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Research article

Stability of multi-population traffic flows

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Abstract: Traffic waves, known also as stop-and-go waves or phantom jams, appear naturally as traffic instabilities, also in confined environments as a ring-road. A multi-population traffic is studied on a ring-road, comprised of drivers with stable and unstable behavior. There exists a critical penetration rate of stable vehicles above which the system is stable, and under which the system is unstable. In the latter case, stop-and-go waves appear, provided enough cars are on the road. The critical penetration rate is explicitly computable, and, in reasonable situations, a small minority of aggressive drivers is enough to destabilize an otherwise very stable flow. This is a source of instability that a single population model would not be able to explain. Also, the multi-population system can be stable below the critical penetration rate if the number of cars is sufficiently small. Instability emerges as the number of cars increases, even if the traffic density remains the same (i.e., number of cars and road size increase similarly). This shows that small experiments could lead to deducing imprecise stability conditions.

Keywords: traffic flow; multi-population; stability; stop-and-go; control

1. Introduction

Traffic jams that appear for seemingly no reason are a phenomenon that everyone has experienced. Even though there are no accidents, no lane reductions, no highway exits, so called stop-and-go waves form. This paradox has a mathematical answer: under certain conditions, traffic equilibrium states are unstable. This phenomenon has been observed and studied in many works. The consequence of such traffic instabilities are important: accelerating and decelerating increase strongly the fuel consumption and the gas emissions compared to the associated equilibrium flow.

In particular the study of [18], and the experiment associated, provided evidence that such waves form on a circular road with only about twenty drivers starting from same spacing and speed. The impact of the waves includes increased fuel consumption as measured in a similar experiment [17]. Reducing these instabilities by acting on the traffic, for instance rendering the equilibrium stable, would have a strong impact. This is why many approaches have been considered to solve the problem, from ramp-metering control [3, 10, 15, 19, 25–27] to the use of connected autonomous vehicles which act as "wave-dampeners" in the traffic (see for instance [4, 17, 20, 22, 23, 28] and [5] for a more detailed review). But, to be able to act on traffic efficiently, one needs to understand how these traffic waves emerge and their stability.

The main trigger for the creation of waves is the collective behavior of human agents on the road, which typically becomes unstable when the equilibrium velocity cross a certain threshold and becomes too low. They have been extensively studied in the literature, see [7, 13, 14], and recently explained theoretically in [4] in a ring-road framework similar to [17, 18]. These works study a single-phase traffic where all drivers follow the same car-following model. However, in real life, it is likely that the instability of equilibrium flows is strongly influenced by the differences in driving characteristics among agents. These differences could occur due to the type of vehicle (trucks, SUV, small cars, etc.) or differences in behavior between drivers. For instance, trucks platooning was studied for its fuel-consumption impact in mixed traffic [6, 24]. Another interesting situation is when some drivers adopt a collaborative way of driving [11]. In this case the traffic can be seen as a system with two populations: the standard drivers and those who adopt the collaborative behavior. Several interesting questions could be asked:

- Can the presence of two or several populations create some instabilities?
- Would a minority of collaborative drivers be able to render a stable traffic that would otherwise be unstable? On the contrary, could a minority of aggressive drivers make an otherwise very stable traffic unstable?

In this paper we provide some answers to these questions. Specifically, a mixed traffic on a ring-road is analyzed, where two or more populations coexists. After characterizing the equilibrium flows, we study the stability of the overall flow depending on the proportion of cars in each class of vehicles. We show that, when an instable and a stable population co-exist, there is a critical penetration rate above which the system is stable and under which the system is unstable, provided there are enough cars on the road. This penetration rate is explicitly computable. We also provide qualitative bounds on the critical penetration rate τ_0, in order to elucidate what makes a collaborating behavior effective from a traffic stability point of view.

For reasonable parameters, the critical penetration rate of stable vehicle is very high. This means that, not taking into account the differences in dynamics, can lead to not understanding the source of instability of the flow. Indeed, we show that a small minority of aggressive drivers, in an otherwise very stable flow, is enough to break the stability. This is something that a single-population model cannot grasp.

Surprisingly, we also show that the ability of a group of drivers to stabilize or destabilize the system only depends on its penetration rate in the traffic and not on the order of the cars. One could think for instance that three trucks in a row is less effective to stabilize the traffic than three trucks equally distributed on the ring-road. But it is not.

Finally, we also show that small experiments, as considered in [17, 18], could lead to underestimate the instability of the traffic flow and the critical penetration rate. More precisely, for a small number of vehicles the system can still be stable even below the critical penetration rate, but becomes unstable when the number of cars increases.

The paper is organized as follows: in Section 2 we present the framework and the system, in Section 3 we state the main results, in Section 4 we provide an analysis of the traffic around its equilibrium flows, while in Section 5 we show our results for a two-phase traffic flow and in Section 6 for a general multi-phase traffic flow. Finally, in Section 7 we provide some numerical simulations to illustrate our results.

2. General traffic model on a ring-road

We study a general traffic model with n vehicles and a ring-road of length L. Mathematically this means that we consider the system on the domain $\mathbb{T} = \mathbb{R}/L\mathbb{Z}$. We denote by $\{x_j(t)\}_{j=1}^n$ the location of the cars. By further denoting the headway and velocity of the cars as

$$h_j(t) = x_{j+1}(t) - x_j(t) \text{ and } v_j(t) = \dot{x}_j(t), \tag{2.1}$$

the traffic is generally described by

$$\begin{cases} \dot{h}_j(t) = v_{j+1}(t) - v_j(t), \\ \dot{v}_j(t) = f_j(h_j(t), \dot{h}_j(t), v_j(t)), \end{cases} \quad \forall j \in \{1, 2, ..., n\} \tag{2.2}$$

where f_j is the car following model for the driver j and with the convention $x_{n+1} = x_1 + L$ on the ring road (or, equivalently, $x_{n+1} = x_1$ in $\mathbb{T}$), which means that

$$\sum_{j=1}^{n} h_j(t) = L, \quad \forall\ t \geq 0. \tag{2.3}$$

For a general car following model we usually have the following physical conditions on $f_j(h, 0, v)$:

$$\frac{\partial}{\partial h} f_j(h, 0, v) > 0, \ \frac{\partial}{\partial v} f_j(h, 0, v) < 0, \ \frac{\partial}{\partial \dot{h}} f_j(h, 0, v) > 0. \tag{2.4}$$

The first condition simply means that, for a given speed, the incentive to accelerate increases with the headway. The second condition means that for a given headway the incentive to accelerate decreases with the speed. And the third condition means that for a given headway and speed, the incentive to accelerate increases if the headway is currently increasing (i.e., if the leading vehicle is moving away). These conditions can be found in nearly all car following models (e.g., Intelligent Driver Models [21], Follow-the-Leader [8], Bando–Follow-the-Leader [2, 4], etc.)

While, in most traffic analysis, the car following model f is chosen to be identical for all cars on the road, in general the car-following model f_j depends on the driving habit of the j-th driver, which may differ from driver to driver. It can also depends on the j-th vehicle itself since different kinds of cars may result in different parameters (e.g., one can simply compare a truck and a mini cooper on the road). We summarize three typical cases below:

- *Unified model*

This is the case that is most commonly considered, where f_j does not depend on $j \in \{1, 2, ..., n\}$ and there is a single type of vehicle on the road. The stability of this system has been studied in [4] and their results are recalled below. Under such a unified setting, a further special case can be the so-called Bando–Follow-the-Leader model (Bando-FTL, or equivalently OV-FTL), that combines a Bando (or Optimal Velocity) part introduced in [2] which represents the preference of a driver to reach its "own" optimal velocity (that depends on the headway), and a "Follow-the-Leader" (FTL) part introduced in [8] which represent the incentive of the driver to mimic its leader. More precisely, we have

$$f_j(h, \dot{h}, v) = a \cdot (V(h) - v) + b \cdot \frac{\dot{h}}{h^2}, \tag{2.5}$$

the Bando part has a weight a and V is the optimal velocity function discussed in the next paragraph, while the FTL part has a weight b. This model was studied for instance in [4, 5, 12].

- *Mixed traffic* or *Collaborative driving*

In a mixed traffic, the functions $\{f_j\}_{j=1}^n$ are chosen from a finite set. In other words, the drivers and the vehicles can be classified in finitely many categories:

$$f_j \in \{F_k : k = 1, 2, ..., m\}, \ \forall j \in \{1, 2, ..., n\} \tag{2.6}$$

with $m \ll n$. A particular example is the two population traffic, where $m = 2$ while n might be large. This situation corresponds for instance to a collaborative driving where some drivers follow a collaborative behaviors and have a "good" function F_1 while the rest of the traffic follows a standard function F_2 that would lead to instabilities and stop-and-go waves. We deal with this case in Section 5. To illustrate the mixed traffic setting, we can look at the case where the $\{f_j\}$ are given by the "general" Bando-FTL model, characterised by

$$f_j(h, \dot{h}, v) = a_j \cdot \Big(V_j(h) - v\Big) + b_j \cdot \frac{\dot{h}}{h^2}, \tag{2.7}$$

$$(a_j, b_j, V_j) \in \{(A_k, B_k, \mathcal{V}_k) : k = 1, 2, ..., m\}, \tag{2.8}$$

In this case, the weights (a_j, b_j) represents driving habit of the driver, and that the "Bando function" $V_j(h)$ represents its velocity preference, which can depend for instance of the type of vehicle (cars, trucks, etc.). This optimal velocity is typically taken as a C^2 bounded and strictly increasing function of the headway (the farther the leading vehicle, is the faster the velocity preference is likely to be, up to a limit value which is our own speed limit in the absence of traffic). We remark that in the literature for Bando model the function $V(h)$ is fixed to a ratio of hyperbolic tangents, however, in reality different types of vehicles may have direct influence, that is the reason we call it "general" Bando and adapt it with different $V_j(h)$ functions. We provide in Section 7 numerical simulations for this particular model.

- *Mixed traffic with common velocity preference*

This is a particular case of mixed traffic where all drivers have the same velocity preference. For instance for the Bando-FTL model this means that $V_j = V$ and is the same across drivers, while a_j

and b_j remain driver-dependent. As we will see in Section 4.1, in this case the equilibrium flows are the same as the equilibrium flows in the *unified model*.

3. Main results

We are interested in the stability of the system (2.2) around its equilibrium flow. A precise description of the equilibrium flows is given below in Section 4.1. In the unified model, where all drivers have the same car-following model $f_j = f$, the stability of the system was studied in [4]. Recall the physical "common sense" condition in (2.4). For any uniform flow $(\bar{h}, \bar{v})$, where $\bar{h}$ and $\bar{v}$ are both constants that do not depend on time, we shall define

$$(\alpha, \beta, \gamma) = \left(\frac{\partial f}{\partial h}, \frac{\partial f}{\partial \dot{h}} - \frac{\partial f}{\partial v}, \frac{\partial f}{\partial \dot{h}} \right) (\bar{h}, 0, \bar{v}), \tag{3.1}$$

satisfying

$$\alpha > 0, \ \beta > \gamma > 0. \tag{3.2}$$

We have used here the notation convention where $(f_1, f_2, f_3)(x, y, z)$ denotes $(f_1(x, y, z), ..., f_3((x, y, z)))$. The authors of [4] showed the following results.

