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Research article

A linearly implicit energy-preserving exponential time differencing scheme for the fractional nonlinear Schrödinger equation

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Abstract: In this paper, we present a new method to solve the fractional nonlinear Schrödinger equation. Our approach combines the invariant energy quadratization method with the exponential time differencing method, resulting in a linearly-implicit energy-preserving scheme. To achieve this, we introduce an auxiliary variable to derive an equivalent system with a modified energy conservation law. The proposed scheme uses stabilized exponential time differencing approximations for time integration and Fourier pseudo-spectral discretization in space to obtain a linearly-implicit, fully-discrete scheme. Compared to the original energy-preserving exponential integrator scheme, our approach is more efficient as it does not require nonlinear iterations. Numerical experiments confirm the effectiveness of our scheme in conserving energy and its efficiency in long-time computations.

Keywords: fractional nonlinear Schrödinger equation; invariant energy quadratization; structure-preserving algorithm; exponential time differencing

1. Introduction

In 2000, Laskin, a Canadian scholar, explored non-Brownian motion as a possible approach to deriving the path integral. He expanded on Feynmann's path integral by incorporating Levy paths, as documented in his works [11, 12], which allowed for the modification of the Schrödinger equation. By utilizing the non-local Laplacian operator $(-\Delta)^{\alpha/2}(1 < \alpha \leq 2)$, he established the Schrödinger model [13], which accurately describes the quantum state variations of non-local physical systems over time, surpassing the classical Schrödinger equation. The model has been successfully applied to various fields [9, 15, 23]. This paper focuses on the fractional nonlinear Schrödinger (NLS) equation with the form

$$\mathrm{i}\frac{\partial u(\boldsymbol{x},t)}{\partial t} - (-\Delta)^{\frac{\alpha}{2}} u(\boldsymbol{x},t) + \beta|u(\boldsymbol{x},t)|^2 u(\boldsymbol{x},t) = 0, \quad \boldsymbol{x} \in \Omega \subset \mathbb{R}^2, \quad t \in (0,T], \tag{1.1}$$

$$u(\boldsymbol{x},0) = u_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \tag{1.2}$$

where $1 < \alpha \leq 2$, $\mathrm{i}^2 = -1$, $u_0(\boldsymbol{x})$ is a given smooth function, β is a real positive constant. The fractional Laplacian $-(-\Delta)^{\frac{\alpha}{2}}$ with period $2L$ can be defined by [22]

$$-(-\Delta)^{\frac{\alpha}{2}} u(\boldsymbol{x},t) = -\sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} |\upsilon_{k_1}^2 + \upsilon_{k_2}^2|^{\frac{\alpha}{2}} \hat{\mathbf{u}}_{\mathbf{k}} e^{\mathrm{i}\boldsymbol{\upsilon}_{\mathbf{k}}\cdot(\boldsymbol{x}+\boldsymbol{L})}, \text{ with } \hat{\mathbf{u}}_{\mathbf{k}} = \frac{1}{(2L)^2}\int_\Omega u(\boldsymbol{x},t) e^{-\mathrm{i}\upsilon_k\cdot(\boldsymbol{x}+\boldsymbol{L})} d\boldsymbol{x}, \tag{1.3}$$

where $\upsilon_{k_1} = \frac{k_1\pi}{L}$, $\upsilon_{k_2} = \frac{k_2\pi}{L}$, $\boldsymbol{\upsilon}_{\mathbf{k}} \cdot (\boldsymbol{x} + L) = \upsilon_{k_1}(x + L) + \upsilon_{k_2}(y + L)$. By setting $u = p + \mathrm{i}q$, we can rewrite system (1.1) as a pair of real-valued equations

$$p_t = (-\Delta)^{\frac{\alpha}{2}} q - \beta(p^2 + q^2)q, \tag{1.4}$$

$$q_t = -(-\Delta)^{\frac{\alpha}{2}} p + \beta(p^2 + q^2)p. \tag{1.5}$$

Guo et al. derive the equation possessing the energy conservation law [6], namely

$$\mathcal{H}(p,q) = \frac{1}{2}\int_\Omega \Big[((-\Delta)^{\frac{\alpha}{4}} p)^2 + ((-\Delta)^{\frac{\alpha}{4}} q)^2\Big] d\boldsymbol{x} - \frac{\beta}{4}\int_\Omega (p^2 + q^2)^2 d\boldsymbol{x}, \tag{1.6}$$

where $\mathcal{H}$ is energy function. From variational derivative formula [4], systems (1.4) and (1.5) can be rewritten as as an Hamiltonian system

$$\frac{dz}{dt} = J^{-1}\frac{\delta\mathcal{H}}{\delta z}, \text{ with } z = (p,q)^T, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It is widely acknowledged that conserving energy is crucial in establishing the existence and uniqueness of solutions for PDEs [3, 19]. Traditional methods fail to maintain these conservation laws, making structure-preserving algorithms essential for the numerical analysis of classical differential equations. Recently, researchers have been exploring structure-preserving numerical schemes for fractional equations [18], with a focus on the fractional NLS equation. For example, Li et al. [16] developed a fast conservative finite element scheme for the strongly coupled fractional NLS equation. Wang et al. [20, 21] also presented different schemes that can exactly conserve energy for various fractional NLS equations. Additional related research can be found in [4] and the references therein. However, most structure-preserving schemes for the fractional NLS equation are fully implicit, requiring iterations to solve algebraic systems at each time step. This leads to a substantial number of calculations.

