in Figure 1 (see [16]). In fact, this cluster is composed of isoperimetric regions and hence should trivially have minimal perimeter among clusters with regions of the same areas. However, it turns out that this cluster has an infinite perimeter; hence all clusters with the same prescribed areas have an infinite perimeter too. Nevertheless, quite curiously, Apollonian packings of a circle give nontrivial examples of infinite isoperimetric clusters for fractional perimeters [5, 7, 8], as shown in Example 4.1. An even simpler example of an infinite isoperimetric planar cluster is given in Example 4.2 (see Figure 1) where the Caccioppoli perimeter is replaced by an anisotropic perimeter functional. For general results about finite clusters minimizing anisotropic perimeters, we refer to [1, 6, 9, 17, 22, 23].

Our main result, Theorem 3.1, states that if $d = 2$ (the planar case), given any sequence of positive numbers $\mathbf{a} = (a_1, a_2, \dots, a_k, \dots)$ such that $\sum_{k=0}^{\infty} \sqrt{a_k} < +\infty$, there exists a minimal cluster $\mathbf{E}$ in $\mathbb{R}^2$ with $\mathbf{m}(\mathbf{E}) = \mathbf{a}$ and with finite perimeter. The assumption on $\mathbf{a}$ in fact is necessary to have at least a competitor cluster with a finite perimeter. The proof relies on two facts that are only available in the planar case: the isodiametric inequality for connected sets and the semicontinuity of the length (i.e., the one-dimensional Hausdorff measure) of connected sets (Gołąb theorem).

In dimension $d > 2$, very few results are currently known. Existence can be obtained only in a generalized sense, as shown in [24], and we cannot even exclude that $\tilde{\partial}\mathbf{E} = \mathbb{R}^d$.

2. Notation and preliminaries

2.1. Perimeters and boundaries

For a set $E \subset \mathbb{R}^d$ with finite perimeter one can define the *reduced boundary* $\partial^* E$ as the set of boundary points x where the outer normal vector $\nu_E(x)$ can be defined. One has $D1_E = \nu_E \cdot \mathcal{H}^{d-1} \llcorner \partial^* E$ where 1_E is the characteristic function of E and $D1_E$ is its distributional derivative (the latter is a vector valued measure and its total variation is denoted by $|D1_E|$). We use a version of the measure theoretic boundary of a measurable set E defined by

$$\tilde{\partial}E := \{x \in \mathbb{R}^d \colon 0 < \left|E \cap B_\rho(x)\right| < \left|B_\rho(x)\right| \quad \text{for all } \rho > 0\}.$$

The respective notions for clusters can be defined by setting

$$\begin{aligned} \partial^*\mathbf{E} &:= \bigcup_{k=1}^{+\infty} \bigcup_{j=0}^{k-1} \partial^* E_k \cap \partial^* E_j, \\ \tilde{\partial}\mathbf{E} &:= \{x \in \mathbb{R}^d \colon 0 < \left|E_k \cap B_\rho(x)\right| < \left|B_\rho(x)\right| \\ &\qquad\qquad \text{for all } \rho > 0 \text{ and some } k = k(\rho, x) \in \mathbb{N}\}. \end{aligned}$$

Clearly $\partial^*\mathbf{E} \subseteq \tilde{\partial}\mathbf{E}$ because given an $x \in \partial^*\mathbf{E}$, there exists a k such that $x \in \partial^* E_k$, while $\tilde{\partial}E_k \subseteq \tilde{\partial}\mathbf{E}$ for all k. Also it is easy to check that $\tilde{\partial}\mathbf{E}$ is closed (and in fact is the closure of the union of all the measure theoretic boundaries $\tilde{\partial}E_k$). Moreover the following result holds true.

Proposition 2.1. *If* $\mathbf{E}$ *is a cluster with finite perimeter, then* $P(\mathbf{E}) = \mathcal{H}^{d-1}(\partial^*\mathbf{E})$.