Theorem 3.1 ([4], *unified model*)**.** *A uniform flow equilibrium* $(\bar{h}, \bar{v})$ *of the system* (2.2) *is*

- *locally stable around this flow, if*

$$\beta^2 - \gamma^2 - 2\alpha \geq 0, \tag{3.3}$$

- *unstable around this flow provided n sufficiently large, if*

$$\beta^2 - \gamma^2 - 2\alpha < 0. \tag{3.4}$$

The definition of local stability in this context is recalled in Definition 4.1. In this paper we investigate what happens when there is not anymore a single population of vehicles but several. In particular, what happens when vehicles with an unstable behavior (i.e., which satisfies Eq (3.4)) coexists with vehicles with a stable behavior (i.e., which satisfies Eq (3.3))?

Our main results are the following: consider first a two population system where the vehicles follow either the car following model f^1 or the car following model f^2. In this case, any stationary state (or equilibrium flow) can be described by $(\bar{h}_i, \bar{v})$ with $i \in \{1, 2\}$ (see Section 4.1). Define

$$(\alpha^i, \beta^i, \gamma^i) = \left(\frac{\partial f^i}{\partial h}, \frac{\partial f^i}{\partial \dot{h}} - \frac{\partial f^i}{\partial v}, \frac{\partial f^i}{\partial \dot{h}} \right) (\bar{h}_i, 0, \bar{v}), \quad i \in \{1, 2\}, \tag{3.5}$$

we have the following theorem when both populations have a stable behavior

Theorem 3.2. *If* $(\beta^1)^2 - (\gamma^1)^2 - 2\alpha^1 \geq 0$ *and* $(\beta^2)^2 - (\gamma^2)^2 - 2\alpha^2 \geq 0$, *then the steady-state* $(\bar{h}_i, \bar{v})_{i\in\{1,2\}}$ *of the ring road system* (2.2) *and* (2.3) *is locally exponentially stable.*

When the population have different behaviors, we denote n_1 and n_2 the number of cars of each population in the road, and we have the following theorem

Theorem 3.3. *Assume that* $\Delta_1 := (\beta^1)^2 - (\gamma^1)^2 - 2\alpha^1 > 0$ *and* $\Delta_2 := (\beta^2)^2 - (\gamma^2)^2 - 2\alpha^2 < 0$. *There exists a critical penetration rate* $\tau_0 \in (0, 1)$ *such that for any pair* $(n_1, n_2) \in \mathbb{N}^2$ *verifying*

$$\frac{n_1}{n_1 + n_2} > \tau_0, \tag{3.6}$$

the ring road traffic system (2.2) *and* (2.3) *is locally exponentially stable around any equilibrium flow* $(\bar{h}_i, \bar{v})_{i\in\{1,2\}}$, *whatever the ordering of the cars.*

On the other hand, for any fixed penetration rate

$$\tau = \frac{n_1}{n_1 + n_2} < \tau_0 \tag{3.7}$$

there exists some $M > 0$ *effectively computable such that for any* $n_1, n_2 \in \mathbb{N}^2$ *satisfying* $n_1 + n_2 > M$, *the ring road traffic system* (2.2) *and* (2.3) *is unstable around the equilibrium flow* $(\bar{h}_i, \bar{v})_{i\in\{1,2\}}$.

Moreover, the critical penetration rate τ_0 *is explicitly given by*

$$\begin{aligned} &\tau_0 = 1 - \left(1 + \max\left\{-\frac{H_2(y)}{H_1(y)} :\ y \in \left(0, \Gamma^2\right]\right\}\right)^{-1}, \\ \textit{where} \quad &H_i(y) : = \log\left(\frac{(\alpha^i)^2 + (\gamma^i)^2 y}{(\alpha^i)^2 + ((\beta^i)^2 - 2\alpha^i)y + y^2}\right), \quad \textit{for } i \in \{1, 2\}, \\ \textit{and} \quad &\Gamma^2 := \frac{-(\alpha^2)^2 + \sqrt{(\alpha^2)^4 - (\alpha^2)^2(\gamma^2)^2\Delta_2}}{(\gamma^2)^2} \in (0, -\Delta_2). \end{aligned} \tag{3.8}$$

Even though τ_0 can be computed easily through a minimization algorithm, we can also give some practical upper and lower bounds for qualitative studies:

$$\begin{aligned} &\tau_0 \geq \frac{-\Delta_2(\alpha^1)^2}{\Delta_1(\alpha^2)^2 - \Delta_2(\alpha^1)^2}, \\ \text{and} \quad &\tau_0 \leq \frac{(-\Delta_2)\left((\alpha^1)^2 + (\gamma^1)^2\Gamma^2\right)\left((\alpha^1)^2 + ((\beta^1)^2 - 2\alpha^1)\Gamma^2 + (\Gamma^2)^2\right)}{(\beta^2)^2(\alpha^1)^2\Delta_1 + (-\Delta_2)\left((\alpha^1)^2 + (\gamma^1)^2\Gamma^2\right)\left((\alpha^1)^2 + ((\beta^1)^2 - 2\alpha^1)\Gamma^2 + (\Gamma^2)^2\right)}. \end{aligned} \tag{3.9}$$

This allows a remark: for a stable class of vehicles to be efficient at stabilizing a mixed traffic flow with a small penetration rate, α^1 should be small. This means that an efficient collaborating behavior for stabilizing traffic flow will be composed of vehicles driving without taking much the headway into consideration (apart for safety reasons).

Finally, when one of the populations has a stable behavior but corresponding to the critical case $(\beta^1)^2 - (\gamma^1)^2 - 2\alpha^1 = 0$ and the other population has an unstable behavior, we have the following theorem

Theorem 3.4. *If* $\Delta_1 := (\beta^1)^2 - (\gamma^1)^2 - 2\alpha^1 = 0$ *and* $\Delta_2 := (\beta^2)^2 - (\gamma^2)^2 - 2\alpha^2 < 0$, *then provided sufficiently many vehicles on the ring road, the traffic system* (2.2)–(2.3) *is unstable around the the equilibrium flow* $(\bar{h}_i, \bar{v})_{i\in\{1,2\}}$.

This can be generalized for an multi-phase traffic with more than two populations (see Theorem 6.1 in Section 6).

4. Analysis of the system

4.1. Characterization of the equilibrium flow

The aim of this section is to describe the stationary states, or equilibrium flows, of the ring road traffic (2.2) and (2.3). Here, "stationary" and "equilibrium" means that the headway and the velocity do not change with respect to time, hence, for an equilibrium flow $(\bar{h}_j, \bar{v}_j)_{j\in\{1,...,n\}}$, we have $h_j(t) = \bar{h}_j$ and $v_j(t) = \bar{v}_j, \quad \forall t \in [0, +\infty)$. Considering the fact that the headway between two cars does not change, we know that the value of $\bar{v}_j$ does not depend on $j \in \{1, 2, ..., n\}$, therefore

$$h_j(t) = \bar{h}_j \text{ and } v_j(t) = \bar{v}, \quad \forall t \in [0, +\infty). \tag{4.1}$$

Before going into the details, let us introduce the "*velocity preferred headway*" of f_j: for a given velocity v, the so-called "velocity preferred headway" is given, when it exists, by h such that

$$f_j(h, 0, v) = 0, \tag{4.2}$$

which, according to Eq (2.4), admits at most a unique value, and we denote it by $g_j(v)$ when it exists.

Since the headway does not change, we have

$$\dot{h}_j(t) = \dot{x}_{j+1}(t) - \dot{x}_j(t) = v_{j+1}(t) - v_j(t), \tag{4.3}$$

which, to be combined with Eq (4.1), imply that there exists a constant $\bar{v}$ such that,

$$v_j(t) = \bar{v}, \ \forall j \in \{1, 2, ..., n\}, \ \forall t \in \mathbb{R}_+ \tag{4.4}$$

$$h_j(t) = \bar{h}_j, \ \forall t \in \mathbb{R}_+. \tag{4.5}$$

Furthermore, thanks to Eq (2.2), we have

$$f_j(\bar{h}_j, 0, \bar{v}) = 0, \tag{4.6}$$

thus $\bar{v}$ needs to be chosen in such a way that $g_j(\bar{v})$ exists for all $j \in \{1, ..n\}$ and that

$$\bar{h}_j = g_j(\bar{v}). \tag{4.7}$$

Therefore, the equilibrium flow of the traffic is given by,

$$(h_j(t), v_j(t)) = (g_j(\bar{v}), \bar{v}) \tag{4.8}$$

on a ring road having length

$$L = \sum_{j=1}^{n} g_j(\bar{v}). \tag{4.9}$$

Keeping in mind the preceding characterization of equilibrium flow, we are able to address the stability of the ring road traffic around such flows.

Definition 4.1 (Local stability around the equilibrium flow). *Let us consider an equilibrium flow, $\{(\bar{x}_j(t), \bar{v})\}_{j=1}^n$. The ring road traffic is said to be exponentially stable around this equilibrium flow, if there exist some constant $C > 0$ and $\lambda > 0$ such that, for any initial state, $(x_1(0), ..., x_n(0), v_1(0), ..., v_n(0))^T$ being sufficiently close to $(\bar{x}_1(0), ..., \bar{x}_n(0), \bar{v}, ..., \bar{v})^T$, the traffic satisfies*

$$|(x_1(t) - \bar{x}_1(t) - c, ..., x_n(t) - \bar{x}_n(t) - c, v_1(t) - \bar{v}, ..., v_n(t) - \bar{v})| \tag{4.10}$$

$$\leq Ce^{-\lambda t}|(x_1(0) - \bar{x}_1(0), ..., x_n(0) - \bar{x}_n(0), v_1(0) - \bar{v}, ..., v_n(0) - \bar{v})| \tag{4.11}$$

with some constant $c \in \mathbb{R}$ depending on the initial state. The constant c comes from the fact that an equilibrium flow defines a headway and a velocity, but the location of the cars is only defined up to a constant.

We remark here that this definition is equivalent to Definition 4.5 given below of the stability of the traffic in terms of headway-velocity.