In [2], the exponential integrators scheme was studied in depth. This scheme has a unique advantage in that it accurately evaluates the contribution of the linear part, allowing for stable and precise computations even when dealing with highly rigid linear terms. As a result, this scheme has been successfully applied to Hamiltonian PDEs [17] and has been found to permit larger step sizes and achieve greater accuracy than non-exponential schemes. Additionally, the invariant energy quadratization (IEQ) method, developed by Yang and colleagues, has been used to create efficient and accurate linearly-implicit schemes for certain gradient models [5] and conservative differential equations [1, 10]. For example, in [1], a linearly implicit scheme for the fractional NLS equation was developed using the IEQ method. However, practical computations using these schemes still require smaller step sizes, making them computationally expensive for long-time numerical simulations. To address these challenges, we propose a novel approach that combines the exponential time differencing method with the IEQ approach to develop an efficient linearly-implicit energy-preserving scheme for the fractional NLS equation. Surprisingly, there has been little research in this area to date. The main goal of this paper is to develop a linearly-implicit exponential integrator scheme for this equation, using the IEQ method. Our contributions can be summarized as follows: (i) we introduce the Hamiltonian form of the equation; (ii) we combine the IEQ method and the exponential integrators technique to develop a linearly implicit scheme that conserves energy; (iii) we demonstrate that the construction process of the proposed scheme can be extended to develop conservative schemes for other differential equations. Moreover, our numerical results show that the scheme inherits the modified energy and has better numerical stability than traditional energy-preserving schemes in practical computations.

The paper is arranged as follows. In Section 2, we obtain a new equivalent system with modified energy via the IEQ method. In Section 3, we construct a linearly-implicit exponential integrator scheme for solving the fractional NLS equation, and prove that the proposed scheme can preserve the discrete energy. A numerical example is presented in Section 4 to illustrate the theoretical results. We draw some conclusions in Section 5.

2. IEQ method for the equation

In this Section, we utilize the IEQ idea to transform Eqs (1.4) and (1.5) into an equivalent system. Following the idea of the IEQ approach, we introduce an auxiliary variable $w := w(t) = \sqrt{G(p,q) + C_0}$, where $G(p,q) = \frac{\beta}{4}(p^2 + q^2)^2$, by choosing a sufficiently large constant C_0, the well-posedness of w is ensured. This results in a modified formalism for the Hamiltonian energy (1.6).

$$\mathcal{E}(t) = \frac{1}{2}\int_{\Omega}\left[((-\Delta)^{\frac{\alpha}{4}}p)^2 + ((-\Delta)^{\frac{\alpha}{4}}q)^2\right]d\boldsymbol{x} - \int_{\Omega} w^2 d\boldsymbol{x}, \tag{2.1}$$

and the corresponding fractional NLS systems (1.4) and (1.5) can be written as

$$p_t = (-\Delta)^{\frac{\alpha}{2}}q - \frac{\beta(p^2+q^2)q}{\sqrt{G(p,q)+C_0}}w, \tag{2.2}$$

$$q_t = -(-\Delta)^{\frac{\alpha}{2}}p + \frac{\beta(p^2+q^2)p}{\sqrt{G(p,q)+C_0}}w, \tag{2.3}$$

$$w_t = \frac{\beta(p^2+q^2)pp_t + \beta(p^2+q^2)qq_t}{2\sqrt{G(p,q)+C_0}}, \tag{2.4}$$

with the consistent initial condition $p_0 = p_0(\boldsymbol{x},0) = \mathrm{Re}(u_0(\boldsymbol{x})), q_0 = q_0(\boldsymbol{x},0) = \mathrm{Im}(u_0(\boldsymbol{x})), w_0 = \sqrt{G(p_0,q_0) + C_0}$, and the periodic boundary condition. It is easy to prove the systems (2.2)–(2.4) possesses the modified energy

$$\mathcal{E}(t) = \mathcal{E}(0). \tag{2.5}$$

3. Construction of conservative scheme

3.1. Spatial discretization

For positive even integers N_x, N_y and N, we define: $h_x := \frac{2L}{N_x}, h_y := \frac{2L}{N_y}$, and set time step $\tau = \frac{T}{N}$. The spatial grid points are given: $\Omega_h = \{(x_i, y_j) | \ i = 0, 1, \cdots, N_x - 1; j = 0, 1, \cdots, N_y - 1\}$, where $x_i = -L + ih_x, 0 \le i \le N_x - 1, y_j = -L + jh_y, 0 \le j \le N_y - 1$. Let $\mathcal{U}_h = \{\boldsymbol{u} | \boldsymbol{u} = (u_{0,0}, \cdots, u_{N_x-1,0}, u_{0,1}, \cdots, u_{N_x-1,1}, \cdots, u_{0,N_y-1}, \cdots, u_{N_x-1,N_y-1})^T\}$. For $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{U}_h$, we define

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_h = h_x h_y \sum_{i=0}^{N_x-1} \sum_{j=0}^{N_y-1} u_{i,j} \bar{v}_{i,j}, \quad \|\boldsymbol{u}\| = \langle \boldsymbol{u}, \boldsymbol{u} \rangle_h^{\frac{1}{2}}, \ \ \|\boldsymbol{u}\|_\infty = \sup_{(x_i, y_j) \in \Omega_h} |u_{i,j}|,$$

where $\bar{v}_{i,j}$ is the conjugate of $v_{i,j}$, and we introduce some operators

$$\delta_t \boldsymbol{u}^n = \frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\tau}, \ \boldsymbol{u}^{n+1/2} = \frac{\boldsymbol{u}^{n+1} + \boldsymbol{u}^n}{2}, \ \ \hat{\boldsymbol{u}}^{n+1/2} = \frac{3\boldsymbol{u}^n - \boldsymbol{u}^{n-1}}{2}.$$

Let $(x_i, y_j) \in \Omega_h$ be the Fourier collocation points. Using the interpolation polynomial $I_{\mathbf{N}} p(x, y)$ to approximate $p_{\mathbf{N}}(x, y)$ of the function $p(x, y)$, and we have

$$I_{\mathbf{N}} p(x, y) = p_{\mathbf{N}}(x, y) = \sum_{k_1=-N_x/2}^{N_x/2} \sum_{k_2=-N_y/2}^{N_y/2} \tilde{p}_{k_1,k_2} e^{\mathrm{i}k_1\mu(x+L)} e^{\mathrm{i}k_2\mu(y+L)},$$

where $\mu = \frac{\pi}{L}$, and the coefficient

$$\tilde{p}_{k_1,k_2} = \frac{1}{N_x c_{k_1}} \frac{1}{N_y c_{k_2}} \sum_{i=0}^{N_x-1} \sum_{j=0}^{N_y-1} p(x_i, y_j) e^{-\mathrm{i}k_1\mu(x+L)} e^{-\mathrm{i}k_2\mu(y+L)},$$

with $c_{k_1} = 1$ for $|k_1| < N_x/2$, $c_{k_2} = 1$ for $|k_2| < N_y/2$, $c_{k_1} = 2$ for $k_1 = \pm N_x/2$, and $c_{k_2} = 2$ for $k_2 = \pm N_y/2$.