Proof. Consider the sets X_n, defined for $1 \le n \le \infty$ by

$$X_n := \left\{x \in \mathbb{R}^d \colon \#\{k \in \mathbb{N} \colon x \in \partial^* E_k\} = n\right\}$$

(notice that $k = 0 \in \mathbb{N}$, the external region, is included in the count). It is clear that $X_n = \emptyset$ for all $n \ge 3$ because in every point of $\partial^* E_k$ there is an approximate tangent hyperplane which can only be shared by two regions.

We claim that $\mathcal{H}^{d-1}(X_1) = 0$. To this aim suppose by contradiction that $\mathcal{H}^{d-1}(X_1) > 0$. Then there exists a $j \in \mathbb{N}$ such that

$$\left|D1_{E_j}\right|(X_1) = \mathcal{H}^{d-1}(X_1 \cap \partial^* E_j) > 0,$$

because X_1 is contained in the countable union $\left(\cup_{j=0}^{\infty} X_1\right) \cap \partial^* E_j$. Hence there is a subset $A \subset X_1 \cap \partial^* E_j$ such that $D1_{E_j}(A) \neq \mathbf{0}$. Notice that $\sum_{k=0}^{\infty} 1_{E_k} = 1$, hence also $\sum_k D1_{E_k} = \mathbf{0}$ in the sense of distributions. Moreover, if $P(\mathbf{E}) < +\infty$ the above convergence holds also in the sense of vector measures (in total variation), hence $\sum_k D1_{E_k}(A) = \mathbf{0}$. Since $D1_{E_j}(A) \neq \mathbf{0}$, there must exist at least another index $k \neq j$ such that $D1_{E_k}(A) \neq \mathbf{0}$, and therefore $\mathcal{H}^{d-1}(A \cap \partial^* E_k) > 0$. But then

$$\emptyset \neq A \cap \partial^* E_k \subset X_1 \cap \partial^* E_j \cap \partial^* E_k, \quad j \neq k,$$

contrary to the definition of X_1, which proves the claim. In conclusion, the union of all the reduced boundaries $\partial^* E_k$ is contained in X_2 up to an $\mathcal{H}^{d-1}$-negligible set. Hence

$$
\begin{aligned}
P(\mathbf{E}) &= \frac{1}{2}\sum_{k=0}^{+\infty} P(E_k) = \frac{1}{2}\sum_{k=0}^{+\infty} \mathcal{H}^{d-1}(\partial^* E_k \cap X_2) = \\
&= \frac{1}{2}\sum_{k=0}^{+\infty}\sum_{j\neq k} \mathcal{H}^{d-1}(\partial^* E_k \cap \partial^* E_j) = \sum_{k=0}^{+\infty}\sum_{j=k+1}^{+\infty} \mathcal{H}^{d-1}(\partial^* E_k \cap \partial^* E_j) = \\
&= \mathcal{H}^{d-1}\left(\bigsqcup_{k=0}^{+\infty}\bigsqcup_{j=k+1}^{+\infty} \partial^* E_k \cap \partial^* E_j\right) = \mathcal{H}^{d-1}(\partial^* \mathbf{E})
\end{aligned}
$$

as claimed. ◻

2.2. Auxiliary results

In the following theorem we collect some known existence and regularity results for *finite* minimal clusters from [19, 21].

Theorem 2.2 (existence and regularity of planar N-clusters)**.** *Let $a_1, a_2, \dots, a_N$ be given positive real numbers. Then there exists a minimal N-cluster $\mathbf{E} = (E_1, \dots E_N)$ in $\mathbb{R}^d$, with $|E_k| = a_k$ for $k = 1, \dots, N$. If $\mathbf{E}$ is a minimal N-cluster and $d = 2$, then $\tilde{\partial}\mathbf{E}$ is a pathwise connected set composed by circular arcs or line segments joining in triples at a finite number of vertices. Moreover in this case $P(\mathbf{E}) = \mathcal{H}^1(\tilde{\partial}\mathbf{E})$.*