Remark 4.1 (Ring road condition). *We remark that for fixed $\{f_j\}_{j=1}^n$ and L there is at most one equilibrium flow, i.e., at most one value of $\bar{v}$ such that Eq* (4.9) *holds. This means that the length of the road imposes the equilibrium flow and the steady velocity $\bar{v}$ and reciprocally that imposing a given speed $\bar{v}$ and a given system of n cars determines the length of the road. Since in reality we may want to study the limit where both n and L go to ∞, what really matters for us is the desired steady velocity. For this reason we do not fix the value of L in this paper but we fix instead the value of $\bar{v}$, which in turns, defines the value of L via Eq* (4.9). *This situation will be named as "ring road condition" in the following.*

Another important property of the "ring road condition" is that the value of L, Eq (4.9), *does not depend on the order of $\{f_j\}_{j=1}^n$, namely if $\{\tilde{f}_j\}_{j=1}^n = \{f_j\}_{j=1}^n$ up to some permutations then their lengths of ring road coincide.*

Remark 4.2 (Number of parameters describing the equilibrium flow). *Looking at Eq* (4.6)*, the stationary headway $\bar{h}_j = g_j(\bar{v})$ only depends on the driving characterization function f_j. This means that it is identical for all vehicles with the same class of parameters. In particular*

- *For unified model, $g_j(\bar{v})$ does not depend on $j \in \{1, 2, ..., n\}$. Thus, only 2 parameters describe the n-vehicles equilibrium flow: $\bar{v}$ and $g(\bar{v})$.*
- *For general collaborative driving as indicated in Eq* (2.6) *or Eq* (2.7) *and* (2.8)*, $g_j(\bar{v})$ has m different choices. Hence, the n-vehicles equilibrium flow is described by m+1 parameters.*
- *For the unified car Bando-FTL model of collaborative driving described by Eqs* (2.7) *and* (2.8) *with $V_j = V$, where the drivers have different driving habits but the same velocity preference, $g_j(\bar{v})$ does not depend on $j \in \{1, 2, ..., n\}$ and the equilibrium flow is still only described by two parameters, despite different driving habits. Actually, in this particular case, we observe from Eq* (2.7) *that $g_j(\bar{v}) = V^{-1}(\bar{v})$. As a direct consequence, the "ring road condition" Eq* (4.9) *simply becomes $L = nV^{-1}(\bar{v})$.*

4.2. Traffic around equilibrium flow

In this section we describe the linearized system around the equilibrium flows presented in the previous section. Let a general traffic model be given by Eqs (2.2) and (2.3). Let $(\bar{h}_i, \bar{v})$ be an equilibrium flow. From the previous Section, $(\bar{h}_i, \bar{v})$ satisfies (4.8) and (4.9). In particular, we remark that $(\bar{h}_i, \bar{v})$ is entirely parametrized by $\bar{v}$.

By denoting the perturbation in headway and velocity as

$$y_j(t) = h_j(t) - \bar{h}_j \text{ and } u_j(t) = v_j(t) - \bar{v}, \tag{4.12}$$

the traffic flow characterized by Eqs (2.2) and (2.3) can be written in terms of (y_j, u_j):

$$\begin{cases} \dot{y}_j = u_{j+1} - u_j, \\ \dot{u}_j = f_j(\bar{h}_j + y_j, u_{j+1} - u_j, \bar{v} + u_j), \end{cases} \tag{4.13}$$

and Eq (2.3) becomes

$$\sum_{j=1}^{n} y_j(t) = \sum_{j=1}^{n} h_j(t) - \sum_{j=1}^{n} \bar{h}_j = 0. \tag{4.14}$$

This condition is equivalent to say that the solutions of the preceding system always stay in the $(2n-1)$ dimensional subspace of $\mathbb{C}^2$:

$$\mathcal{H} := \left\{ (y_1, y_2, ..., y_n, u_1, u_2, ..., u_n) \in \mathbb{C}^{2n} : \sum_{k=1}^{n} y_k = 0 \right\}, \tag{4.15}$$

in particular, when the initial state takes value from $\mathcal{H} \cap \mathbb{R}^{2n}$, the solution also stays in $\mathcal{H} \cap \mathbb{R}^{2n}$. Here, we introduce complex valued spaces for the ease of notations when considering eigenvectors and eigenvalues.

Therefore the system (4.13) and (4.14) can be expressed as

$$\begin{cases} \dot{y}_j = u_{j+1} - u_j, \\ \dot{u}_j = f_j(\bar{h}_j + y_j, u_{j+1} - u_j, \bar{v} + u_j), \\ (y_j(t), u_j(t)) \in \mathcal{H}. \end{cases} \tag{4.16}$$

Linearized traffic around equilibrium flow

Next, standard linearization yields the linearized ring road traffic system

$$\begin{cases} \dot{y}_j = u_{j+1} - u_j, \\ \dot{u}_j = \alpha_j y_j - \beta_j u_j + \gamma_j u_{j+1}, \\ (y_j(t), u_j(t)) \in \mathcal{H}, \end{cases} \tag{4.17}$$

where $(\alpha_j, \beta_j, \gamma_j)$ (independent of time) are given by

$$(\alpha_j, \beta_j, \gamma_j) = \left(\frac{\partial f_j}{\partial h}, \frac{\partial f_j}{\partial \dot{h}} - \frac{\partial f_j}{\partial v}, \frac{\partial f_j}{\partial \dot{h}} \right) (g_j(\bar{v}), 0, \bar{v}) \tag{4.18}$$

which satisfies, from Eq (2.4),

$$\alpha_j > 0, \ \beta_j > \gamma_j > 0. \tag{4.19}$$

Form now on, for the ease of notations, we will denote by Λ_i the set of parameters describing the car following model f_i and Δ_j the quantity describing its stability around the equilibrium flow.

Definition 4.2. *For any given trio*

$$\Lambda_j := (\alpha_j, \beta_j, \gamma_j) \tag{4.20}$$

we define its discriminant as

$$\Delta_j := (\beta_j)^2 - (\gamma_j)^2 - 2\alpha_j. \tag{4.21}$$

The expression of Δ_j comes from [4] (see Theorem 3.1), it describes the stability of f_j around an equilibrium flow in a single-phase traffic, i.e., when all cars have the same car-following model $f := f_j$. What we will see in the following is that Δ_j is also a good indicator of the stability in a multi-phase traffic. For this reason, we introduce the following definition, inspired by Theorem 3.1,

Definition 4.3. *We classify a trio,* $\Lambda := (\alpha, \beta, \gamma) \in \mathbb{R}^3$ *satisfying the "common sense" condition* (3.2)*, by the value of its discriminant as follows. We say that* Λ *is*

- **stable***, if* $\Delta := \beta^2 - \gamma^2 - 2\alpha > 0$.
- **critical***, if* $\Delta = 0$.
- **unstable***, if* $\Delta < 0$.

Finally, for a general traffic model (2.2) with n cars satisfying the ring road condition (4.1), note that there is at most n different set of parameters Λ_j (one per car). However, if we restrict it into some m-phase mixed traffic model (2.6), where all the vehicles can be classified in m different types, then, thanks to Eqs (4.6)–(4.8), Λ_j only have at most m different values.

Let us now denote

$$z(t) = (y_1, ..., y_n, u_1, ...u_n)^T(t) \in \mathcal{H} \tag{4.22}$$

the linearized traffic system (4.17) becomes

$$\dot{z}(t) = M_n z(t), \quad z(t) \in \mathcal{H}, \tag{4.23}$$

with

$$M_n := \begin{pmatrix} O_n & A_n \\ C_n & B_n \end{pmatrix} \tag{4.24}$$

where

$$O_n := \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & \dots & \dots & \\ & & & & 0 \end{pmatrix}, \quad A_n := \begin{pmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & \dots & \dots & \\ 1 & & & & -1 \end{pmatrix}. \tag{4.25}$$

$$C_n := \begin{pmatrix} \alpha_1 & & & & \\ & \alpha_2 & & & \\ & & \alpha_3 & & \\ & & \dots & \dots & \\ & & & & \alpha_n \end{pmatrix}, \quad B_n := \begin{pmatrix} -\beta_1 & \gamma_1 & & & \\ & -\beta_2 & \gamma_2 & & \\ & & -\beta_3 & \gamma_3 & \\ & & \dots & \dots & \\ \gamma_n & & & & -\beta_n \end{pmatrix}. \tag{4.26}$$

The following definitions describe stability properties of both the linearized and nonlinear systems around equilibrium flows.

Definition 4.4 (Linearized stability around the equilibrium flow)**.** *Let us consider an equilibrium flow on the ring road:* $\{(\bar{h}_j, \bar{v})\}_{j=1}^n$. *The linearized ring road traffic* (4.17) *is said to be exponentially stable around this equilibrium flow, if there exist some* $\lambda > 0$ *and* $C > 0$ *such that for any initial state,* $z(0) = (y_1(0), ..., y_n(0), u_1(0), ..., u_n(0))^T \in \mathcal{H}$*, the solution of the Cauchy problem* (4.23) *(by recalling Eqs* (4.17)–(4.26)*) satisfies*

$$|z(t)| \leq Ce^{-\lambda t}|z(0)|, \forall t \in \mathbb{R}^+. \tag{4.27}$$

The preceding definition describes the stability of the linearized system (4.17). Actually, thanks to the standard linearization argument, when the linearized system is stable in the sense of Definition 4.4, the original nonlinear system (4.13) is automatically locally stable in $\mathcal{H}$ in the following sense:

Definition 4.5 (Local stability around the equilibrium flow: an alternative definition)**.** *Let us consider an equilibrium flow of the ring road:* $\{(\bar{h}_j, \bar{v})\}_{j=1}^n$. *The ring road traffic* (4.16) *is said to be locally exponentially stable around this equilibrium flow, if there exist some* $\varepsilon > 0$, $\lambda > 0$ *and* $C > 0$ *such that for any initial state,* $z(0) = (y_1(0), ..., y_n(0), u_1(0), ..., u_n(0))^T \in \mathcal{H}$*, satisfying*

$$|z(0)| \leq \varepsilon, \tag{4.28}$$

the solution of the Cauchy problem (4.13) *satisfies*

$$|z(t)| \leq Ce^{-\lambda t}|z(0)|, \forall t \in \mathbb{R}^+. \tag{4.29}$$

Let us remark that in the preceding two definitions, it is equivalent to express the stability in terms of position-velocity around the equilibrium flow, $\{(\bar{x}_j(t), \bar{v})\}_{j=1}^n$, like Definition 4.1.

Remark 4.3. *One can note that we do not look here at the well-posedness in general of the nonlinear system. The reason is that we look at the local stability around steady-states, and the well-posedness in this framework is immediate if the stability guaranteed. However, it would be interesting to know whether the nonlinear system is well-posed for any initial condition or any reasonable initial condition. When using Bando-Follow the leader model, the single population system has been shown to be well-posed as long as* h_j *are initially above a lower bound in [9] (for the IDM the situation is more complicated, one can look at [1] for instance). It would be interesting to investigate whether the same type of argument would work in the multi-population case.*

4.3. On the characterization of eigenvalues (counting multiplicity)

Since the local exponential stability of the nonlinear system (4.16) can be directly deduced from the exponential stability of the linearized system (4.23) we are going to focus on the latter. Using Routh–Hurwitz criterion the exponential stability of the linearized system (4.23) depends on the spectrum of the operator

$$\begin{aligned} \mathcal{L} : \mathcal{H} &\to \mathcal{H} \\ x &\mapsto M_n x. \end{aligned} \tag{4.30}$$

Therefore we investigate the spectrum of the matrix M_n on $\mathcal{H}$.