Then the $-(-\Delta)^{\frac{\alpha}{2}}$ on $p(x_i, y_j)$ can be discretized by [8]

$$\begin{aligned}(D^\alpha \mathbf{p})_{i,j} : &= -(-\Delta)^{\frac{\alpha}{2}} p_{\mathbf{N}}(x_i, y_j) \\ &= -\frac{1}{N_x N_y} \sum_{k_1=0}^{N_x-1} \sum_{k_2=0}^{N_y-1} d_{k_1,k_2}^{(s)} \Big(\sum_{i=0}^{N_x-1} \sum_{j=0}^{N_y-1} p_{i,j} e^{-2\pi \mathbf{i}(ik_1 + jk_2)/N_x} \Big) e^{2\pi \mathbf{i}(ik_1 + jk_2)/N_y},\end{aligned} \tag{3.1}$$

where

$$d_{k_1,k_2}^{(\alpha)} = \begin{cases} \mu^\alpha (k_1^2 + k_2^2)^{\alpha/2}, & 0 \le k_1 \le N_x/2, \quad 0 \le k_2 \le N_y/2, \\ \mu^\alpha (k_1^2 + (k_2 - N_2)^2)^{\alpha/2}, & 0 \le k_1 \le N_x/2, \quad N_y/2 \le k_2 \le N_y - 1, \\ \mu^\alpha ((k_1 - N_1)^2 + k_2^2)^{\alpha/2}, & N_x/2 \le k_1 \le N_x - 1, \quad 0 \le k_2 \le N_y/2, \\ \mu^\alpha ((k_1 - N_1)^2 + (k_2 - N_2)^2)^{\alpha/2}, & N_x/2 \le k_1 \le N_x - 1, \quad N_y/2 \le k_2 \le N_y - 1. \end{cases}$$

Especially for $N = N_x = N_y$, the negative definite and symmetric matrix D^s is given

$$D^s = -(\mathbf{F} \otimes \mathbf{F})^{-1} \Lambda (\mathbf{F} \otimes \mathbf{F}), \quad \Lambda = \mathrm{diag}(\mathbf{d}^{(s)}),$$

where $\mathbf{F}$ is a orthogonal matrix with the entries

$$(\mathbf{F})_{j,k} = \frac{1}{\sqrt{N}} \Big[e^{\frac{-2\pi \mathbf{i}(j-1)(k-1)}{N}} \Big], k = 1, 2, \cdots, N.$$

One can observe that the expression $(\mathbf{F} \otimes \mathbf{F})^{-1} \Lambda (\mathbf{F} \otimes \mathbf{F}) \mathbf{p}$ can be efficiently calculated in $O(N^2 \log N)$ operations using the built-in function `ifft2(Λ. * fft2(p))` in Matlab.

3.2. Construction of the conservative exponential scheme

By applying the Fourier pseudo-spectral to Eqs (2.2)–(2.4), we obtain

$$\boldsymbol{p}_t = -D^{\alpha}\boldsymbol{q} - \frac{g_2(\boldsymbol{p},\boldsymbol{q})}{\sqrt{G(\boldsymbol{p},\boldsymbol{q})+C_0}}\boldsymbol{w}, \tag{3.2}$$

$$\boldsymbol{q}_t = D^{\alpha}\boldsymbol{p} + \frac{g_1(\boldsymbol{p},\boldsymbol{q})}{\sqrt{G(\boldsymbol{p},\boldsymbol{q})+C_0}}\boldsymbol{w}, \tag{3.3}$$

$$\boldsymbol{w}_t = \frac{g_1(\boldsymbol{p},\boldsymbol{q})\boldsymbol{p}_t + g_2(\boldsymbol{p},\boldsymbol{q})\boldsymbol{q}_t}{2\sqrt{G(\boldsymbol{p},\boldsymbol{q})+C_0}}, \tag{3.4}$$

where $g_1(\boldsymbol{p},\boldsymbol{q}) = \beta(\boldsymbol{p}^2+\boldsymbol{q}^2)\boldsymbol{p}$, $g_2(\boldsymbol{p},\boldsymbol{q}) = \beta(\boldsymbol{p}^2+\boldsymbol{q}^2)\boldsymbol{q}$.

Let

$$z(t) = \big(\boldsymbol{p}(t),\boldsymbol{q}(t)\big)^T,\quad f(\boldsymbol{p}(t),\boldsymbol{q}(t),\boldsymbol{w}(t)) = \Big(\frac{g_1(\boldsymbol{p},\boldsymbol{q})}{\sqrt{G(\boldsymbol{p},\boldsymbol{q})+C_0}}\boldsymbol{w}, \frac{g_2(\boldsymbol{p},\boldsymbol{q})}{\sqrt{G(\boldsymbol{p},\boldsymbol{q})+C_0}}\boldsymbol{w}\Big)^T,$$

$$\mathcal{S} = \begin{pmatrix} \boldsymbol{O} & -\boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{O} \end{pmatrix},\quad \mathcal{D} = \begin{pmatrix} D^{\alpha} & \boldsymbol{O} \\ \boldsymbol{O} & D^{\alpha} \end{pmatrix},$$

then, systems (3.2)–(3.4) can be rewritten as

$$z_t = \mathcal{S}\mathcal{D}z(t) + \mathcal{S}f \tag{3.5}$$

$$\boldsymbol{w}_t = \frac{g_1(\boldsymbol{p},\boldsymbol{q})\boldsymbol{p}_t + g_2(\boldsymbol{p},\boldsymbol{q})\boldsymbol{q}_t}{2\sqrt{G(\boldsymbol{p},\boldsymbol{q})+C_0}}. \tag{3.6}$$