Proposition 2.3 (cluster truncation)**.** *Let $\mathbf{E} = (E_1, \dots, E_k, \dots)$ be a (finite or infinite) cluster and let $T_N\mathbf{E}$ be the N-cluster $(E_1, \dots, E_N)$. Then*

$$P(T_N\mathbf{E}) \leq P(\mathbf{E}).$$

Proof. For measurable sets E, F the inequality

$$P(E \cup F) + P(E \cap F) \leq P(E) + P(F)$$

holds, hence if $|E \cap F| = 0$, one has

$$P(E) = P((E \cup F) \cap (\mathbb{R}^d \setminus F)) \leq P(E \cup F) + P(F).$$

It follows that

$$
\begin{aligned}
2P(T_N\mathbf{E}) &= \sum_{k=1}^{n} P(E_k) + P\left(\bigcup_{k=1}^{n} E_k\right) \\
&\leq \sum_{k=1}^{n} P(E_k) + P\left(\bigcup_{k=1}^{\infty} E_k\right) + P\left(\bigcup_{k=n+1}^{\infty} E_k\right) \\
&\leq \sum_{k=1}^{n} P(E_k) + P\left(\bigcup_{k=1}^{\infty} E_k\right) + \sum_{k=n+1}^{\infty} P(E_k) \\
&= \sum_{k=1}^{\infty} P(E_k) + P\left(\bigcup_{k=1}^{\infty} E_k\right) = 2P(\mathbf{E})
\end{aligned}
$$

as claimed. ◻

Lemma 2.4. *Let E be a measurable set and Ω be an open connected set. If $\tilde{\partial}E \cap \Omega = \emptyset$, then either $|\Omega \cap E| = 0$ or $|\Omega \setminus E| = 0$.*

Proof. Notice that $\Omega \setminus \tilde{\partial}E = A_0 \cup A_1$, where

$$A_0 := \left\{x \in \Omega \colon \left|B_\rho(x) \cap E\right| = 0 \quad \text{for some } \rho > 0\right\},$$
$$A_1 := \left\{x \in \Omega \colon \left|B_\rho(x) \setminus E\right| = 0 \quad \text{for some } \rho > 0\right\}.$$

It is clear that A_0 and A_1 are open disjoint sets, and if $\tilde{\partial}E \cap \Omega = \emptyset$, then their union is the whole set Ω. If Ω is connected, it implies that either A_0 or A_1 is equal to Ω which means that either $|\Omega \cap E| = 0$ or $|\Omega \setminus E| = 0$. □

3. Main result

The statement below provides existence of infinite planar isoperimetric clusters.

Theorem 3.1 (existence). *Let $\mathbf{a} = (a_1, \dots, a_k, \dots)$ be a sequence of nonnegative numbers such that $\sum_{k=1}^\infty \sqrt{a_k} < \infty$. Then there exists a minimal cluster $\mathbf{E}$ in $\mathbb{R}^2$, with $\mathbf{m}(\mathbf{E}) = \mathbf{a}$ satisfying additionally*

$$\bigcup_{k=1}^\infty E_k \text{ is bounded,} \tag{3.1}$$

$$\tilde{\partial}\mathbf{E} \text{ is pathwise connected,} \tag{3.2}$$

$$\mathcal{H}^1(\tilde{\partial}\mathbf{E} \setminus \partial^*\mathbf{E}) = 0. \tag{3.3}$$

Remark 3.2. In view of Eq (3.3) and Proposition 2.1, for the minimal cluster provided by Theorem 3.1, one has

$$P(\mathbf{E}) = \mathcal{H}^1(\tilde{\partial}\mathbf{E}) = \mathcal{H}^1(\partial^*\mathbf{E}). \tag{3.4}$$

Of course there exists a set with finite perimeter E such that $P(E) < \mathcal{H}^1(\tilde{\partial}E)$, so that Eq (3.4) is false for general clusters that are not minimal.