In this section we provide a two-step approach to find the eigenvalues of the matrix M_n. In particular, the second approach gives a full description of the multiplicity of the eigenvalues. We first look at the spectrum of M_n on $\mathbb{C}^{2n}$ and then see which eigenvalue remains when we restrict M_n to $\mathcal{H}$.

The spectrum of M_n on $\mathbb{C}^{2n}$.

A first method consists in considering the eigen-pairs (ω, z) of the matrix M_n:

$$M_n z = \omega z. \tag{4.31}$$

We know from the preceding equation that, for any $j \in \{1, 2, ..., n\}$,

$$\omega y_j = u_{j+1} - u_j, \tag{4.32}$$

$$\omega u_j = \alpha_j y_j - \beta_j u_j + \gamma_j u_{j+1}, \tag{4.33}$$

which further implies

$$(\omega^2 + \beta_j \omega + \alpha_j) u_j = (\gamma_j \omega + \alpha_j) u_{j+1}. \tag{4.34}$$

Therefore

$$\prod_{j=1}^{n} F_j(\omega) = 1, \quad F_j(\omega) := \frac{\gamma_j \omega + \alpha_j}{\omega^2 + \beta_j \omega + \alpha_j}, \tag{4.35}$$

which algebraically provides $2n$ solutions (with multiplicity). However, this approach does not yet give any information on the multiplicity of the eigenvalues: indeed, we only know that any eigenvalue ω should satisfy Eq (4.35). Thus, we use a second method to obtain this piece of information.

Recall that the full spectrum information of a matrix is given by the roots of the characteristic polynomial. Thus we shall this direct method to find all eigenvalues (counting multiplicity) of the matrix M_n. We are going to compute

$$\chi(\lambda) := \det \begin{pmatrix} \lambda I_n & -A_n \\ -C_n & \lambda I_n - B_n \end{pmatrix} \tag{4.36}$$

Notice that $\lambda I_n C_n = C_n \lambda I_n$, we know that

$$\det \begin{pmatrix} \lambda I_n & -A_n \\ -C_n & \lambda I_n - B_n \end{pmatrix} = \det\left((\lambda I_n)(\lambda I_n - B_n) - (-C_n)(-A_n)\right) = \det(\lambda^2 I_n - \lambda B_n - C_n A_n) = \det(\widetilde{M}_n), \tag{4.37}$$

where, $\widetilde{M}_n$ is given by

$$\begin{pmatrix} \lambda^2 + \beta_1 \lambda + \alpha_1 & -\gamma_1 \lambda - \alpha_1 & & & \\ & \lambda^2 + \beta_2 \lambda + \alpha_2 & -\gamma_2 \lambda - \alpha_2 & & \\ & & \lambda^2 + \beta_3 \lambda + \alpha_3 & -\gamma_3 \lambda - \alpha_3 & \\ & & \cdots & \cdots & \\ -\gamma_n \lambda - \alpha_n & & & & \lambda^2 + \beta_n \lambda + \alpha_n \end{pmatrix}. \tag{4.38}$$

By considering the first column of the matrix, its determinant read as

$$\begin{aligned} \chi(\lambda) = \det(\widetilde{M}_n) &= \prod_{i=1}^{n} (\lambda^2 + \beta_i \lambda + \alpha_i) + (-1)^{n-1} (-\gamma_n \lambda - \alpha_n) \prod_{j=1}^{n-1} (-\gamma_j \lambda - \alpha_j) \\ &= \prod_{i=1}^{n} (\lambda^2 + \beta_i \lambda + \alpha_i) - \prod_{j=1}^{n} (\gamma_j \lambda + \alpha_j). \end{aligned} \tag{4.39}$$

Therefore, all the eigenvalues are given by

$$\prod_{j=1}^{n} \frac{\gamma_j \lambda + \alpha_j}{\lambda^2 + \beta_j \lambda + \alpha_j} = 1, \tag{4.40}$$

which is coincident with Eq (4.35).

Remark 4.4 (Independence of the stability with the order of the cars)**.** *Similarly to the length of the road for a given steady-state (see Remark 4.1), again, we remark that the spectrum information of the traffic does not depend on the order of* $\{f_j\}_{j=1}^n$*. Namely, for a given desired steady velocity* $\bar{v}$*, if* $\{\tilde{f}_j\}_{j=1}^n = \{f_j\}_{j=1}^n$ *up to some permutations, then their stability around the related equilibrium flows coincide. This means that, in the mixed-traffic setting, the stability only depends on the penetration rate of the different types of cars.*

The spectrum of M_n on $\mathcal{H}$

By looking at Eq (4.40), it is easy to observe that $\lambda = 0$ is an eigenvalue of the matrix M_n. In the following, we prove that $\lambda = 0$ is a simple eigenvalue of the matrix M_n acting on $\mathbb{C}^{2n}$ however, it is not a eigenvalue of M_n acting on $\mathcal{H}$. This is an important point as, otherwise, we would not be able to deduce the stability of the system (4.23) from the eigenvalue analysis.

Let us start by showing that $\lambda = 0$ is a simple eigenvalue of the matrix M_n. By comparing the coefficients of the characteristic polynomial $\chi(\lambda)$ given by Eq (4.39), we immediately notice that $\chi(0) = 0$. Then it suffices to show that $\chi'(0) \neq 0$. Indeed, suppose that $\lambda = 0$ has (at least) multiplicity two, then the characteristic polynomial can be written as $\chi(\lambda) = \lambda^2 P(\lambda)$ where P is again a polynomial, and consequently $\chi'(0) = 0$. Proving that $\chi'(0) \neq 0$ is equivalent to prove that

$$\sum_{i=1}^{n} \beta_i \left(\frac{\prod_{k=1}^{n} \alpha_k}{\alpha_i} \right) \neq \sum_{i=1}^{n} \gamma_i \left(\frac{\prod_{k=1}^{n} \alpha_k}{\alpha_i} \right), \tag{4.41}$$

which is guaranteed by the "common sense" condition (4.19).

Next, we show that even though 0 is a simple eigenvalue of the matrix M_n, it is not included in the finite spectrum of the operator M_n acting on $\mathcal{H}$. Indeed, suppose by contradiction that there exists $z = (y_1, ..., y_n, u_1, ..., u_n)^T \in \mathcal{H}$ such that z is an eigenvector of M_n associated to the eigenvalue 0. We have

$$M_n z = 0. \tag{4.42}$$

As $z \neq 0$ and using Eqs (4.25) and (4.26) we deduce that there exists some $C \neq 0$ such that for any $j \in \{1, 2, ..., n\}$,

$$u_j = C, \ y_j = \frac{\beta_j - \gamma_j}{\alpha_j} C. \tag{4.43}$$

Without loss of generality, we assume that $C > 0$. By recalling the "common sense" condition (4.19), this implies that

$$\sum_{j=1}^{n} y_j > 0, \tag{4.44}$$

but as $z \in \mathcal{H}$ we know from Eq (4.15) that

$$\sum_{j=1}^{n} y_j = 0, \tag{4.45}$$

which leads to a contradiction. This implies that

$$\mathrm{Sp}_{\mathcal{H}}(M_n) \subseteq \mathrm{Sp}_{\mathbb{C}^{2n}}(M_n) \setminus \{0\}, \tag{4.46}$$

where $\mathrm{Sp}_{\mathcal{H}}(M_n)$ is the spectrum of M_n on $\mathcal{H}$ and $\mathrm{Sp}_{\mathbb{C}^{2n}}(M_n)$ is the spectrum of M_n on $\mathbb{C}^{2n}$. On the other hand, for any $\lambda \in \mathrm{Sp}_{\mathbb{C}^{2n}}(M_n) \setminus \{0\}$ we can find at least one related eigenvector $z \in \mathbb{C}^{2n}$. It is clear that $M_n z \in \mathcal{H}$, thus $z \in \mathcal{H}$. Therefore,

$$\mathrm{Sp}_{\mathcal{H}}(M_n) = \mathrm{Sp}_{\mathbb{C}^{2n}}(M_n) \setminus \{0\}. \tag{4.47}$$

5. Two-phase traffic flow

In this section, we study the stability of the equilibrium flows in a two-phase traffic flow. This situation represents for instance two class of vehicles such as trucks and cars, or also the coexistence of vehicles with and without a collaborative driving behavior. In the following, these two classes of vehicles will be called *Type 1 vehicles* and *Type 2 vehicles*. Let $\bar{v}$ be an equilibrium velocity, from Eq (4.8) this imposes the equilibrium headway $\bar{h}_1$ (*resp.* $\bar{h}_2$) of the *Type 1 vehicles* (*resp. Type 2 vehicles*). Thus, we are looking at a situation where, for every $j \in \{1, 2, ..., n\}$, using the notation (4.20) and (4.21),

$$\Lambda_j = (\alpha_j, \beta_j, \gamma_j) \in \{\Lambda^1, \Lambda^2\} = \{(\alpha^1, \beta^1, \gamma^1), (\alpha^2, \beta^2, \gamma^2)\}. \tag{5.1}$$

Let us denote by n_1 the number of *Type 1 vehicles* with parameters Λ^1 and n_2 the number of *Type 2 vehicles* with parameters Λ^2 such that the total number of vehicles is $n = n_1 + n_2$.

Suppose that the mixed traffic on road is represented by the "ordering" $(a_1, a_2, ..., a_n)$ that belongs to

$$\mathcal{K} := \left\{(a_1, a_2, ..., a_n) : a_k \in \{1, 2\}\, \forall 1 \le k \le n, \sum_{k=1}^{n} a_k = n_1 + 2n_2\right\}, \tag{5.2}$$

where $a_k \in \{1, 2\}$ implies that the k-th vehicle on the road is of *Type* a_k: because there is no lane changing in a single ring road, $(a_1, a_2, ..., a_n)$ is invariant with respect to time. Consequently, there is a unique equilibrium flow corresponding to $\bar{v}$ (or equivalently there is a unique equilibrium flow corresponding to L, from Remark 4.1): the headway before the k-th vehicle is given by $\bar{h}_{a_k}$. Furthermore, from Section 4.2, the linearized system around this equilibrium flow is

$$\begin{cases} \dot{y}_j = u_{j+1} - u_j, \\ \dot{u}_j = \alpha_j y_j - \beta_j u_j + \gamma_j u_{j+1}, \\ (\alpha_j, \beta_j, \gamma_j) = (\alpha^{a_j}, \beta^{a_j}, \gamma^{a_j}), \\ (y_j(t), u_j(t))_{j \in \{1,...,n\}} \in \mathcal{H}, \end{cases} \tag{5.3}$$

which can be further represented in forms of Eqs (4.23)–(4.26).