Let $\mathcal{V} = \tau\mathcal{S}\mathcal{D}$, then integrating systems (3.5) and (3.6) from t^n to t^{n+1}, we can obtain

$$\boldsymbol{z}^{n+1} = \exp(\mathcal{V})\boldsymbol{z}^n + \tau\int_0^1 \exp[(1-\xi)\mathcal{V}]\mathcal{S}f(\boldsymbol{p}(t^n+\xi\tau),\boldsymbol{q}(t^n+\xi\tau),\boldsymbol{w}(t^n+\xi\tau))d\xi, \tag{3.7}$$

$$\boldsymbol{w}^{n+1} = \boldsymbol{w}^n + \tau\int_0^1 \frac{g_1\big(\boldsymbol{p}(t^n+\xi\tau),\boldsymbol{q}(t^n+\xi\tau)\big)\boldsymbol{p}_t + g_2\big(\boldsymbol{p}(t^n+\xi\tau),\boldsymbol{q}(t^n+\xi\tau)\big)\boldsymbol{q}_t}{2\sqrt{G\big(\boldsymbol{p}(t^n+\xi\tau),\boldsymbol{q}(t^n+\xi\tau)\big)+C_0}}d\xi. \tag{3.8}$$

Then, we can use $f\big(\boldsymbol{p}(t^n+\xi\tau),\boldsymbol{q}(t^n+\xi\tau),\boldsymbol{w}(t^n+\xi\tau)\big)$ and

$$\frac{g_1\big(\boldsymbol{p}(t^n+\xi\tau),\boldsymbol{q}(t^n+\xi\tau)\big)\boldsymbol{p}_t + g_2\big(\boldsymbol{p}(t^n+\xi\tau),\boldsymbol{q}(t^n+\xi\tau)\big)\boldsymbol{q}_t}{2\sqrt{G\big(\boldsymbol{p}(t^n+\xi\tau),\boldsymbol{q}(t^n+\xi\tau)\big)+C_0}}$$

to approximate $f(\hat{\boldsymbol{p}}^{n+\frac{1}{2}},\hat{\boldsymbol{q}}^{n+\frac{1}{2}},\boldsymbol{w}^{n+\frac{1}{2}})$ and

$$\frac{g_1(\hat{\boldsymbol{p}}^{n+\frac{1}{2}},\hat{\boldsymbol{q}}^{n+\frac{1}{2}})\delta_t\boldsymbol{p}^n + g_2(\hat{\boldsymbol{p}}^{n+\frac{1}{2}},\hat{\boldsymbol{q}}^{n+\frac{1}{2}})\delta_t\boldsymbol{q}^n}{2\sqrt{G(\hat{\boldsymbol{p}}^{n+\frac{1}{2}},\hat{\boldsymbol{q}}^{n+\frac{1}{2}})+C_0}},$$

respectively, and obtain the following new scheme

$$\boldsymbol{z}^{n+1} = \exp(\mathcal{V})\boldsymbol{z}^n + \tau\phi(\mathcal{V})S\,f(\hat{\boldsymbol{p}}^{n+\frac{1}{2}},\hat{\boldsymbol{q}}^{n+\frac{1}{2}},\boldsymbol{w}^{n+\frac{1}{2}}), \tag{3.9}$$

$$\boldsymbol{w}^{n+1} = \boldsymbol{w}^n + \tau \frac{g_1(\hat{\boldsymbol{p}}^{n+\frac{1}{2}}, \hat{\boldsymbol{q}}^{n+\frac{1}{2}})\delta_t \boldsymbol{p}^n + g_2(\hat{\boldsymbol{p}}^{n+\frac{1}{2}}, \hat{\boldsymbol{q}}^{n+\frac{1}{2}})\delta_t \boldsymbol{q}^n}{2\sqrt{G(\hat{\boldsymbol{p}}^{n+\frac{1}{2}}, \hat{\boldsymbol{q}}^{n+\frac{1}{2}}) + C_0}}, \tag{3.10}$$

where

$$\phi(\mathcal{V}) = \int_0^1 \exp[(1-\xi)\mathcal{V}]d\xi,$$

$$f(\hat{\boldsymbol{p}}^{n+\frac{1}{2}}, \hat{\boldsymbol{q}}^{n+\frac{1}{2}}, \boldsymbol{w}^{n+\frac{1}{2}}) = \Big(\frac{g_1(\hat{\boldsymbol{p}}^{n+\frac{1}{2}}, \hat{\boldsymbol{q}}^{n+\frac{1}{2}})\boldsymbol{w}^{n+\frac{1}{2}}}{\sqrt{G(\hat{\boldsymbol{p}}^{n+\frac{1}{2}}, \hat{\boldsymbol{q}}^{n+\frac{1}{2}}) + C_0}}, \frac{g_2(\hat{\boldsymbol{p}}^{n+\frac{1}{2}}, \hat{\boldsymbol{q}}^{n+\frac{1}{2}})\boldsymbol{w}^{n+\frac{1}{2}}}{\sqrt{G(\hat{\boldsymbol{p}}^{n+\frac{1}{2}}, \hat{\boldsymbol{q}}^{n+\frac{1}{2}}) + C_0}} \Big)^T, \quad n = 1, 2, \cdots.$$

Remark 3.1. It should be noted that the proposed schemes (3.9) and (3.10) are a three level method, therefore, we can calculate the $\boldsymbol{z}^1$ and $\boldsymbol{w}^1$ by using the $\boldsymbol{z}^0$ instead of $\hat{\boldsymbol{z}}^{\frac{1}{2}}$ for the first step.