It is interesting to note that, as shown in Example 4.3, there exists a finite cluster $\mathbf{E}$ satisfying Eq (3.4), for which one does not have $P(E_k) = \mathcal{H}^1(\tilde{\partial}E_k)$ for all k. It would be interesting to see whether these equalities hold for minimal clusters.

Proof. Let $\bar{p} := 2\sqrt{\pi}\sum_{k=1}^\infty \sqrt{a_k} < +\infty$, and

$$p := \inf\{P(\mathbf{E}) \colon \mathbf{E} \text{ cluster in } \mathbb{R}^2 \text{ with } |E_k| = a_k,\ k = 1, 2, \dots, n, \dots\},$$
$$p_n := \inf\{P(\mathbf{E}) \colon \mathbf{E}\ n\text{-cluster in } \mathbb{R}^2 \text{ with } |E_k| = a_k,\ k = 1, \dots, n\},$$

so that a cluster $\mathbf{E}$ with measures $\mathbf{m}(\mathbf{E}) = \mathbf{a}$ is minimal, if and only if $P(\mathbf{E}) = p$, while an n-cluster $\mathbf{E}$ with measures $|E_k| = a_k$ for $k = 1, \dots, n$ is minimal, if and only if $P(\mathbf{E}) = p_n$.

If $\mathbf{E}$ is a competitor for p, then $T_n\mathbf{E}$ is a competitor for p_n and, by Proposition 2.3, one has $P(T_n\mathbf{E}) \le P(\mathbf{E})$. Hence $p_n \le p$. Moreover one can build a competitor for p which is composed by circular disjoint regions $(B_1, \dots, B_j, \dots)$, where B_j are disjoint balls of radii $\sqrt{\frac{a_j}{\pi}}$, to find that $p \le \bar{p} < +\infty$.

For each $n \geq 1$ consider a minimal n-cluster $\mathbf{F}^n$ with $|F^n_k| = a_k$ for $k \leq n$, and $F^n_k := \emptyset$ for $k > n$, so that $P(\mathbf{F}^n) = p_n$. By the regularity properties of minimal clusters (Theorem 2.2), the boundary $\tilde{\partial}\mathbf{F}^n$ is connected and composed by a finite union of circular arcs, hence $\operatorname{diam}\tilde{\partial}\mathbf{F}^n \leq p_n \leq \bar{p}$. Up to translations, we shall suppose that all the regions F^n_k of all the clusters $\mathbf{F}^n$ are contained in a ball of radius $\bar{p}$. In fact,

$$\bar{p} \geq p \geq \sup_n p_n = \sup_n P(\mathbf{F}^n) \geq \sup_n \operatorname{diam}\tilde{\partial}\mathbf{F}^n.$$

Up to a subsequence we can hence assume that the first regions F^n_1 converge to a set E_1 in the sense that their characteristic functions $\mathbf{1}_{F^n_1}$ converge to the characteristic function $\mathbf{1}_{E_1}$ in the Lebesgue space $L^1(\mathbb{R}^2)$ as $n \to \infty$ (we call this convergence L^1 convergence of sets). Analogously, up to a sub-subsequence also the second regions F^n_2 converge in L^1 sense to a set E_2. In this way we define inductively the sets E_k for all $k \geq 1$. Then there exists a diagonal subsequence with indices n_j such that for all k one has $F^{n_j}_k \to E_k$ in L^1 for all $k \geq 1$ as $j \to +\infty$.