We introduce the following Lemma

Lemma 5.1. *Let given* $(n_1, n_2) \in \mathbb{N}^2$. *If the following inequality holds,*

$$\left(\frac{(\alpha^1)^2 + (\gamma^1)^2 x^2}{(\alpha^1)^2 + ((\beta^1)^2 - 2\alpha^1)x^2 + x^4}\right)^{n_1} \left(\frac{(\alpha^2)^2 + (\gamma^2)^2 x^2}{(\alpha^2)^2 + ((\beta^2)^2 - 2\alpha^2)x^2 + x^4}\right)^{n_2} < 1, \forall x \in \mathbb{R} \setminus \{0\}, \tag{5.4}$$

then System (5.2) *and* (5.3) *is exponentially stable in the sense of Definition 4.4.*

On the other hand, if for some $x \in \mathbb{R}$ *we have*

$$\left(\frac{(\alpha^1)^2 + (\gamma^1)^2 x^2}{(\alpha^1)^2 + ((\beta^1)^2 - 2\alpha^1)x^2 + x^4}\right)^{n_1} \left(\frac{(\alpha^2)^2 + (\gamma^2)^2 x^2}{(\alpha^2)^2 + ((\beta^2)^2 - 2\alpha^2)x^2 + x^4}\right)^{n_2} > 1, \tag{5.5}$$

then there exists some $M_0 \in \mathbb{N}$ *effectively computable such that for any integer* $M \geq M_0$, *the System* (5.2) *and* (5.3) *with* (n_1, n_2) *being replaced by* (Mn_1, Mn_2) *is unstable.*

Proof. This proof is essentially the same as the one given by [4, Section II] in a *unified models* framework (namely $n_2 = 0$). For readers' convenience we sketch its proof as follows.

By representing System (5.2) and (5.3) in form of (4.23)–(4.26), thank to Section 4.3, the eigenvalues (counting multiplicity) are explicitely characterized by Eq (4.40):

$$\left(\frac{\gamma^1\lambda + \alpha^1}{\lambda^2 + \beta^1\lambda + \alpha^1}\right)^{n_1} \left(\frac{\gamma^2\lambda + \alpha^2}{\lambda^2 + \beta^2\lambda + \alpha^2}\right)^{n_2} = 1. \tag{5.6}$$

Inspired by [4, Section II], we consider the following meromorphic function

$$G(z) = \left(\frac{\gamma^1 z + \alpha^1}{z^2 + \beta^1 z + \alpha^1}\right)^{n_1} \left(\frac{\gamma^2 z + \alpha^2}{z^2 + \beta^2 z + \alpha^2}\right)^{n_2}, \ \forall z \in \mathbb{C}. \tag{5.7}$$

Since all the poles are located on the left half plane, $G(z)$ is holomorphic on the right half plane $\mathbb{C}^+ = \{z \in \mathbb{C} : \Re(z) \geq 0\}$. We notice that $|G(z)|$ tends to 0 as $|z|$ tends to $+\infty$. Then, thanks to the maximum principle of holomorphic functions, the maximum of $|G(z)|$ over $\mathbb{C}^+$ must takes place at the imaginary axis. By considering $z = ix$ we get

$$|G(z)|^2 = \left(\frac{(\alpha^1)^2 + (\gamma^1)^2 x^2}{(\alpha^1)^2 + ((\beta^1)^2 - 2\alpha^1)x^2 + x^4}\right)^{n_1} \left(\frac{(\alpha^2)^2 + (\gamma^2)^2 x^2}{(\alpha^2)^2 + ((\beta^2)^2 - 2\alpha^2)x^2 + x^4}\right)^{n_2}, \tag{5.8}$$

which explains the inequality (5.4).

If Condition (5.4) is satisfied, then we know that $|G(z)| \leq 1$ in $\mathbb{C}^+$ with $|G(z)|$ equals to 1 only at $z = 0$. Therefore, all the eigenvalues are located in $\{z \in \mathbb{C} : \Re(z) < 0\} \cup \{0\}$, which, to be combined with Eq (4.47), yields the required exponential stability.

On the other hand, if Condition (5.5) is satisfied, then $|G(z)|$ is not always smaller than 1. As $n_1 + n_2 \neq 0$, without loss of generality we assume that $n_2 \neq 0$ and we define $\mu = n_1/n_2$. Condition (5.5) implies

$$\left(\frac{(\alpha^1)^2 + (\gamma^1)^2 x^2}{(\alpha^1)^2 + ((\beta^1)^2 - 2\alpha^1)x^2 + x^4}\right)^{\mu} \left(\frac{(\alpha^2)^2 + (\gamma^2)^2 x^2}{(\alpha^2)^2 + ((\beta^2)^2 - 2\alpha^2)x^2 + x^4}\right) > 1. \tag{5.9}$$

We denote

$$G_\mu(z) = \left(\frac{\gamma^1 z + \alpha^1}{z^2 + \beta^1 z + \alpha^1}\right)^{\mu} \left(\frac{\gamma^2 z + \alpha^2}{z^2 + \beta^2 z + \alpha^2}\right), \ \forall z \in \mathbb{C}. \tag{5.10}$$

We consider the curve $C \subset \mathbb{C}$:

$$C := \{z \in \mathbb{C} : |G_\mu(z)| = 1\}. \tag{5.11}$$

Let us further define an open subset of C by

$$C^+ := \{z \in C : \Re(z) > 0\}, \tag{5.12}$$

which is not empty thanks to Condition (5.9), the fact that $\lim_{\mathrm{Re}(z)\to+\infty} |G_\mu(z)| = 0$ and the continuity of G_μ. It is natural to consider the continuous function on C^+ defined as

$$\begin{aligned} F_1 \ &: C^+ \to \mathbb{S}^1 \\ &z \mapsto F_1(z) := G_\mu(z), \end{aligned} \tag{5.13}$$

where $\mathbb{S}^1$ denotes the unit circle in the complex plane $\mathbb{C}$. We can find a connected open set $O \subset \mathbb{S}^1$ such that

$$O \subset F_1(C^+). \tag{5.14}$$

By denoting the length of O as $|O|$, the value of M_0 can be chosen as

$$M_0 := \left[\frac{2\pi}{|O|}\right] + 1. \tag{5.15}$$

Indeed, for any $M \geq M_0$, we know from the definition of M that

$$\frac{2\pi}{M} < |O|. \tag{5.16}$$

Therefore, the set

$$\{e^{2k\pi/M} : k = 1, 2, ..., M\} \cap O \tag{5.17}$$

is not empty. We assume that for some k, the point $e^{2k\pi/M}$ belongs to O. Thus, by the definition of O there exists some $z_0 \in C^+$ such that

$$G_\mu(z_0) = F_1(z_0) = e^{2k\pi/M}. \tag{5.18}$$

Meanwhile, we recall that for the ring road traffic with $(M\mu)$ *Type 1 vehicles* and *M Type 2 vehicles* the stability of the System (5.2) and (5.3) is determined by the solutions of $G_M(z) = 1$:

$$G_M(z) = \left(\frac{\gamma^1 z + \alpha^1}{z^2 + \beta^1 z + \alpha^1}\right)^{M\mu} \left(\frac{\gamma^2 z + \alpha^2}{z^2 + \beta^2 z + \alpha^2}\right)^{M}. \tag{5.19}$$

The preceding equations immediately yield $G_M(z_0) = 1$ with $\Re(z_0) > 0$: the system is unstable. □

This lemma allows us to show the Theorems 3.2–3.4 that we reformulate here for convenience in Theorem 5.1,5.2:

Theorem 5.1. *Let given $\bar{v} > 0$. If $\Delta_1 \geq 0$ and $\Delta_2 \geq 0$, then for any $(n_1, n_2) \in \mathbb{N}^2$ and any ordering of the vehicles on the road, the ring road traffic system* (4.16) *is locally exponentially stable around the equilibrium flow.*

Proof of Theorem 5.1. The proof is straightforward. Indeed, by the definition of Δ_1 and the assumption that $\Delta \geq 0$ we know that for any $x \in \mathbb{R} \setminus \{0\}$ there is

$$\left(\frac{(\alpha^1)^2 + (\gamma^1)^2 x^2}{(\alpha^1)^2 + ((\beta^1)^2 - 2\alpha^1)x^2 + x^4}\right)^{n_1} \left(\frac{(\alpha^2)^2 + (\gamma^2)^2 x^2}{(\alpha^2)^2 + ((\beta^2)^2 - 2\alpha^2)x^2 + x^4}\right)^{n_2} < 1.$$

Similarly,

$$\left(\frac{(\alpha^2)^2 + (\gamma^2)^2 x^2}{(\alpha^2)^2 + ((\beta^2)^2 - 2\alpha^2)x^2 + x^4}\right)^{n_2} < 1,$$

thus Inequality (5.4) holds. The proof of this theorem concludes by applying Lemma 5.1. □

Remark 5.1. *Note that, in the critical case, i.e. one or both of Δ_1 and Δ_2 equals to 0, the system is still exponentially stable.*

Theorem 5.2. *Let given $\bar{v} > 0$. We assume that $\Delta_1 \geq 0$ and $\Delta_2 < 0$.*

(1) If $\Delta_1 > 0$ then there exists some effectively computable threshold constant $\tau_0 \in (0, 1)$ depending on Λ^1 and Λ^2 and given by Eq (3.8) *such that for any pair $(n_1, n_2) \in \mathbb{N}^2$ verifying*

$$\frac{n_1}{n_1 + n_2} > \tau_0, \tag{5.20}$$

the inequality (5.4) *is satisfied. In other words, for any ordering of the vehicles on the road $(a_1, a_2, ..., a_n) \in \mathcal{K}$, the ring road traffic system* (4.16) *is locally exponentially stable around the equilibrium flow associated to $\bar{v}$.*

On the other hand, for any penetration rate

$$\tau := \frac{n_1}{n_1 + n_2} < \tau_0 \tag{5.21}$$

there exists $M > 0$ such that for any $n_1, n_2 \in \mathbb{N}^2$ satisfying $n_1 + n_2 > M$, the ring road traffic system (2.2) *and* (2.3) *is unstable around the equilibrium flow $(\bar{h}_i, \bar{v})_{i \in \{1,2\}}$.*

(2) If $\Delta_1 = 0$ (namely, Λ^1 is critical), then for any penetration rate $\tau = n_1/(n_1 + n_2)$ there exists $M > 0$ effectively computable such that if $n > M$ the ring road traffic system (4.16) *is unstable around the equilibrium flow associated to $\bar{v}$.*

Finally, even though τ_0 can be easily calculated with the help of a minimization algorithm we show some simpler upper and lower bounds.