Theorem 3.1. Schemes (3.4) and (3.5) conserve the modified energy

$$E^{n+1} = E^n, \text{ with } E^n = \frac{1}{2}(\boldsymbol{z}^n)^T \mathcal{D} \boldsymbol{z}^n + (\boldsymbol{w}^n)^2, \quad n = 0, 1, \cdots.$$

To this end, we first give some lemmas.

Lemma 3.1. [7] Assuming that $g(x)$ is a sufficiently smooth function, in the vicinity of zero, with 0 being a removable singularity, we can define $g(0)$ as the limit of $g(x)$ as x approaches zero.

$$g(x) = \sum_{k=0}^{\infty} \frac{g^k(0)}{k!} x^k,$$

and for a matrix $\mathcal{A}$, the matrix valued function is defined by $g(\mathcal{A}) = \sum\limits_{k=0}^{\infty} \frac{g^k(0)}{k!} \mathcal{A}^k$.

Lemma 3.2. [17] If $\mathcal{D}$ is a symmetric matrix and τ is a non-negative scalar, then the matrix $\mathcal{A}$ defined by $\mathcal{A} = \exp(\mathcal{V})^T \mathcal{D} \exp(\mathcal{V}) - \mathcal{D}$ is a nilpotent matrix, where $\mathcal{V} = \tau \mathcal{S}\mathcal{D}$ and $\mathcal{S}$ is a skew-symmetric matrix.

Next, we show that the proposed scheme can preserve the modified energy.

Proof. If $\mathcal{D}$ is a singular matrix, let $\{\mathcal{D}_\epsilon\}$ be a series of symmetric and nonsingular matrices, which converge to $\mathcal{D}$ when $\epsilon \to 0$. We assume $\boldsymbol{z}_\epsilon^n, \boldsymbol{w}_\epsilon^n$ satisfy the perturbed scheme

$$\boldsymbol{z}_\epsilon^{n+1} = \exp(\mathcal{V}_\epsilon)\boldsymbol{z}_\epsilon^n + \tau g(\mathcal{V}_\epsilon)\mathcal{S} f(\hat{\boldsymbol{z}}_\epsilon^{n+\frac{1}{2}}, \boldsymbol{w}_\epsilon^{n+\frac{1}{2}}), \tag{3.11}$$

$$\boldsymbol{w}_\epsilon^{n+1} = \boldsymbol{w}_\epsilon^n + \tau \frac{g_1(\hat{\boldsymbol{p}}_\epsilon{}^{n+\frac{1}{2}}, \hat{\boldsymbol{q}}_\epsilon{}^{n+\frac{1}{2}})\delta_t \boldsymbol{p}_\epsilon{}^n + g_2(\hat{\boldsymbol{p}}_\epsilon{}^{n+\frac{1}{2}}, \hat{\boldsymbol{q}}_\epsilon{}^{n+\frac{1}{2}})\delta_t \boldsymbol{q}_\epsilon{}^n}{2\sqrt{G(\hat{\boldsymbol{p}}_\epsilon{}^{n+\frac{1}{2}}, \hat{\boldsymbol{q}}_\epsilon{}^{n+\frac{1}{2}}) + C_0}}, \tag{3.12}$$

where $\mathcal{V}_\epsilon = \tau \mathcal{S} \mathcal{D}_\epsilon$, $n = 0, 1, \cdots$. Then, we denote

$$\tilde{f}_\epsilon := \mathcal{D}_\epsilon^{-1} f_\epsilon = \mathcal{D}_\epsilon^{-1} f(\hat{\boldsymbol{z}}_\epsilon^{n+\frac{1}{2}}, \boldsymbol{w}_\epsilon^{n+\frac{1}{2}}),$$

and rewrite E^n as

$$E_\epsilon^n = \frac{1}{2}(\boldsymbol{z}_\epsilon^n)^T \mathcal{D}_\epsilon \boldsymbol{z}_\epsilon^n + (\boldsymbol{w}_\epsilon^n)^2. \tag{3.13}$$

According to Eq (3.4), we have

$$
\begin{aligned}
\frac{1}{2}\left(z_\epsilon^{n+1}\right)^T \mathcal{D}_\epsilon z_\epsilon^{n+1} &= \frac{1}{2}[\,(z_\epsilon^n)^T \exp(\mathcal{V}_\epsilon)^T + \tau f_\epsilon^T \mathcal{S}^T \phi(\mathcal{V}_\epsilon)^T]\mathcal{D}_\epsilon[\,\exp(\mathcal{V}_\epsilon)\, z_\epsilon^n + \tau\phi(\mathcal{V}_\epsilon)\mathcal{S} f_\epsilon]\\
&= \frac{1}{2}(z_\epsilon^n)^T \exp(\mathcal{V}_\epsilon)^T \mathcal{D}_\epsilon \exp(\mathcal{V}_\epsilon)\, z_\epsilon^n + (z_\epsilon^n)^T \exp(\mathcal{V}_\epsilon)^T \mathcal{D}_\epsilon\,[\exp(\mathcal{V}_\epsilon) - I]\,\tilde{f}_\epsilon\\
&\quad + \frac{1}{2}\tilde{f}_\epsilon^T[\,\exp(\mathcal{V}_\epsilon)^T - I]\mathcal{D}_\epsilon[\,\exp(\mathcal{V}_\epsilon) - I]\tilde{f}_\epsilon\\
&= \frac{1}{2}[\,(z_\epsilon^n)^T \exp(\mathcal{V}_\epsilon)^T + \tau f_\epsilon^T \mathcal{S}^T \phi(\mathcal{V}_\epsilon)^T]\mathcal{D}_\epsilon[\,\exp(\mathcal{V}_\epsilon)\, z_\epsilon^n + \tau\phi(\mathcal{V}_\epsilon)\mathcal{S} f_\epsilon]\\
&= \frac{1}{2}(z_\epsilon^n)^T \exp(\mathcal{V}_\epsilon)^T \mathcal{D}_\epsilon \exp(\mathcal{V}_\epsilon)\, z_\epsilon^n + (z_\epsilon^n)^T \exp(V_\epsilon)^T \mathcal{D}_\epsilon\,[\exp(\mathcal{V}_\epsilon) - I]\,\tilde{f}_\epsilon\\
&\quad + \frac{1}{2}\tilde{f}_\epsilon^T[\,\exp(\mathcal{V}_\epsilon)^T - I]\mathcal{D}_\epsilon[\,\exp(\mathcal{V}_\epsilon) - I]\tilde{f}_\epsilon.
\end{aligned}
\tag{3.14}
$$