Consider the cluster $\mathbf{E}$ with the regions E_k defined above. By continuity we have $\mathbf{m}(\mathbf{E}) = \mathbf{a}$ because $F^{n_j}_k \to E_k$ in L^1 as $j \to \infty$ and $\left|F^{n_j}_k\right| = a_k$ for all j. We claim that the union of all the regions of $\mathbf{F}^{n_j}$ also converges to the union of all the regions of $\mathbf{E}$. For all $\varepsilon > 0$ take N such that $\sum_{k=N+1}^{\infty} a_k \leq \varepsilon$ and notice that

$$\left(\bigcup_{k=1}^{\infty} E_k\right) \triangle \left(\bigcup_{k=1}^{\infty} F^{n_j}_k\right) \subseteq \bigcup_{k=1}^{N}\left(E_k \triangle F^{n_j}_k\right) \cup \bigcup_{k=N+1}^{\infty} E_k \cup \bigcup_{k=N+1}^{\infty} F^{n_j}_k.$$

Hence

$$\limsup_j \left|\bigcup_{k=1}^{\infty} E_k \triangle \bigcup_{k=1}^{\infty} F^{n_j}_k\right| \leq \lim_j \sum_{k=1}^{N} \left|E_k \triangle F^{n_j}_k\right| + 2\varepsilon = 2\varepsilon.$$

Letting $\varepsilon \to 0$ we obtain the claim.

By the lower semicontinuity of the perimeter, we have

$$P(E_k) \leq \liminf_{j\to+\infty} P(F^{n_j}_k) \qquad \text{and} \qquad P\left(\bigcup_{k=1}^{+\infty} E_k\right) \leq \liminf_{j\to+\infty} P\left(\bigcup_{k=1}^{+\infty} F^{n_j}_k\right),$$

and hence $P(\mathbf{E}) \leq \liminf_j P(\mathbf{F}^{n_j}) \leq p$, proving that $\mathbf{E}$ is actually a minimal cluster. Since all the regions F^n_k are equibounded, we obtain Eq (3.1).

We are going to prove Eq (3.3). By Theorem 2.2 the minimal n-cluster $\mathbf{F}^n$ has a measure theoretic boundary $\tilde{\partial}\mathbf{F}^n$ which is a compact and connected set such that $P(\mathbf{F}^n) = \mathcal{H}^1(\tilde{\partial}\mathbf{F}^n)$. Up to a subsequence, the compact sets $\tilde{\partial}\mathbf{F}^{n_j}$, being uniformly bounded, converge with respect to the Hausdorff distance, to a compact set K. Without loss of generality suppose n_j is labeling this new subsequence.

We claim that $\tilde{\partial}\mathbf{E} \subseteq K$. In fact for any given $x \in \tilde{\partial}\mathbf{E}$ and any $\rho > 0$ there exists $k = k(\rho)$ such that $B_\rho(x) \cap E_k$ and $B_\rho(x) \setminus E_k$ both have positive Lebesgue measure. Since $\left|B_\rho(x) \cap F^{n_j}_k\right| \to \left|B_\rho(x) \cap E_k\right| > 0$ and $\left|B_\rho(x) \setminus F^{n_j}_k\right| \to \left|B_\rho(x) \setminus E_k\right| > 0$ for $j = j(\rho)$ sufficiently large by Lemma 2.4, there is a point $x^j_k \in B_\rho(x) \cap \tilde{\partial}F^{n_j}_k$. As $\rho \to 0$ the sequence x^j_k converges to x, and since $\tilde{\partial}F^{n_j}_k \subseteq \tilde{\partial}\mathbf{F}^{n_j}$ we conclude that $x \in K$.

The sets $\tilde{\partial}\mathbf{F}^n$ are connected, and therefore, by the classical Gołąb theorem on semicontinuity of one-dimensional Hausdorff measure over sequences of connected sets (see [3, Theorem 4.4.17] or [25, theorem 3.3] for its most general statement and a complete proof), one has

$$\mathcal{H}^1(K) \leq \liminf_n \mathcal{H}^1(\tilde{\partial}\mathbf{F}^n)$$

and K is itself connected. Summing up and using Proposition 2.1, we get

$$\begin{aligned} P(\mathbf{E}) &= \mathcal{H}^1(\partial^*\mathbf{E}) \le \mathcal{H}^1(\tilde{\partial}\mathbf{E}) \le \mathcal{H}^1(K) \\ &\le \liminf_n P(\mathbf{F}^n) \le \limsup_n p_n \le p \le P(\mathbf{E}), \end{aligned} \tag{3.5}$$

hence $\mathcal{H}^1(\partial^*\mathbf{E}) = \mathcal{H}^1(\tilde{\partial}\mathbf{E}) = \mathcal{H}^1(K)$, $p_n \to p$, and Eq (3.3) follows.