Corollary 5.3. *The critical penetration rate τ_0 defined in Theorem 5.2 satisfies*

$$\begin{aligned} &\tau_0 \geq \frac{-\Delta_2(\alpha^1)^2}{\Delta_1(\alpha^2)^2 - \Delta_2(\alpha^1)^2}, \\ \textit{and} \quad &\tau_0 \leq \frac{(-\Delta_2)\left((\alpha^1)^2 + (\gamma^1)^2\Gamma^2\right)\left((\alpha^1)^2 + ((\beta^1)^2 - 2\alpha^1)\Gamma^2 + (\Gamma^2)^2\right)}{(\beta^2)^2(\alpha^1)^2\Delta_1 + (-\Delta_2)\left((\alpha^1)^2 + (\gamma^1)^2\Gamma^2\right)\left((\alpha^1)^2 + ((\beta^1)^2 - 2\alpha^1)\Gamma^2 + (\Gamma^2)^2\right)}. \end{aligned} \tag{5.22}$$

As we can see that Theorem 3.2–3.4 in Section 3 are direct consequences of the more detailed Theorems 5.1,5.2, in the following we only give the proofs of the latter theorems.

Proof of Theorem 5.2. Note that it suffices to study the exponential stability of the linearized system (5.3) since the local exponential stability of the nonlinear system follows: At first we prove point (1) of this theorem. Looking at Lemma 5.1, it is thus sufficient to investigate Eq (5.4). Let us first get an intuition about what happens depending on the proportion of stable and unstable vehicle. To simplify the condition, we can set $y = x^2$ and $\mu = n_1/n_2$, namely $\mu/(1+\mu)$ is the proportion of stable vehicle in the traffic. The condition (5.4) becomes

$$\left(\frac{(\alpha^1)^2 + (\gamma^1)^2 y}{(\alpha^1)^2 + ((\beta^1)^2 - 2\alpha^1)y + y^2}\right)^{\mu}\left(\frac{(\alpha^2)^2 + (\gamma^2)^2 y}{(\alpha^2)^2 + ((\beta^2)^2 - 2\alpha^2)y + y^2}\right) < 1, \forall y \in \mathbb{R}_+^*. \tag{5.23}$$

We set

$$h_1(y) = \left(\frac{(\alpha^1)^2 + (\gamma^1)^2 y}{(\alpha^1)^2 + ((\beta^1)^2 - 2\alpha^1)y + y^2}\right), \forall y \in \mathbb{R}_+^*. \tag{5.24}$$

and we h_2 similarly. We observe that, for $i \in \{1, 2\}$,

$$h_i'(y) = \frac{-(\gamma^i)^2 y^2 - 2(\alpha^i)^2 y - (\alpha^i)^2 \Delta^i}{\left((\alpha^i)^2 + ((\beta^i)^2 - 2\alpha^i)y + y^2\right)^2}, \forall y \in \mathbb{R}_+^*. \tag{5.25}$$

where, we recall that $\Delta^i = (\beta^i)^2 - (\gamma^i)^2 - 2\alpha^i$. This means that h_i has at most two points where its derivative vanishes and these potential points are given by

$$y_\pm = -\frac{(\alpha^i)^2}{(\gamma^i)^2}\left(1 \mp \sqrt{1 - \frac{(\gamma^i)^2}{(\alpha^i)^2}\Delta^i}\right). \tag{5.26}$$

However, note that we are only interested in the values of h_i on $[0, +\infty)$. This gives some insight about what happens when Δ^i moves from a positive value (stable region) to a negative value (possibly unstable region): when Δ^i is positive there is no non-negative vanishing points of h_i', which means that h_i start at the value $h_i(0) = 1$ and then decrease strictly continuously until it reaches the limit $\lim_{y\to+\infty} h_i(y) = 0$. When Δ_i is negative, then y_+ is the only positive vanishing point of h' which means that h_i still starts at the value $h_i(0) = 1$ but increase strictly up to $y = y_+$ and becomes larger than 1. The critical case $\Delta^i = 0$ corresponds to the special case where $y_+ = 0$ and therefore h still decreases strictly on $[0, +\infty)$. Let us now prove (1) of Theorem 5.2.

Quantitative characterization of the optimal choice of τ_0

We define

$$H_i(y) = \log(h_i) = \log\left(\frac{(\alpha^i)^2 + (\gamma^i)^2 y}{(\alpha^i)^2 + ((\beta^i)^2 - 2\alpha^i)y + y^2}\right), i \in \{1, 2\}. \tag{5.27}$$

then for any given $y > 0$, (5.4) is equivalent to

$$\mu H_1(y) + H_2(y) < 0, \quad \forall\ y > 0. \tag{5.28}$$

Since $H_1(y) < 0\ \forall y > 0$, the preceding condition is equivalent to having

$$\mu > -\frac{H_2(y)}{H_1(y)}, \quad \forall\ y > 0. \tag{5.29}$$

Therefore, it leads us to introduce the quantity

$$K := \sup\left\{-\frac{H_2(y)}{H_1(y)} :\ y \in (0, +\infty)\right\}, \tag{5.30}$$

and to show that this quantity is finite. If so, it suffices to choose $\mu > K$ to conclude the exponential stability. We will show the following: define

$$N_0 = \max\left\{-\frac{H_2(y)}{H_1(y)} :\ y \in \left(0, \Gamma^2\right]\right\} < +\infty, \tag{5.31}$$

$$\Gamma^2 := \frac{-(\alpha^2)^2 + \sqrt{(\alpha^2)^4 - (\alpha^2)^2(\gamma^2)^2\Delta_2}}{(\gamma^2)^2} \in (0, -\Delta_2), \tag{5.32}$$

$$\tau_0 := \frac{N_0}{N_0 + 1}, \tag{5.33}$$

then $K = N_0 < +\infty$ and if

$$\frac{n_1}{n_1 + n_2} > \tau_0, \tag{5.34}$$

the condition (5.29) is satisfied and consequently the system is exponentially stable around the considered equilibrium flow.

From (5.25)

$$H_1'(y) = \frac{-(\gamma^1)^2 y^2 - 2(\alpha^1)^2 y - (\alpha^1)^2\Delta_1}{((\alpha^1)^2 + (\gamma^1)^2 y)\left((\alpha^1)^2 + ((\beta^1)^2 - 2\alpha^1)y + y^2\right)}, \forall y \in \mathbb{R}_+^*, \tag{5.35}$$

$$H_2'(y) = \frac{-(\gamma^2)^2 y^2 - 2(\alpha^2)^2 y - (\alpha^2)^2\Delta_2}{((\alpha^2)^2 + (\gamma^2)^2 y)\left((\alpha^2)^2 + ((\beta^2)^2 - 2\alpha^2)y + y^2\right)} \forall y \in \mathbb{R}_+^*. \tag{5.36}$$

And using this together with Eq (5.27), we deduce that

$$H_1(0) = H_2(0) = 0, \tag{5.37}$$

$$H_1(y) < 0, \quad H_1'(y) < 0, \quad \forall y \in (0, +\infty), \tag{5.38}$$

$$H_2(y) < 0, \quad \forall y \in (-\Delta_2, +\infty). \tag{5.39}$$

Concerning H_2 observe that we have the following key symmetry

$$H_2\left(\frac{(\alpha^2)^2(-\Delta_2 - y)}{(\alpha^2)^2 + (\gamma^2)^2 y}\right) = H_2(y), \quad \forall y \in [0, -\Delta_2]. \tag{5.40}$$

This implies, by recalling the definition of Γ^2 in Eq (5.32),

$$\sup\left\{-\frac{H_2(y)}{H_1(y)} :\ y \in \left(0, \Gamma^2\right]\right\} = \sup\left\{-\frac{H_2(y)}{H_1(y)};\ y \in (0, +\infty)\right\}, \tag{5.41}$$

or equivalently $N_0 = K$. Considering the fact H_2/H_1 is a continuous function, in order to prove that N_0 is bounded it only remains to show that there exists a finite limit as y tends to 0^+. From Eq (5.35),

$$\lim_{y \to 0^+} -\frac{H_2(y)}{H_1(y)} = -\frac{H_2'(0)}{H_1'(0)} = \frac{-\Delta_2(\alpha^1)^2}{\Delta_1(\alpha^2)^2} \in (0, +\infty), \tag{5.42}$$

which concludes that $N_0 < +\infty$. As $n_1 = \mu n_2$, Eq (5.34) is equivalent to $\mu > N_0$.

To show such a choice of τ_0 is optimal, is suffices to observe that if $\mu < N_0$ (or equivalently $n_1/(n_1 + n_2) < \tau_0$) then by continuity there exists a subset $[y_1, y_2] \subset (0, \Gamma^2]$ with $y_1 \neq y_2$ such that

$$\mu < -\frac{H_1(y)}{H_2(y)}, \quad \text{for any } y \in [y_1, y_2], \tag{5.43}$$

which implies that for any $y \in [y_1, y_2]$, $y > 0$ and

$$\left(\frac{(\alpha^1)^2 + (\gamma^1)^2 y}{(\alpha^1)^2 + ((\beta^1)^2 - 2\alpha^1)y + y^2}\right)^{\mu} \left(\frac{(\alpha^2)^2 + (\gamma^2)^2 y}{(\alpha^2)^2 + ((\beta^2)^2 - 2\alpha^2)y + y^2}\right) > 1. \tag{5.44}$$

Setting $x = \sqrt{y}$, Eq (5.44) together with Lemma 5.1 and Eq (5.5) allows us to conclude that there exists M large enough such that for any $n_1 > M$ and $n_2 > M$ satisfying $n_1/(n_1 + n_2) = \tau$, the system (4.16) is unstable around the equilibrium flow $(\bar{h}_i, \bar{v})_{i\in\{1,2\}}$.

□

Proof of Corollary 5.3. We have now proved the existence of the critical penetration rate τ_0 and we obtained a quantitative characterization. Note that τ_0 can be easily computed by a minimization algorithm using Eqs (5.31)–(5.33) and provides the optimal penetration rate of stable cars to stabilize the traffic. In the following, for the qualitative study convenience, we also present some lower and upper bounds of τ_0 (or equivalently, some lower and upper bounds of N_0).

A simple lower bound of τ_0.

Thanks to Eq (5.42), we see that

$$N_0 \geq \frac{-\Delta_2(\alpha^1)^2}{\Delta_1(\alpha^2)^2}, \tag{5.45}$$

hence a lower bound of τ_0 can be expressed by

$$B_l := \frac{-\Delta_2(\alpha^1)^2}{\Delta_1(\alpha^2)^2 - \Delta_2(\alpha^1)^2}. \tag{5.46}$$

Recall that $-\Delta_2$ is strictly greater than 0, so $B_l \in (0, 1)$.

A simple upper bound of τ_0.