On the other hand, from Eq (3.12), we have

$$
\begin{aligned}
\left(\boldsymbol{w}_\epsilon^{n+1}\right)^2 - (\boldsymbol{w}_\epsilon^n)^2 &= \frac{g_1(\hat{\boldsymbol{p}}_\epsilon^{\,n+\frac{1}{2}}, \hat{\boldsymbol{q}}_\epsilon^{\,n+\frac{1}{2}})(\boldsymbol{p}_\epsilon^{n+1} - \boldsymbol{p}_\epsilon^n) + g_2(\hat{\boldsymbol{p}}_\epsilon^{\,n+\frac{1}{2}}, \hat{\boldsymbol{q}}_\epsilon^{\,n+\frac{1}{2}})(\boldsymbol{q}_\epsilon^{n+1} - \boldsymbol{q}_\epsilon^n)}{2\sqrt{G(\hat{\boldsymbol{p}}_\epsilon^{\,n+\frac{1}{2}}, \hat{\boldsymbol{q}}_\epsilon^{\,n+\frac{1}{2}}) + C_0}}\boldsymbol{w}_\epsilon^{n+\frac{1}{2}}\\
&= \left(\left(z_\epsilon^{n+1}\right)^T - (z_\epsilon^n)^T\right) f_\epsilon\\
&= (z_\epsilon^n)^T \left[\exp(\mathcal{V}_\epsilon)^T - I\right] f_\epsilon + \tau f_\epsilon^T \mathcal{S}^T \phi(\mathcal{V}_\epsilon)^T f_\epsilon\\
&= (z_\epsilon^n)^T \left[\exp(\mathcal{V}_\epsilon)^T - I\right] \mathcal{D}\tilde{f}_\epsilon + \tau f_\epsilon^T \mathcal{S}^T \phi(\mathcal{V}_\epsilon)^T \mathcal{D}\tilde{f}_\epsilon\\
&= (z_\epsilon^n)^T \left[\exp(\mathcal{V}_\epsilon)^T \mathcal{D}_\epsilon - \mathcal{D}_\epsilon\right] \tilde{f}_\epsilon + \tilde{f}_\epsilon^T \mathcal{V}_\epsilon^T \phi(\mathcal{V}_\epsilon)^T \mathcal{D}_\epsilon\tilde{f}_\epsilon\\
&= (z_\epsilon^n)^T \left[\exp(\mathcal{V}_\epsilon)^T \mathcal{D}_\epsilon - \mathcal{D}_\epsilon\right] \tilde{f}_\epsilon + \tilde{f}_\epsilon^T \left[\exp(\mathcal{V}_\epsilon)^T \mathcal{D}_\epsilon - \mathcal{D}_\epsilon\right] \tilde{f}_\epsilon.
\end{aligned}
\tag{3.15}
$$

This together with Eq (3.9), we can deduce

$$
\begin{aligned}
E_\epsilon^{n+1} - E_\epsilon^n &= \frac{1}{2}\left(z_\epsilon^{n+1}\right)^T \mathcal{D}_\epsilon z_\epsilon^{n+1} - \frac{1}{2}(z_\epsilon^n)^T \mathcal{D}_\epsilon z_\epsilon^n + \left(\boldsymbol{w}_\epsilon^{n+1}\right)^2 - (\boldsymbol{w}_\epsilon^n)^2\\
&= \frac{1}{2}(z_\epsilon^n)^T \left[\exp(\mathcal{V}_\epsilon)^T \mathcal{D}_\epsilon \exp(\mathcal{V}_\epsilon) - \mathcal{D}_\epsilon\right] z_\epsilon^n + (z_\epsilon^n)^T \left[\exp(\mathcal{V}_\epsilon)^T \mathcal{D}_\epsilon \exp(\mathcal{V}_\epsilon) - \mathcal{D}_\epsilon\right] \tilde{f}_\epsilon\\
&\quad + \frac{1}{2}\tilde{f}_\epsilon^T \left[\exp(\mathcal{V}_\epsilon)^T \mathcal{D}_\epsilon \exp(\mathcal{V}_\epsilon) - \mathcal{D}_\epsilon\right] \tilde{f}_\epsilon + \frac{1}{2}\tilde{f}_\epsilon^T \left[\exp(\mathcal{V}_\epsilon)^T \mathcal{D}_\epsilon - \mathcal{D}_\epsilon \exp(V_\epsilon)\right] \tilde{f}_\epsilon\\
&= \frac{1}{2}\tilde{f}_\epsilon^T \mathcal{B}_\epsilon \tilde{f}_\epsilon + \frac{1}{2}\left(z_\epsilon^n + \tilde{f}_\epsilon\right)^T \mathcal{A}_\epsilon \left(z_\epsilon^n + \tilde{f}_\epsilon\right) = 0,
\end{aligned}
\tag{3.16}
$$

where

$$
\mathcal{B}_\epsilon = \exp(\mathcal{V}_\epsilon)^T \mathcal{D}_\epsilon - \mathcal{D}_\epsilon \exp(\mathcal{V}_\epsilon),
$$

$$
\mathcal{A}_\epsilon = \exp(\mathcal{V}_\epsilon)^T \mathcal{D}_\epsilon \exp(\mathcal{V}_\epsilon) - \mathcal{D}_\epsilon.
$$

The last equality is based on Lemma 3.1 and the skew symmetry of the matrix $\mathcal{B}$. Thus, from Eq (3.9), when $\epsilon \to 0, z_\epsilon^n \to z^n, \boldsymbol{w}_\epsilon^n \to \boldsymbol{w}^n$, we obtain $E^{n+1} = E^n$.