Finally, to prove that $\tilde{\partial}\mathbf{E}$ is connected, it is enough to show $\tilde{\partial}\mathbf{E} = K$. We already know that $\tilde{\partial}\mathbf{E} \subseteq K$ so we suppose by contradiction that there exists an $x \in K \setminus \tilde{\partial}\mathbf{E}$. Take any $y \in K$. The set K is arcwise connected by rectifiable arcs, since it is a compact connected set of finite one-dimensional Hausdorff measure (see e.g., [10, lemma 3.11] or [3, theorem 4.4.7]), in other words, there exists an injective continuous curve $\gamma\colon [0,1] \to K$ with $\gamma(0) = x$ and $\gamma(1) = y$. Since $\tilde{\partial}\mathbf{E}$ is closed in K there is a small $\varepsilon > 0$ such that $\gamma([0,\varepsilon]) \subset K \setminus \tilde{\partial}\mathbf{E}$, and hence $\mathcal{H}^1(K \setminus \tilde{\partial}\mathbf{E}) > 0$ contrary to $\mathcal{H}^1(K) = \mathcal{H}^1(\tilde{\partial}\mathbf{E})$. This contradiction shows the last claim and hence concludes the proof. □

4. Some examples

We collect here some interesting examples of infinite planar clusters.

Example 4.1 (Apollonian packing). A cluster $\mathbf{E}$, as depicted in Figure 1, can be constructed so that each region $E_k = B_{r_k}(x_k)$, $k \neq 0$, is a ball contained in the ball $B := B_1(0)$. The balls can be chosen to be pairwise disjoint and such that $\left|B \setminus \bigcup_{k=1}^\infty E_k\right| = 0$ (see [16]).

Clearly such a cluster must be minimal because each region E_k has the minimum possible perimeter among sets with the given area and the same is true for the complement of the exterior region E_0 which is their union. However, one has $P(\mathbf{E}) = +\infty$. In fact, $\mathcal{H}^1(\partial^*\mathbf{E}) = 0$ since $\partial^* E_k \cap \partial^* E_j$ is either empty or a singleton for all $k \neq j$, and by Proposition 2.1 if $P(\mathbf{E}) < +\infty$ one would have $P(\mathbf{E}) = \mathcal{H}^1(\partial^*\mathbf{E}) = 0$, while on the other hand $P(\mathbf{E}) > P(E_0)/2 > 0$, a contradiction.

Note that this is a quite pathological example of an infinite cluster for which $\mathcal{H}^1(\partial^*\mathbf{E}) = 0$ but both $P(\mathbf{E}) = +\infty$ and $\mathcal{H}^1(\tilde{\partial}\mathbf{E}) = +\infty$. In fact, the measure theoretic boundary $\tilde{\partial}\mathbf{E}$ of this cluster is the *residual set*, i.e. the set of zero measure which remains when the balls E_k are removed from the large ball $\overline{B}$:

$$\tilde{\partial}\mathbf{E} = \overline{B} \setminus \bigcup_{k=1}^{+\infty} B_{r_k}(x_k). \tag{4.1}$$

This set has Hausdorff dimension $d > 1$ (see [14]) and hence $\mathcal{H}^1(\tilde{\partial}\mathbf{E}) = +\infty$.