The precise value of N_0 is given by Eq (5.31), but it is rather difficult to determine by hand. Indeed, to do so we would need to compare the extreme points of $-H_2/H_1$, which are given by

$$\mathcal{E} := \{z \in (0, \Gamma^2) : H_2'(z)H_1(z) - H_2(z)H_1'(z) = 0\}. \tag{5.47}$$

In terms of those extreme points, N_0 is further given by

$$N_0 = \max\left\{-\frac{H_2(y)}{H_1(y)} : y \in \mathcal{E} \cup \{0, \Gamma^2\}\right\}, \tag{5.48}$$

where

$$-\frac{H_2(0)}{H_1(0)} := -\frac{H_2'(0)}{H_1'(0)} = \frac{-\Delta_2(\alpha_1^1)^2}{\Delta_1(\alpha_1^2)^2}. \tag{5.49}$$

Actually, we can exclude Γ^2 in $\{0, \Gamma^2\}$ from the expression (5.48): suppose that $\Gamma^2 \in \mathcal{E}$ then $\mathcal{E}\cup\{0, \Gamma^2\} = \mathcal{E}\cup\{0\}$; otherwise, there exists some $y_0 \in (\Gamma^2-\delta, \Gamma^2+\delta)$ such that $-H_2(y_0)/H_1(y_0)$ is bigger than those of Γ^2, then thanks to the symmetric property of H_2 and the monotonous property fo H_1, Eqs (5.37)–(5.40), there exists some $y_1 \in (\Gamma^2 - \delta, \Gamma^2)$ such that $-H_2(y_1)/H_1(y_1)$ is bigger than those of Γ^2. Therefore,

$$N_0 = \max\left\{-\frac{H_2(y)}{H_1(y)} : y \in \mathcal{E} \cup \{0\}\right\}. \tag{5.50}$$

Next, notice that for any extreme point $z \in \mathcal{E}$ we have

$$-\frac{H_2(z)}{H_1(z)} = -\frac{H_2'(z)}{H_1'(z)}, \tag{5.51}$$

which, to be combined with Eq (5.49), yields

$$N_0 = \max\left\{-\frac{H_2'(y)}{H_1'(y)} : y \in \mathcal{E} \cup \{0\}\right\}. \tag{5.52}$$

We remark here that, though it is relatively easy to get the maximum value of $-H_2'/H_1'$ in $[0, \Gamma^2]$ (as its extreme points can be calculated explicitly via polynomials), this value is not necessarily equivalent to N_0. More precisely,

$$N_0 \le \max\left\{-\frac{H_2'(y)}{H_1'(y)} : y \in [0, \Gamma^2]\right\} =: \widetilde{N}_0. \tag{5.53}$$

The value of $\widetilde{N}_0$ can also be expressed in terms of extreme points:

$$\widetilde{N}_0 = \max\left\{-\frac{H_2'(y)}{H_1'(y)} : y \in \widetilde{\mathcal{E}}\right\}, \tag{5.54}$$

$$\widetilde{\mathcal{E}} := \{z \in [0, \Gamma^2] : H_2''(z)H_1'(z) - H_2'(z)H_1''(z) = 0\}. \tag{5.55}$$

In the following we present a simple upper bound for $\widetilde{N}_0$. Observe that $-H_2'(y)/H_1'(y)$ is characterized by

$$\frac{-(\gamma^2)^2y^2 - 2(\alpha^2)^2y - (\alpha^2)^2\Delta_2}{((\alpha^2)^2 + (\gamma^2)^2y)\,((\alpha^2)^2 + ((\beta^2)^2 - 2\alpha^2)y + y^2)} \cdot \frac{\left((\alpha^1)^2 + (\gamma^1)^2y\right)\left((\alpha^1)^2 + ((\beta^1)^2 - 2\alpha^1)y + y^2\right)}{(\gamma^1)^2y^2 + 2(\alpha^1)^2y + (\alpha^1)^2\Delta_1}, \tag{5.56}$$

while for $y \in [0, \Gamma^2]$ there are

$$(\gamma^1)^2y^2 + 2(\alpha^1)^2y + (\alpha^1)^2\Delta_1 \ge (\alpha^1)^2\Delta_1, \tag{5.57}$$

$$\left((\alpha^1)^2 + (\gamma^1)^2y\right)\left((\alpha^1)^2 + ((\beta^1)^2 - 2\alpha^1)y + y^2\right) \le \left((\alpha^1)^2 + (\gamma^1)^2\Gamma^2\right)\left((\alpha^1)^2 + ((\beta^1)^2 - 2\alpha^1)\Gamma^2 + (\Gamma^2)^2\right), \tag{5.58}$$

$$-(\gamma^2)^2y^2 - 2(\alpha^2)^2y - (\alpha^2)^2\Delta_2 \le -(\alpha^2)^2\Delta_2, \tag{5.59}$$

$$\left((\alpha^2)^2 + (\gamma^2)^2y\right)\left((\alpha^2)^2 + ((\beta^2)^2 - 2\alpha^2)y + y^2\right) \ge (\alpha^2)^2(\beta^2)^2. \tag{5.60}$$

Consequently,

$$\widetilde{N}_0 \le \frac{(-\Delta_2)\left((\alpha^1)^2 + (\gamma^1)^2\Gamma^2\right)\left((\alpha^1)^2 + ((\beta^1)^2 - 2\alpha^1)\Gamma^2 + (\Gamma^2)^2\right)}{(\beta^2)^2(\alpha^1)^2\Delta_1}, \tag{5.61}$$

which further implies the following upper bound of τ_0:

$$B_u := \frac{(-\Delta_2)\Big((\alpha^1)^2 + (\gamma^1)^2\Gamma^2\Big)\Big((\alpha^1)^2 + ((\beta^1)^2 - 2\alpha^1)\Gamma^2 + (\Gamma^2)^2\Big)}{(\beta^2)^2(\alpha^1)^2\Delta_1 + (-\Delta_2)\left((\alpha^1)^2 + (\gamma^1)^2\Gamma^2\right)\left((\alpha^1)^2 + ((\beta^1)^2 - 2\alpha^1)\Gamma^2 + (\Gamma^2)^2\right)}. \tag{5.62}$$

Let us now prove point (2) of Theorem 5.2. We define again $\mu = n_1/n_2$. Suppose that $\Delta_1 = 0$ and $\Delta_2 < 0$. Thanks to Eq (5.35), we know that

$$\mu H_1'(0) + H_2'(0) = -\frac{\Delta_2}{(\alpha^2)^2} > 0, \tag{5.63}$$

which, to be combined with the fact that $\mu H_1(0) + H_2(0) = 0$, imply the existence of $y > 0$ such that

$$\mu H_1(y) + H_2(y) > 0. \tag{5.64}$$

As a direct consequence, the system is not stable.

□

6. Multi-phase collaborative driving

In this section, analogy to the preceding Section, we study the stability of the equilibrium flows in a multi-phase mixed traffic flow. For any given equilibrium velocity $\bar{v}$, keeping the notation Λ_j and Eq (4.21), we have that

$$\Lambda_j = (\alpha_j,\beta_j,\gamma_j) \in \{\Lambda^1, ..., \Lambda^m\} = \{(\alpha^1,\beta^1,\gamma^1), ..., (\alpha^m,\beta^m,\gamma^m)\}. \tag{6.1}$$

Again, for $k \in \{1, 2, ..., m\}$ we denote by n_k the number of *Type k vehicles* with parameters Λ^k such that the total number of vehicles is $n = \sum_{j=1}^m n_j$.

For any ordering of the vehicle on road, i.e., the j-th vehicle is of *Type* a_j, the linearized system around the unique equilibrium flow is

$$\begin{cases} \dot{y}_j = u_{j+1} - u_j, \\ \dot{u}_j = \alpha_j y_j - \beta_j u_j + \gamma_j u_{j+1}, \\ (\alpha_j,\beta_j,\gamma_j) = (\alpha^{a_j},\beta^{a_j},\gamma^{a_j}), \\ (y_j(t), u_j(t))_{j\in\{1,...,n\}} \in \mathcal{H}. \end{cases} \tag{6.2}$$

Similar to Theorem 5.1 and Theorem 5.2 we have the following theorem concerning the stability of the m-phase mixed ring road traffic.

Theorem 6.1. *Let given $\bar{v} > 0$. We assume without loss of generality that $\Delta_1 \geq \Delta_2 \geq ... \geq \Delta^m$.*

(1) If $\Delta^m \geq 0$, then for any $(n_1, n_2, n_3, ..., n_m) \in \mathbb{N}^m$ and any ordering of the vehicles on the road, the ring road traffic system (4.16) *is locally exponentially stable around the equilibrium flow.*

(2) *If* $\Delta_1 > 0$ *and* $\Delta^m < 0$*, then there exists some effectively computable threshold constant* $\tau_1 \in (0,1)$ *depending on* $\{\Lambda^1, ..., \Lambda^m\}$ *such that for any m-tuple* $(n_1, n_2, ..., n_m) \in \mathbb{N}^2$ *verifying*

$$\frac{n_1}{\sum_{k=1}^m n_k} > \tau_1, \tag{6.3}$$

in other words, for any ordering of the vehicles on the road $(a_1, a_2, ..., a_n) \in \mathcal{K}$*, the ring road traffic system* (4.16) *is locally exponentially stable around the equilibrium flow associated to* $\bar{v}$*.*
On the other hand, we further assume that for some $p \in \{1, ..., m-1\}$ *there is* $\Delta^p > 0 \geq \Delta^{p+1}$*. Then, there exists some* $\tau_2 \in (0,1)$ *effectively computable such that for any fixed penetration rates*

$$\left(\frac{n_1}{\sum_{j=1}^m n_j}, ..., \frac{n_m}{\sum_{j=1}^m n_j}\right) \ \textit{satisfying} \ \frac{\sum_{j=1}^P n_j}{\sum_{j=1}^m n_j} < \tau_2 \tag{6.4}$$

there exists $M > 0$ *such that for any* $n_1, ..., n_m \in \mathbb{N}^m$ *satisfying* $\sum_{k=1}^m n_k > M$*, the ring road traffic system* (4.16) *is unstable around the equilibrium flow associated to* $\bar{v}$*.*

(3) *If* $\Delta_1 = 0$ *(namely,* Λ^1 *is critical), then for any fixed penetration rate of the cars the ring road traffic system* (4.16) *is unstable around the equilibrium flow associated to* $\bar{v}$ *provided sufficiently many cars on road.*

Indeed, similar to the two-phase traffic, we look at the ring road traffic with m populations having respectively $(n_1, n_2, ..., n_m)$ many vehicles. The local exponential stability of this system is still equivalent to the study of the real part of eigenvalues of the linearized system that are explicitly given by the roots of the polynomial (4.35) presented in Section 4.3:

$$\prod_{j=1}^n F_j(\omega) = 1, \quad F_j(\omega) := \frac{\gamma_j \omega + \alpha_j}{\omega^2 + \beta_j \omega + \alpha_j}. \tag{6.5}$$

It becomes more delicate to calculate the exact roots of the polynomial. Instead, again, we investigate the value of the polynomial on the imaginary axis. The question becomes:

$$\text{whether} \ \prod_{k=1}^m \left(\frac{(\alpha^k)^2 + (\gamma^k)^2 x^2}{(\alpha^k)^2 + ((\beta^k)^2 - 2\alpha^k)x^2 + x^4}\right)^{n_k} < 1 \ \text{ holds for any } \ x \in \mathbb{R} \setminus \{0\}? \tag{6.6}$$

Thanks to the same reasoning as in Lemma 5.1, we get the following conclusion.