If $\mathcal{D}$ is a nonsingular matrix, the proof is similar to above process, for brevity, we omit it. The proof is completed.

4. Numerical example

Two examples are given to confirm the efficiency, conservation and stability of the proposed scheme. We use the formula

$$\text{Rate} = \frac{\ln(\text{error}_1/\text{error}_2)}{\ln(\tau_1/\tau_2)} \tag{4.1}$$

to evaluate the convergence rate, where $\tau_i, \text{error}_i, (i = 1, 2)$ are step sizes and L^∞-norm errors with the step size τ_i, respectively. The relative energy error is defined as

$$RE^n = |\frac{E^n - E^0}{E^0}|, \quad RH^n = |\frac{H^n - H^0}{H^0}|, \tag{4.2}$$

where E^n and E^n denote the conservation laws at $t_n = n\tau$.

Example 4.1. Considering the equation with

$$u(x, 0) = \exp(-x^2)\exp(-\mathrm{i}x), \quad x \in \Omega = [-40, 40]. \tag{4.3}$$

The exact solution is not given, we use

$$\|e\|_\infty = \|u(h, \tau) - u(h, \frac{\tau}{2})\|_\infty,$$

to obtain numerical errors.

In the follow-up of the article, the **E-IEQ** is the developed scheme in our work, **CN-IEQ** is a linearly implicit Crank-Nicolson IEQ scheme [1], **E-AVF** scheme is a fully-implicit exponential AVF scheme, and the **E-SAV** scheme [14] represents a linearly implicit exponential scheme based on the SAV approach. Yet the general, we first take $\alpha = 1.8$ to demonstrate the accuracy, efficiency and conservation properties of **E-IEQ** scheme. In Table 1, numerical errors and convergence orders implies that four schemes have second-order accuracy in the time. Noting that 'NaN' implies the **CN-IEQ** scheme cannot be implemented with a large time step, but the **E-IEQ** scheme can. Figure 1 shows that linearly implicit schemes are more efficient than the **E-AVF** scheme, and the **E-IEQ** scheme is most efficient of the four schemes.

Table 1. Temporal accuracy of different schemes with, $h = 80/512$.

Scheme		$\tau = \frac{1}{40}$	$\tau = \frac{1}{80}$	$\tau = \frac{1}{160}$	$\tau = \frac{1}{320}$
CN-IEQ	$\|\|e\|\|_\infty$	NaN	NaN	9.81e-03	2.54e-03
	Rate	*	*	*	1.95
E-IEQ	$\|\|e\|\|_\infty$	3.32e-02	7.64e-03	1.82e-04	4.46e-04
	Rate	*	2.12	2.07	2.03
E-SAV [14]	$\|\|e\|\|_\infty$	2.91e-02	7.26e-03	1.81e-03	4.53e-04
	Rate	*	2.00	2.00	2.00
EAVF [17]	$\|\|e\|\|_\infty$	1.68e-02	4.19e-03	1.04e-03	2.62e-04
	Rate	*	1.99	2.00	2.00

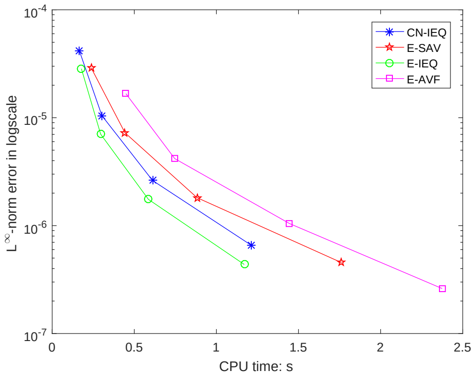
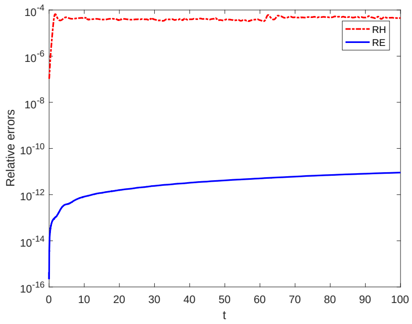
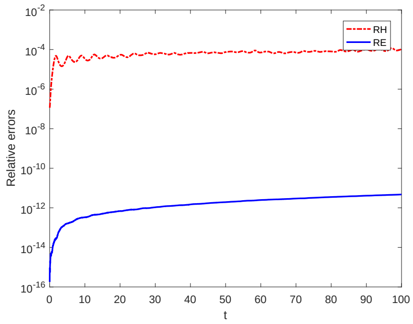


Figure 1. The numerical error versus the CPU time using the four numerical schemes.

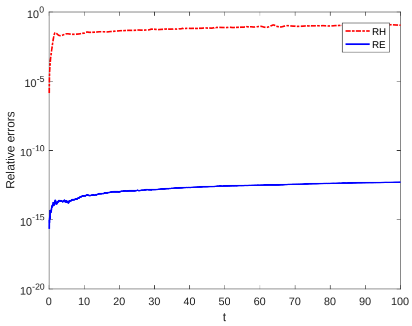
In Figure 2, the relative errors of the modified energy and original energy of the equation are presented. From the results, we can see three linearly implicit schemes only can conserve the modified energy, the **E-AVF** scheme can preserve the inherent energy of the equation. Finally, we focus on the correlation between the fractional order α and the soliton's shape. Figure 3 displays numerical outcomes for various α values. It is evident that α has a significant impact on the soliton's form. As α shifts from 1.3 to 2, the soliton's appearance changes dramatically. The findings reveal that the solitary wave can endure stable propagation under finite initial circumstances, even after long periods of propagation.