However we can consider the fractional (nonlocal) perimeter P_s defined by

$$P_s(E) := \int_E \int_{\mathbb{R}^2 \setminus E} \frac{1}{|x-y|^{2+s}}\, dx\, dy$$

which induces the respective nonlocal perimeter $P_s(\mathbf{E})$ of the cluster $\mathbf{E}$ by means of definition (1.2) with P_s in place of P. If r_k is the radius of the k-th disk of the cluster it turns out (see [4]) that the infimum of all α, such that the series $\sum_k r_k^\alpha$ converges, is equal to d, the Hausdorff dimension of $\tilde{\partial}\mathbf{E}$. Since $d \in (1,2)$ for all $s < 2-d$, we have

$$\sum_k r_k^{2-s} < +\infty,$$

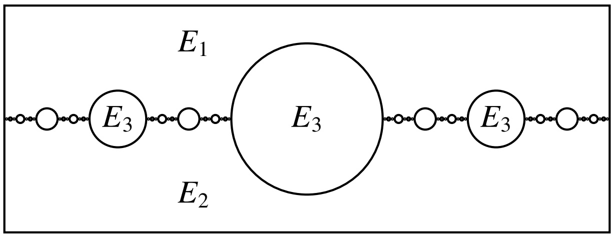

Figure 2. An example of a cluster $\mathbf{E}$ with finite perimeter such that $P(\mathbf{E}) = \mathcal{H}^1(\tilde{\partial}\mathbf{E})$, but $P(E_3) < \mathcal{H}^1(\tilde{\partial}E_3)$.

and since $P_s(B_r) = C \cdot r^{2-s}$ (with $0 < C < +\infty$), we obtain $P_s(\mathbf{E}) < +\infty$ for such s. It is well known (see [12]) that the solution to the fractional isoperimetric problem is given by balls, hence $\mathbf{E}$ provides an example of an infinite minimal cluster with respect to the fractional perimeter P_s.

Example 4.2 (Anisotropic isoperimetric packing). We can find a similar example, if we consider an anisotropic perimeter such that the isoperimetric problem has the square (instead of the circle) as a solution. If ϕ is any norm on $\mathbb{R}^2$, one can define the perimeter P_ϕ which is the relaxation of the functional P_ϕ defined on regular sets $E \subset \mathbb{R}^2$ by the formula

$$P_\phi(E) := \int_{\partial^* E} \phi(\nu_E(x))\, d\mathcal{H}^1(x),$$

where $\nu_E(x)$ is the exterior unit normal vector to the reduced boundary $\partial^* E$ in x. If $\phi(x, y) = |x| + |y|$ (the *Manhattan* norm) it is well known that the P_ϕ-minimal set with prescribed area (i.e., the *Wulff shape*) is a square with sides parallel to the coordinate axes (which is the ball for the dual norm). It is then easy to construct an infinite cluster $\mathbf{E} = (E_1, \ldots, E_k \ldots)$, where each E_k is a square and also the union of all such squares is a square, see Figure 1. By iterating such a construction it is not difficult to understand that given any sequence a_k, $k = 1, 2, \ldots$ of numbers such that their sum is equal to 1 and each number is a power of $\frac{1}{4}$, it is possible to find a cluster $\mathbf{E}$ with $\mathbf{m}(\mathbf{E}) = \mathbf{a}$ such that each E_k is a square and the union $\bigcup_k E_k$ is the unit square.

Example 4.3 (Cantor circles). See Figure 2 and [2, example 2 pp. 59]. Take a rectangle R divided in two by a segment S on its axis. Let C be a Cantor set with positive measure constructed on S. Consider the set E_3 which is the union of the balls with diameter on the intervals composing the complementary set $S \setminus C$. Let E_1 and E_2 be the two connected components of $R \setminus \overline{E_3}$. It turns out that the 3-cluster $\mathbf{E} = (E_1, E_2, E_3)$ has finite perimeter and the perimeter of $\mathbf{E}$ is represented by the Hausdorff measure of the boundary

$$P(\mathbf{E}) = \mathcal{H}^1(\tilde{\partial}\mathbf{E}).$$

However the same is not true for each region. In fact the boundary $\tilde{\partial}E_3$ of the region E_3 includes C and hence

$$P(E_3) < \mathcal{H}^1(\tilde{\partial}E_3).$$

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Conflict of interest

The authors declare that there is no conflict of interest.

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