- If Condition (6.6) is satisfied, then all the roots of the polynomial excluding 0 are distributed on the left complex region i.e., $\{z \in \mathbb{C} : \Re(z) < 0\}$. Hence, the linearized system is exponentially stable.
- If Condition (6.6) is not verified, and if further there exists some $x_0 \in \mathbb{R}$ such that the value of the function in (6.6) is strictly larger than 1, then the ring road traffic system with these penetration rates of vehicles is not stable provided sufficiently many cars. This is obtained using the same reasoning as in the two population case (see Eqs (5.9)–(5.19)).

Recalling Definitions 4.2, 4.3 concerning the classification of Δ^k, we know that

- if $\Delta^k > 0$, then

$$\left(\frac{(\alpha^k)^2 + (\gamma^k)^2 x^2}{(\alpha^k)^2 + ((\beta^k)^2 - 2\alpha^k)x^2 + x^4}\right)^{n_k} < 1, \forall x \in \mathbb{R} \setminus \{0\}; \tag{6.7}$$

- if $\Delta^k = 0$, then Inequality (6.7) is also satisfied;
- if $\Delta^k < 0$, then

$$\left(\frac{(\alpha^k)^2 + (\gamma^k)^2 x^2}{(\alpha^k)^2 + ((\beta^k)^2 - 2\alpha^k)x^2 + x^4}\right)^{n_k} > 1, \text{ for some } x \in \mathbb{R} \setminus \{0\}. \tag{6.8}$$

This observation, together with Condition (6.6), finally leads to Theorem 6.1 in a way that is very similar to the proofs of Theorem 5.1, 5.2. The easier cases (1) and (3) are direct consequences of the preceding observations. Concerning the mixed case (2) such that both stable and unstable vehicles coexist, from a heuristic point of view the more stable cars there are on the road, the more likely it is that Condition (6.6) is satisfied, i.e., that for any ordering of the vehicles on the road $(a_1, a_2, ..., a_n) \in \mathcal{K}$, the ring road traffic system (4.16) is locally exponentially stable around the equilibrium flow associated to $\bar{v}$. Otherwise with fewer stable cars on road the unstable cars will dominate the traffic to prevent us from getting Condition (6.6). This comes from the fact that we get the following condition for the stability instead of Eq (5.28)

$$\sum_{i=1}^{n} n_i H_i(y) < 0, \quad \forall y \geq 0, \tag{6.9}$$

where $H_i(y) = \log\left(\frac{(\alpha^i)^2+(\gamma^i)^2 y}{(\alpha^i)^2+((\beta^i)^2-2\alpha^i)y+y^2}\right)$ is defined similarly as in Eq (5.27); in this specific case there is

$$H_i(0) = 0 \text{ and } H_i'(0) < 0, \forall i \in \{1, 2, ..., p\}, \tag{6.10}$$

$$H_i(0) = 0 \text{ and } H_i'(0) > 0, \forall i \in \{p+1, p+2, ..., m\}. \tag{6.11}$$

7. Numerical experiments

In this Section we present some numerical experiments to illustrate Theorems 3.2–3.4 with two type of populations described by some set of parameters Λ_1 and Λ_2.

We assume for this example that the vehicles are described by the nonlinear Bando-FTL model (2.5) and we consider two populations described by the parameters (a_1, b_1) and (a_2, b_2). We assume that they have the same velocity preference V given by

$$V(h) = V_{\max} \frac{\tanh(\frac{h-l_v}{d_0} - 2) + \tanh(2)}{1 + \tanh(2)}, h \in (0, +\infty), \tag{7.1}$$

which is a usual choice for Bando-FTL [2]. The quantity $V_{\max}$ corresponds to the maximal velocity preference for a given car. It corresponds to the inner speed limit of a given driver with a given car (which can be different from the official speed limit). l_v is the length of the vehicle, while $2d_0$ is a safety distance below which the feeling of unsafeness would make the driver wants to stop. Here, their values are respectively taken as: $V_{\max} = 9.25\ m.s^{-1}$, $l_v = 4.5\ m$ and $d_0 = 2.5$. In all the numerical simulations the cars are initially placed at an equal distance on the road with a random ordering between the two types of drivers. The initial velocity is close to half the velocity preference with a small random perturbation (between 0 and 0.3 $m.s^{-1}$).

We study a case where the first population has a very stable behavior on the road with $\Delta_1 > 0$ while the second population is made of slightly more aggressive driver that have a slightly unstable behavior such that $\Delta_2 < 0$ but $|\Delta_2| < |\Delta_1|$. To do so, we choose the parameters $a_1 = 4$, $b_1 = 20$, $a_2 = 0.5$, $b_2 = 20$, thus the instability of the second population will simply come from a higher sensitivity to the velocity preference rather than the "Follow-the-leader" behavior. We choose L and N such that $L/N = 10.4\ m$, which is a steady-state value similar to the setting of the real life experiment described in [17]. Note that parameters (a_2, b_2) are typical values that were obtained in [16] after calibration on data from real life experiments. From Theorem 5.2 we deduce the following:

$$\Delta_1 = 7.28, \quad \Delta_2 = -0.84, \quad \tau_0 = 0.881 \tag{7.2}$$

as we can see, although $|\Delta_1|$ is one order of magnitude above $|\Delta_2|$, the penetration rate of stable vehicle needed for having a stable flow is very high and above 88%. This means that even a very low proportion of slightly more aggressive drivers in a large road can completely destabilize a traffic that would be very stable otherwise. In Figure 1 (left) we show the instantaneous speed variance across drivers of a traffic flow with 500 cars, a penetration rate of stable cars of 80% ($\tau = 0.802$), and we see that the system becomes quickly unstable as the speed variance only increase during the entire simulation. In Figure 1 (right) we show the instantaneous speed variance of the same traffic when the penetration rate of stable car is 88.2% instead and we see that the speed variance, already low at initial time, decreases exponentially fast. Here we define the instantaneous speed variance across drivers as the variance of the drivers' velocity at a given time t, i.e., the quantity

$$V_s(t) = \frac{1}{N}\sum_{i=1}^{N}\left(v_i(t) - \frac{1}{N}\sum_{i=j}^{N} v_j(t)\right)^2. \tag{7.3}$$

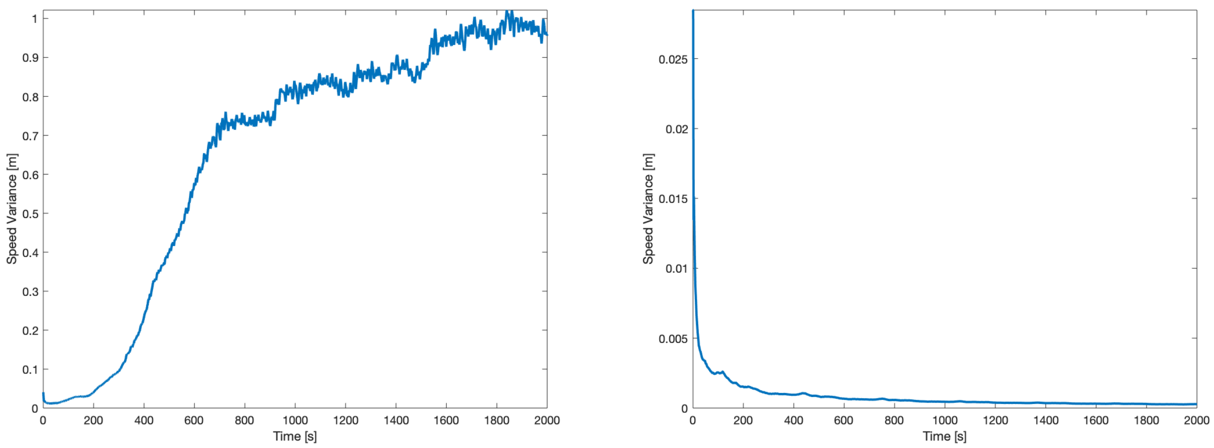

Figure 1. Speed variance in $m^2.s^{-2}$ over time in a two population traffic flow: drivers with stable behaviour ($\Delta_1 = 7.28$) and drivers with unstable behavior ($\Delta_2 = -0.84$). Predicted critical penetration rate: $\tau_0 = 0.881$. **Left:** 80% drivers with stable behavior; **Right:** 88.2% drivers with stable behavior.

However, as highlighted in Theorem 3.3, this instability might only occur with a large enough number of cars and might be missed when looking at experiments with a small number of cars such as [17,18] rather than a real freeway with sufficiently many cars. To get some insight on this phenomenon, we ran several numerical experiments where we kept the same steady state described by $L/N = 10.4$ m, hence the same $\tau_0 = 0.881$, but we varied the number of cars N from 10 to 120. Moreover, for each number of cars N we also ran several simulations with different proportions of cars exhibiting a stable behavior. In Figure 2 we show for each fixed number of cars N, the minimal penetration rate of cars with stable behavior above which we observe a terminal instantaneous speed variance of the system (i.e., the variance of speeds taken across all cars of the system at the given final time) below 0.01 $m^2.s^{-2}$ after 2000s of simulations. Such a negligible terminal speed variance indicates the strong stability of the whole system while a non-negligible speed variance indicates that the cars are not in the uniform flow equilibrium at final time. We see that the effective penetration rate of stable cars above which the total system is stable is significantly lower than τ_0 for small number of cars, and this value becomes quickly close to the theoretical value $\tau_0 = 0.881$ above $N = 40$ (to be compared with the values $N \sim 20$ of [17,18]).

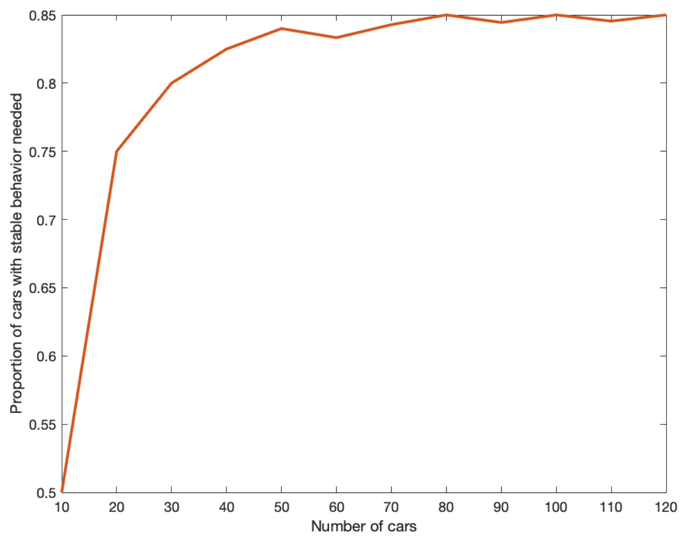

Figure 2. Minimal penetration rate of cars with stable behavior to have a terminal instantaneous speed variance of the system (i.e., taken across cars at the final time and not across time) below 0.01 $m^2.s^{-2}$ after 2000 s.

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Conflict of interest

The authors declare there is no conflict of interest.

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