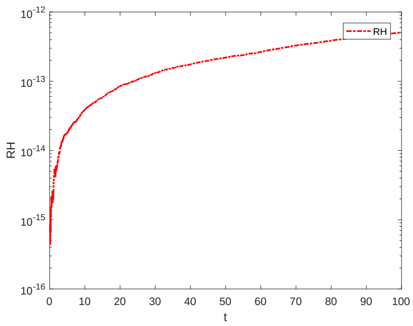
(a) **CN-IEQ**



(b) **E-IEQ**

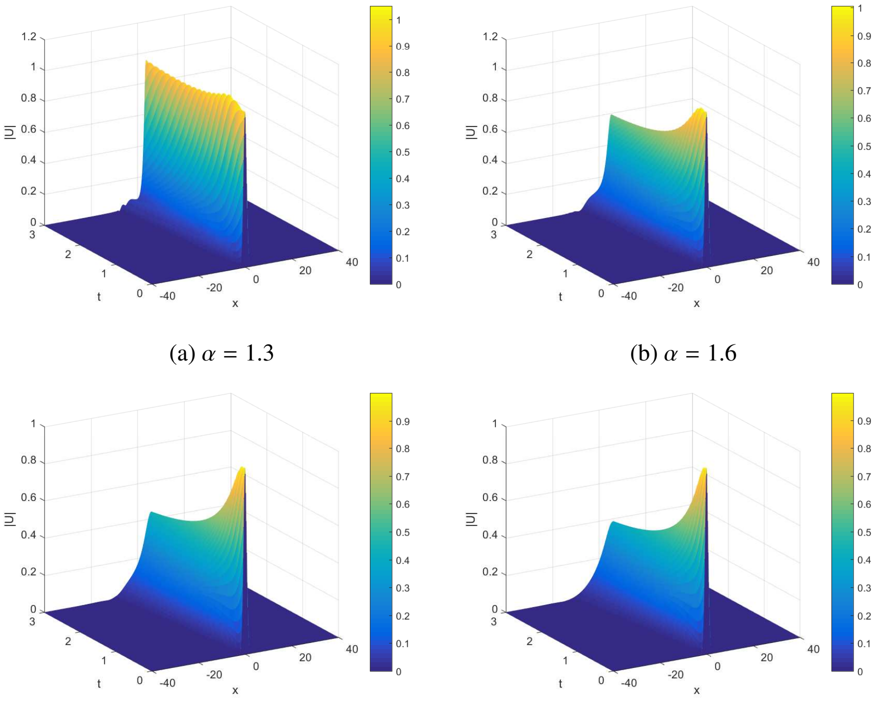


(c) **E-SAV**



(d) **E-AVF**

Figure 2. Relative errors of conservation laws for different schemes with $h = 80/512$, $\tau = 0.01$.



(a) $\alpha = 1.3$

(b) $\alpha = 1.6$

(c) $\alpha = 1.8$

(d) $\alpha = 2$

Figure 3. Evolution of the solitons with $N = 512, \tau = 0.01$ for different order α.

Example 4.2. We consider the equation on $(x, y) = (-\pi, \pi) \times (-\pi, \pi)$ and has the form

$$u(x, y, t) = A\exp(\mathrm{i}(\lambda_1 x + \lambda_2 y - \omega t)), \tag{4.4}$$

with

$$\omega = ({\lambda_1}^2 + {\lambda_2}^2)^{\frac{\alpha}{2}} - \beta|A|^2. \tag{4.5}$$

where the amplitude $A = 1$, the wave numbers $\lambda_1 = \lambda_2 = 2$ and $\beta = 2$.

This example takes $\alpha = 1.4, 1.7, 2$. Figure 4 shows that the proposed scheme has second-order in time. We then run a long-time simulation till $T = 50$ and plot the relative energy deviation in Figure 5 which indicates that the new scheme can preserve the modified energy exactly in discrete scene.

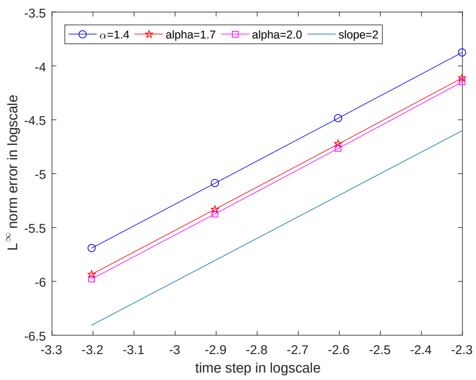


Figure 4. Convergence orders in time for different order α with $h = 2\pi/16$.

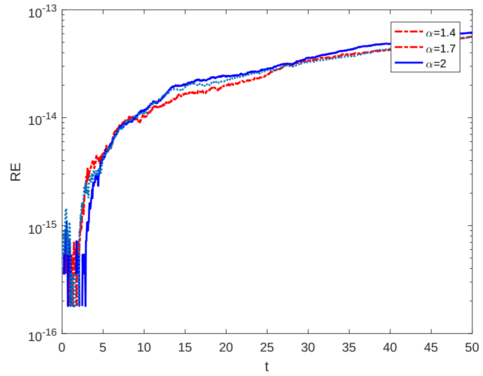


Figure 5. Convergence orders in time for different order α with $h = 2\pi/16$.

5. Conclusions

In this paper, we introduce a unique technique to solve the fractional nonlinear Schrödinger equation while conserving energy. We achieve this through a combination of the invariant energy quadratization method and the exponential time differencing method. Our proposed scheme is highly efficient and performs well even with large time steps during long-time computations. In discrete settings, the modified energy is precisely preserved. To demonstrate our approach, we present a numerical example. Furthermore, this technique can be applied to other fractional Hamiltonian PDEs, including the fractional nonlinear wave equation and the fractional Klein-Gordon-Schrödinger equation, by following a similar process.

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Conflict of interest

The authors declare there is no conflict of interest.

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